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A TRIANGULATION OF $\mathrm{GL}(n, F)$

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Abstract. Let F be a non-Archimedean field. We prove that each open and compact subset of $\mathrm{GL}_n(F)$ can be decomposed into finitely many open, compact, and self-conjugate subsets. As a corollary, we obtain a short, elementary proof of a well-known theorem of I.M. Gelfand and D.A. Kazhdan.

1. Introduction

Notation. Throughout this paper, F will denote a non-archimedean local field. By ν, $\mathcal{O}_F$, ω, we denote the valuation, the ring of integers, and the uniformizer of F, respectively.

Let $K = \mathrm{GL}(n, \mathcal{O}_F)$ be the maximal compact subgroup of the general linear group $G = \mathrm{GL}(n, F)$. By $\mathrm{End}(n, F)$, $\mathrm{End}(n, \mathcal{O}_F)$ we denote the set of $n \times n$ matrices with entries in F, $\mathcal{O}_F$, respectively.

The transpose of a matrix $a \in G$ is denoted by $a^\top$. Similarly, we denote by $\mathcal{A}^\top$ the action of the transposing operator on a subset $\mathcal{A} \subset \mathrm{End}(n, F)$.

For a given representation π of G we let $\widetilde{\pi}$ denote its contragredient representation.

Definition 1. A set $\mathcal{A} \subset \mathrm{End}(n, F)$ is called self-conjugate under the transposing operator if there exists at least one element $x \in G$ such that $\mathcal{A}^\top = x\mathcal{A}x^{-1}$.

The goal of this paper is to prove the following triangulation theorem.

Theorem 1. *Let $C \subset G$ be an open and compact set. There exist finitely many open and compact sets $C_1, \dots, C_s \subset C$ such that*

1) $C = \bigcup_{i=1}^s C_i$ *and* $C_i \cap C_j = \emptyset$ *if* $i \neq j$;
2) *each* C_i *is self-conjugate under the transposing operator.*

The theorem will be proved in the next section. As an application of Theorem 1 we shall give a new proof of the next classical result.

Theorem 2. *Let π be an irreducible admissible representation of G. If $\widehat{\pi}$ is the representation given by $\widehat{\pi}(g) = \pi(g^\top)^{-1}$, then $\widetilde{\pi}$ and $\widehat{\pi}$ are isomorphic.*

Theorem 2 goes back to Gelfand and Kazhdan. In fact, in [4], the more general case of a regular group G is discussed. Later, Bernstein and Zelevinsky gave a different proof for $\mathrm{GL}(n, F)$ (see [1], Ch. III, §7). A key tool for Gelfand and Kazhdan's arguments is a result concerning the existence of certain geometrical factor spaces for the action of an algebraic group on an algebraic variety (see Deligne's

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Theorem 5.8.1 in [3], or Rosenlicht's theorem in [5]). Bernstein and Zelevinsky do not make use of this result, the core of their arguments being based on Gelfand and Kazhdan's theory of derivations of representations. A nice exposition of these ideas, together with a proof for the case of GL$(2, F)$, can also be found in [2, pp. 433–451].

The proof we give for Theorem 2 is elementary and self-contained. It starts from the observation that if a compact open $\mathcal{A} \subset \mathrm{GL}(n, F)$ is self-conjugate under the transposing operator, then the operators $\pi(\mathrm{char}_{\mathcal{A}})$ and $\pi(\mathrm{char}_{\mathcal{A}^\top})$ have the same trace. Using this fact together with a standard argument (see §3), we show that Theorem 2 is a consequence of Theorem 1.

The organization of this paper is as follows. In §2 we introduce the necessary tools and give the proof of Theorem 1. In §3 we show how Theorem 1 implies Theorem 2. At the end of the same section we give a very simple proof of Theorem 2, independent of Theorem 1, for the case $n = 2$. This proof is based on the observation that in GL$(2, F)$, a matrix and its transpose are in the same K-orbit with respect to the action by conjugation. The appendix contains a known algebraic result which is used in an essential way in §2.

2. Proofs of results

The proof of Theorem 1 is organized along the following lines. In Lemma 1 we introduce a set of conditions under which one can describe a fundamental system of self-conjugate neighborhoods for a given element $a \in G$. In Lemmas 2 and 3 we show that these conditions can always be satisfied. The purpose of Lemma 4 is to show that one can partition $G = G^0 \cup G^1 \cup \ldots$, such that for each $i \geq 0$ and $a, b \in G^i$, the fundamental systems of self-conjugate neighborhoods for a, b, constructed via Lemmas 1, 2, and 3, are translates of each other. This idea is then used to write any open and compact C as a union of self-conjugate, relatively disjoint parts.

