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A Tutorial on Design Analysis for Random Vibration

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The von Mises stress is often used as the metric for evaluating design margins, particularly for structures made of ductile materials. While computing the von Mises stress distribution in a structural system due to a deterministic load condition may be straightforward, difficulties arise when considering random vibration environments. As a result, alternate methods are used in practice. One such method involves resolving the random vibration environment to an equivalent static load. This technique, however, is only appropriate for a very small class of problems and can easily be used incorrectly. Monte Carlo sampling of numerical realizations that reproduce the second order statistics of the input is another method used to address this problem. This technique proves computationally inefficient and provides no insight as to the character of the distribution of von Mises stress.

This tutorial describes a new methodology to investigate the design reliability of structural systems in a random vibration environment. The method provides analytic expressions for root mean square (RMS) von Mises stress and for the probability distributions of von Mises stress which can be evaluated efficiently and with good numerical precision. Further, this new approach has the important advantage of providing the asymptotic properties of the probability distribution. A brief overview of the theoretical development of the methodology is presented, followed by detailed instructions on how to implement the technique on engineering applications. As an example, the method is applied to a complex finite element model of a Global Positioning Satellite (GPS) system. This tutorial presents an efficient and accurate methodology for correctly applying the von Mises stress criterion to complex computational models. The von Mises criterion is the traditional method for determination of structural reliability issues in industry.

Introduction

The primary purpose of finite element stress analysis is to estimate the reliability of engineering designs. In structural applications, the von Mises stress due to a given load is often used as the metric for evaluating design margins. For deterministic loads, both static and dynamic, the calculation of von Mises stress is straightforward, e.g. [6]. For loads modeled as random processes, the task is different; the responses to such loads are themselves random processes and the properties must be determined in terms of those of both the loads and the system. Hence, both the input and output quantities exhibit statistical behavior.

There are many ways to analyze such systems (see for example [3], [7] or [2]). Typically, when considering a linear system subject to ductile failure, input forces are specified by their auto spectral densities. In the case of multiple force inputs, the forces may be specified by a cross spectral density matrix. Herein, it is demonstrated how that information can be used to calculate the prob-

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ability distributions for the von Mises stress at different locations on the body. Computation of the root mean squared (RMS) von Mises stress may be accomplished as part of that calculation.

Theoretical Development

A detailed theoretical development is available in [4,5]; only the equations required for understanding the analysis procedure are presented here. Methods to calculate the RMS von Mises stress are discussed, followed by the primary relations for development of the probability distribution function (PDF) of von Mises stress at arbitrary locations in the model.

RMS value

When the applied random load involves either forces applied at several locations or forces applied at one location but in more than one direction, the loads are usually represented by the cross spectral density matrix [1]

$$S_{FF}(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} E[\bar{F}(\omega, T) F(\omega, T)^T], \tag{1}$$

where $F(\omega, T)$ is the finite Fourier transform of the vector of force components sampled over a period T; $(\bullet)^T$ denotes the matrix transpose; $(\bar{\bullet})$ denotes the complex conjugate; and $E[\bullet]$ is the operator of mathematical expectation. In the case of a single scalar input force, this reduces to the auto spectral density.

The stress at the point in question can be assembled from the contributions of each mode:

$$\sigma(t, x) = \sum_n q_n(t) \Psi_n(x), \tag{2}$$

where q_n is the nth modal coordinate and $\Psi_n(x)$ is the stress vector at location x associated with that mode, comprised of the six non-redundant terms for the stress tensor.

The square of the von Mises stress can be expressed as a quadratic operator on the stress vector

$$p^2(t, x) = \sigma(t, x)^T A\, \sigma(t, x), \tag{3}$$

where

$$\sigma(t,x) = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{4}$$

It can be shown (as in [4]) that the RMS von Mises stress can be written as

$$\langle p^2 \rangle = \sum_{i,j}^{\#\text{ modes}} W_{ij} B_{ij}, \tag{5}$$

where B is composed of the modal contributions of the components of the stress modes

$$B_{mn} = \Psi_m^T A \Psi_n, \tag{6}$$

and

$$W_{ij} = \sum_{a,a'}^{N_{force}} \varphi_{ai} \varphi_{a'j} \frac{1}{T} \sum_{n}^{N_\omega} \overline{d}_i(n) d_j(n) S_{FF}(n)_{a,a'}. \tag{7}$$

Here, φ_{ai} is the displacement eigenvector for mode i and degree of freedom a, T is the sampling time, $S_{FF}(n)_{a,a'}$ represents the cross correlation input matrix term and

$$d_i(n) = \frac{1}{\omega_i^2 - \omega_n^2 + 2j\gamma_i \omega_n \omega_i}, \tag{8}$$

with ω_i the eigenfrequency, ω_n the sample frequency and γ_i the modal damping.

Equation 7 represents all the frequency response of the system and needs to be computed only once for each model. For a single input, this expression reduces to a frequency weighted sum of the autospectral density.

This RMS von Mises stress provides an excellent metric for failure of most ductile materials. Traditionally, a 'three-sigma' rule is applied to this value to arrive at a low probability of failure. Because the probability distribution function for von Mises stress is not Gaussian, this is only an approximation. However, it can be shown to be conservative, and is sufficient for many applications.

Probability distribution

To obtain the probability distribution of von Mises stress, begin with the zero time-lag covariance matrix of modal coordinates $\Gamma_{qq} = E[q(t)q(t)^T]$, which may be obtained directly from $S_{FF}(\omega)$ and the modal response of the structure [7].

