A TRANSPLANTATION THEOREM FOR THE HANKEL TRANSFORM ON THE HARDY SPACE

YUICHI KANJIN*

Abstract. The transplantation operators for the Hankel transform are considered and their boundedness on the real Hardy space is established. As its application, we obtain the Hörmander-Mihlin type multiplier theorem for the Hankel transform on the real Hardy space.

1. Introduction. The Hankel transform $\mathcal{H}_\mu f$ of order μ of a function f on the open half line $(0, \infty)$ is defined by

$$\mathcal{H}_\mu f(y) = \int_0^\infty f(t)\sqrt{yt}\, J_\mu(yt)dt\,, \quad y > 0\,,$$

where J_μ is the Bessel function of the first kind of order μ. The Bessel functions with $\mu = -1/2$ and $\mu = 1/2$ are

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z\,, \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z$$

and the Hankel transforms $\mathcal{H}_{-1/2}f$ and $\mathcal{H}_{1/2}f$ are the cosine and sine transforms:

$$\mathcal{H}_{-1/2}f(y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos ytdt\,, \quad \mathcal{H}_{1/2}f(y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin ytdt\,.$$

It is known that for $\mu \geq -1/2$, $\mathcal{H}_\mu$ is an isometry on $L^2(0, \infty)$ (Parseval's theorem for the Hankel transform) and $\mathcal{H}_\mu \mathcal{H}_\mu = I$ (The inversion formula for the Hankel transform), and

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty \mathcal{H}_\mu f(x)\mathcal{H}_\mu g(x)dx$$

for $f, g \in L^2(0, \infty)$ (Plancherel's theorem for the Hankel transform), where I is the identity operator and $L^2(0, \infty)$ is the Lebesgue space of functions on $(0, \infty)$ such that $\|f\|_2 = (\int_0^\infty |f(x)|^2\, dx)^{1/2} < \infty$.

We shall consider the composite

$$\mathcal{T}_\mu^\nu = \mathcal{H}_\mu \mathcal{H}_\nu\,,$$

which is an isometry on $L^2(0, \infty)$ for $\mu, \nu \geq -1/2$. For $f \in L^1(0, \infty)$ with $\mathcal{H}_\nu f \in L^1(0, \infty)$, $\mathcal{T}_\mu^\nu f$ has the integral representation

$$\mathcal{T}_\mu^\nu f(x) = \int_0^\infty \int_0^\infty f(t)\sqrt{yt}J_\nu(yt)dt \sqrt{xy}J_\mu(xy)dy, \quad x > 0.$$

We call $\mathcal{T}_\mu^\nu$ the transplantation operator from ν to μ. The aim of this paper is to prove that the transplantation operators $\mathcal{T}_\mu^\nu$ are bounded on the real Hardy space. As an application, we shall obtain the Hörmander-Mihlin type multiplier theorem for the Hankel transform on the real Hardy space.

   The main tools of our proofs are the atomic decomposition and the molecular character-ization of the real Hardy space, and Schindler's integral representation [13] of $\mathcal{T}_\mu^\nu$. We recall the representation here. Let $T_{\mu,\nu}$ be an operator defined by

$$\begin{aligned}
(1) \qquad T_{\mu,\nu}f(x) &= \lim_{\delta \to +0} \int_{|x-y|>\delta} f(y)\tilde{I}_{\mu,\nu}(x, y)dy + k(\mu, \nu)f(x), \\
k(\mu, \nu) &= \cos((\mu - \nu)\pi/2),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{I}_{\mu,\nu}(x, y) \\
(2) \qquad &= K_{\mu,\nu}\sqrt{xy}\left(\frac{y}{x}\right)^\nu \frac{1}{x^2 - y^2} F\left(\frac{\nu - \mu}{2}, \frac{\mu + \nu}{2}; \nu + 1; \frac{y^2}{x^2}\right) \\
&= 2^{-1}K_{\mu,\nu}\left(\frac{y}{x}\right)^{\nu+1/2}\left(\frac{1}{x-y} + \frac{1}{x+y}\right)F\left(\frac{\nu - \mu}{2}, \frac{\mu + \nu}{2}; \nu + 1; \frac{y^2}{x^2}\right), \\
K_{\mu,\nu} &= \frac{2\Gamma((\mu + \nu + 2)/2)}{\Gamma(\nu + 1)\Gamma((\mu - \nu)/2)}
\end{aligned}$$

for $0 < y < x$, and

$$\tilde{I}_{\mu,\nu}(x, y) = \tilde{I}_{\nu,\mu}(y, x)$$

for $y > x > 0$. Here, $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function, that is,

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^\infty \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k, \quad |z| < 1,$$

where $(\lambda)_0 = 1$, $(\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1)$, $k \geq 1$. If $\mu = \nu+2k$ and $k = 0, 1, 2, \ldots$, then $\tilde{I}_{\mu,\nu}(x, y) = 0$ for $y > x > 0$. If $k = 0, -1, -2, \ldots$, then $\tilde{I}_{\mu,\nu}(x, y) = 0$ for $x > y > 0$. In these cases, $\tilde{I}_{\mu,\nu}(x, y)$ have more elementary forms (see [13]). Schindler proved that if $\mu, \nu \geq -1/2$, then the following (A) and (B) hold:

   (A)   For $f \in C_c^\infty(0, \infty)$, $\mathcal{T}_\mu^\nu f(x) = T_{\mu,\nu}f(x)$ a.e. $x > 0$, where $C_c^\infty(0, \infty)$ is the space of infinitely differentiable functions of compact support in $(0, \infty)$;

   (B)   Let $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$. If $\int_0^\infty |f(x)|^p x^{\alpha p}dx < \infty$, then the value $T_{\mu,\nu}f(x)$ exists for a.e. $x > 0$, and

$$\int_0^\infty |T_{\mu,\nu}f(x)|^p x^{\alpha p}dx \leq C \int_0^\infty |f(x)|^p x^{\alpha p}dx,$$

with a constant C independent of f.

Guy [6] proved that the operators $\mathcal{T}_\mu^\nu$, initially defined on $L^2(0, \infty)$, are extendable to bounded operators on the L^p-spaces, $1 < p < \infty$, and this is the first of the transplantation theorem for classical expansions. Schindler [13] showed a refined version of Guy's result by getting the explicit formula of $\mathcal{T}_\mu^\nu$ as we recalled above.

To consider the transplantation operators $\mathcal{T}_\mu^\nu$ for the case $p = 1$ is our problem, and the main result of this paper is that the operators $\mathcal{T}_\mu^\nu$ are bounded on the real Hardy space, which gives us the Hörmander-Mihlin type multiplier theorem for the Hankel transform on the real Hardy space.

There are transplantation theorems for other orthogonal expansions. Askey and Wainger [2] gave a transplantation theorem for the ultraspherical series, and Askey [1] generalized their theorem to the Jacobi series. Some transplantation theorems are in Gilbert [5] and in Muckenhoupt [12]. The Laguerre series case is in Kanjin [7]. Miyachi [10] and [11] quite recently obtained a transplantation theorem for the Jacobi series in weighted Hardy spaces.