Lemma 1. *Let $\mathcal{A} \subset \mathrm{End}(n, \mathcal{O}_F)$ be a compact and open set for which $0 \in \mathcal{A}$, and let $a \in G$. If there exists an element $x \in G$ satisfying the conditions*

$$a^\top = xax^{-1}, \;\; \mathcal{A}^\top = x\mathcal{A}x^{-1},$$

then the set $a(1 + \omega^\mu \mathcal{A})$ $(\mu \geq 1)$, is a compact, open neighborhood of a in G which is self-conjugate under the transposing operator.

Proof. We only need to check that $a(1 + \omega^\mu \mathcal{A})$ is self-conjugate under the transposing operator. Indeed,

$$\begin{aligned}(a(1 + \omega^\mu \mathcal{A}))^\top = (1 + \omega^\mu \mathcal{A})^\top a^\top &= x(1 + \omega^\mu \mathcal{A})ax^{-1} \\ &= (xa^{-1})a(1 + \omega^\mu \mathcal{A})(xa^{-1})^{-1}.\end{aligned}$$

□

Lemma 2. *Let $a \in G$. Then there exists a symmetric matrix $x \in G$, such that $a^\top = xax^{-1}$.*

Proof. The desired matrix identity can be rewritten as

$$a^\top x = xa,$$

and using the symmetry condition on x, the above equation becomes a system in the variables x_{ij} $(i \leq j)$. Denote by $x_1, \ldots, x_r$ the independent variables of this system, all the other variables being linear combinations over F, of $x_1, \ldots, x_r$. We want to get an F-solution of this system satisfying the extra-condition that the matrix x has a nonzero determinant. This translates into a nonvanishing condition for a polynomial in the variables $x_1, \ldots, x_r$, say

$$P(x_1, \ldots, x_r) \neq 0, \quad (P \in F[X_1, \ldots, X_r]).$$

If by absurd, P vanishes completely over F, then $P \equiv 0$, so P will vanish completely over $\overline{F}$. This contradicts the appendix lemma. The proof is complete. $\square$

Lemma 3. *Let $x \in G$ be a symmetric matrix. Then the compact, open subset $\mathcal{A}(x) \subset \mathrm{End}(n, \mathcal{O}_F)$ given by*

$$\mathcal{A}(x) := (x^{-1} \cdot \mathrm{End}(n, \mathcal{O}_F) \cdot x) \cap \mathrm{End}(n, \mathcal{O}_F),$$

satisfies the identity $\mathcal{A}(x)^\top = x\mathcal{A}(x)x^{-1}$.

Proof. Since x is symmetric, we have

$$\mathcal{A}(x)^\top = (x \cdot \mathrm{End}(n, \mathcal{O}_F) \cdot x^{-1}) \cap \mathrm{End}(n, \mathcal{O}_F) = x\mathcal{A}(x)x^{-1}.$$

$\square$

Lemma 4. *For every $m \geq 1$, the following family is finite:*

$$\mathcal{F}_m := \{\mathcal{A}(x) \ \ : \ \ x \in G \cap \mathrm{End}(n, \mathcal{O}_F), x - \textit{symmetric}, \nu(\det x) = m\},$$

where $\mathcal{A}(x)$ is the set defined in Lemma 3.

Proof. Let $x \in G$ be a symmetric matrix. By the Cartan decomposition, we have

$$x = \alpha d \beta$$

where $\alpha, \beta \in K$, $d = \mathrm{diag}(\omega^{\mu_1}, \ldots, \omega^{\mu_n})$, and $0 \leq \mu_1 \leq \ldots \leq \mu_n$. Using the fact that $\mathrm{End}(n, \mathcal{O}_F)$ is invariant under the multiplicative action of K, the subset $\mathcal{A}(x)$ can be written

$$\mathcal{A}(x) = \beta^{-1}((d^{-1} \cdot \mathrm{End}(n, \mathcal{O}_F) \cdot d) \cap \mathrm{End}(n, \mathcal{O}_F))\beta = \beta^{-1}\mathcal{A}(d)\beta.$$

Moreover, $\mathcal{A}(d)$ is invariant under the multiplicative action of $(d^{-1}Kd) \cap K$, the latter being a subgroup of finite index in K. Therefore, one can conclude that for a fixed diagonal matrix d, the family

$$\{\mathcal{A}(x) \ \ : \ \ x = \alpha d \beta, \ \alpha, \beta \in K\},$$

is finite, so $\mathcal{F}_m$ is finite. $\square$

Proof of Theorem 1. We may assume that $C = cK_{\lambda_0}$, where $c \in G, \lambda_0 \geq 1$. In particular, $aK_{\lambda_0} \subseteq C$ for all $a \in C$. Here $K_{\lambda_0} \subset K$ is the kernel of the reduction map

$$K \ni k \longmapsto k \pmod{\omega^{\lambda_0}} \in \mathrm{GL}(n, \mathcal{O}_F/\omega^{\lambda_0}\mathcal{O}_F).$$