A singular value decomposition of Γ_{qq} can be used to map the modal coordinates into uncorrelated variables[8]

$$\Gamma_{qq} = QX^2Q^T, \tag{9}$$

where X is a diagonal matrix whose dimension is the rank of Γ_{qq}, Q is a rectangular matrix having the property that $Q^TQ = I_{\Gamma_{qq}}$, and $I_{\Gamma_{qq}}$ is the identity whose dimension is the rank of Γ_{qq}. Here, only the nonzero terms of the diagonal matrix and the corresponding columns of the rotation matrix are retained. The rank of X defines the number of independent modal responses that participate in the excitation. Defining

$$\beta = X^{-1}Q^Tq, \tag{10}$$

it is obvious that the components of β are independent, identically distributed Gaussian processes, each with unit variance. In the new coordinates, β, the square of the von Mises stress is

$$p^2 = \beta^TC\beta, \tag{11}$$

where

$$C = XQ^TBQX. \tag{12}$$

Matrix C is square, having dimensionality equal to the rank of Γ_{qq} but possibly much lower rank. The rank of C is the minimum of the rank of the matrices in the product on the right hand side of equation 12.

Because C is symmetric, positive semi-definite, it exhibits the following singular value decomposition

$$C = RD^2R^T, \tag{13}$$

where the matrix D is diagonal and has dimension equal to the rank of C, R is a rectangular matrix having property that $R^TR = I_C$, and I_C is the identity matrix whose dimension is the rank of C. With another change of variable, $y = R^T\beta$, the square of the von Mises stress becomes

$$p^2 = y^T D^2 y = \sum_n y_n^2 D_n^2. \tag{14}$$

The dimension of D is the number of independent "stress processes" at the location of interest.

The statistics of the von Mises stress are determined via appropriate integration over the joint probability distribution of the y_k's. The mean square of the von Mises stress is

$$\langle p^2 \rangle = \sum_r D_r^2, \tag{15}$$

which is an alternative method of arriving at the RMS von Mises stress. Note that it is computationally more complex than equation 5, but both expressions possess common features. They both bear two terms, the first of which involves a modal sum and expectation value over the input range. This term (equation 7 or 9) needs to be calculated only once. The second portion integrates the stress into the solution at each output location.

When a detailed probability of failure is required, one can calculate the probability of the von Mises stress being less than some value Y:

$$P(p < Y) = \int_{Z(\{D\}, Y)} S \prod \rho_r(y_r) \prod dy_r \tag{16}$$

where $Z(\{D\}, Y)$ is the N-dimensional ellipsoid containing points y associated with von Mises stress less than Y, *i.e.*

$$Z(\{D\}, Y) = \{y:((y^T D^2 y) < Y^2)\}, \tag{17}$$

and N is the rank of D. The integral of equation (16) is generally impossible to evaluate exactly, but an approximate box quadrature method has been developed. That approximation is described in detail in [5], but a numerical implementation is enumerated in the example section below. Numerical experiments with the procedure indicate that the box quadrature converges with error on the order of h^3 where h is the maximum characteristic dimension of each box.

Numerical Implementation

In this section, example calculations to compute the RMS von Mises stress and the probability of exceeding some maximum stress are presented. MSC/Nastran is the analysis code used to compute the normal mode response of the structure, and code segments from Matlab or C are utilized to demonstrate each of the steps of the process; the example calculations are presented for clarity at the expense of numerical efficiency. The calculations can be divided into two principal steps, A) the computation of RMS stress at each point (which is computational quite inexpensive), and B) computation of the probability of failure. Because the second calculation involves a significant level of computational effort, it is typically only performed at portions of the model that exhibit a stress response that is close to yield during step A.

The principal steps in the calculation of the RMS von Mises stress are:

1. Compute the normal mode response of the structure including both output displacements, Φ, and natural (not principal) stress vectors Ψ.
2. Convert the modal displacements and stresses into a format readable by Matlab.
3. Determine the input power spectral density function, $PSD(\omega)$.
4. Compute the modal participation matrix W_{ij}.
5. For each element in the model, compute B_{ij} and the term by term product of WB. The sum of all terms in this matrix is the square of the von Mises stress.

The principal steps in the calculation of probability of failure follow a similar track (the first three steps are identical to the RMS calculations and need not be repeated):

6. Using the input PSD and the modal displacements, compute the modal participation through singular value decomposition. Truncate modes as appropriate.
7. At each desired output location, transform to new coordinates and compute the probability of failure.

Note that both calculations are characterized by a two-part analysis: computation of the coupling of the input force with the modes (which is performed once), and computing the effect at each output location.

Following are some comments regarding each step in the procedure:

1. Normal Mode Response

The normal mode response should cover the frequency range of interest. Typically, all the modes in the frequency range are used. The modes which do not contribute significantly to the response are eliminated in later sections of the process. The natural stresses must also be output on a mode by mode basis.

2. Convert Data into a Computational Format

All the operations that are performed on the data could be done in Nastran in a DMAP routine. For simplicity and flexibility, we have chosen an alternative route. Thus, the data must be translated into a format readable by the computational tool of choice. The translation is of course dependent on the analysis package as well as the computational tool, and we therefore do not provide tools for this here. In the nastran environment, this translation can be performed on either the `output2` file, or on results sent to the punch file. Table 1 lists the contents of the resulting Matlab file for the m distinct modes.

3. Determine the Power Spectral Density

For illustration, a single PSD is assumed and applied at a single input degree of freedom. Clearly this can be generalized to many different, independent PSDs. The PSD is typically supplied in the requirements for the analysis. We define this PSD using two matlab vector variables, `fval`, which represents the frequencies of definition, and `PSD`, which represents the magnitude of the PSD at each frequency value.

4. *Compute the modal participation matrix* W_{ij}.