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2. Results. Let $H^1(\mathbf{R})$ be the real Hardy space, that is, the space of the boundary functions $f(x) = \Re F(x)$ of the real parts $\Re F(z)$ of functions $F(z)$ in the Hardy space $H^1(\mathbf{R}_+^2) = \{F(z);\ \text{analytic in} \mathbf{R}_+^2 \text{ and } \|F\|_{H^1(\mathbf{R}_+^2)} = \sup_{t>0} \int_{-\infty}^{\infty} |F(x + it)| dx < \infty\}$ on the upper half plane $\mathbf{R}_+^2 = \{z = x + it\,;\ t > 0\}$, with the norm $\|f\|_{H^1(\mathbf{R})} = \|F\|_{H^1(\mathbf{R}_+^2)}$.

We shall work on the space $H^1(0, \infty)$ defined by

$$H^1(0, \infty) = \{h|_{(0,\infty)}\,;\ h \in H^1(\mathbf{R}),\ \operatorname{supp} h \subset [0, \infty)\},$$

where $[0, \infty)$ is the closed half line, and we endow the space with the norm $\|f\|_{H^1(0,\infty)} = \|h\|_{H^1(\mathbf{R})}$, where $h \in H^1(\mathbf{R})$, $\operatorname{supp} h \subset [0, \infty)$ and $f = h|_{(0,\infty)}$. We remark that

$$H^1(0, \infty) = \{h|_{(0,\infty)}\,;\ h \in H^1(\mathbf{R}),\ \text{even}\}$$

and $c_1\|h\|_{H^1(\mathbf{R})} \le \|f\|_{H^1(0,\infty)} \le c_2\|h\|_{H^1(\mathbf{R})}$ with positive constants c_1 and c_2, where $f = h|_{(0,\infty)}$ and $h \in H^1(\mathbf{R})$ is even. This fact is in [4, Chapter III, Lemma 7.40].

Our theorem is as follows:

THEOREM. (i) *Let* $\mu \ge -1/2$ *and* $\nu > -1/2$. *Then* $\mathcal{T}_\mu^\nu$, *initially defined on* $H^1(0, \infty) \cap L^2(0, \infty)$, *is uniquely extended to a bounded operator on* $H^1(0, \infty)$, *and if we still denote it by* $\mathcal{T}_\mu^\nu$, *then*

$$\|\mathcal{T}_\mu^\nu f\|_{H^1(0,\infty)} \le C\|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0, \infty)$ *with a constant* C *depending only on* μ *and* ν.

(ii) *If* $\mu \ge -1/2$, *then* $\mathcal{T}_\mu^{-1/2}$ *is uniquely extended to a bounded operator from* $H^1(0, \infty)$ *to* $L^1(0, \infty)$, *that is,*

$$\|\mathcal{T}_\mu^{-1/2} f\|_{L^1(0,\infty)} \le C\|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0, \infty)$ *with a constant* C *depending only on* μ *and* ν.

As an application of our theorem we deal with the Hörmander-Mihlin type multiplier theorem for the Hankel transform. Let $\mu \geq -1/2$ and $\phi \in L^\infty(0, \infty)$. We define a Hankel multiplier operator $\mathcal{M}_\phi^\mu$ with multiplier ϕ by

$$\mathcal{M}_\phi^\mu f = \mathcal{H}_\mu(\phi \mathcal{H}_\mu(f))$$

for $f \in L^2(0, \infty)$. Since $\mathcal{H}_\mu$ is an isometry on $L^2(0, \infty)$, the multiplier operator $\mathcal{M}_\phi^\mu$ is a bounded operator on $L^2(0, \infty)$ with the operator norm $\|\phi\|_\infty$. We also define a Fourier multiplier operator $\mathcal{M}_m$ with multiplier $m \in L^\infty(\mathbf{R})$ by

$$\mathcal{M}_m h = \mathcal{F}^{-1}(m\mathcal{F}(h))$$

for $h \in L^2(\mathbf{R})$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively:

$$\mathcal{F}(h)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} h(x)e^{-ix\xi}dx, \quad \mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(\xi)e^{ix\xi}d\xi.$$

The Hörmander-Mihlin multiplier theorem for $H^1(\mathbf{R})$ says that, if m with $\|m\|_{L^\infty(\mathbf{R})} \leq A$ satisfies the condition

$$(3) \qquad \left(\frac{1}{R} \int_{R < |\xi| \leq 2R} \left| \frac{dm(\xi)}{d\xi} \right|^2 d\xi \right)^{1/2} \leq AR^{-1}$$

for $R > 0$, where A is independent of R, then the Fourier multiplier operator $\mathcal{M}_m$, initially defined on $H^1(\mathbf{R}) \cap L^2(\mathbf{R})$, is uniquely extended to a bounded operator on $H^1(\mathbf{R})$. If we still denote it by $\mathcal{M}_m$, then $\|\mathcal{M}_m h\|_{H^1(\mathbf{R})} \leq CA\|h\|_{H^1(\mathbf{R})}$ for $h \in H^1(\mathbf{R})$ with C independent of h and m (cf. [4, Chapter III, Theorem 7.30]). We may refer to [14, Chapter IV, §3, §6] and [4, Chapter II, Theorem 6.3] for the L^p-space case.

COROLLARY. *Let* $\mu \geq -1/2$. *Suppose that* ϕ *with* $\|\phi\|_{L^\infty(0,\infty)} \leq A$ *satisfies the condition*

$$(4) \qquad \left(\frac{1}{R} \int_{R < y \leq 2R} \left| \frac{d\phi(y)}{dy} \right|^2 dy \right)^{1/2} \leq AR^{-1}$$

for $R > 0$, *where* A *is independent of* R. *Then the Hankel multiplier operator* $\mathcal{M}_\phi^\mu$, *initially defined on* $H^1(0, \infty) \cap L^2(0, \infty)$, *is uniquely extended to a bounded operator from* $H^1(0, \infty)$ *to* $L^1(0, \infty)$. *If we also denote it by* $\mathcal{M}_\phi^\mu$, *then*

$$\|\mathcal{M}_\phi^\mu f\|_{L^1(0,\infty)} \leq CA\|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0, \infty)$ *with* C *independent of* f *and* ϕ.

The corollary is deduced from the theorem as follows. Let $\phi \in L^\infty(0, \infty)$ satisfy the condition (4), and let $f \in H^1(0, \infty) \cap C_c^\infty(0, \infty)$. We extend ϕ and f to the functions on $\mathbf{R}$ as even functions, and we denote them by ϕ_e and f_e. Since the function ϕ_e satisfies the condition (3), the Fourier multiplier operator $\mathcal{M}_{\phi_e}$ is a bounded operator on $H^1(\mathbf{R})$. Since $\mathcal{H}_{-1/2} f(y) = \mathcal{F} f_e(y)$, $y > 0$, we see that $\mathcal{M}_{\phi_e} f_e(x) = \mathcal{M}_\phi^{-1/2} f(x)$, $x > 0$. Further,

$\mathcal{M}_{\phi_e} f_e$ is an even function. Thus, $\mathcal{M}_\phi^{-1/2}$ has a unique bounded extension on $H^1(0, \infty)$. The inequality $\|g\|_{L^1(0,\infty)} \le \|g\|_{H^1(0,\infty)}$ holds, and so $\mathcal{M}_\phi^{-1/2}$ is uniquely extended to a bounded operator from $H^1(0, \infty)$ to $L^1(0, \infty)$. Let $\mu > -1/2$. It follows from the theorem that $\mathcal{T}_{-1/2}^\mu$ is a bounded operator on $H^1(0, \infty)$ and $\mathcal{T}_\mu^{-1/2}$ is a bounded operator from $H^1(0, \infty)$ to $L^1(0, \infty)$. Therefore, the identity $\mathcal{M}_\phi^\mu = \mathcal{T}_\mu^{-1/2} \mathcal{M}_\phi^{-1/2} \mathcal{T}_{-1/2}^\mu$ on $L^2(0, \infty)$ implies the corollary.

