Tôhoku Math. J.
43 (1991), 537–555

A TRANSPLANTATION THEOREM FOR LAGUERRE SERIES

YUICHI KANJIN

(Received July 20, 1990, revised January 4, 1991)

1. Introduction. Let $L_n^\alpha(x)$, $\alpha > -1$, be the Laguerre polynomial of degree n and of order α defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \left(\frac{d}{dx} \right)^n (e^{-x} x^{n+\alpha}) .$$

Then the functions $\tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2}$, $n = 0, 1, 2, \cdots$, are orthonormal on the interval $(0, \infty)$ with respect to the ordinary Lebesgue measure dx, where

$$(\tau_n^\alpha)^{-2} = \int_0^\infty \{L_n^\alpha(x)\}^2 e^{-x} x^\alpha \, dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} .$$

This orthonormal system leads us to the formal expansion of a function $f(x)$ on $(0, \infty)$:

$$f(x) \sim \sum_{n=0}^\infty a_n^\alpha(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} ,$$

where $a_n^\alpha(f)$ is the n-th Fourier-Laguerre coefficient of order α of $f(x)$ defined by

$$a_n^\alpha(f) = \int_0^\infty f(x) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} \, dx .$$

We note that the integral converges and $a_n^\alpha(f)$ is finite if $\alpha \geq 0$ and $1 \leq p \leq \infty$, or if $-1 < \alpha < 0$ and $(1+\alpha/2)^{-1} < p \leq \infty$.

Our theorem is as follows:

THEOREM. *Let $\alpha, \beta > -1$ and $\gamma = \min\{\alpha, \beta\}$. If $\gamma \geq 0$, then*

$$\int_0^\infty \left| \sum_{n=0}^\infty a_n^\beta(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} \right|^p dx \leq C \int_0^\infty |f(x)|^p \, dx \tag{1.1}$$

for $1 < p < \infty$, where C is a constant independent of f. If $-1 < \gamma < 0$, then (1.1) *holds for $(1+\gamma/2)^{-1} < p < -2/\gamma$.*

Historically, Guy [11] proved a transplantation theorem for Hankel transforms. Schindler [14] proved Guy's theorem showing an explicit integral representation. For other classical expansions, Askey and Wainger [3], [4] gave transplantation theorems for ultraspherical coefficients and its dual. Furthermore, Askey [1], [2] generalized their theorems to Jacobi polynomial expansions. In [1], he also proved a transplantation theorem [1, Theorem 3] for Laguerre coefficients of orders α and $\beta=\alpha+2$. Our theorem is the dual to his theorem with arbitrary β. Some other related transplantation theorems are found in Gilbert [10] and in Muckenhoupt [13].

An advantage of our transplantation theorem is that, if a norm inequality of Laguerre series of some order α is proved, then the corresponding norm inequality of any other order β holds automatically. We give an application. Let $\Lambda=\{\lambda_n\}_{n=0}^{\infty}$ be a bounded sequence. We define a multiplier operator $\mathscr{M}_{\Lambda}^{\alpha}$ with multiplier Λ by $a_n^{\alpha}(\mathscr{M}_{\Lambda}^{\alpha}(f))=\lambda_n a_n^{\alpha}(f)$, $n=0, 1, 2, \cdots$, that is,

$$\mathscr{M}_{\Lambda}^{\alpha}(f)\sim\sum_{n=0}^{\infty}\lambda_n a_n^{\alpha}(f)\tau_n^{\alpha}L_n^{\alpha}(x)e^{-x/2}x^{\alpha/2}\,.$$

We denote by L^p the Lebesgue space of all measurable functions $f(x)$ defined on $(0, \infty)$ such that $\|f\|_p=\{\int_0^{\infty}|f(x)|^p dx\}^{1/p}<\infty$. We may denote $\|f\|_p$ by $\|f(x)\|_p$.

COROLLARY. *Let $\lambda(x)$ be a four times differentiable function on $(0, \infty)$ satisfying $\sup_{x>0}|\lambda^{(j)}(x)x^j|\leq B$ $(j=0, 1, 2, 3, 4)$, and let $\Lambda=\{\lambda(s(2n+1))\}_{n=0}^{\infty}$ for $s>0$. Then, $\|\mathscr{M}_{\Lambda}^{\alpha}(f)\|_p\leq CB\|f\|_p$ $(f\in L^p)$ if $\alpha\geq 0$ and $1<p<\infty$, or if $-1<\alpha<0$ and $(1+\alpha/2)^{-1}<p<-2/\alpha$, where C is a constant depending only on α and p.*

The corollary is obtained instantly by applying our theorem to the following result due to Długosz [6].

(A) *Długosz's criterion (cf. [6, §1]). Let $\lambda(x)$ and Λ be a function and a sequence given in the corollary. If $\alpha=0, 1, 2, \cdots$, then $\|\mathscr{M}_{\Lambda}^{\alpha}(f)\|_p\leq CB\|f\|_p$ $(f\in L^p)$ for $1<p<\infty$, where C is a constant depending only on p.*

We use this criterion to prove our main theorem.

In §2, we shall extend the parameter β of $T_{\alpha}^{\beta}(f)$ to complex β, where $T_{\alpha}^{\beta}(f)$ is the function defined by the series $\sum_{n=0}^{\infty}a_n^{\beta}(f)\tau_n^{\alpha}L_n^{\alpha}(x)e^{-x/2}x^{\alpha/2}$. Using (A), we shall reduce the estimate of the L^p-norm of $T_{\alpha}^{\beta}(f)$ to that of the operator $T_{\alpha,\varphi}^{\beta}(f)$ defined in (2.8) which is easier to treat. We extend the parameter β to complex numbers $\beta=\alpha+k+i\theta$, $k=0, 2$, $-\infty<\theta<\infty$, and apply an interpolation theorem. In §3, we shall estimate the L^p-norm of $T_{\alpha,\varphi}^{\alpha+i\theta}(f)$. To do so, we deal with $T_{\alpha,\varphi}^{\alpha+\varepsilon+i\theta}(f)$, $\varepsilon>0$. We shall modify $T_{\alpha,\varphi}^{\alpha+\varepsilon+i\theta}(f)$. The essential part of our proof is to use the formula

$$L_n^{\alpha+\varepsilon+i\theta}(y)=\frac{\Gamma(n+\alpha+\varepsilon+i\theta+1)}{\Gamma(\varepsilon+i\theta)\Gamma(n+\alpha+1)}\int_0^1 v^{\alpha}(1-v)^{\varepsilon-1+i\theta}L_n^{\alpha}(vy)\,dv\,,$$

and estimate the L^p-norm of the operator by the singular integral operator theory and Hardy's inequality. In §4, the L^p norm of $T_{\alpha,\varphi}^{\alpha+2+i\theta}(f)$ will be evaluated by an argument similar to that of §3.

2. Reduction. We extend the definition of the n-th Fourier-Laguerre coefficient

$a_n^\beta(f)$ to complex β as follows. By the explicit representation

$$L_n^\beta(x)=\sum_{k=0}^{n}\binom{n+\beta}{n-k}\frac{(-x)^k}{k!},$$

the definition of the Laguerre polynomial is extended to complex β. $L_n^\beta(x)$ is analytic in β except at the points $\beta=-n-1,\ -n-2,\cdots$ for fixed x. The coefficient $\tau_n^\beta=\{\Gamma(n+1)/\Gamma(n+\beta+1)\}^{1/2}$ is analytic in β in the cut plane $|\arg(\beta+n+1)|<\pi$, where we take the branch of the square root equal to $+1$ for $\beta=0$. Let C_c^∞ be the space of infinitely differentiable functions with compact support in $(0,\infty)$. For $f\in C_c^\infty$, the definition of $a_n^\beta(f)$ is extended to complex β and is analytic in $|\arg(\beta+n+1)|<\pi$.

LEMMA 1. *Let $f\in C_c^\infty$. Let $\alpha>-1$ and $\Delta>0$. Then, for every $j=1,2,3,\cdots$, there are a constant C and a number n_0 such that*

$$|a_n^\beta(f)|\le C(1+|\theta|^j)e^{\pi|\theta|/2}n^{(\Delta-j)/2+1/4} \tag{2.1}$$

for $n\ge n_0$, $-\infty<\theta<\infty$ and $0\le\delta\le\Delta$, where $\beta=\alpha+\delta+i\theta$.