With the eigenvectors and eigenvalues available from the modal analysis, one can compute the modal participation matrix as described in equation 7. This is computed once per model.

```
fval=deltaf*(1:num_freq);          % sampling frequencies
omega=2*pi*fval;
omegaj=2*pi*freqj;                 % natural frequencies
for j=1:num_modes
     D(j,:)=ones(size(omega))./ (omegaj(j)*omegaj(j) - ...
            omega.^2 + 2*sqrt(-1)*omegaj(j)*gamma(j)*omega);
end
% fdof is the degree of freedom where force is applied
for i=1:num_modes
  for j=1:num_modes
     W(i,j)=phi(fdof,i)*phi(fdof,j)*1/deltaf* ..
            sum( D(i,:).*conj(D(j,:)).*PSD);
  end
end
```

5. *For each element in the model, compute* B_{ij} *and the product of* BW.

At each output location, the matrix B is computed from the stress eigenvectors, as outlined in equation 6. The two matrices are combined term by term, and the square of the von Mises stress is determined from the sum of the terms.

```
for elem=1:num_elems         % loop thru all elements
  elem_dof=elem*6:elem*6+6;
  for i=1:num_modes
     for j=1:num_nodes
       B(i,j)=psi(elem_dof,i)'*A*psi(elem_dof,j);
     end
  end
  p2=sum(sum(B.*W));
  RMS_stress=sqrt(p2);
end
```

matlab variable	symbol	description
phi	Φ	*m* eigenvectors, each of length 3 x number_of_nodes
psi	Ψ	*m* stress eigenvectors, each of length 6 x number of elements
freqj	$\omega / 2\pi$	*m* - eigenfrequencies

Table 1: Translation Data

For computation of the probability of failure, only the last steps change. If the probability of failure is to be computed at all output locations, the RMS stress can be computed as a by product of the calculation.

6. Compute the modal participation

The expectation value of the modal coordinates is then computed as follows.

```
[X2,Q]=vm_expect(phi(fdof,:),freq,damp,fval,PSD);
```

```
function [X2,Q,alpha]=vm_expect(phi,freq,dmp,fval, PSD)
% computes the Singular Value Decomposition of the expectation values of
% the modal coordinates. This is done in the frequency domain.
% Returns:  X2 and Q where
%   [Q, X2, Q']=svd( E(alpha*alpha') )
M=size(phi,2);                      % M is # modes
Nval=size(fval,1);
w=fval*2*pi;
wr=freq*2*pi;
ww=zeros(Nval,M);
df=fval(2)-fval(1);
sPSD=sqrt(abs(PSD)*df) .* exp(sqrt(-1)*2*pi*rand(size(PSD)));
% frequency terms
for j=1:M
  ww(:,j)=sPSD./(wr(j)^2-w.^2+sqrt(-1)*2*dmp(j)*wr(j)*w);
end

P=zeros(M,M);
for i=1:M
  for j=1:M
    P(i,j)=sum(real(conj(ww(:,i)).*ww(:,j)));
  end
end
P=P .* (PSD'*PSD);

[Q, X2, Qt]=svd(P);
```

The vector $X2$ is the square of the new coordinate, β, described in equation (10). Modal truncation can be performed safely at this point. Typically, only components with large contributions need be kept, say for example, those terms larger than 1/1000 of the largest term in $X2$.

```
tmp=diag(X2);
n=sum(tmp>tmp(1)/1000);    % n is the new rank
X2=X2(1:n,1:n);
Q=Q(:,1:n);
```

7. At each desired output location, transform to new coordinates and compute the probability of failure.

As described in equation. (10), we transform to a new coordinate system.

```
Beta=Q*sqrt(X2);
```

At each output location, the probability of failure can be computed. This is an n-dimensional integral which must be numerically evaluated. The integral is illustrated for two dimensions in Fig. 1, where the probability of failure is the integral of the gaussian functions outside the elliptical integration limits. A very accurate representation of this integral uses the C language subroutine shown below, which approximates the region by a combination of boxes.