REMARK. Let $\mu > -1/2$. Assume that $\mathcal{M}_\phi^\mu$ is a bounded operator on $H^1(0, \infty)$. Then, $\phi = 0$ if we assume additionally that ϕ satisfies $\phi = \mathcal{H}_{-1/2}\Phi$ for some $\Phi \in L^1(0, \infty)$. For, we first note that $\mathcal{M}_\phi^{1/2}$ is a bounded operator on $H^1(0, \infty)$ by the identity $\mathcal{M}_\phi^{1/2} = \mathcal{T}_{1/2}^\mu \mathcal{M}_\phi^\mu \mathcal{T}_\mu^{1/2}$ and the theorem. Let $f \in H^1(0, \infty) \cap C_c^\infty(0, \infty)$. Since $\mathcal{M}_\phi^{1/2} f \in H^1(0, \infty)$, $\mathcal{M}_\phi^{1/2} f$ has the vanishing mean property:

$$\int_0^\infty \mathcal{M}_\phi^{1/2} f(x)dx = 0.$$

We extend ϕ and Φ to the even functions on $\mathbf{R}$, and denote them by ϕ_e and Φ_e. We note that $\phi_e = \mathcal{F}\Phi_e$. Further, we extend f to the odd function on $\mathbf{R}$, which is denoted by f_o. Since $-i\mathcal{H}_{1/2}f(y) = \mathcal{F}f_o(y)$, $y > 0$, we see that $\mathcal{M}_\phi^{1/2} f(x) = \mathcal{M}_{\phi_e} f_o(x)$, $x > 0$. The identity $\mathcal{M}_{\phi_e} f_o = \Phi_e * f_o$ holds. Therefore, we have

$$\begin{aligned}
0 &= \int_0^\infty \int_{-\infty}^\infty \Phi_e(y) f_o(x - y)dydx \\
&= \int_{-\infty}^\infty \Phi_e(y) \int_0^\infty f_o(x - y)dxdy \\
&= \int_0^\infty \Phi_e(y) \int_{-y}^\infty f_o(u)dudy + \int_{-\infty}^0 \Phi_e(y) \int_{-y}^\infty f_o(u)dudy \\
&= -2\int_0^\infty \Phi(y) \int_0^y f(u)dudy = -2\int_0^\infty f(u) \int_u^\infty \Phi(y)dydu,
\end{aligned}$$

that is, $\int_0^\infty f(u) \int_u^\infty \Phi(y)dydu = 0$ for all $f \in H^1(0, \infty) \cap C_c^\infty(0, \infty)$, which implies that $\int_u^\infty \Phi(y)dy$ is a constant function in u, and so $\Phi(y) = 0$ for a.e. $y > 0$. We conclude $\phi = 0$.

We conjecture that without the additional condition the above statement holds, that is, if $\mathcal{M}_\phi^\mu$ with $\phi \in L^\infty(0, \infty)$ is a bounded operator on $H^1(0, \infty)$, then ϕ is constant, where $\mu > -1/2$.

The theorem will be proved in the next section. The atomic decomposition and the molecular characterization of the real Hardy space will play important roles in our proof. A real-valued function a is called an atom centered at c if (i) $a(x)$ is supported in an interval $[c - h/2, c + h/2]$, (ii) $\|a\|_2 \le h^{-1/2}$, and (iii) $\int_{\mathbf{R}} a(x)dx = 0$. The space $H^1(\mathbf{R})$ is characterized in terms of atoms: $f \in H^1(\mathbf{R})$ if and only if $f = \sum_{j=0}^\infty \lambda_j a_j$, where each a_j is an atom and $\sum_{j=0}^\infty |\lambda_j| < \infty$. Further, the H^1-norm $\|f\|_{H^1(\mathbf{R})}$ is equivalent to $\inf \sum_{j=0}^\infty |\lambda_j|$,
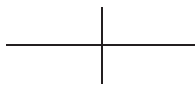
the infimum being taken over all decompositions, and the series $\sum_{j=0}^{\infty} \lambda_j a_j$ converges in H^1-norm.

We deal with the functions $f \in H^1(0, \infty)$. These functions are also characterized as follows (see [4, Chapter III, Lemma 7.40]): $f \in H^1(0, \infty)$ if and only if $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where each a_j is an atom with supp $a_j \subset [0, \infty)$ and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. Moreover, the norm $\|f\|_{H^1[0,\infty)}$ is equivalent to inf $\sum_{j=0}^{\infty} |\lambda_j|$, the infimum being taken over all such decompositions. By this decomposition, we see that $H^1(0, \infty) \cap L^2(0, \infty)$ is dense in $H^1(0, \infty)$.

We call a real-valued function M a molecule centered at c if M satisfies the following conditions: (i)

$$N(M) = \|M\|_{L^2(\mathbf{R})}^{1/2} \| \, | \cdot - c \, | \, M \|_{L^2(\mathbf{R})}^{1/2} < \infty \, ;$$

(ii) $\int_{\mathbf{R}} M(x) dx = 0$. We call $N(M)$ the molecular norm of $M(x)$. The molecular characterization asserts that if $f = \sum_j M_j$ with molecules M_j and $\sum_j N(M_j) < \infty$, then $f \in H^1(\mathbf{R})$ and $\|f\|_{H^1(\mathbf{R})} \leq C \sum_j N(M_j)$ with an absolute constant C. For the atomic decomposition and the molecular characterization, we may refer to [4, III].

3. Proofs. The theorem will be proved by the following two lemmas.

LEMMA 1. *If $\nu > -1/2$, then $\mathcal{T}_\nu^{\nu+2}$ and $\mathcal{T}_{\nu+2}^\nu$ are uniquely extended to bounded operators on $H^1(0, \infty)$, that is,*

$$\|\mathcal{T}_\nu^{\nu+2} f\|_{H^1(0,\infty)} \leq C \|f\|_{H^1(0,\infty)} \, ,$$
$$\|\mathcal{T}_{\nu+2}^\nu f\|_{H^1(0,\infty)} \leq C \|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0, \infty)$ with a constant C depending only on ν.

LEMMA 2. (i) *If $\mu \geq -1/2$ and $\nu \geq 1/2$, then $\mathcal{T}_\mu^\nu$ is uniquely extended to a bounded operator on $H^1(0, \infty)$, that is,*

$$\|\mathcal{T}_\mu^\nu f\|_{H^1(0,\infty)} \leq C \|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0, \infty)$ with a constant C depending only on μ and ν.

(ii) *If $\mu \geq -1/2$, then $\mathcal{T}_\mu^{-1/2}$ is uniquely extended to a bounded operator from $H^1(0, \infty)$ to $L^1(0, \infty)$, that is,*

$$\|\mathcal{T}_\mu^{-1/2} f\|_{L^1(0,\infty)} \leq C \|f\|_{H^1(0,\infty)}$$

for $f \in H^1(0, \infty)$ with a constant C depending only on μ.