PROOF. By the formula (cf. [7, 10.12 (28)])

$$L_n^\beta(y)e^{-y}y^\beta=\frac{(n-j)!}{n!}\left(\frac{d}{dy}\right)^j\{L_{n-j}^{\beta+j}(y)e^{-y}y^{\beta+j}\},$$

we have

$$a_n^\beta(f)=\tau_n^\beta\frac{(n-j)!}{n!}\int_0^\infty f(y)\ e^{y/2}y^{-\beta/2}\left(\frac{d}{dy}\right)^j\{L_{n-j}^{\beta+j}(y)e^{-y}y^{\beta+j}\}\,dy.$$

By integration by parts

$$a_n^\beta(f)=\tau_n^\beta\frac{(n-j)!}{n!}(-1)^j\int_0^\infty\left\{\left(\frac{d}{dy}\right)^j(f(y)e^{y/2}y^{-\beta/2})\right\}L_{n-j}^{\beta+j}(y)e^{-y}y^{\beta+j}dy.$$

Since f is a function with compact support in $(0,\infty)$, we may assume $\operatorname{supp} f\subset[a,b]$, $0<a<b<\infty$. Thus

$$|a_n^\beta(f)|\le C(1+|\theta|^j)|\tau_n^\beta|n^{-j}\int_a^b|L_{n-j}^{\beta+j}(y)|\,dy,$$

where C is a constant independent of n and θ, and is bounded in $0\le\delta\le\Delta$. We apply the formula (cf. [7, 10.12 (30)])

$$L_m^{\mu+\nu}(y)=\frac{\Gamma(m+\mu+\nu+1)}{\Gamma(\nu)\Gamma(m+\mu+1)}\int_0^1 v^\mu(1-v)^{\nu-1}L_m^\mu(vy)\,dv, \tag{2.2}$$

$\operatorname{Re}\mu>-1$, $\operatorname{Re}\nu>0$ with $\mu=\alpha+j-1$, $\nu=1+\delta+i\theta$ and $m=n-j$ to the integrand $L_{n-j}^{\beta+j}(y)$. Then, we have

(2.3) $$|a_n^\beta(f)|\le\frac{C(1+|\theta|^j)}{|\Gamma(1+\delta+i\theta)|}n^{-j}A_{n,\beta}\int_a^b\int_0^1 v^{\alpha+j-1}|L_{n-j}^{\alpha+j-1}(vy)|\,dv\,dy\,,$$

where $A_{n,\beta}=|\tau_n^\beta|\,|\Gamma(n+\beta+1)|/|\Gamma(n+\alpha)|$. Since $\Gamma(1+\delta+i\theta)^{-1}=B(1/2+\delta,1/2+i\theta)\cdot\{\Gamma(1/2+\delta)\Gamma(1/2+i\theta)\}^{-1}$ and $|\Gamma(1/2+i\theta)|^2=\pi/\cosh\pi\theta$, we have

(2.4) $$\frac{1}{|\Gamma(1+\delta+i\theta)|}\le\frac{B(1/2,1/2)}{\inf_{x\ge1/2}\Gamma(x)}\left\{\frac{\cosh\pi\theta}{\pi}\right\}^{1/2}\le Ae^{\pi|\theta|/2}$$

for $-\infty<\theta<\infty$ and $\delta\ge0$, where A is an absolute constant. We estimate $A_{n,\beta}$. We have

$$A_{n,\beta}=\left\{\frac{|\Gamma(n+1)\Gamma(n+\alpha+\delta+1)|}{|\Gamma(n+\alpha)|^2}\left|\frac{\Gamma(n+\alpha+\delta+1+i\theta)}{\Gamma(n+\alpha+\delta+1)}\right|\right\}^{1/2}.$$

It follows from the identity (cf. [7, 1.3 (3)])

(2.5) $$\frac{\Gamma(x+iy)}{\Gamma(x)}=e^{-i\gamma y}\frac{x}{x+iy}\prod_{k=1}^{\infty}\frac{e^{iy/k}}{1+iy/(k+x)}$$

with $x=n+\alpha+\delta+1$ and $y=\theta$ that $|\Gamma(n+\alpha+\delta+1+i\theta)/\Gamma(n+\alpha+\delta+1)|\le1$ for $n=0,1,2,\cdots$, $\delta\ge0$ and $-\infty<\theta<\infty$. Thus, we have

(2.6) $$A_{n,\beta}\le Cn^{(\Delta-\alpha+2)/2}$$

for $n\ge n_0$, where the constant C and n_0 depend only on Δ and α. To estimate the integral on the right side of the inequality (2.3), we note that the integral is independent of β. We have

$$\int_a^b\int_0^1 v^{\alpha+j-1}|L_{n-j}^{\alpha+j-1}(vy)|\,dv\,dy=\int_0^1\int_{va}^{vb}|L_{n-j}^{\alpha+j-1}(t)|\,dt\,v^{\alpha+j-2}\,dv$$

$$=\left\{\int_0^{1/(n-j)}+\int_{1/(n-j)}^1\right\}\int_{va}^{vb}|L_{n-j}^{\alpha+j-1}(t)|\,dt\,v^{\alpha+j-2}\,dv=D_1+D_2\,,$$

say. It follows from the asymptotic formula [15, (7.6.8)] of the Laguerre polynomial that

(2.7) $$\begin{aligned}D_1&\le\int_0^{1/(n-j)}\int_{va}^{vb}(Cn^{\alpha+j-1})\,dt\,v^{\alpha+j-2}\,dv\le Cn^{-1}\,,\\ D_2&\le\int_{1/(n-j)}^1\int_{va}^{vb}C(n/t)^{(\alpha+j-1)/2}(nt)^{-1/4}\,dt\,v^{\alpha+j-2}\,dv\le Cn^{(\alpha+j)/2-3/4}\end{aligned}$$

for large n, where C is a constant independent of n. Combining (2.4), (2.6) and (2.7), we complete the proof. q.e.d.

Let $\alpha>-1$ and $\mathrm{Re}\,\beta>-1$. We define an operator T_α^β by $a_n^\alpha(T_\alpha^\beta(f))=a_n^\beta(f)$,

$n=0, 1, 2, \cdots$, for $f\in C_c^\infty$, that is,

$$T_\alpha^\beta(f)\sim \sum_{n=0}^\infty a_n^\beta(f)\tau_n^\alpha L_n^\alpha(x)e^{-x/2}x^{\alpha/2}.$$

It follows from (2.1) that $T_\alpha^\beta(f)\in L^2(0,\infty)$ for $f\in C_c^\infty$. Let $\{\varphi_n\}$ be the sequence defined by

$$\varphi_n=\left\{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+1+i\theta)}\right\}^{1/2}, \qquad n=0,1,2,\cdots, \ -\infty<\theta<\infty .$$

We choose the branch of the square root which is equal to $+1$ for $\theta=0$. We define also an operator $T_{\alpha,\varphi}^\beta$ by

$$T_{\alpha,\varphi}^\beta(f)\sim \sum_{n=0}^\infty \varphi_n a_n^\beta(f)\tau_n^\alpha L_n^\alpha(x)e^{-x/2}x^{\alpha/2} \tag{2.8}$$

for $f\in C_c^\infty$. Since the sequence $\{\varphi_n\}$ is bounded for every θ, we have $T_{\alpha,\varphi}^\beta(f)\in L^2(0,\infty)$ for $f\in C_c^\infty$. We state here two propositions. Theorem will follow from Proposition 1, which in turn is deduced from Proposition 2.

Proposition 1. (I) *If* $\alpha=0, 1, 2, \cdots$, *then*

$$\| T_\alpha^{\alpha+k+i\theta}(f)\|_p\le M(\theta)\|f\|_p \qquad (f\in C_c^\infty)$$

for $1<p<\infty$, $-\infty<\theta<\infty$ *and* $k=0, 2$ *where* $M(\theta)$ *is independent of* f *and satisfies the condition*

$$\sup_{-\infty<\theta<\infty} e^{-\kappa|\theta|}\log M(2\theta)<\infty \qquad \textit{for some} \quad \kappa<\pi . \tag{\#}$$

(II) *If* $\alpha\ge 0$, *then*

$$\| T_\alpha^{\alpha+2}(f)\|_p\le C\|f\|_p \qquad (f\in C_c^\infty) \tag{2.9}$$

for $1<p<\infty$, *where* C *is a constant independent of* f. *If* $-1<\alpha<0$, *then* (2.9) *holds for* $(1+\alpha/2)^{-1}<p<-2/\alpha$.

Proposition 2. *If* $\alpha\ge 0$, *then*

$$\| T_{\alpha,\varphi}^{\alpha+k+i\theta}(f)\|_p\le M(\theta)\|f\|_p \qquad (f\in C_c^\infty) \tag{2.10}$$

for $1<p<\infty$, $-\infty<\theta<\infty$ *and* $k=0, 2$, *where* $M(\theta)$ *is independent of* f *and satisfies* (#). *If* $-1<\alpha<0$, *then* (2.10) *with* $k=2$ *holds for* $(1+\alpha/2)^{-1}<p<-2/\alpha$.

We show first that Proposition 2 implies Proposition 1. Since $\varphi_n=1$ for $\theta=0$, Proposition 1, (II) is a special case of Proposition 2. We apply Długosz's criterion to the function

$$\lambda(x)=\left\{\frac{\Gamma(x+\alpha+1/2+i\theta)}{\Gamma(x+\alpha+1/2)}\right\}^{1/2}, \qquad \alpha>-1/2 .$$

We have $\lambda((2n+1)/2)=\varphi_n^{-1}$. Here, the branch of the root is so chosen that $\lambda(x)=+1$ for $\theta=0$. Let $\Lambda=\{\varphi_n^{-1}\}_{n=0}^{\infty}$. Then we have $\mathscr{M}_\Lambda^\alpha(T_{\alpha,\varphi}^{\alpha+k+i\theta}(f))=T_\alpha^{\alpha+k+i\theta}(f)$. Thus, Proposition 1, (I) follows from Proposition 2 by Lemma 2 below which shows that the function $\lambda(x)$ satisfies Długosz's condition with the constant $B=C(1+\theta^4)$, where C is a constant depending only on α.

LEMMA 2. *Let $\alpha>-1/2$ and $j=0, 1, 2, \cdots$. Then,*

$$\sup_{x>0}|\lambda^{(j)}(x)x^j|\leq C(1+|\theta|^j)$$

for $-\infty<\theta<\infty$, where C is a constant independent of θ.