By Lemma 2, for each element $a \in C$ there exists a symmetric matrix $x_a \in G$ with the property that $a^\top = x_a a x_a^{-1}$. Multiplying by an appropriate scalar, we may assume that all entries of x_a are in $\mathcal{O}_F$ and their gcd is 1. If $i \geq 0$, we define $C^i \subset C$ to be the set

$$C^i = \{a \in C \ \ : \ \ \nu(\det x_a) = i\}.$$

Since $\mathcal{F}_0 = \{\mathrm{End}(n, \mathcal{O}_F)\}$ (by Lemma 3), we shall associate to all $a \in C^0$ the group $H_a (= K_{\lambda_0})$,

$$H_a := 1 + \omega^{\lambda_0}\mathrm{End}(n, \mathcal{O}_F) = 1 + \omega^{\lambda_0}\mathcal{A}(x_a).$$

By Lemmas 1, 2, and 3, the set $N_a := aH_a$ is a compact, open neighborhood of a in C which is self-conjugate under the transposing operator.

Now let $\mathcal{F}_1 = \{\mathcal{A}_1, \dots, \mathcal{A}_{r_1}\}$. Since all $\mathcal{A}(x)$ are compact neighborhoods of 0, we can find some weights $\lambda_1, \dots, \lambda_{r_1}$, so that the following inclusions are true:

$$\omega^{\lambda_0}\mathrm{End}(n, \mathcal{O}_F) \supset \omega^{\lambda_1}\mathcal{A}_1 \supset \omega^{\lambda_2}\mathcal{A}_2 \supset \dots \omega^{\lambda_{r_1}}\mathcal{A}_{r_1}.$$

To each $a \in C^1$ we associate the subgroup $H_a \subset K_{\lambda_0}$,

$$H_a := 1 + \omega^{\lambda_i}\mathcal{A}_i,$$

where $\mathcal{A}_i = \mathcal{A}(x_a)$. As before, $N_a := aH_a$ is a compact, open neighborhood of a in C which is self-conjugate under the transposing operator.

Inductively on i, we can associate to each $a \in C^i$ a neighborhood $N_a = aH_a$ which is self-adjoint under the transposing operator, where H_a is a compact, open subgroup of K_{λ_0}. Moreover, with respect to the inclusion relation $\supset$, the family

$$\{H_a \ : \ a \in C\}$$

is a lattice isomorphic to $(\mathbb{N}, \leq)$. This way we can conclude that

$$N_{a_1} \cap N_{a_2} \neq \emptyset \text{ if and only if } N_{a_1} \subset N_{a_2} \text{ or } N_{a_2} \subset N_{a_1}. \tag{1}$$

By (1) and the compactness of C, one can find a finite relatively disjoint cover of C with subsets of the form N_a, which proves the theorem. □

3. Applications

3.1. **Proof of Theorem 2.** We start this section by showing how Theorem 2 can be reduced to Theorem 1. This is accomplished by using the following lemma (see [4](2), pp. 103–104.

Lemma 5. *Let π be an irreducible admissible representation of G. Then $\widetilde{\pi}$ is isomorphic to $\widehat{\pi}$ if and only if for all open compact $C \subset G$, we have*

$$\mathrm{Tr}\left(\pi(\mathrm{char}_C)\right) = \mathrm{Tr}\left(\pi(\mathrm{char}_{C^\top})\right).$$

Proof. Since π is irreducible, the representations $\widehat{\pi}$ and $\widetilde{\pi}$ will be irreducible. In order to show that $\widehat{\pi}$ and $\widetilde{\pi}$ are isomorphic, it is enough to prove that their characters are the same, i.e.,

$$\mathrm{Tr}(\widehat{\pi}(f)) = \mathrm{Tr}(\widetilde{\pi}(f)), \quad \text{for all } \ f \in \mathcal{H}(G), \tag{2}$$

where $\mathcal{H}(G)$ is the Hecke algebra of G. If for $f \in \mathcal{H}(G)$ one defines $f^-(g) = f(g^{-1})$, then it is known that $\mathrm{Tr}(\widetilde{\pi}(f)) = \mathrm{Tr}(\pi(f^-))$, while a change of variables shows that $\widehat{\pi}(f) = \pi((f^-)^\top)$, where $f^\top(g) = f(g^\top)$. Thus relation (2) is equivalent to

$$\mathrm{Tr}(\pi(f)) = \mathrm{Tr}(\pi(f^\top)), \quad \text{for all } \ f \in \mathcal{H}(G). \tag{3}$$

Our conclusion follows from the fact that f is a linear combination of characteristic functions of compact open subsets of G. □

Proof of Theorem 2. By Theorem 1 and Lemma 5, it is sufficient to show that if $C \subseteq G$ is compact open and $C^\top = cCc^{-1}$, then

$$\pi(\mathrm{char}_{C^\top}) = \pi(c)\pi(\mathrm{char}_C)\pi(c)^{-1}.$$

This is a simple change of variables. □

3.2. **The case $n = 2$.** The following lemma is the key tool when $n = 2$. It is not true for higher n, thus the argument works only in this particular case.