We see here that the theorem is deduced from these lemmas. We first note that the identity $\mathcal{T}_\mu^\tau \mathcal{T}_\tau^\nu = \mathcal{T}_\mu^\nu$ on $L^2(0, \infty)$ holds since

$$\mathcal{T}_\mu^\tau \mathcal{T}_\tau^\nu = \mathcal{H}_\mu \mathcal{H}_\tau \mathcal{H}_\tau \mathcal{H}_\nu = \mathcal{H}_\mu \mathcal{H}_\nu = \mathcal{T}_\mu^\nu \, .$$

Let us prove the part (i) of the theorem. Let $\mu \geq -1/2$ and $\nu > -1/2$. It follows from Lemma 1 that $\mathcal{T}_{\nu+2}^\nu$, initially defined on $H^1(0, \infty) \cap L^2(0, \infty)$, is uniquely extended to a bounded operator on $H^1(0, \infty)$. Since $\nu + 2 \geq 1/2$, it follows from the part (i) of Lemma 2 that the operator $\mathcal{T}_\mu^{\nu+2}$ is uniquely extended to a bounded operator on $H^1(0, \infty)$. Because

of the fact $\mathcal{T}_\mu^\nu = \mathcal{T}_\mu^{\nu+2}\mathcal{T}_{\nu+2}^\nu$ on $H^1(0, \infty) \cap L^2(0, \infty)$, we see that $\mathcal{T}_\mu^\nu$ has a unique bounded extension on $H^1(0, \infty)$. The part (ii) of the theorem is the part (ii) of Lemma 2 itself.

We turn to the proof of Lemma 1. Let $\nu > -1/2$, and put

$$U^{(\nu)}f(x) = \int_x^\infty \left(\frac{x}{t}\right)^{\nu+1/2} f(t)\frac{dt}{t}, \quad S^{(\nu)}f(x) = \frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\nu+1/2} f(t)dt$$

for $x > 0$. Then, we see that

$$\mathcal{T}_\nu^{\nu+2}f = 2(\nu+1)U^{(\nu)}f - f, \quad \mathcal{T}_{\nu+2}^\nu f = 2(\nu+1)S^{(\nu)}f - f$$

for $f \in L^2(0, \infty)$ by [13, p. 383, line 5 from below and p. 381, line 8 from below]. In [8, Proposition], we proved that $U^{(\nu)}$ and $S^{(\nu)}$ are extended to bounded operators on $H^1(0, \infty)$ for $\nu > -1/2$, and thus, $\mathcal{T}_\nu^{\nu+2}$ and $\mathcal{T}_{\nu+2}^\nu$ have the same boundedness, which is Lemma 1.

Lemma 2 will be reduced to the following Lemma 3 and Lemma 4.

LEMMA 3. *Assume that* (μ, ν) *satisfies* $\mu \geq -1/2$, $\nu \geq 1/2$ *or* $\mu \geq -1/2$, $\nu = -1/2$. *Let* a *be an atom centered at* c *with* $\operatorname{supp} a \subset [0, \infty)$, *and we regard* $\mathcal{T}_\mu^\nu a$ *as* $\mathcal{T}_\mu^\nu a(x) = 0$ *for* $x \leq 0$. *Then, there exists a constant* C *depending only on* μ *and* ν *such that*

$$(5) \qquad N(\mathcal{T}_\mu^\nu a) = \|\mathcal{T}_\mu^\nu a\|_2^{1/2} \, \| \, |\cdot - c| \, \mathcal{T}_\mu^\nu a \, \|_2^{1/2} \leq C.$$

LEMMA 4. *Let* $\mu \geq -1/2$ *and* $\nu \geq 1/2$. *Then,* $\int_0^\infty \mathcal{T}_\mu^\nu a(x)dx = 0$ *for every atom* a *with* $\operatorname{supp} a \subset [0, \infty)$.

We show first that Lemma 2 is obtained by Lemma 3 and Lemma 4. Let $f \in H^1(0, \infty) \cap L^2(0, \infty)$. Let $f = \sum_{j=0}^\infty \lambda_j a_j$ be an atomic decomposition of f such that $\sum_{j=0}^\infty |\lambda_j| \leq C\|f\|_{H^1(0,\infty)}$, where C is independent of f. To prove Lemma 2, we shall first show that

$$(6) \qquad \mathcal{T}_\mu^\nu f(x) = \sum_{j=0}^\infty \lambda_j \mathcal{T}_\mu^\nu a_j(x) \quad \text{a.e. } x > 0$$

for $\mu \geq -1/2$, $\nu \geq 1/2$ or $\mu \geq -1/2$, $\nu = -1/2$. Let $g \in C_c^\infty(0, \infty)$. Then we have

$$\int_0^\infty \mathcal{T}_\mu^\nu f(x)g(x)dx = \int_0^\infty \mathcal{H}_\mu \mathcal{H}_\nu f(x)g(x)dx = \int_0^\infty f(x)\mathcal{H}_\nu \mathcal{H}_\mu g(x)dx$$

by Plancherel's theorem and the inversion formula. The inequality $|\mathcal{H}_\nu \mathcal{H}_\mu g(x)| \leq C\|\mathcal{H}_\mu g\|_{L^1(0,\infty)}$ holds, and $\|\mathcal{H}_\mu g\|_{L^1(0,\infty)} < \infty$ since $g \in C_c^\infty(0, \infty)$. For every atom a_j, we have $\|a_j\|_{L^1(0,\infty)} \leq 1$. Thus we have

$$\int_0^\infty f(x)\mathcal{H}_\nu \mathcal{H}_\mu g(x)dx = \int_0^\infty \sum_{j=0}^\infty \lambda_j a_j(x)\mathcal{H}_\nu \mathcal{H}_\mu g(x)dx$$

$$= \sum_{j=0}^\infty \lambda_j \int_0^\infty a_j(x)\mathcal{H}_\nu \mathcal{H}_\mu g(x)dx$$

$$= \sum_{j=0}^\infty \lambda_j \int_0^\infty \mathcal{H}_\mu \mathcal{H}_\nu a_j(x)g(x)dx \, .$$

We remark that the inequality $\|\psi\|_{L^1(0,\infty)} \leq 2^{3/2} N(\psi)$ holds (cf. [4, Chapter III, Lemma 7.11]). It follows from Lemma 3 that

$$\|\mathcal{H}_\mu \mathcal{H}_\nu a_j\|_{L^1(0,\infty)} = \|\mathcal{T}_\mu^\nu a_j\|_{L^1(0,\infty)} \leq CN(\mathcal{T}_\mu^\nu a_j) \leq C \,.$$

Here and below, C denotes a positive constant which may differ at each different occurrence. Thus, the last sum is equal to

$$\int_0^\infty \sum_{j=0}^\infty \lambda_j \mathcal{H}_\mu \mathcal{H}_\nu a_j(x) g(x) dx \,,$$

which leads to

$$\int_0^\infty \mathcal{T}_\mu^\nu f(x) g(x) dx = \int_0^\infty \sum_{j=0}^\infty \lambda_j \mathcal{T}_\mu^\nu a_j(x) g(x) dx$$

for all $g \in C_c^\infty(0,\infty)$, and we get (6).