PROOF. By (2.5), we have $\sup_{x>0}|\lambda(x)|\leq 1$. Let $\psi(z)$ be the logarithmic derivative of $\Gamma(z)$, that is, $\psi(z)=\Gamma'(z)/\Gamma(z)$. Let $u=x+\alpha+1/2$. We note $u>0$. We have $\lambda'(x)=\lambda(x)\{\psi(u+i\theta)-\psi(u)\}/2$. Differentiating in x both sides of the identity j times, we have

$$\lambda^{(j+1)}(x)=\frac{1}{2}\sum_{k=0}^{j}\binom{j}{k}\lambda^{(j-k)}(x)\{\psi^{(k)}(u+i\theta)-\psi^{(k)}(u)\}\,.$$

We see by the identity that it is enough to show that

$$\sup_{x>0}|\psi^{(k)}(u+i\theta)-\psi^{(k)}(u)|\,x^{k+1}\leq C|\theta|$$

for every $k=0, 1, 2, \cdots$. For $k=0$, we use the formula (cf. [7, 1.7 (24)])

$$\psi(z)=\log z+\int_0^\infty\left\{\frac{1}{1-e^t}+\frac{1}{t}-1\right\}e^{-tz}\,dt$$

for $\operatorname{Re}z>0$. We have

$$|\psi(u+i\theta)-\psi(u)|\leq\log\left|1+i\frac{\theta}{u}\right|+\left|\tan^{-1}\frac{\theta}{u}\right| +2|\theta|\int_0^\infty\left|\frac{1}{1-e^t}+\frac{1}{t}-1\right|e^{-tu}t\,dt\leq C\frac{|\theta|}{u}\,.$$

Since

$$\psi^{(k)}(z)=(-1)^{k+1}k!\sum_{m=0}^{\infty}\frac{1}{(z+m)^{k+1}}$$

(cf. [7, 1.17 (9)]), we have

$$\psi^{(k)}(u+i\theta)-\psi^{(k)}(u)=(-1)^{k+1}k!\sum_{m=0}^{\infty}\left\{\frac{1}{u+i\theta+m}-\frac{1}{u+m}\right\}$$
$$\cdot\left\{\frac{1}{(u+i\theta+m)^k}+\frac{1}{(u+i\theta+m)^{k-1}(u+m)}+\cdots+\frac{1}{(u+m)^k}\right\}.$$

Thus, we have

$$|\psi^{(k)}(u+i\theta)-\psi^{(k)}(u)|\cdot x^{k+1}\leq k!|\theta|x^{k+1}\sum_{m=0}^{\infty}\frac{1}{(u+m)^2}\frac{k+1}{(u+m)^k}\leq C|\theta|,$$

where C is a constant not depending on θ and x. q.e.d.

We show that Proposition 1 implies Theorem. Let f and g be in C_c^∞, and let z satisfy $0\leq \operatorname{Re} z\leq 1$. We write $z=\delta+i\theta$. We define

$$\Phi_\alpha(z)=\int_0^\infty T_\alpha^{\alpha+2z}(f)(x)g(x)dx$$

for $\alpha>-1$. Then, it follows from (2.1) with $j=4$ that

$$|\Phi_\alpha(z)|^2=|\sum_{n=0}^{\infty}a_n^{\alpha+2z}(f)a_n^\alpha(g)|^2$$
$$\leq\sum_{n=0}^{\infty}|a_n^{\alpha+2z}(f)|^2\sum_{n=0}^{\infty}|a_n^\alpha(g)|^2\leq C(1+\theta^4)^2e^{2\pi|\theta|}\|g\|_2^2,$$

for $0\leq\delta\leq 1$ and $-\infty<\theta<\infty$, where C is a constant not depending on δ and θ. Thus, $\Phi_\alpha(z)$ is analytic in the strip $0<\delta<1$, and continuous in the closed strip, and of admissible growth there, that is, $\sup\{e^{-\kappa|\theta|}\log|\Phi_\alpha(z)|;\ 0\leq\delta\leq 1,\ -\infty<\theta<\infty\}<\infty$ for some $\kappa<\pi$. Let $\alpha=0,1,2,\cdots$, $1<p<\infty$ and $1/p+1/q=1$. By Proposition 1, (I), we have $|\Phi_\alpha(k+i\theta)|\leq M(2\theta)$, $k=0, 1$, for $\|f\|_p=\|g\|_q=1$. It follows from the lemma of Hirschman [12, Lemma 1] that $|\Phi_\alpha(\delta)|\leq C$ for $0\leq\delta\leq 1$, where C is a constant depending on δ. Thus, we have

$$\|T_\alpha^\beta(f)\|_p\leq C\|f\|_p\qquad(f\in C_c^\infty)\tag{2.11}$$

for α, β and p satisfying the condition

(*) $\alpha=0,1,2,\cdots,\quad \alpha\leq\beta\leq\alpha+2\quad$ and $\quad 1<p<\infty$,

where C is a constant not depending on f. Note that we may obtain the above inequality by using a special case of Stein's complex interpolation theorem. Since $\int T_\alpha^\beta(f)g=\sum a_n^\beta(f)a_n^\alpha(g)=\int fT_\beta^\alpha(g)$, it follows from the duality argument that

$$\|T_\beta^\alpha(f)\|_p\leq C\|f\|_p\qquad(f\in C_c^\infty)\tag{2.12}$$

for α, β and p with (*). By the standard density argument, T_α^β is extended to the whole space L^p. We denote the extension also by T_α^β. Then, (2.11) and (2.12) hold for all $f\in L^p$

and for α, β and p with $(*)$. We note that $a_n^\alpha(T_\alpha^\beta(f))=a_n^\beta(f)$ for all $f\in L^p$ and for α, β and p with $(*)$. The same argument is applicable to $T_\alpha^{\alpha+2}$, $\alpha>-1$. It follows that

$$\| T_\alpha^\beta(f)\|_p\le C\|f\|_p \qquad (f\in L^p)\,, \tag{2.13}$$

and $a_n^\alpha(T_\alpha^\beta(f))=a_n^\beta(f)$ $(f\in L^p)$ if $0\le\gamma=\min\{\alpha,\beta\}$, $|\alpha-\beta|=2$ and $1<p<\infty$, or if $-1<\gamma<0$, $|\alpha-\beta|=2$ and $(1+\gamma/2)^{-1}<p<-2/\gamma$. By duality it is enough to show $\| T_\alpha^\beta(f)\|_p\le C\|f\|_p$ $(f\in L^p)$ in the following three cases (i) $0\le\alpha<\beta$, $1<p<\infty$, (ii) $-1<\alpha<0\le\beta$, $(1+\alpha/2)^{-1}<p<-2/\alpha$ and (iii) $-1<\alpha<\beta<0$, $(1+\alpha/2)^{-1}<p<-2/\alpha$. We use the property $T_\alpha^\beta\circ T_\beta^\zeta(f)=T_\alpha^\zeta(f)$ $(f\in L^p)$ for suitable α, β, ζ and p which follows from $a_n^\alpha(T_\alpha^\beta(f))=a_n^\beta(f)$ $(f\in L^p)$. We show only the case (ii) since the other cases are proved by a similar argument. Let N be the integer such that $2N\le\beta<2(N+1)$. It follows that $T_\alpha^\beta=T_\alpha^{\alpha+2}\circ T_{\alpha+2}^0\circ T_0^2\circ\cdots\circ T_{2N}^\beta$. We have the desired inequality by applying (2.13) to the operators $T_\alpha^{\alpha+2}$, $T_0^2,\cdots,T_{2(N-1)}^{2N}$, and (2.12) to $T_{\alpha+2}^0$, and (2.11) to T_{2N}^β. Therefore, we see that Proposition 1 implies the theorem.

The rest of the paper is devoted to the proof of Proposition 2. We shall estimate the L^p norm of $T_{\alpha,\varphi}^{\alpha+i\theta}(f)$, $\alpha\ge0$, in §3 and that of $T_{\alpha,\varphi}^{\alpha+2+i\theta}(f)$, $\alpha>-1$, in §4.

3. Estimate of L^p norm of $T_{\alpha,\varphi}^{\alpha+i\theta}(f)$, $\alpha\ge0$.

Let $\varepsilon>0$. We define

$$G_\varepsilon^\theta(f)(x)=\sum_{n=0}^\infty \varphi_n\omega_n^\alpha a_n^{\alpha+\varepsilon+i\theta}(f)\tau_n^\alpha L_n^\alpha(x)e^{-x/2}x^{\alpha/2} \tag{3.1}$$

for $\alpha>-1$, $f\in C_c^\infty$ and $x>0$, where

$$\omega_n^\alpha=\{\Gamma(n+\alpha+i\theta+1)/\Gamma(n+\alpha+i\theta+1+\varepsilon)\}^{1/2}\,. \tag{3.2}$$

We take the branch of the square root which is positive for $\theta=0$. It follows from Lemma 1 that $\lim_{\varepsilon\to+0}G_\varepsilon^\theta(f)(x)=T_{\alpha,\varphi}^{\alpha+i\theta}(f)(x)$ for every $x>0$. We shall show that

$$\| G_\varepsilon^\theta(f)\|_p\le M(\theta)(\|f(x)x^{\varepsilon/2}\|_p+\|f(x)x^{-\varepsilon/2}\|_p) \tag{3.3}$$

for $\alpha\ge0$, $1<p<\infty$, $0<\varepsilon<1$, $-\infty<\theta<\infty$ and $f\in C_c^\infty$, where $M(\theta)$ is independent of f and ε, and satisfies the condition (#) in Proposition 1. Then, letting $\varepsilon\to+0$, we have $\| T_{\alpha,\varphi}^{\alpha+i\theta}(f)\|_p\le M(\theta)\|f\|_p$ for $\alpha\ge0$, $1<p<\infty$, $-\infty<\theta<\infty$ and $f\in C_c^\infty$ by Fatou's lemma and Lebesgue's convergence theorem. This is the inequality to be proved for $T_{\alpha,\varphi}^{\alpha+i\theta}(f)$.