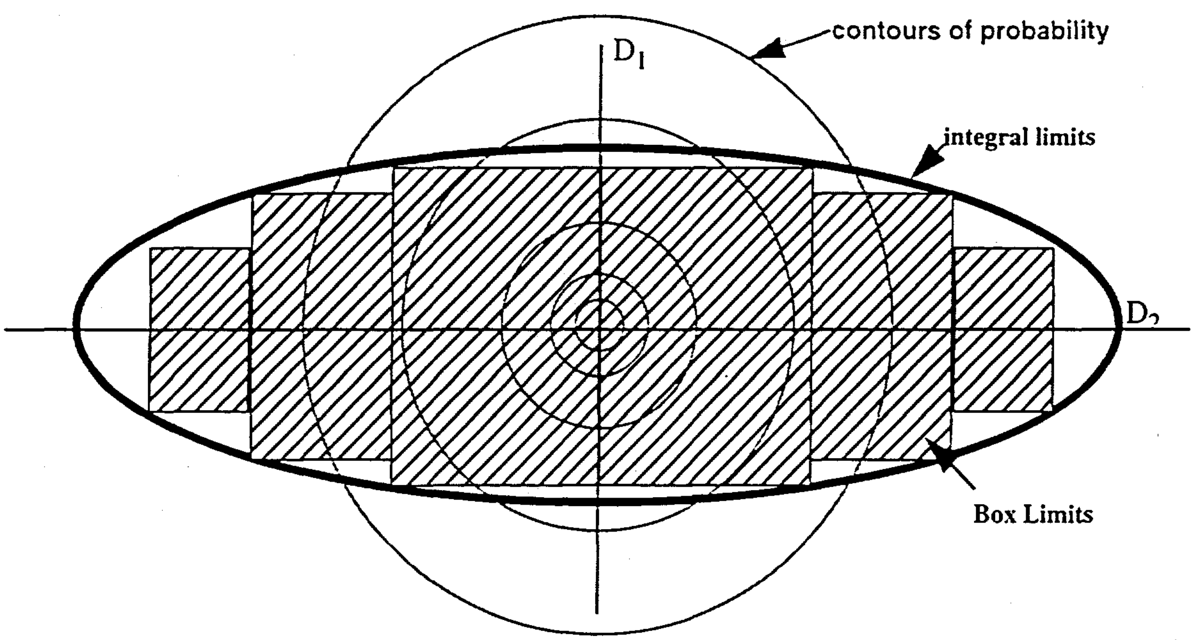

Figure 1: Box integral approximation.

```
// recursive routine to calculate a lower bound for the integral
double slabL(double *D,     int generation,
            double remain, double *xi, int Inner)
{
  double ymax=sqrt(remain)/D[generation];
  if(generation==4)                     return(erf(ymax/root2));
  if(D[generation+1] < D[0]*0.01)       return(erf(ymax/root2));

  double sum=0;
  double y1, y2;
  y1 = 0;
  int i;
  // in the following, it is assumed that xi[Inner] < 1;
  for(i=0; i<Inner; i++){
    y1 = xi[i]   *ymax;
    y2 = xi[i+1]*ymax;
    double remain2 = remain - (y2*D[generation])*(y2*D[generation]);
    sum += (erf(y2/root2) - erf(y1/root2))*
      slabL( D,     generation+1,
            remain2, xi, Inner);
  }
  return(sum);
}
```

This can be called with a driver routine like the following.

```
// INNER is the number of boxes for the numerical integration
#define INNER   32
double D[5];
double Yfail;          // the failure criteria
// add code here to define the values of D[] and Yfail.
double xi[INNER+2];
for(i=0; i<INNER+2; i++)
                xi[i] = double(i)/double(INNER+1);
double prob_fail= slabL(D,0,Yfail*Yfail,xi,INNER);
```

Example Application

To illustrate the methods outlined herein on an engineering application, consider the design of a Global Positioning Satellite (GPS) system, which must withstand launch environments specified by random vibration loads. A computational model of the GPS system, shown in Fig. 2, was created to assess the design of the assembly during the launch cycle. Shown in Fig. 3 are the contours of RMS von Mises stress when the input PSD, shown in Fig. 4, is applied in the Z-direction at the base.

Assume that the structure is made of a single material that has a yield stress of 50 *ksi* and that the system specifications are such that the probability that the von Mises stress exceeds the yield stress anywhere in the structure must be less than 1 in 1000, *e.g.*,

$$P(p > 50\ ksi) < \frac{1}{1000}. \tag{18}$$

Note that the methods employed here are in no way limited to systems of a single material. As illustrated in Fig. 3, 3 times the RMS von Mises stress at the accelerometer mount is very close to the yield stress. As previously discussed, the output von Mises stress does not exhibit Gaussian behavior. Therefore, the "3-σ rule" is not an appropriate means of determining the reliability of the structure. However, it can be shown to be a conservative measure, indicating that a more precise determination of the probability of exceeding the yield stress in warranted.

To determine the exact probability of exceeding the yield stress, one must compute the distribution of von Mises stress at all finite elements in the accelerometer mount. Shown in Fig. 5 is the probability distribution function (PDF) of von Mises stress for the element in the mount that exhibits the greatest RMS value. Note the extremely non-Gaussian behavior. From this plot, one computes the probability of exceeding the yield stress at this element to be 0.00974. In addition,

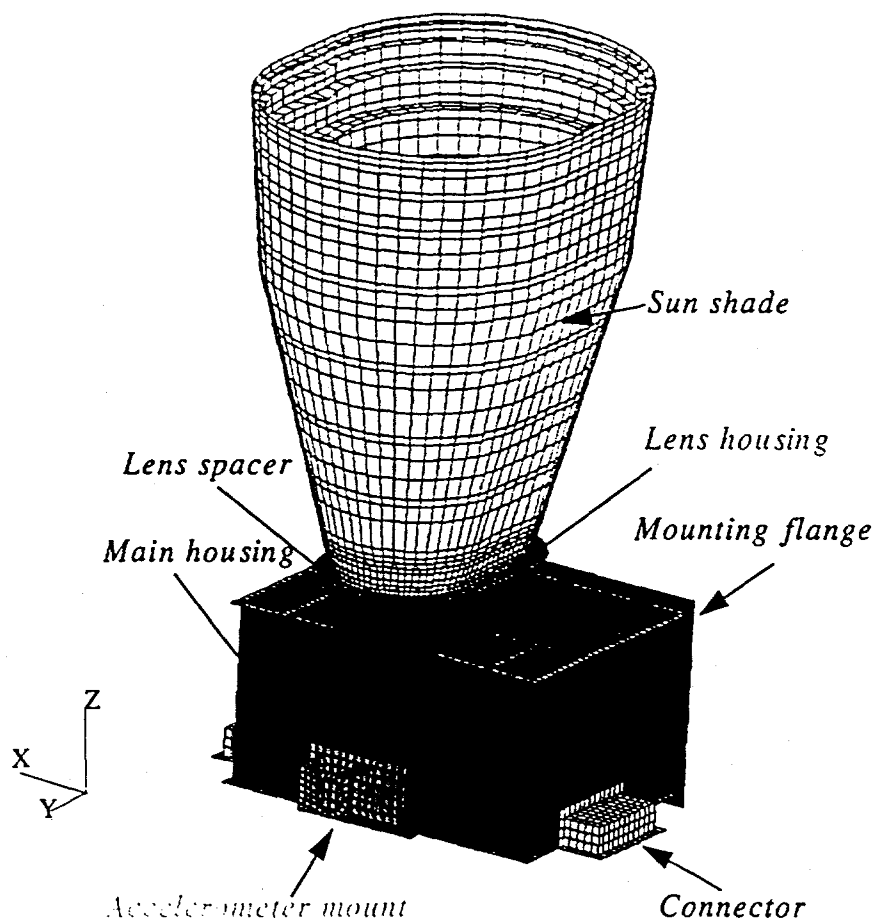

Figure 2: Computational model of GPS assembly.