Because of (6), we have

$$\|\mathcal{T}_\mu^\nu f\|_{H^1(0,\infty)} \leq C \sum_{j=0}^\infty N(\lambda_j \mathcal{T}_\mu^\nu a_j) \leq C \sum_{j=0}^\infty |\lambda_j| N(\mathcal{T}_\mu^\nu a_j)$$

$$\leq C \sum_{j=0}^\infty |\lambda_j| \leq C \|f\|_{H^1(0,\infty)}$$

for $\mu \geq -1/2, \nu \geq 1/2$ by Lemma 3, Lemma 4 and the molecular characterization. If $\mu \geq -1/2, \nu = -1/2$, then

$$\|\mathcal{T}_\mu^\nu f\|_{L^1(0,\infty)} \leq \sum_{j=0}^\infty |\lambda_j| \, \|\mathcal{T}_\mu^\nu a_j\|_{L^1(0,\infty)} \leq C \sum_{j=0}^\infty |\lambda_j| N(\mathcal{T}_\mu^\nu a_j)$$

$$\leq C \sum_{j=0}^\infty |\lambda_j| \leq C \|f\|_{H^1(0,\infty)} \,.$$

These inequalities allow us to use the standard density argument, and we obtain Lemma 2.

We now come to the proofs of Lemma 3 and Lemma 4.

PROOF OF LEMMA 3.   Let a be an atom centered at c with supp $a \subset [0,\infty)$. Let $Q = [c - h/2, c + h/2] \subset [0,\infty)$ be the smallest interval containing supp a. Since $\mathcal{T}_\mu^\nu$ is an isometry on $L^2(0,\infty)$, it follows that

(7) $\|\mathcal{T}_\mu^\nu a\|_2 = \|a\|_2 \leq h^{-1/2} \,.$

To prove (5), it is enough to show that $\| \,|\cdot -c| \, \mathcal{T}_\mu^\nu a \,\|_2 \leq Ch^{1/2}$. We put $\tilde{Q} = [c - h, c + h]$. We write

$$\| \,|\cdot -c| \, \mathcal{T}_\mu^\nu a \,\|_2^2 = \left\{ \int_{(0,\infty) \cap \tilde{Q}} + \int_{(0,\infty) \setminus \tilde{Q}} \right\} |x - c|^2 |\mathcal{T}_\mu^\nu a(x)|^2 dx$$

$$= V_1 + V_2 \,, \quad \text{say} \,.$$

For V_1, we have by (7),

$$V_1 \leq h^2 \|\mathcal{T}_\mu^\nu a\|_2^2 \leq h \, .$$

An essential part of the proof is to show $V_2 \leq Ch$. By Schindler's results (A) and (B), we see that $\mathcal{T}_\mu^\nu = T_{\mu,\nu}$ on $L^2(0, \infty)$. Thus, Schindler's integral representation (1) leads us to

$$\mathcal{T}_\mu^\nu a(x) = \lim_{\delta \to +0} \int_{|x-y|>\delta} a(y)\tilde{I}(x, y)dy + k(\mu, \nu)a(x) \quad a.e. \ x > 0 \, ,$$

where we put $\tilde{I}(x, y) = \tilde{I}_{\mu,\nu}(x, y)$ for simplicity. For $x \in (0, \infty) \setminus \tilde{Q}$, we have

$$\mathcal{T}_\mu^\nu a(x) = \int_Q a(y)\tilde{I}(x, y)dy \, ,$$

and thus,

$$V_2 = \int_{(0,\infty)\setminus\tilde{Q}} |x - c|^2 \left| \int_Q a(y)\tilde{I}(x, y)dy \right|^2 dx \, .$$

The Taylor expansion of $\tilde{I}(x, y)$ in y at c and the cancellation property of atoms imply

$$\int_Q a(y)\tilde{I}(x, y)dy = \int_Q a(y)\frac{\partial \tilde{I}}{\partial y}(x, c + \theta(y - c))(y - c)dy, \quad 0 < \theta < 1 \, .$$

If we show

(8) $$\left| \frac{\partial \tilde{I}}{\partial y}(x, \xi) \right| \leq \frac{C}{|x - c|^2}, \quad \xi = c + \theta(y - c), \ 0 < \theta < 1, \ y \in Q, \ x \in (0, \infty) \setminus \tilde{Q}$$

with C depending only on μ and ν, then

$$\left| \int_Q a(y)\tilde{I}(x, y)dy \right| \leq \frac{C}{|x - c|^2} \int_Q |a(y)| \, |y - c|dy$$

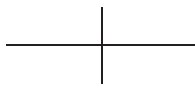$$\leq \frac{C}{|x - c|^2} \|a\|_2 h^{3/2} \leq \frac{C}{|x - c|^2} h \, ,$$

which leads to the desired inequality

$$V_2 \leq Ch^2 \int_{(0,\infty)\setminus\tilde{Q}} \frac{dx}{|x - c|^2} \leq Ch \, .$$

The rest of the proof is devoted to proving (8). We divide the matter into two cases; Case I: $c + h < x$ and $y \in Q$; Case II: $0 < x < c - h$ and $y \in Q$.

We begin with Case I. Since $0 < y < x$, it follows from (2) that

$$\frac{\partial \tilde{I}}{\partial y}(x, y) = 2^{-1}K_{\mu,\nu}\{W_1^+(x, y) + W_2^+(x, y) + W_3^+(x, y)\} \, ,$$

where

$$W_1^+(x, y) = \left(v + \frac{1}{2}\right)\left(\frac{y}{x}\right)^{v-1/2}\frac{1}{x}\left(\frac{1}{x-y} + \frac{1}{x+y}\right)F\left(\frac{v-\mu}{2}, \frac{v+\mu}{2}; v+1; \frac{y^2}{x^2}\right),$$

$$W_2^+(x, y) = \left(\frac{y}{x}\right)^{v+1/2}\left(\frac{1}{(x-y)^2} + \frac{-1}{(x+y)^2}\right)F\left(\frac{v-\mu}{2}, \frac{v+\mu}{2}; v+1; \frac{y^2}{x^2}\right),$$

and

$$W_3^+(x, y) = \left(\frac{y}{x}\right)^{v+1/2}\left(\frac{1}{x-y} + \frac{1}{x+y}\right)\frac{\partial}{\partial y}\left\{F\left(\frac{v-\mu}{2}, \frac{v+\mu}{2}; v+1; \frac{y^2}{x^2}\right)\right\}$$

$$= \frac{v^2 - \mu^2}{2(v+1)}\left(\frac{y}{x}\right)^{v+3/2}\frac{1}{x}\left(\frac{1}{x-y} + \frac{1}{x+y}\right)F\left(\frac{v-\mu+2}{2}, \frac{v+\mu+2}{2}; v+2; \frac{y^2}{x^2}\right)$$

from the formula $(d/dz)F(\alpha, \beta; \gamma; z) = (\alpha\beta/\gamma)F(\alpha+1, \beta+1; \gamma+1; z)$. We shall show

(9) $$|W_j^+(x, \xi)| \leq \frac{C}{|x-c|^2}, \quad j = 1, 2, 3$$

with C depending only on μ and v.