To prove (3.3), we shall express $G_\varepsilon^\theta(f)$ in an integral form (3.10) for $\alpha>-1$, $\varepsilon>0$ and $-\infty<\theta<\infty$. The expression for $G_\varepsilon^\theta(f)$ with $\varepsilon=2$ and $\alpha>-1$ will be used also in §4. We define

$$G_{\varepsilon,r}^\theta(f)(x)=\sum_{n=0}^\infty r^n\varphi_n\omega_n^\alpha a_n^{\alpha+\varepsilon+i\theta}(f)\tau_n^\alpha L_n^\alpha(x)e^{-x/2}x^{\alpha/2}\,,$$

for $0<r\le1$. We note that $\lim_{r\to1-}G_{\varepsilon,r}^\theta(f)(x)=G_\varepsilon^\theta(f)(x)$ for every x. By the formula (cf. [15, (5.4.1)])

$$L_n^\alpha(x)=\frac{e^x x^{-\alpha/2}}{n!}\int_0^\infty e^{-t}t^{n+\alpha/2}J_\alpha(2(tx)^{1/2})dt\,,\quad \alpha>-1\,,$$

we have

$$G_{\varepsilon,r}^\theta(f)(x)=e^{x/2}\sum_{n=0}^\infty \varphi_n\omega_n^\alpha a_n^{\alpha+\varepsilon+i\theta}(f)\tau_n^\alpha\int_0^\infty \frac{(rt)^n}{n!}t^{\alpha/2}J_\alpha(2(tx)^{1/2})e^{-t}dt$$

$$=e^{x/2}\int_0^\infty\left\{\sum_{n=0}^\infty \varphi_n\omega_n^\alpha a_n^{\alpha+\varepsilon+i\theta}(f)\tau_n^\alpha\frac{(rt)^n}{n!}\right\}J_\alpha(2(tx)^{1/2})\,e^{-t}t^{\alpha/2}\,dt$$

for $\alpha>-1$ and $0<r\leq 1$. We remark that $a_n^{\alpha+\varepsilon+i\theta}(f)=O(n^{-j})$ $(n\to\infty)$ for large j by (2.1). It follows from the definition of $a_n^{\alpha+\varepsilon+i\theta}(f)$ that

$$G_{\varepsilon,r}^\theta(f)(x)=e^{x/2}\int_0^\infty\int_0^\infty\left\{\sum_{n=0}^\infty \varphi_n\omega_n^\alpha\tau_n^{\alpha+\varepsilon+i\theta}\tau_n^\alpha\frac{(rt)^n}{n!}L_n^{\alpha+\varepsilon+i\theta}(y)\right\}\tag{3.4}$$

$$\cdot f(y)e^{-y/2}y^{(\alpha+\varepsilon+i\theta)/2}J_\alpha(2(tx)^{1/2})e^{-t}t^{\alpha/2}\,dydt$$

for $\alpha>-1$ and $0<r\leq 1$. We apply the formula (2.2) with $\mu=\alpha$, $\nu=\varepsilon+i\theta$ and $m=n$ to $L_n^{\alpha+\varepsilon+i\theta}(y)$. Then we have

$$G_{\varepsilon,r}^\theta(f)(x)=\frac{e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 v^\alpha(1-v)^{\varepsilon-1+i\theta}\sum_{n=0}^\infty\frac{(rt)^n}{\Gamma(n+\alpha+1)}L_n^\alpha(vy)$$

$$\cdot f(y)e^{-y/2}y^{(\alpha+\varepsilon+i\theta)/2}J_\alpha(2(tx)^{1/2})e^{-t}t^{\alpha/2}\,dvdydt$$

for $\alpha>-1$ and $0<r\leq 1$. By the formula (cf. [15, (5.1.16)])

$$\sum_{n=0}^\infty\frac{w^n}{\Gamma(n+\alpha+1)}L_n^\alpha(x)=e^w(xw)^{-\alpha/2}J_\alpha(2(xw)^{1/2})\tag{3.5}$$

and a change of variables, we have

$$G_{\varepsilon,r}^\theta(f)(x)=\frac{2e^{x/2}}{r^{\alpha/2}\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+1}(1-u^2)^{\varepsilon-1+i\theta}J_\alpha(r^{1/2}suz)\tag{3.6}$$

$$\cdot g_\varepsilon^\theta(z)J_\alpha((2x)^{1/2}s)e^{-(1-r)s^2/2}s\,dudzds\,,$$

where $g_\varepsilon^\theta(z)=f(z^2/2)e^{-z^2/4}(z^2/2)^{(\varepsilon+i\theta)/2}z$ for $\alpha>-1$ and $0<r\leq 1$. We remark that this identity with $\alpha+2$ in place of α will be referred to in §4. In the rest of this section, we assume $0<r<1$. We can change the order of integration in the above triple integral. Since $g_\varepsilon^\theta\in C_c^\infty$, it is enough to show that $h(u,s)$ is integrable in (u,s) for fixed z, where $h(u,s)=u^{\alpha+1}(1-u^2)^{\varepsilon-1}|J_\alpha(r^{1/2}suz)|\,|J_\alpha((2x)^{1/2}s)|e^{-(1-r)s^2/2}s$. We write $\int_0^\infty\int_0^1 h(u,s)duds=\{\int_0^1\int_0^1+\int_1^\infty\int_0^{1/s}+\int_0^\infty\int_{1/s}^1\}h(u,s)duds=H_1+H_2+H_3$, say. By the asymptotic formulas

$$J_\alpha(t)\sim t^\alpha(t\to+0)\qquad\text{for}\quad\alpha>-1,\ \text{and}\ J_\alpha(t)=O(t^{-1/2})\ (t\to\infty)\,,\tag{3.7}$$

we have $H_1 \leq C\int_0^1\int_0^1 u^{\alpha+1}(1-u^2)^{\varepsilon-1}(su)^{\alpha}s^{\alpha}e^{-(1-r)s^2/2}s\,duds<\infty$ for $\alpha>-1$, and $H_2 \leq C\int_1^{\infty}\int_0^{1/s}u^{\alpha+1}(1-u^2)^{\varepsilon-1}(su)^{\alpha}s^{-1/2}e^{-(1-r)s^2/2}s\,duds<\infty$ for $\alpha>-1$, and $H_3 \leq C\int_1^{\infty}\int_{1/s}^1 u^{\alpha+1}\cdot(1-u^2)^{\varepsilon-1}(su)^{-1/2}s^{-1/2}e^{-(1-r)s^2/2}s\,duds<\infty$ for arbitrary α. By inverting the order of integration and changing variables, we have

$$G_{\varepsilon,r}^{\theta}(f)(x)=\frac{2e^{x/2}}{r^{(\alpha+1)/2}\Gamma(\varepsilon+i\theta)}\int_0^1 u^{\alpha}(1-u^2)^{\varepsilon-1+i\theta}\int_0^{\infty} g_{\varepsilon}^{\theta}(w/(r^{1/2}u))X_{1-r}(w,(2x)^{1/2})\,dwdu$$

for $\alpha>-1$, where

$$X_{\gamma}(w,t)=\int_0^{\infty}J_{\alpha}(ws)J_{\alpha}(ts)e^{-\gamma s^2/2}s\,ds\,,\qquad 0<\gamma<1\,.$$

It follows from the formulas [7, 7.7 (25) and 7.14 (27)] that

$$X_{\gamma}(w,t)=\frac{1}{\gamma}\exp\left(-\frac{w^2+t^2}{2\gamma}\right)I_{\alpha}\left(\frac{wt}{\gamma}\right),$$

and

$$\int_0^{\infty}X_{\gamma}(w,t)dw=\left(\frac{\pi}{2\gamma}\right)^{1/2}\exp\left(-\frac{t^2}{4\gamma}\right)I_{\alpha/2}\left(\frac{t^2}{4\gamma}\right)=W_{\gamma}(t)\,,$$

say, for $\alpha>-1$, where I_{α} is the modified Bessel function. By the asymptotic formula

$$I_{\alpha}(z)=(2\pi z)^{-1/2}[e^{z}\{1+O(|z|^{-1})\}+ie^{-z+\alpha\pi i}\{1+O(|z|^{-1})\}]\,,\tag{3.8}$$