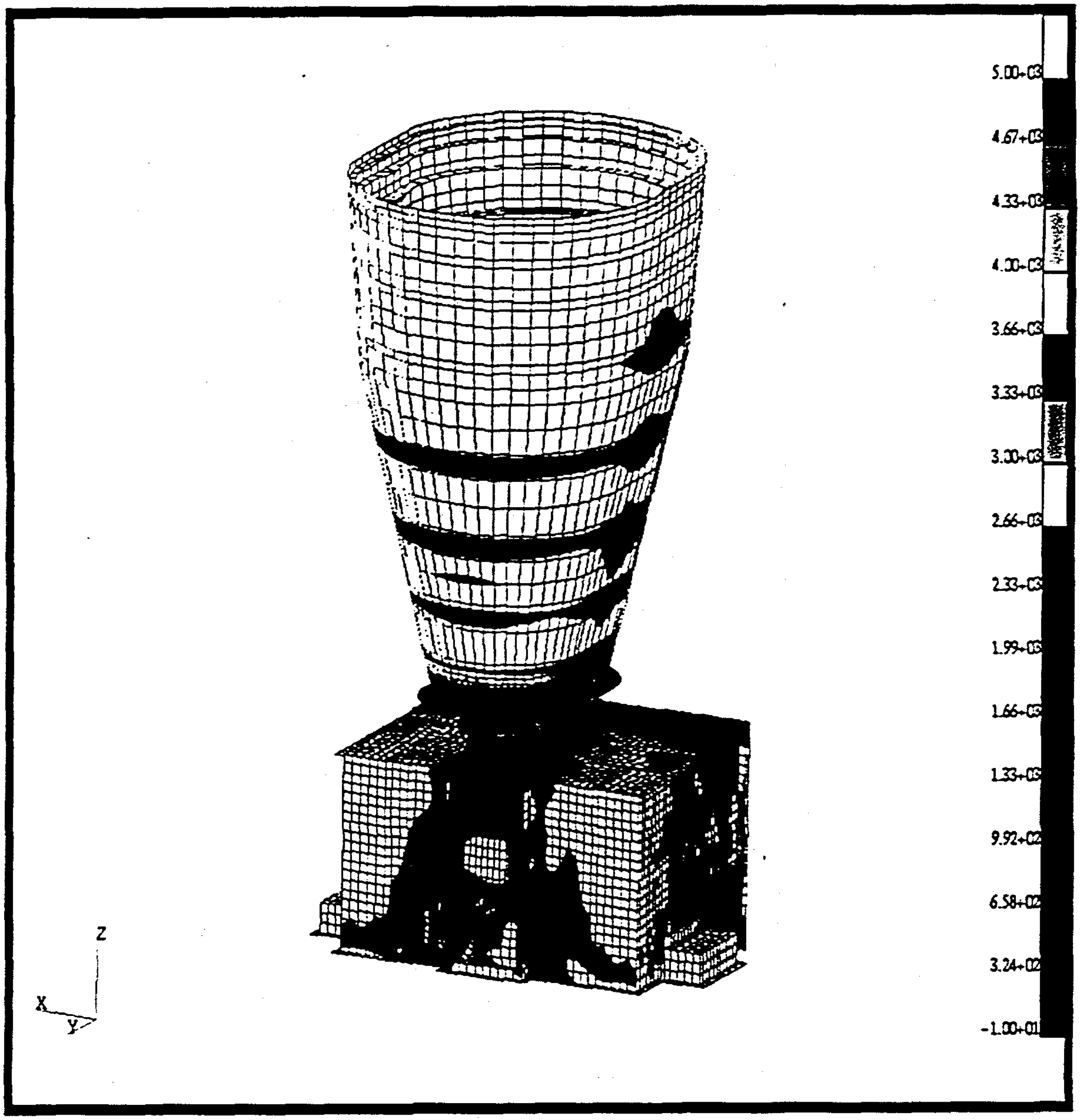

Figure 3: RMS von Mises stress contours (psi).

Fig. 6 illustrates the probability of exceeding the yield stress for each element in the accelerometer bracket. The results are shown as contours of this probability on a logarithmic scale (*e.g.*, a value of -4 denotes that the probability of exceeding the yield stress is 1e-4).

These results clearly indicate that a re-design of the accelerometer mount is necessary if the mount is to survive the launch environment with the prescribed level of reliability. Figure 7 illustrates a simple re-design that should solve the problem. Upon performing all the calculations a second time with the modified model, it is evident that 3 times the RMS von Mises stress is now far below the yield stress, as shown in Fig. 8. In addition, Figs. 9 and 10 illustrate the PDF of von Mises stress at the same element and the contours of the log of the probability of exceeding the yield stress, respectively. From these results, it is apparent that the probability of exceeding the