Since

$$\lim_{z \to 1-} F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

for $\Re(\gamma - \alpha - \beta) > 0$ (cf. [9, (9.3.4)]), it follows from $v + 1 - (v - \mu)/2 - (v + \mu)/2 = 1$ and $\xi < x$ that

$$\left|F\left(\frac{v-\mu}{2}, \frac{v+\mu}{2}; v+1; \frac{\xi^2}{x^2}\right)\right| \leq C$$

for $0 < y < x$ with a constant C depending only on μ, v. We see that $(\xi/x)^{v-1/2} \leq 1$ for $0 < y < x$ when $v - 1/2 \geq 0$, and that

$$|W_1^+(x, \xi)| \leq C\frac{1}{x}\left(\frac{1}{|x-\xi|} + \frac{1}{x+\xi}\right).$$

Since $\xi \in Q$ and $c + h < x$, it follows that $|x - \xi| \geq |x - c|/2$. Also, $x + \xi > x > |x - c|$. These imply the inequality (9) with $j = 1$. We note that the term W_1^+ does not appear in $\partial\tilde{I}/\partial y$ when $v = -1/2$.

For $W_2^+(x, \xi)$, in a similar way, we have

$$|W_2^+(x, \xi)| \leq C\left(\frac{1}{|x-\xi|^2} + \frac{1}{(x+\xi)^2}\right) \leq \frac{C}{|x-c|^2}$$

for $v \geq -1/2$, which is the inequality (9) with $j = 2$.

To estimate $W_3^+(x, \xi)$, we use the formula (cf. [9, (9.2.6)]):

$$\gamma(1 - z)F(\alpha, \beta; \gamma; z) - \gamma F(\alpha - 1, \beta; \gamma; z) + (\gamma - \beta)zF(\alpha, \beta; \gamma + 1; z) = 0.$$

The substitution $\alpha = (\nu - \mu + 2)/2$, $\beta = (\nu + \mu + 2)/2$, $\gamma = \nu + 2$, $z = y^2/x^2$ gives

$$F\left(\frac{\nu - \mu + 2}{2}, \frac{\nu + \mu + 2}{2}; \nu + 2; \frac{y^2}{x^2}\right) = \frac{x^2}{x^2 - y^2}F_1 - \frac{\nu - \mu + 2}{2(\nu + 2)}\frac{y^2}{x^2 - y^2}F_2,$$

where

$$F_1 = F\left(\frac{\nu - \mu}{2}, \frac{\nu + \mu + 2}{2}; \nu + 2; \frac{y^2}{x^2}\right),$$

$$F_2 = F\left(\frac{\nu - \mu + 2}{2}, \frac{\nu + \mu + 2}{2}; \nu + 3; \frac{y^2}{x^2}\right).$$

This implies

$$W_3^+(x, \xi) = c_{\mu,\nu}\left(\frac{\xi}{x}\right)^{\nu+3/2}\left(\frac{1}{x - \xi} + \frac{1}{x + \xi}\right)^2 F_1|_{y=\xi}$$

$$+ c'_{\mu,\nu}\left(\frac{\xi}{x}\right)^{\nu+5/2}\left(\frac{1}{(x - \xi)^2} - \frac{1}{(x + \xi)^2}\right) F_2|_{y=\xi},$$

where $c_{\mu,\nu}$ and $c'_{\mu,\nu}$ are some constants depending only on μ and ν. We note that $|F_1|, |F_2| \leq C$ for $0 < y < x$ since $\nu + 2 - (\nu - \mu)/2 - (\nu + \mu + 2)/2 = 1$ and $\nu + 3 - (\nu - \mu + 2)/2 - (\nu + \mu + 2)/2 = 1$. Thus, in the same way as in the above cases, we have the inequality (9) with $j = 3$, which completes Case I.

We turn to Case II. It follows from $0 < x < y$ that

$$\frac{\partial \tilde{I}}{\partial y}(x, y) = 2^{-1}K_{\nu,\mu}\{W_1^-(x, y) + W_2^-(x, y) + W_3^-(x, y)\},$$

$$W_1^-(x, y) = -\left(\mu + \frac{1}{2}\right)\left(\frac{x}{y}\right)^{\mu+1/2}\frac{1}{y}\left(\frac{1}{y - x} + \frac{1}{y + x}\right)F\left(\frac{\mu - \nu}{2}, \frac{\nu + \mu}{2}; \mu + 1; \frac{x^2}{y^2}\right),$$

$$W_2^-(x, y) = \left(\frac{x}{y}\right)^{\mu+1/2}\left(\frac{-1}{(y - x)^2} + \frac{-1}{(y + x)^2}\right)F\left(\frac{\mu - \nu}{2}, \frac{\nu + \mu}{2}; \mu + 1; \frac{x^2}{y^2}\right)$$

and

$$W_3^-(x, y) = \left(\frac{x}{y}\right)^{\mu+1/2}\left(\frac{1}{y - x} + \frac{1}{y + x}\right)\frac{\partial}{\partial y}\left\{F\left(\frac{\mu - \nu}{2}, \frac{\nu + \mu}{2}; \mu + 1; \frac{x^2}{y^2}\right)\right\}$$

$$= -\frac{\mu^2 - \nu^2}{2(\mu + 1)}\left(\frac{x}{y}\right)^{\mu+5/2}\frac{1}{y}\left(\frac{1}{y - x} + \frac{1}{y + x}\right)$$

$$\times F\left(\frac{\mu - \nu + 2}{2}, \frac{\nu + \mu + 2}{2}; \mu + 2; \frac{x^2}{y^2}\right)$$

$$= -c_{\nu,\mu}\left(\frac{x}{y}\right)^{\mu+5/2}\left(\frac{1}{y - x} + \frac{1}{y + x}\right)^2 F_3$$

$$- c'_{\nu,\mu}\left(\frac{x}{y}\right)^{\mu+7/2}\left(\frac{1}{(y - x)^2} - \frac{1}{(y + x)^2}\right) F_4,$$

where

$$F_3 = F\left(\frac{\mu - \nu}{2}, \frac{\nu + \mu + 2}{2}; \mu + 2; \frac{x^2}{y^2}\right),$$

$$F_4 = F\left(\frac{\mu - \nu + 2}{2}, \frac{\nu + \mu + 2}{2}; \mu + 3; \frac{x^2}{y^2}\right).$$

Since $0 < h < c$, it follows that $\xi \geq c - h/2 \geq c/2 \geq |x - c|/2$, which implies $1/(x + \xi) \leq 1/\xi \leq 2/|x - c|$. This inequality and $1/|x - \xi| \leq 2/|x - c|$ allow us to follow the lines of the proof of Case I if $\mu \geq -1/2$, and get the inequality (8) in Case II. We complete the proof of Lemma 3.