$-\pi/2<\arg z<3\pi/2$ (cf. [7, 7.13 (5)]), we have $W_{\gamma}(t)=O(1)$ $(\gamma\to+0)$ for fixed t. Thus, we have $|\int_0^{\infty}g_{\varepsilon}^{\theta}(w/(r^{1/2}u))X_{1-r}(w,(2x)^{1/2})dw|=O(1)$ $(r\to1-)$ uniformly in u for fixed x. By Lebesgue's convergence theorem, we have

$$\lim_{r\to1-}G_{\varepsilon,r}^{\theta}(f)(x)=\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^1 u^{\alpha}(1-u^2)^{\varepsilon-1+i\theta}\cdot\lim_{r\to1-}\left\{\int_0^{\infty}g_{\varepsilon}^{\theta}(w/(r^{1/2}u))X_{1-r}(w,(2x)^{1/2})dw\right\}du\tag{3.9}$$

for $\alpha>-1$. Let $Z_{\gamma}(w,t)=W_{\gamma}(t)^{-1}X_{\gamma}(w,t)$ and let $0<a<b<\infty$. Then, by (3.8), we have $Z_{\gamma}(w,t)\leq C\gamma^{-1/2}\exp(-(w-t)^2/(2\gamma))$ for $a\leq t, w\leq b$, where C is a constant independent of t, w and γ. This leads to the fact that the family $\{Z_{\gamma}\}_{\gamma}$ is a summability kernel in $a\leq t, w\leq b$.Thus we see that the limit on the right side of (3.9) is $(2x)^{-1/2}g_{\varepsilon}^{\theta}((2x)^{1/2}/u)$ by $W_{\gamma}(t)\to1/t$ $(\gamma\to+0)$ and $g_{\varepsilon}^{\theta}\in C_c^{\infty}$. Therefore, by the definition of g_{ε}^{θ} and a change of variables, we have an integral representation

$$G_{\varepsilon}^{\theta}(f)(x)=\frac{e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^1 v^{\alpha/2-1}(1-v)^{\varepsilon-1+i\theta}f(x/v)e^{-x/(2v)}(x/v)^{(\varepsilon+i\theta)/2}dv\tag{3.10}$$

for $\alpha>-1$, $\varepsilon>0$ and $-\infty<\theta<\infty$.

We shall evaluate the L^p-norm of $G_\varepsilon^\theta(f)$ for $\alpha \ge 0$. Let

$$I_\varepsilon^\theta(f)(x) = \frac{1}{\Gamma(\varepsilon+i\theta)} \int_x^\infty \frac{f(t)}{t} e^{-(t-x)/2} t^{(\varepsilon+i\theta)/2} \left(1-\frac{x}{t}\right)^{\varepsilon-1+i\theta} dt,$$

$$J_\varepsilon^\theta(f)(x) = \frac{1}{\Gamma(\varepsilon+i\theta)} \int_x^\infty \frac{f(t)}{t} e^{-(t-x)/2} t^{(\varepsilon+i\theta)/2} \left\{\left(\frac{x}{t}\right)^{\alpha/2} - 1\right\} \left(1-\frac{x}{t}\right)^{\varepsilon-1+i\theta} dt$$

for $f \in C_c^\infty$. Then, we note that $G_\varepsilon^\theta(f) = I_\varepsilon^\theta(f) + J_\varepsilon^\theta(f)$. Since $\alpha \ge 0$, it follows that $|\{(x/t)^{\alpha/2} - 1\}(1-x/t)^{\varepsilon-1+i\theta}| \le C$ for $t > x$, where C is a constant depending only on α. We have $|J_\varepsilon^\theta(f)(x)| \le C|\Gamma(\varepsilon+i\theta)|^{-1} \int_x^\infty |f(t)| t^{\varepsilon/2-1} dt$. We remark that we have $|\Gamma(\varepsilon+i\theta)|^{-1} \le A(1+|\theta|)e^{\pi|\theta|/2}$ by (2.4) and $\Gamma(\varepsilon+i\theta)^{-1} = (\varepsilon+i\theta)\Gamma(1+\varepsilon+i\theta)^{-1}$, where A is an absolute constant. It follows from Hardy's inequality that

$$\|J_\varepsilon^\theta(f)\|_p \le C(1+|\theta|)e^{\pi|\theta|/2} \|f(x)x^{\varepsilon/2}\|_p \tag{3.11}$$

for $1 < p < \infty$, $0 < \varepsilon < 1$, $\alpha \ge 0$ and $f \in C_c^\infty$, where C is a constant independent of ε, θ and f.

We next treat $I_\varepsilon^\theta(f)$. We extend $f \in C_c^\infty$ to the function on $(-\infty, \infty)$ which coincides with f on $(0, \infty)$ and vanishes on $(-\infty, 0]$. We also denote the function by f. We define

$$\tilde{I}_\varepsilon^\theta(f)(x) = \int_{-\infty}^\infty f(t)Q(x-t)\,dt, \qquad -\infty < x < \infty,$$

where

$$Q(u) = \frac{1}{\Gamma(\varepsilon+i\theta)} e^{-|u|/2} |u|^{\varepsilon-1+i\theta} \chi_{(-\infty,0)}(u).$$

The function $\chi_{(-\infty,0)}(u)$ is the characteristic function of the interval $(-\infty, 0)$. We note that $\tilde{I}_\varepsilon^\theta(f(t)|t|^{-(\varepsilon+i\theta)/2})(x) = I_\varepsilon^\theta(f)(x)$ for $x > 0$. We shall show that $\tilde{I}_\varepsilon^\theta$ is a singular integral operator. It follows from the formulas [8,1.4 (7) and 2.4 (7)] that the Fourier transform $\hat{Q}(y) = \int_{-\infty}^\infty Q(u)e^{-iuy}du$ is given by

$$\hat{Q}(y) = (1/4 + y^2)^{-(\varepsilon+i\theta)/2} e^{i\varepsilon\tan^{-1}2y} e^{-\theta\tan^{-1}2y}.$$

Therefore, we have $|\hat{Q}(y)| \le 2e^{\pi|\theta|/2} = B_1$, say. We easily see that

$$\left|\frac{d}{du}Q(u)\right| \le A\frac{(1+|\theta|)}{|\Gamma(\varepsilon+i\theta)|}u^{-2} \le A(1+|\theta|)^2 e^{\pi|\theta|/2} u^{-2} = B_2 u^{-2},$$

say, for $u \ne 0$, where A is an absolute constant. By the Calderón-Zygmund theory of singular integrals (cf. [9, II.5 Theorem 5.7]), we see that the Lebesgue measure of $\{x \in \boldsymbol{R};\ |\tilde{I}_\varepsilon^\theta(f)(x)| > \lambda\}$ is bounded by $A_1(B_1^2 + B_2 + 1)\lambda^{-1}\|f\|_1 \le A_2 e^{\pi|\theta|}\lambda^{-1}\|f\|_1 = B_3\lambda^{-1}\|f\|_1$, say, where A_1 and A_2 are absolute constants. The Marcinkiewicz interpolation theorem (cf. [9, II.2 Theorem 2.11]) leads to

$$\left\{\int_{-\infty}^\infty |\tilde{I}_\varepsilon^\theta(f)(x)|^p dx\right\}^{1/p} \le \left\{\frac{2pB_3}{p-1} + \frac{4pB_1^2}{2-p}\right\}^{1/p} \|f\|_p = M_p(\theta)\|f\|_p, \tag{3.12}$$

say, for $1<p<2$. We note that $M_p(\theta)$ is independent of ε and f, and satisfies the condition (#) in Proposition 1. By the duality argument, we see that (3.12) holds for $2<p<\infty$ with $M_q(\theta)$, $1/p+1/q=1$. Since $\| I_\varepsilon^\theta(f) \|_p \le \{\int_{-\infty}^{\infty} |\tilde{I}_\varepsilon^\theta(f(t)|t|^{-(\varepsilon+i\theta)/2})(x)|^p dx\}^{1/p}$, it follows that

$$\| I_\varepsilon^\theta(f) \|_p \le M(\theta) \| f(x) x^{-\varepsilon/2} \|_p , \tag{3.13}$$

for $1<p<\infty$, $0<\varepsilon<1$, $-\infty<\theta<\infty$, where $M(\theta)$ is independent of ε and f, and satisfies (#). By (3.11) and (3.13), we have the inequality (3.3) to be proved.

4. L^p-estimate of $T_{\alpha,\varphi}^{\alpha+2+i\theta}(f)$, $\alpha>-1$. Let

$$\rho_n = \alpha + \frac{3}{2} + i\theta + \frac{-1/4}{\{(n+\alpha+2+i\theta)(n+\alpha+1+i\theta)\}^{1/2} + (n+\alpha+3/2+i\theta)} ,$$

$$\sigma_n = \left\{ \frac{\Gamma(n+\alpha+3+i\theta)}{\Gamma(n+\alpha+1)} \right\}^{1/2} ,$$

for $\alpha>-1$, $-\infty<\theta<\infty$ and $n=0, 1, 2, \cdots$. For the above square roots, we choose the branches positive for $\theta=0$. We define

$$U^\theta(f)(x) = \sum_{n=0}^{\infty} (\rho_n/\sigma_n) a_n^{\alpha+2+i\theta}(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} ,$$

$$V^\theta(f)(x) = \sum_{n=0}^{\infty} (n/\sigma_n) a_n^{\alpha+2+i\theta}(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} ,$$

for $\alpha>-1$, $-\infty<\theta<\infty$, $f\in C_c^\infty$ and $x>0$. Then, we have $T_{\alpha,\varphi}^{\alpha+2+i\theta}(f) = U^\theta(f) + V^\theta(f)$. We shall estimate the L^p-norms of $U^\theta(f)$ and $V^\theta(f)$.