Lemma 6. *If $a \in \mathrm{End}(2, F)$, then there exists $x \in K$ satisfying $a^\top = xax^{-1}$.*

Proof. The equation $xa = a^\top x$ can be written as a four-dimensional linear system:

$$\begin{cases} a_{11}x_{11} + a_{21}x_{12} = a_{11}x_{11} + a_{21}x_{21}, \\ a_{12}x_{11} + a_{22}x_{12} = a_{11}x_{12} + a_{21}x_{22}, \\ a_{11}x_{21} + a_{21}x_{22} = a_{12}x_{11} + a_{22}x_{21}, \\ a_{12}x_{21} + a_{22}x_{22} = a_{12}x_{12} + a_{22}x_{22}, \end{cases}$$

where $a = (a_{ij}), x = (x_{ij})$, $(1 \le i, j \le 2)$. By imposing the symmetry condition $x_{21} = x_{12}$ we can see that the first and the fourth equations are true, and the second and the third are the same. So the system reduces to the equation

$$(a_{22} - a_{11})x_{12} + a_{12}x_{11} - a_{21}x_{22} = 0.$$

This can be solved in $\mathcal{O}_F$ as follows.

Case 1: $\nu(a_{22} - a_{11}) \ge \min\{\nu(a_{12}), \nu(a_{21})\}$. Without loss of generality, we may assume that $\nu(a_{12}) \ge \nu(a_{21})$. Then the numbers

$$x_{12} = 1,\ x_{11} = 0,\ x_{22} = a_{21}^{-1}(a_{22} - a_{11}),$$

form a solution to our equation.

Case 2: $\nu(a_{11} - a_{22}) < \min\{\nu(a_{12}), \nu(a_{21})\}$. Then the numbers

$$x_{12} = (a_{11} - a_{22})^{-1} \cdot (a_{12} - a_{21}),\ x_{11} = 1,\ x_{22} = 1,$$

form a solution to our equation. It is now easy to see that in both cases the discriminant of x is a unit $\mathcal{O}_F$, i.e., $x \in \mathrm{GL}(2, \mathcal{O}_F)$ as required. $\square$

Lemma 7. *Let $a \in G$. Then aK_λ and $(aK_\lambda)^\top$ are conjugate for all $\lambda \ge 1$.*

Proof. By Lemma 6, there exists $x \in K$ such that $a^\top = xax^{-1}$. Then

$$(aK_\lambda)^\top = K_\lambda^\top a^\top = K_\lambda xax^{-1} = xK_\lambda ax^{-1} = (xa^{-1})aK_\lambda(xa^{-1})^{-1}.$$

$\square$

Proof of Theorem 2 (under the assumption $n = 2$). Replace Theorem 1 by Lemma 7 in the corresponding argument of §3.1. $\square$

4. Appendix

The following lemma is part of the mathematical folklore. We include it for the sake of completeness.

Lemma 8. *Let $a \in \mathrm{GL}(n, F)$. Let $\overline{F}$ be an algebraic closure of F. There exists a symmetric matrix $x \in \mathrm{GL}(n, \overline{F})$ satisfying the identity*

$$a^\top = xax^{-1}.$$

Proof. Over the field $\overline{F}$, the matrix a can be brought to a Jordan canonical form with $i \times i$ diagonal cells of the form

$$J_\lambda := \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 1 & \lambda & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda \end{pmatrix} \quad (\lambda \in \overline{F},\ 1 \le i \le n).$$

If w is the $i \times i$ matrix with 1 on the anti-diagonal and zero in the rest, then the following identity is true:

$$J_\lambda^\top = wJ_\lambda w^{-1}.$$

We can conclude that there exist matrices $\gamma \in \mathrm{GL}(n, \overline{F})$, $\alpha \in \mathrm{GL}(n, \mathbb{Z})$, such that α is symmetric, $\gamma a \gamma^{-1}$ is the Jordan canonical form of a, and

$$(\gamma a \gamma^{-1})^\top = \alpha(\gamma a \gamma^{-1})\alpha^{-1}.$$

But this is equivalent to $a^\top = (\gamma^\top \alpha\gamma)a(\gamma^\top\alpha\gamma)^{-1}$. □

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