PROOF OF LEMMA 4. Let a be an atom with $\text{supp}\, a \subset [0, \infty)$. It follows from Lemma 3 and the inequality $\|\mathcal{T}_\mu^\nu a\|_{L^1(0,\infty)} \leq 2^{3/2} N(\mathcal{T}_\mu^\nu a)$ that $\mathcal{T}_\mu^\nu a$ is integrable for $\mu \geq -1/2, \nu \geq 1/2$ or $\mu \geq -1/2, \nu = -1/2$. Thus, for these μ, ν, we have

$$\int_0^\infty \mathcal{T}_\mu^\nu a(x)dx = \lim_{\varepsilon \to +0} \int_0^\infty e^{-\varepsilon x^2} \mathcal{T}_\mu^\nu a(x)dx.$$

By the fact

$$\mathcal{T}_\mu^\nu a(x) = \lim_{M \to \infty} \int_0^M \mathcal{H}_\nu a(y)\sqrt{xy} J_\mu(xy)dy$$

in $L^2(0, \infty)$, we have

$$\int_0^\infty \mathcal{T}_\mu^\nu a(x)dx = \lim_{\varepsilon \to +0} \lim_{M \to \infty} \int_0^\infty e^{-\varepsilon x^2} \int_0^M \mathcal{H}_\nu a(y)\sqrt{xy} J_\mu(xy)dydx$$

$$= \lim_{\varepsilon \to +0} \lim_{M \to \infty} \int_0^\infty e^{-\varepsilon x^2} \int_0^M \int_0^\infty a(t)\sqrt{yt} J_\nu(yt)dt\sqrt{xy} J_\mu(xy)dydx.$$

Since $|\sqrt{z} J_\alpha(z)| \leq C, z > 0$ for $\alpha \geq -1/2$ and $e^{-\varepsilon x^2} a(t)$ is integrable in (x, y, t) on $(0, \infty) \times (0, M) \times (0, \infty)$, it follows that

$$\int_0^\infty \mathcal{T}_\mu^\nu a(x)dx = \lim_{\varepsilon \to +0} \lim_{M \to \infty} \int_0^\infty a(t) B_M^{(\varepsilon)}(t)dt$$

for $\mu, \nu \geq -1/2$, where

$$B_M^{(\varepsilon)}(t) = \int_0^M D_t^{(\varepsilon)}(y)dy,$$

$$D_t^{(\varepsilon)}(y) = \int_0^\infty e^{-\varepsilon x^2} \sqrt{xy} J_\mu(xy)dx \sqrt{ty} J_\nu(ty).$$

To prove $\int_0^\infty \mathcal{T}_\mu^\nu a(x)dx = 0$, we shall show the following:

(I)   Let $t > 0, 0 < \varepsilon < 1$ and $1 < M$. If $\mu > -3/2, \neq -1$ and $\nu > -1/2$, then $|B_M^{(\varepsilon)}(t)| \leq C$, where C depends only on μ and ν.

(II)   For every $t > 0$, $\lim_{\varepsilon \to +0} \lim_{M \to \infty} B_M^{(\varepsilon)}(t) = C_{\mu,\nu}$, where

$$C_{\mu,\nu} = \frac{\Gamma(\mu/2 + 4/3)\Gamma(\nu/2 + 1/4)}{\Gamma(\mu/2 + 1/4)\Gamma(\nu/2 + 3/4)}$$

when $\mu > -3/2, \neq -1$ and $\nu > -1/2$.

If we show (I) and (II), then by the Lebesgue dominated convergence theorem we shall get

$$\int_0^\infty \mathcal{T}_\mu^\nu a(x)dx = \int_0^\infty a(t) \lim_{\varepsilon \to +0} \lim_{M \to \infty} B_M^{(\varepsilon)}(t)dt = C_{\mu,\nu} \int_0^\infty a(t)dt = 0$$

for $\mu \geq -1/2$ and $\nu \geq 1/2$, and the proof of Lemma 4 will be completed.

Let us prove (I) and (II). We shall use the formula (cf. [15, 13.3(3), p. 394])

$$\int_0^\infty e^{-\varepsilon x^2} \sqrt{xy} J_\mu(xy)dx$$
$$= \frac{y^{\mu+1/2}\Gamma((\mu+3/2)/2)}{2^{\mu+1}\varepsilon^{(\mu+3/2)/2}\Gamma(\mu+1)} e^{-y^2/(4\varepsilon)} \Phi((\mu+1/2)/2; \mu+1; y^2/(4\varepsilon)),$$

where $\mu > -3/2$ and $\Phi(\alpha; \gamma; z)$ is Kummer's confluent hypergeometric series defined by $\Phi(\alpha; \gamma; z) = \sum_{k=0}^\infty [(\alpha)_k/(\gamma)_k][z^k/k!]$ for $z, \alpha, \gamma \in C$, $\gamma \neq 0, -1, -2, \ldots$. Since $\Phi(\alpha; \gamma; z)$ is an entire function of z, it follows that for $0 < y \leq 2\sqrt{\varepsilon}$,

$$(10) \qquad \left| \int_0^\infty e^{-\varepsilon x^2} \sqrt{xy} J_\mu(xy)dx \right| \leq C\varepsilon^{-\mu/2-3/4}y^{\mu+1/2},$$

when $\mu > -3/2$. The asymptotic formula (cf. [3, 6.13.1(3), Vol. 1, p. 278])

$$\Phi(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha-\gamma}[1 + O(|z|^{-1})], \quad \Re z \to \infty, \quad \gamma \neq 0, -1, -2, \ldots$$

gives, for $2\sqrt{\varepsilon} \leq y$,

$$(11) \qquad \int_0^\infty e^{-\varepsilon x^2} \sqrt{xy} J_\mu(xy)dx = C_\mu y^{-1} + R_\varepsilon(y), \quad |R_\varepsilon(y)| \leq C\varepsilon y^{-3},$$

if $\mu > -3/2, \neq -1$, where C depends only on μ and $C_\mu = 2^{1/2}\Gamma((\mu+3/2)/2)/\Gamma((\mu+1/2)/2)$.

Let $t > 0, 0 < \varepsilon < 1$ and $1 < M$. We divide the integral $B_M^{(\varepsilon)}(t) = \int_0^M D_t^{(\varepsilon)}(y)dy$ into two parts:

$$B_M^{(\varepsilon)}(t) = \left\{ \int_0^{2\sqrt{\varepsilon}} + \int_{2\sqrt{\varepsilon}}^M \right\} D_t^{(\varepsilon)}(y)dy.$$

We begin with estimating the integral $\int_0^{2\sqrt{\varepsilon}} D_t^{(\varepsilon)}(y)dy$. By (10) and $|\sqrt{z}J_\nu(z)| \leq C, z > 0$ for $\nu \geq -1/2$, we have

$$(12) \qquad \left| \int_0^{2\sqrt{\varepsilon}} D_t^{(\varepsilon)}(y)dy \right| \leq \int_0^{2\sqrt{\varepsilon}} |D_t^{(\varepsilon)}(y)|dy \leq C\varepsilon^{-\mu/2-3/4} \int_0^{2\sqrt{\varepsilon}} y^{\mu+1/2}dy = C$$

for $\mu > -3/2$ and $\nu \geq -1/2$, where C depends only on μ and ν. Let $t > 0$ be fixed and let $\varepsilon > 0$ be sufficiently small so that $2\sqrt{\varepsilon} < 1/t$. By (10) and the fact $J_\nu(z) = O(z^\nu)$ $(z \to 0)$

for $\nu \neq -1, -2, \ldots$, we have

$$\left| \int_0^{2\sqrt{\varepsilon}} D_t^{(\varepsilon)}(y) dy \right| \leq C \varepsilon^{-\mu/2 - 3/4} \int_0^{2\sqrt{\varepsilon}} y^{\mu + 1/2} (ty)^{\nu + 1/2} dy = C t^{\nu + 1/2} \varepsilon^{\nu/2 + 1/4}.$$

Thus, for every $t > 0$, we have

$$(13) \qquad \lim_{\varepsilon \to 0} \int_0^{2\sqrt{\varepsilon}} D_t^{(\varepsilon)}(y) dy = 0$$

when $\mu > -3/2$ and $\nu > -1/2$.