We first deal with $U^\theta(f)$. Let $\Lambda=\{\rho_n\}_{n=0}^{\infty}$. Then, by the definition (3.1) of $G_\varepsilon^\theta(f)$ we see that $U^\theta(f) = \mathcal{M}_\Lambda^\alpha(G_2^\theta(f))$. Since Λ is a quasi-convex sequence, $\mathcal{M}_\Lambda^\alpha$ is bounded in L^p. Indeed, let $\| \Lambda \|_{\mathrm{bqc}} = \sum_{n=0}^{\infty} (n+1) |\Delta^2 \rho_n| + \lim_{n\to\infty} |\rho_n|$, where $\Delta^2\rho_n = \rho_n - 2\rho_{n+1} + \rho_{n+2}$. It follows from the result of Butzer, Nessel and Trebels [5, Theorem 3.2 and p. 139] that

$$\| \mathcal{M}_\Lambda^\alpha(G_2^\theta(f)) \|_p \le C \| \Lambda \|_{\mathrm{bqc}} \| G_2^\theta(f) \|_p \tag{4.1}$$

if $\alpha\ge0$ and $1<p<\infty$, or if $-1<\alpha<0$ and $(1+\alpha/2)^{-1}<p<-2/\alpha$, where C is a constant depending only on α and p. We have to estimate $\| \Lambda \|_{\mathrm{bqc}}$.

LEMMA 3. *If $\alpha>-1$, then $\| \Lambda \|_{\mathrm{bqc}} \le C(1+|\theta|)$, where C is a constant depending only on α.*

PROOF. Let $\eta(x) = \{(x+a)(x+a+1)\}^{1/2}$, $a=\alpha+1+i\theta$, where the branch is so chosen that $\eta(x)$ is positive for $\theta=0$. Let $\rho(x) = (\eta(x)+x+b)^{-1}$ with $b=(2a+1)/2$. We have $\rho''(x) = \eta(x)^{-3}$. Thus, $|\Delta^2\rho_n| \le 2\max\{|\rho''(x)|;\ n\le x\le n+2\} \le 2(n+\alpha+1)^{-3}$. Since $\lim_{n\to\infty}\rho_n = \alpha+3/2+i\theta$, we complete the proof. q.e.d.

To get the desired inequality for $U^\theta(f)$, it is enough to evaluate the L^p-norm of $G_2^\theta(f)$ for $\alpha>-1$. Applying Minkowski's inequality to the integral representation (3.10) of $G_\varepsilon^\theta(f)$ with $\varepsilon=2$, and changing variables, we have

$$\| G_2^\theta(f)\|_p \le \frac{1}{|\Gamma(2+i\theta)|}\int_0^1 v^{\alpha/2-2}(1-v)\left\{\int_0^\infty |f(x/v)e^{x(1-1/v)/2}x|^p dx\right\}^{1/p} dv$$
$$\le \frac{1}{|\Gamma(2+i\theta)|}\int_0^1 v^{\alpha/2+1/p-1}(1-v)\left\{\int_0^\infty |f(t)e^{t(v-1)/2}t|^p dt\right\}^{1/p} dv .$$

Since $e^{t(v-1)/2}t\le 2e^{-1}(1-v)^{-1}$ for $t>0$ and $0<v<1$, it follows that $\| G_2^\theta(f)\|_p\le 2e^{-1}\cdot |\Gamma(2+i\theta)|^{-1}\|f\|_p\int_0^1 v^{\alpha/2+1/p-1}dv$. We remark that the integral on the right side is finite if $\alpha\ge 0$ and $1\le p$, or if $-1<\alpha<0$ and $p<-2/\alpha$. Combining this inequality with Lemma 3 and (4.1), we get

$$\| U^\theta(f)\|_p \le C(1+|\theta|)e^{\pi|\theta|/2}\|f\|_p , \tag{4.2}$$

for $-\infty<\theta<\infty$ and $f\in C_c^\infty$ if $\alpha\ge 0$ and $1<p<\infty$, or if $-1<\alpha<0$ and $(1+\alpha/2)^{-1}<p<-2/\alpha$. Here, C is a constant depending only on α and p.

We now estimate the L^p-norm of $V^\theta(f)$. Define

$$V_\varepsilon^\theta(f)(x)=\sum_{n=0}^\infty (n\omega_n^{\alpha+2}/\sigma_n)a_n^{\alpha+2+\varepsilon+i\theta}(f)\tau_n^\alpha L_n^\alpha(x)e^{-x/2}x^{\alpha/2} ,$$

for $\alpha>-1$, $\varepsilon>0$, $-\infty<\theta<\infty$, $f\in C_c^\infty$ and $x>0$, where $\omega_n^{\alpha+2}$ is as given in (3.2). Since $\lim_{\varepsilon\to+0} V_\varepsilon^\theta(f)(x)=V^\theta(f)(x)$ for every $x>0$, it is enough to show that

$$\| V_\varepsilon^\theta(f)\|_p \le M(\theta)\{\|f(x)x^{\varepsilon/2}\|_p+\|f(x)x^{-\varepsilon/2}\|_p\} , \tag{4.3}$$

for $-\infty<\theta<\infty$ and $f\in C_c^\infty$ if $\alpha\ge 0$ and $1<p<\infty$, or if $-1<p<0$ and $(1+\alpha/2)^{-1}<p<-2/\alpha$, where $M(\theta)$ is independent of f and ε, and satisfies the condition (#) in Proposition 1. We note that $V_\varepsilon^\theta(f)$ is $G_\varepsilon^\theta(f)$ with $n\omega_n^{\alpha+2}/\sigma_n$ and $\varepsilon+2$ in place of $\varphi_n\omega_n^\alpha$ and ε, respectively. Thus, $V_\varepsilon^\theta(f)$ has the form on the right side of (3.4) with $r=1$ and with substitutions as above. We apply the formula (2.2) with $\mu=\alpha+2$, $\nu=\varepsilon+i\theta$ and $m=n$ to $L_n^{\alpha+2+\varepsilon+i\theta}(y)$ in the representation. Thus, we have

$$V_\varepsilon^\theta(f)(x)=\frac{e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 v^{\alpha+2}(1-v)^{\varepsilon-1+i\theta}\sum_{n=0}^\infty\frac{nt^n}{\Gamma(n+\alpha+2+1)}L_n^{\alpha+2}(vy)$$
$$\cdot f(y)e^{-y/2}y^{(\alpha+2+\varepsilon+i\theta)/2}J_\alpha(2(tx)^{1/2})e^{-t}t^{\alpha/2}dvdydt$$

for $\alpha>-1$. The formula (3.5) leads to

$$\sum_{n=0}^\infty\frac{nw^n}{\Gamma(n+\alpha+1)}L_n^\alpha(x)=e^w(xw)^{-\alpha/2}\{wJ_\alpha(2(xw)^{1/2})-(xw)^{1/2}J_{\alpha+1}(2(xw)^{1/2})\} .$$

Using this identity and changing variables, we have

$$V_\varepsilon^\theta(f)(x)=\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+3}(1-u^2)^{\varepsilon-1+i\theta}$$
$$\cdot\{sJ_{\alpha+2}(suz)-uzJ_{\alpha+3}(suz)\}g_\varepsilon^\theta(z)J_\alpha((2x)^{1/2}s)\,dudzds\,,$$

where $g_\varepsilon^\theta(z)$ is as given in (3.6). Let

$$W_\varepsilon^\theta(f)(x)=\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+3}(1-u^2)^{\varepsilon-1+i\theta}sJ_{\alpha+2}(suz)g_\varepsilon^\theta(z)J_\alpha((2x)^{1/2}s)\,dudzds\,,$$

$$F_\varepsilon^\theta(f)(x)=\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+4}(1-u^2)^{\varepsilon-1+i\theta}zJ_{\alpha+3}(suz)g_\varepsilon^\theta(z)J_\alpha((2x)^{1/2}s)\,dudzds\,,$$

for $\alpha>-1$, $\varepsilon>0$, $-\infty<\theta<\infty$, $f\in C_c^\infty$ and $x>0$. We see that $V_\varepsilon^\theta(f)=W_\varepsilon^\theta(f)-F_\varepsilon^\theta(f)$, since the iterated integrals in $W_\varepsilon^\theta(f)$ and $F_\varepsilon^\theta(f)$ are finite. Indeed, we can change the order of the integrals in z and u. Furthermore, if $\beta\geq 0$ and $h\in C_c^\infty$, then

$$\left|\int_0^\infty h(z)J_\beta(suz)\,dz\right|\leq C(su)^{-2}\ \ (su\geq 1)\quad\text{and}\quad \leq C\ \ (su<1)\,, \tag{4.4}$$

which is easily proved by integration by parts and the formula $(d/dt)(t^{\beta+1}J_{\beta+1}(at))=at^{\beta+1}J_\beta(at)$ (cf. [7, 7.2 (50)]).