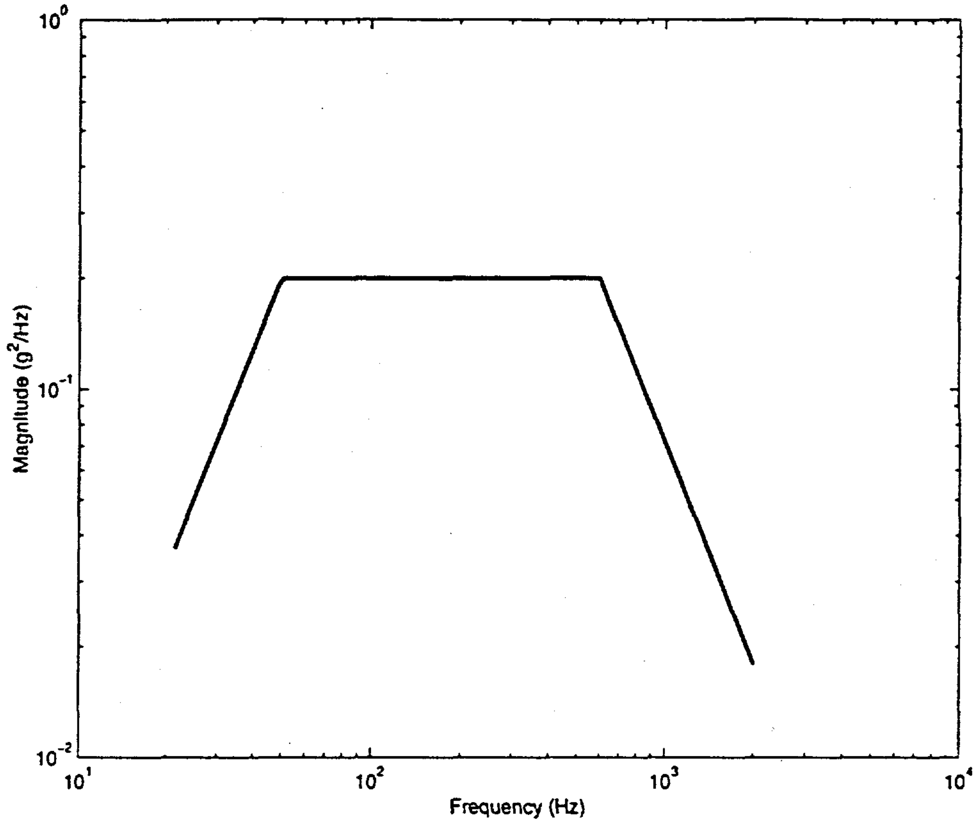

Figure 4: Qualifying launch specifications for GPS system.

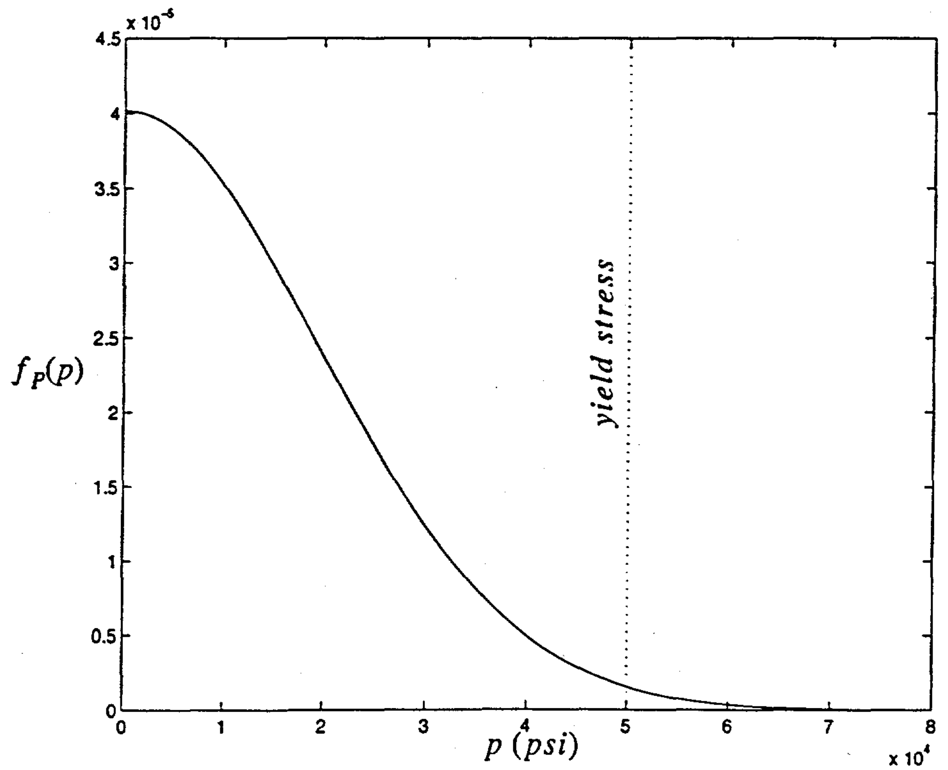

Figure 5: Probability density function of von Mises stress at element with the greatest RMS value.