We next estimate the integral $\int_{2\sqrt{\varepsilon}}^M D_t^{(\varepsilon)}(y) dy$. By (11), we have

$$\int_{2\sqrt{\varepsilon}}^M D_t^{(\varepsilon)}(y) dy = C_\mu U_1 + U_2,$$

where

$$U_1 = \int_{2\sqrt{\varepsilon}}^M \sqrt{ty} J_\nu(ty) y^{-1} dy, \quad U_2 = \int_{2\sqrt{\varepsilon}}^M \sqrt{ty} J_\nu(ty) R_\varepsilon(y) dy$$

for $\mu > -3/2, \neq -1$.

The integral U_2 is estimated by (11) and $|\sqrt{z} J_\nu(z)| \leq C, z > 0$ for $\nu \geq -1/2$. We have

$$(14) \qquad |U_2| \leq C \varepsilon \int_{2\sqrt{\varepsilon}}^\infty y^{-3} dy \leq C$$

for $\mu > -3/2, \neq -1$ and $\nu \geq -1/2$. Let $t > 0$ be fixed, and let $\varepsilon > 0$ be sufficiently small and M be sufficiently large so that $2\sqrt{\varepsilon} < 1/t < M$. We divide the integral as follows:
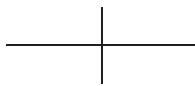
$$U_2 = \left\{ \int_{2\sqrt{\varepsilon}}^{1/t} + \int_{1/t}^M \right\} \sqrt{ty} J_\nu(ty) R_\varepsilon(y) dy = U_2^1 + U_2^2, \quad \text{say}.$$

By the fact $J_\nu(z) = O(z^\nu) \, (z \to 0)$ for $\nu \neq -1, -2, \ldots$, we have

$$|U_2^1| \leq C \int_{2\sqrt{\varepsilon}}^{1/t} (ty)^{\nu + 1/2} \varepsilon y^{-3} dy$$

$$\leq \begin{cases} C \int_0^{1/t} (ty)^{\nu + 1/2} \varepsilon y^{-3} dy \leq C t^2 \varepsilon & (\nu > 3/2), \\ C \int_{2\sqrt{\varepsilon}}^{1/t} t^2 \varepsilon y^{-1} dy \leq C t^2 \varepsilon (|\log t| + \log(1/\varepsilon)) & (\nu = 3/2), \\ C \int_{2\sqrt{\varepsilon}}^\infty (ty)^{\nu + 1/2} \varepsilon y^{-3} dy \leq C t^{\nu + 1/2} \varepsilon^{(\nu + 1/2)/2} & (\nu < 3/2) \end{cases}$$

for $\mu > -3/2, \neq -1$ and $\nu \neq -1, -2, \ldots$. It follows from the fact $\sqrt{z} J_\alpha(z) = O(1) (z \to \infty)$ that

$$|U_2^2| \leq C \int_{1/t}^\infty \varepsilon y^{-3} dy \leq C t^2 \varepsilon.$$

Therefore, we have

$$\text{(15)} \qquad \lim_{\varepsilon \to 0} \lim_{M \to \infty} U_2 = 0$$

for $\mu > -3/2, \neq -1$ and $\nu > -1/2$.

We turn to estimating U_1. We first deal with the case $2\sqrt{\varepsilon} \le 1/t \le M$ and divide the integral:

$$U_1 = \left\{ \int_{2\sqrt{\varepsilon}}^{1/t} + \int_{1/t}^{M} \right\} \sqrt{ty} J_\nu(ty) y^{-1} dy = U_1^1 + U_1^2, \quad \text{say}.$$

By the fact $J_\nu(z) = O(z^\nu)\,(z \to 0)$ for $\nu \neq -1, -2, \ldots$, we have $|U_1^1| \le C \int_0^{1/t} (ty)^{\nu+1/2} y^{-1} dy$. Thus, if $\nu > -1/2$, then $|U_1^1| \le C$. Let us evaluate U_1^2. The Bessel function satisfies $(d/dz) z^{-\alpha} J_\alpha(z) = -z^{-\alpha} J_{\alpha+1}(z)$. This and integration by parts lead to

$$\begin{aligned} U_1^2 &= (-t^{\nu-3/2}) \int_{1/t}^{M} y^{\nu-3/2} \frac{d}{dy}\left((ty)^{-\nu+1} J_{\nu-1}(ty) \right) dy \\ &= (-t^{\nu-3/2})[y^{\nu-3/2}(ty)^{-\nu+1} J_{\nu-1}(ty)]_{1/t}^{M} \\ &\quad - (-t^{\nu-3/2}) \int_{1/t}^{M} (ty)^{-\nu+1} J_{\nu-1}(ty) \frac{d}{dy}(y^{\nu-3/2}) dy = U_1^{2,1} + U_1^{2,2}, \quad \text{say}. \end{aligned}$$

The first term $U_1^{2,1} = -(tM)^{-1/2} J_{\nu-1}(tM) + J_{\nu-1}(1)$ satisfies $|U_1^{2,1}| \le C$ since $\sqrt{z} J_\alpha(z) = O(1)(z \to \infty)$ and $1 \le tM$. The second term

$$U_1^{2,2} = \left(\nu - \frac{3}{2}\right) \frac{1}{t} \int_{1/t}^{M} y^{-2} \sqrt{ty} J_{\nu-1}(ty) dy$$

is evaluated as follows: $|U_1^{2,2}| \le Ct^{-1} \int_{1/t}^{\infty} y^{-2} dy \le C$. Thus, we have $|U_1^2| \le C$ and then $|U_1| \le C$ in the case $2\sqrt{\varepsilon} \le 1/t \le M$. In the case $1/t < 2\sqrt{\varepsilon}$, we have $|U_1| \le C$ in the same way as in the estimation of U_1^2, and in the case $M < 1/t$, we also have $|U_1| \le C$ in the same way as in the estimation of U_1^1. Therefore, these and (14) imply

$$\text{(16)} \qquad \left| \int_{2\sqrt{\varepsilon}}^{M} D_t^{(\varepsilon)}(y) dy \right| \le C$$

for $\mu > -3/2, \neq -1$ and $\nu > -1/2$.

Combining (12) and (16), we have (I). The statement (II) is proved as follows: By (13) and (15), we have

$$\begin{aligned} \lim_{\varepsilon \to +0} \lim_{M \to \infty} B_M^{(\varepsilon)}(t) &= C_\mu \lim_{\varepsilon \to +0} \lim_{M \to \infty} U_1 = C_\mu \int_0^{\infty} \sqrt{ty} J_\nu(ty) y^{-1} dy \\ &= C_\mu \int_0^{\infty} J_\nu(u) u^{-1/2} du = C_{\mu,\nu} \end{aligned}$$

for every $t > 0$ when $\mu > -3/2, \neq -1$ and $\nu > -1/2$. We here used $\int_0^{\infty} J_\nu(u) u^{-1/2} du = \Gamma(\nu/2 + 1/4)/(\Gamma(\nu/2 + 3/4)\sqrt{2})$ for $\nu > -1/2$.

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DEPARTMENT OF MECHANICAL SYSTEMS ENGINEERING
FACULTY OF ENGINEERING
KANAZAWA UNIVERSITY
KANAZAWA 920–8667
JAPAN

E-mail address: kanjin@kenroku.kanazawa-u.ac.jp