We express $W_\varepsilon^\theta(f)$ as a sum of two integrals. Define

$$D_\varepsilon^\theta(f)(x)=\frac{4(\alpha+1)e^{x/2}}{\Gamma(\varepsilon+i\theta)(2x)^{1/2}}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+3}(1-u^2)^{\varepsilon-1+i\theta}J_{\alpha+2}(suz)g_\varepsilon^\theta(z)$$
$$\cdot J_{\alpha+1}((2x)^{1/2}s)\,dudzds\,,$$

$$E_\varepsilon^\theta(f)(x)=\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+3}(1-u^2)^{\varepsilon-1+i\theta}sJ_{\alpha+2}(suz)g_\varepsilon^\theta(z)$$
$$\cdot J_{\alpha+2}((2x)^{1/2}s)\,dudzds\,,$$

for $\alpha>-1$, $\varepsilon>0$, $-\infty<\theta<\infty$, $f\in C_c^\infty$ and $x>0$. Then, it follows from the identity $J_\alpha((2x)^{1/2}s)=2(\alpha+1)(2x)^{-1/2}s^{-1}J_{\alpha+1}((2x)^{1/2}\ s)-J_{\alpha+2}((2x)^{1/2}s)$ (cf. [7, 7.2 (56)]) that $W_\varepsilon^\theta(f)=D_\varepsilon^\theta(f)-E_\varepsilon^\theta(f)$. We note also that the integrals in $D_\varepsilon^\theta(f)$ and $E_\varepsilon^\theta(f)$ are finite. Therefore, we have $V_\varepsilon^\theta(f)=D_\varepsilon^\theta(f)-E_\varepsilon^\theta(f)-F_\varepsilon^\theta(f)$.

We first evaluate the L^p-norm of $E_\varepsilon^\theta(f)$. We see by (3.6) that $E_\varepsilon^\theta(f)$ is equal to $G_\varepsilon^\theta(f)$ with $\alpha+2$ in place of α. Since $\alpha+2>1\geq 0$ when $\alpha>-1$, we can use the inequality (3.3), and so we have

$$\|E_\varepsilon^\theta(f)\|_p\leq M(\theta)\{\|f(x)x^{\varepsilon/2}\|_p+\|f(x)x^{-\varepsilon/2}\|_p\} \tag{4.5}$$

for $\alpha>-1$, $1<p<\infty$, $0<\varepsilon<1$, $-\infty<\theta<\infty$ and $f\in C_c^\infty$, where $M(\theta)$ is independent of f and ε, and satisfies the condition (#) in Proposition 1.

We next express $F_\varepsilon^\theta(f)$ as an integral. By the inequality (4.4) and the asymptotic formula (3.7), we see that

$$F_\varepsilon^\theta(f)(x)=\lim_{\lambda\to+0}\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha+4}(1-u^2)^{\varepsilon-1+i\theta}zJ_{\alpha+3}(suz)g_\varepsilon^\theta(z)$$
$$\cdot J_\alpha((2x)^{1/2}s)s^{-\lambda}dudzds\,.$$

The factor $s^{-\lambda}$, $1>\lambda>0$, enables us to invert the order of integration in the above iterated integral. This fact is obtained by an argument analogous to that for integral (3.6). It follows that

$$F_\varepsilon^\theta(f)(x)=\lim_{\lambda\to+0}\frac{2e^{x/2}}{\Gamma(\varepsilon+i\theta)}\int_0^1\int_0^\infty u^{\alpha+4}(1-u^2)^{\varepsilon-1+i\theta}g_\varepsilon^\theta(z)zR_\lambda(u,z;x)dzdu\,,$$

where

$$R_\lambda(u,z;x)=\int_0^\infty J_{\alpha+3}(uzs)J_\alpha((2x)^{1/2}s)s^{-\lambda}ds\,.$$

By the Weber-Schafheitlin integral [16, 13.4 (2)], we have

$$R_\lambda(u,z;x)=\frac{(2x)^{\alpha/2}\Gamma(\alpha+2-\lambda/2)}{2^\lambda(uz)^{\alpha+1-\lambda}\Gamma(\alpha+1)\Gamma(2+\lambda/2)}\cdot{}_2F_1\left(\alpha+2-\lambda/2,\ -1-\lambda/2;\ \alpha+1;\ \frac{2x}{(uz)^2}\right)\quad\text{for}\quad(2x)^{1/2}<uz\,,\tag{4.6}$$

$$R_\lambda(u,z;x)=\frac{(uz)^{\alpha+3}\Gamma(\alpha+2-\lambda/2)}{2^\lambda(2x)^{(\alpha+4-\lambda)/2}\Gamma(\alpha+4)\Gamma(\lambda/2-1)}\cdot{}_2F_1\left(\alpha+2-\lambda/2,\ 2-\lambda/2;\ \alpha+4;\ \frac{(uz)^2}{2x}\right)\quad\text{for}\quad(2x)^{1/2}>uz\,,\tag{4.7}$$

when $(\alpha+3)+\alpha+1>\lambda>-1$. To invert the order of the limit $\lim_{\lambda\to+0}$ and the integral $\int_0^1\int_0^\infty dudz$ in $F_\varepsilon^\theta(f)$, it is enough to show that, for fixed x and $0<a<b<\infty$,

$$|R_\lambda(u,z;x)|\le C\quad(0\le u\le 1,\ a<z<b,\ (2x)^{1/2}>uz)\,,\tag{4.8}$$

$$|R_\lambda(u,z;x)|\le C\left\{\log\left(1-\frac{2x}{(uz)^2}\right)^{-1}+1\right\}\quad(0\le u\le 1,\ a<z<b,\ (2x)^{1/2}<uz)\,,\tag{4.8'}$$

for $0<\lambda\le 2(\alpha+1)$, where C is a constant depending only on α. In the case $(2x)^{1/2}>uz$, we have by the formula [7, 2.12 (1)] that

$${}_2F_1\left(\alpha+2-\lambda/2,\ 2-\lambda/2;\ \alpha+4;\ \frac{(uz)^2}{2x}\right)$$
$$=\frac{\Gamma(\alpha+4)}{\Gamma(2-\lambda/2)\Gamma(\alpha+2+\lambda/2)}\int_0^1 t^{1-\lambda/2}(1-t)^{\alpha+1+\lambda/2}\left\{1-t\frac{(uz)^2}{2x}\right\}^{-\alpha-2+\lambda/2}dt$$

$$\leq \frac{\Gamma(\alpha+4)}{\Gamma(2-\lambda/2)\Gamma(\alpha+2+\lambda/2)} B(2-\lambda/2, \lambda) = \frac{\Gamma(\alpha+4)\Gamma(\lambda)}{\Gamma(2+\lambda/2)\Gamma(\alpha+2+\lambda/2)}$$

when $\alpha+4>2-\lambda/2>0$. This inequality and (4.7) give (4.8). In the case $(2x)^{1/2}<uz$, we use the formula [7, 2.12 (2)]. It follows that

$$\begin{aligned} {}_2F_1\left(\alpha+2-\lambda/2, -1-\lambda/2; \alpha+1; \frac{2x}{(uz)^2}\right) &= \frac{ie^{i\pi(1-\lambda/2)}\Gamma(\alpha+1)\Gamma(2-\lambda/2)}{2\pi\Gamma(\alpha+2-\lambda/2)} \\ &\cdot \int_0^{(1+)} t^{\alpha+1-\lambda/2}(1-t)^{-2+\lambda/2}\left\{1-t\frac{2x}{(uz)^2}\right\}^{1+\lambda/2} dt \end{aligned}$$

when $\alpha+2-\lambda/2>0$ and $-1+\lambda/2\neq 1, 2, 3, \cdots$. Here, the integral is taken along a contour which starts from the origin, encircles the point 1 once counter-clockwise and returns to the origin. All singularities of the integrand except 1 are outside the contour. This formula and (4.6) lead us to (4.8'). Since ${}_2F_1(\alpha, \beta; \gamma; z)$ is a continuous function in (α, β) for fixed z and γ, we have

$$\begin{aligned} \lim_{\lambda\to+0} R_\lambda(u, z; x) &= \frac{(\alpha+1)(2x)^{\alpha/2}}{(uz)^{\alpha+1}} {}_2F_1\left(\alpha+2, -1; \alpha+1; \frac{2x}{(uz)^2}\right) \\ &= \frac{(\alpha+1)(2x)^{\alpha/2}}{(uz)^{\alpha+1}}\left\{1-\frac{\alpha+2}{\alpha+1}\frac{2x}{(uz)^2}\right\} \end{aligned}$$

for $(2x)^{1/2}<uz$, and $\lim_{\lambda\to+0} R_\lambda(u, z)=0$ for $(2x)^{1/2}>uz$. Therefore, we have

$$\begin{aligned} F_\varepsilon^\theta(f)(x) &= \frac{2^{1+\alpha/2}(\alpha+1)e^{x/2}x^{\alpha/2}}{\Gamma(\varepsilon+i\theta)} \\ &\cdot \int_0^1 \int_{(2x)^{1/2}/u}^\infty u^3(1-u^2)^{\varepsilon-1+i\theta} g_\varepsilon^\theta(z) z^{-\alpha}\left\{1-\frac{\alpha+2}{\alpha+1}\frac{2x}{(uz)^2}\right\} dzdu \\ &= \frac{(\alpha+1)e^{x/2}}{\Gamma(\varepsilon+i\theta)} \int_0^1 \int_{x/v}^\infty v(1-v)^{\varepsilon-1+i\theta} f(y) e^{-y/2} y^{(\varepsilon+i\theta)/2}(x/y)^{\alpha/2}\left\{1-\frac{\alpha+2}{\alpha+1}\frac{x}{yv}\right\} dydv . \end{aligned}$$