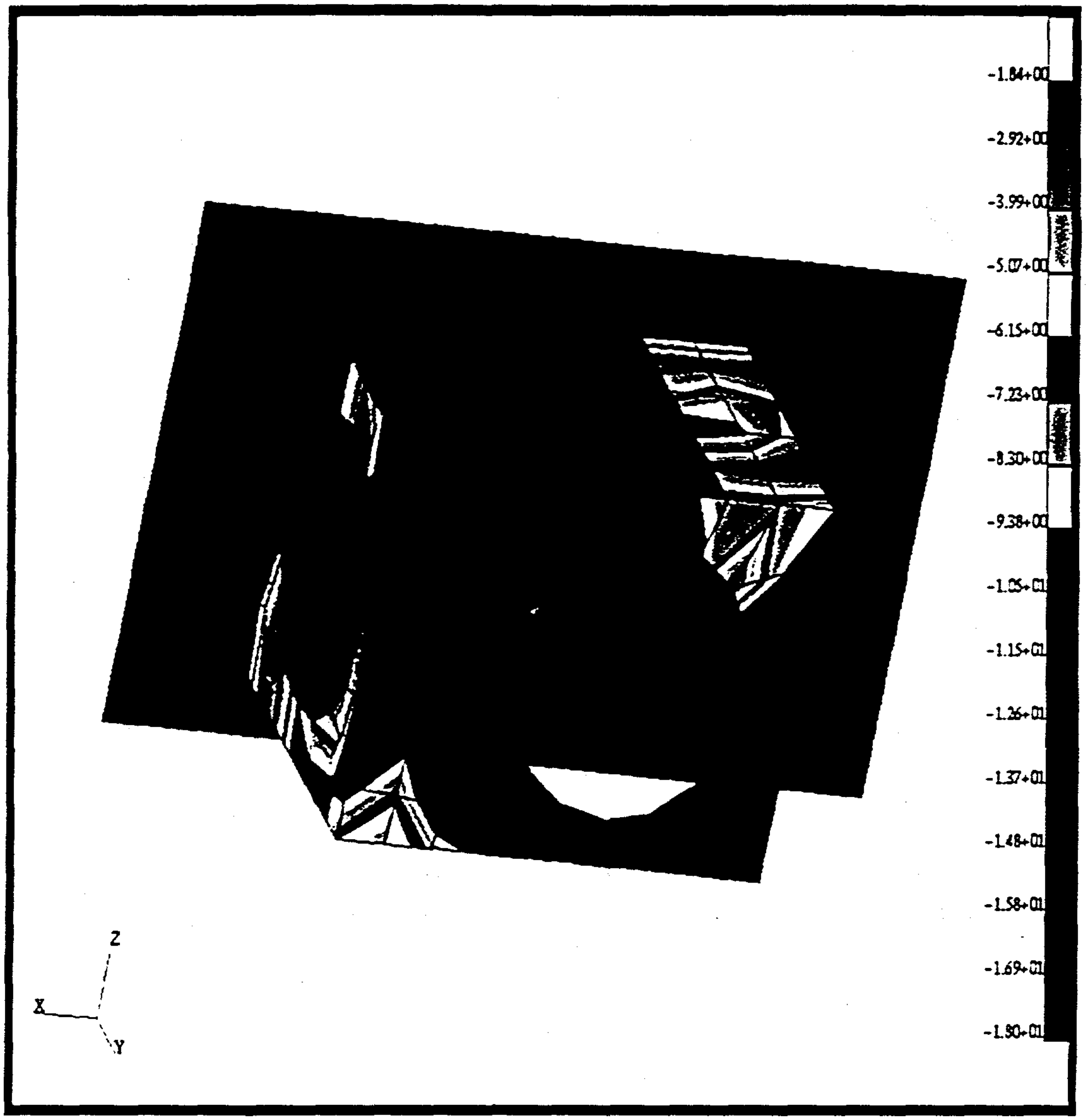

Figure 6: Log of the probability of exceeding the yield stress.

yield stress has reduced to zero (within the limits of numerical precision), thereby achieving the desired degree of reliability.

An alternative method used to address this problem is to assume single degree-of-freedom response of the structure, choosing a single mode (typically the one with the highest modal effective mass within the bandwidth of the input) to compute an "equivalent static g-field" using Miles' relation. Response contributions from other modes are ignored. To the extent that single degree-of-freedom behavior is not realized, this method is inaccurate for ascertaining the global dynamic stress response. Although a careful and proper application of the Miles' procedure to highly localized regions may capture most of the problem areas (*i.e.*, applying the method to localized modes of appendages and other irregularities in the structure), this is rarely done in practice and it is easy to omit regions of potential failure from the analyses. In contrast to the method of computing the RMS and probability distributions of von Mises stress presented herein, the application of Miles'

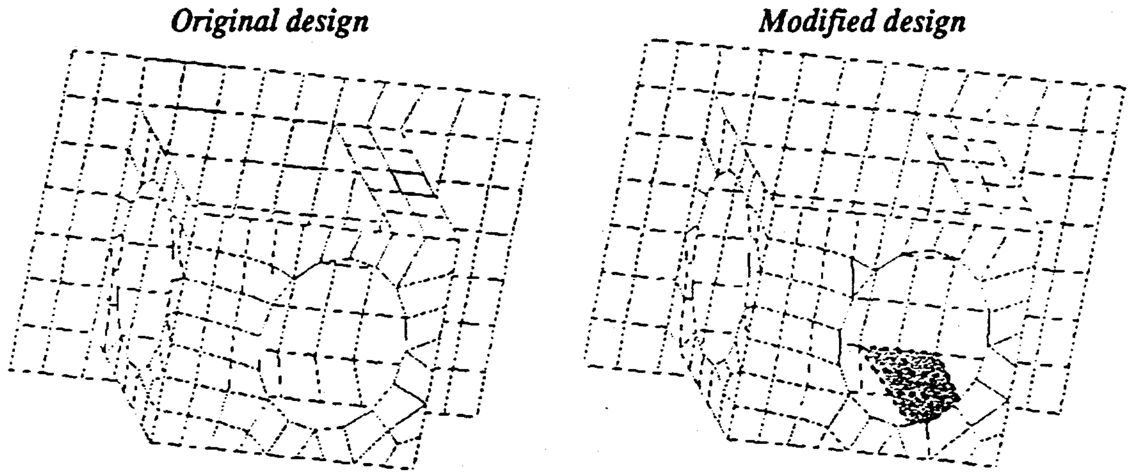

Figure 7: Finite element mesh of accelerometer mount.