Changing the order of integration, we get

$$F_\varepsilon^\theta(f)(x) = \frac{(\alpha+1)e^{x/2}}{\Gamma(\varepsilon+1+i\theta)} \int_x^\infty S_\varepsilon^\theta(x, y) f(y) e^{-y/2} y^{(\varepsilon+i\theta)/2}(x/y)^{\alpha/2} dy , \tag{4.9}$$

$$S_\varepsilon^\theta(x, y) = \left(1-\frac{x}{y}\right)^{\varepsilon+i\theta}\left\{1-\frac{\alpha+2}{\alpha+1}\frac{x}{y}-\frac{\varepsilon+i\theta}{\varepsilon+1+i\theta}\left(1-\frac{x}{y}\right)\right\},$$

and thus,

$$|F_\varepsilon^\theta(f)(x)| \leq \frac{C}{|\Gamma(\varepsilon+1+i\theta)|}\int_x^\infty |f(y)|\, y^{\varepsilon/2}\left(\frac{x}{y}\right)^{\alpha/2} e^{-(y-x)/2} dy , \tag{4.10}$$

for $\alpha > -1$, $1 > \varepsilon > 0$, $-\infty < \theta < \infty$, $x > 0$ and $f \in C_c^\infty$, where C is a constant depending only on α. We define a convolution operator K by $K(h)(x) = \int_{-\infty}^{\infty} h(y)k(x-y)dy$ for h on $(-\infty, \infty)$, where $k(u) = e^{-|u|/2}\chi_{(-\infty,0]}(u)$. We reduce the estimate for $F_\varepsilon^\theta(f)$ to that for $K(h)$.

By (4.10) we have

$$(4.11) \qquad |F_\varepsilon^\theta(f)(x)| \le \begin{cases} \dfrac{C}{|\Gamma(\varepsilon+1+i\theta)|} K(|f(y)|\,|y|^{\varepsilon/2})(x) & (\alpha \ge 0) \\ \dfrac{C}{|\Gamma(\varepsilon+1+i\theta)|} K(|f(y)|\,|y|^{(\varepsilon-\alpha)/2})(x) \cdot |x|^{\alpha/2} & (-1 < \alpha < 0) \end{cases}$$

for $x > 0$. Here, f is extended to the whole real line so that $f(x) = 0$ for $x < 0$. When $\alpha \ge 0$, we have by Minkowski's inequality that

$$(4.12) \qquad \left\{ \int_{-\infty}^{\infty} |K(|f(y)|\,|y|^{\varepsilon/2})(x)|^p dx \right\}^{1/p} \le C \, \| f(x) x^{\varepsilon/2} \|_p ,$$

for $1 \le p < \infty$, where C is a constant depending only on p. For $-1 < \alpha < 0$, we use a weighted norm inequality for a regular convolution transform. Since $|x|^\eta$ is an A_p-weight for $-1 < \eta < p-1$, we have

$$(4.13) \qquad \int_{-\infty}^{\infty} |K(|f(y)|\,|y|^{(\varepsilon-\alpha)/2})(x)|^p |x|^{\alpha p/2} dx$$
$$\le C \int_{-\infty}^{\infty} \{|f(x)|\,|x|^{(\varepsilon-\alpha)/2}\}^p |x|^{\alpha p/2} dx = C \, \| f(x) x^{\varepsilon/2} \|_p^p$$

for $-1 < \alpha p/2 < p-1$ and $1 < p < \infty$, where C is a constant depending only on α and p (cf. [9, IV.3 Theorem 3.1]). Note that we may also obtain (4.13) by dividing the integral on the right side of (4.10) to a sum of the integrals over $(2^k x, 2^{k+1} x)$, $k = 0, 1, 2, \cdots$ and estimating them pointwise. It follows from (4.11), (4.12) and (4.13) that

$$(4.14) \qquad \| F_\varepsilon^\theta(f) \|_p \le \frac{C}{|\Gamma(\varepsilon+1+i\theta)|} \| f(x) x^{\varepsilon/2} \|_p$$

for $1 > \varepsilon > 0$, $-\infty < \theta < \infty$ and $f \in C_c^\infty$ if $\alpha \ge 0$ and $1 \le p < \infty$, or if $-1 < \alpha < 0$ and $1 < p < -2/\alpha$, where C is a constant independent of ε, θ and f.

We obtain the integral representation

$$D_\varepsilon^\theta(f)(x) = \frac{(\alpha+1)}{\Gamma(\varepsilon+1+i\theta)} \int_x^\infty f(y) e^{-(y-x)/2} y^{(\varepsilon+i\theta)/2} \left(\frac{x}{y}\right)^{\alpha/2} \left(1 - \frac{x}{y}\right)^{\varepsilon+i\theta} \frac{dy}{y}$$

for $D_\varepsilon^\theta(f)$ and thus, we have

$$|D_\varepsilon^\theta(f)(x)| \le \frac{2(\alpha+1)}{|\Gamma(\varepsilon+1+i\theta)|} \int_x^\infty |f(y)|\, y^{\varepsilon/2} \left(\frac{x}{y}\right)^{\alpha/2} \frac{dy}{y},$$

for $\alpha > -1$, $1 > \varepsilon > 0$, $-\infty < \theta < \infty$, $x > 0$ and $f \in C_c^\infty$. The proof is similar to that of (4.9) and is omitted. By Hardy's inequality, we have

$$\int_0^\infty \left\{ \int_x^\infty |f(y)|\, y^{\varepsilon/2} \left(\frac{x}{y}\right)^{\alpha/2} \frac{dy}{y} \right\}^p dx = \int_0^\infty \left\{ \int_x^\infty |f(y)|\, y^{(\varepsilon-\alpha)/2-1} dy \right\}^p x^{\alpha p/2} dx$$
$$\le \left\{ \frac{p}{1+\alpha p/2} \right\}^p \int_0^\infty x^{p+\alpha p/2} |f(x)|^p x^{(\varepsilon-\alpha)p/2-p} dx = C \| f(x) x^{\varepsilon/2} \|_p^p$$

and thus,

$$\| D_\varepsilon^\theta(f) \|_p \le \frac{C}{|\Gamma(\varepsilon+1+i\theta)|} \| f(x) x^{\varepsilon/2} \|_p$$

for $1 \le p < \infty$, $\alpha p/2 > -1$ and $f \in C_c^\infty$, where C is a constant depending only on α and p. Therefore, by (4.5), (4.14) and the last inequality, we have (4.3). The inequalities (4.2) and (4.3) lead us to the desired estimate $\| T_{\alpha,\varphi}^{\alpha+2+i\theta}(f) \|_p \le M(\theta) \| f \|_p$ for $-\infty < \theta < \infty$ and $f \in C_c^\infty$ if $\alpha \ge 0$ and $1 < p < \infty$, or if $-1 < \alpha < 0$ and $(1+\alpha/2)^{-1} < p < -2/\alpha$, where $M(\theta)$ is independent of f, and satisfies the condition (#) in Proposition 1.

REFERENCES

[1] R. ASKEY, A transplantation theorem for Jacobi coefficients, Pacific J. Math. 21 (1967), 393–404.
[2] R. ASKEY, A transplantation theorem for Jacobi series, Illinois J. Math. 13 (1969), 583–590.
[3] R. ASKEY AND S. WAINGER, A transplantation theorem for ultraspherical coefficients, Pacific J. Math. 16 (1966), 393–405.
[4] R. ASKEY AND S. WAINGER, A transplantation theorem between ultraspherical series, Illinois J. Math. 10 (1966), 322–344.
[5] P. L. BUTZER, R. J. NESSEL AND W. TREBELS, On summation processes of Fourier expansions in Banach spaces. I: Comparison theorems, Tôhoku Math. J. 24 (1972), 127–140.
[6] J. DŁUGOSZ, L^p-multipliers for the Laguerre expansions, Colloq. Math. 54 (1987), 287–293.
[7] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, Higher transcendental functions, 3 volumes, McGraw-Hill, New York, 1953.
[8] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, Tables of integral transforms, Vol. I, McGraw-Hill, New York, 1954.
[9] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, Weighted norm inequalities and related topics, North-Holland, Amsterdam, New York, Oxford, 1985.
[10] J. E. GILBERT, Maximal theorems for some orthogonal series I, Trans. Amer. Math. Soc. 145 (1969), 495–515.
[11] D. L. GUY, Hankel multiplier transforms and weighted p-norms, Trans. Amer. Math. Soc. 95 (1960), 137–189.
[12] I. I. HIRSCHMAN, JR., A convexity theorem for certain groups of transformations, J. Analyse Math. 2 (1952/53), 209–218.
[13] B. MUCKENHOUPT, Transplantation theorems and multiplier theorems for Jacobi series, Mem. Amer.

Math. Soc. 64 (1986), No. 356.
[14] S. SCHINDLER, Explicit integral transform proofs of some transplantation theorems for the Hankel transform, SIAM J. Math. Anal. 4 (1973), 367–384.
[15] G. SZEGÖ, Orthogonal polynomials, Amer. Math. Soc. Colloquium Publications, 1975.
[16] G. N. Watson, A treaties on the theory of Bessel functions, Cambridge University Press, London, 1966.

DEPARTMENT OF MATHEMATICS
COLLEGE OF LIBERAL ARTS
KANAZAWA UNIVERSITY
KANAZAWA, 920
JAPAN