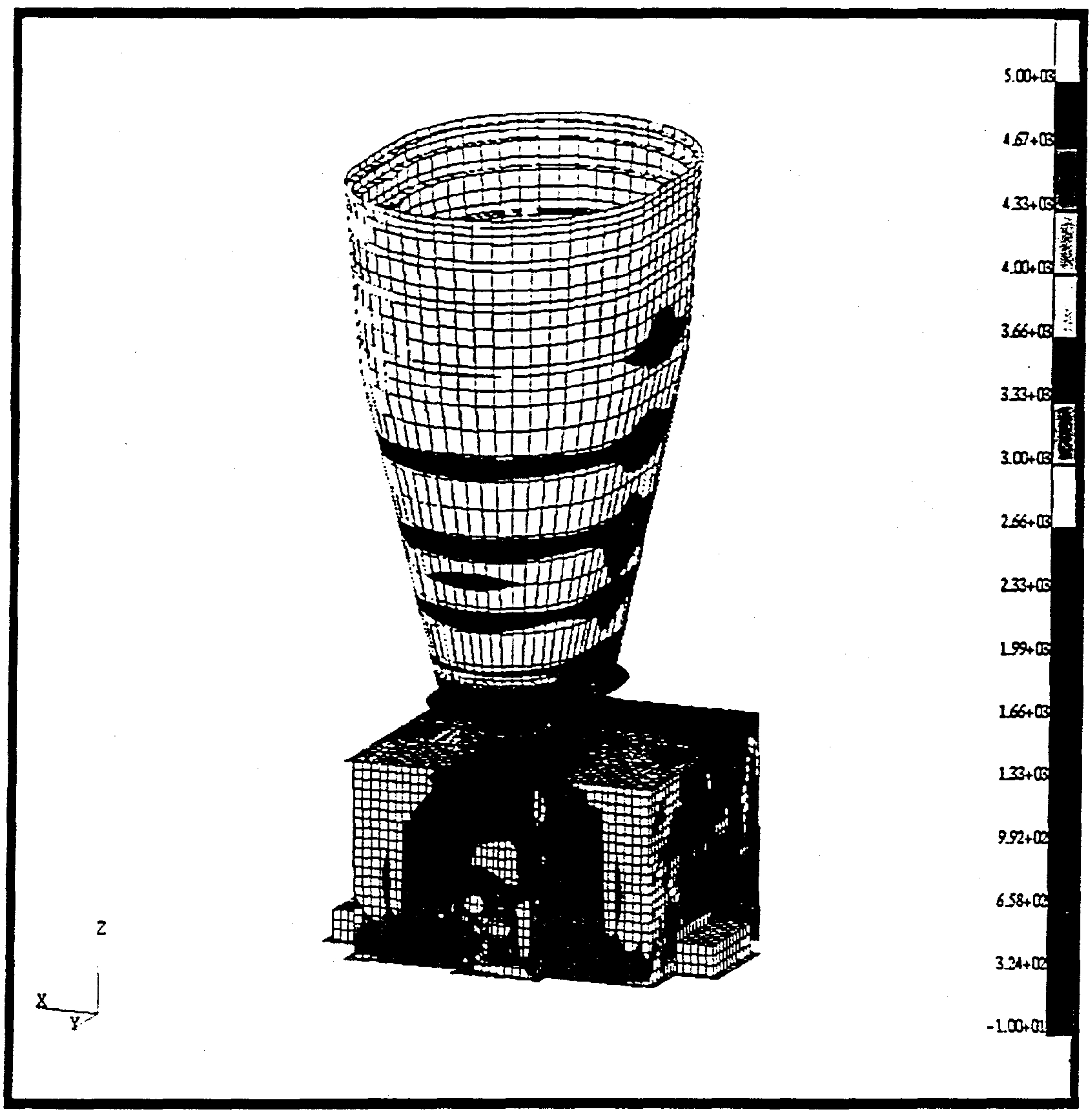

Figure 8: RMS von Mises stress contours for modified design (psi).

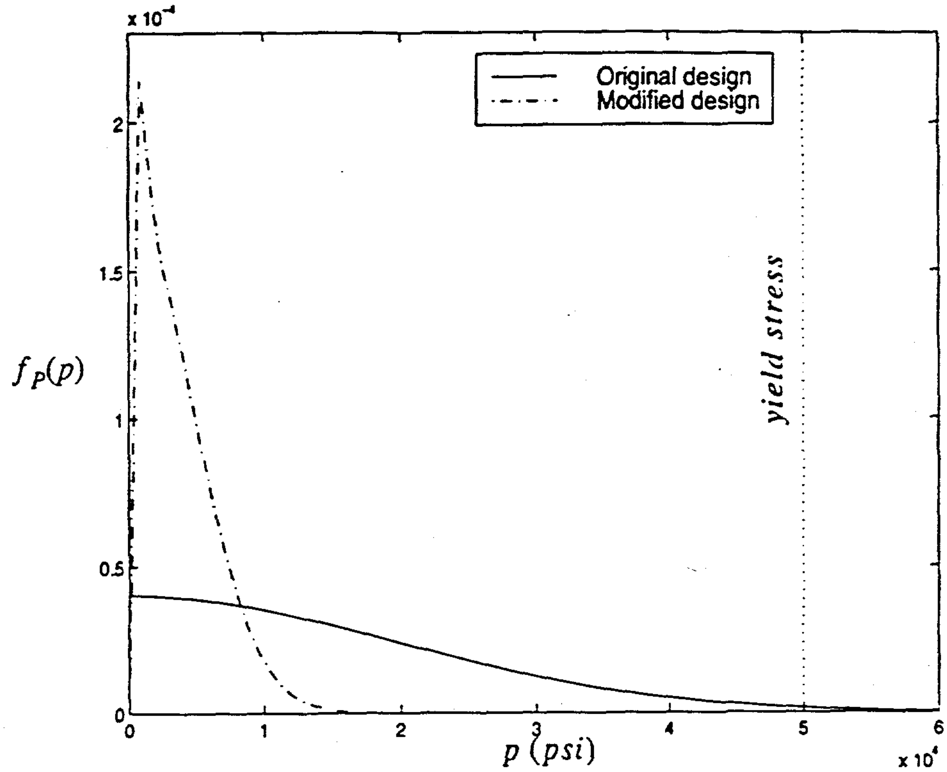

Figure 9: Probability density function of von Mises stress at element with the greatest RMS value, original and modified design

relation to complex structures is a subjective art, not an inclusive and quantitative global procedure.

Summary

A process for determining the RMS von Mises stress as well as the probability distributions of this stress in a linear structure has been outlined. The method has been demonstrated using a complicated finite element model of a GPS system as an example application. The method accurately computes the stress values at desired locations in the structure using a modal superposition approach.

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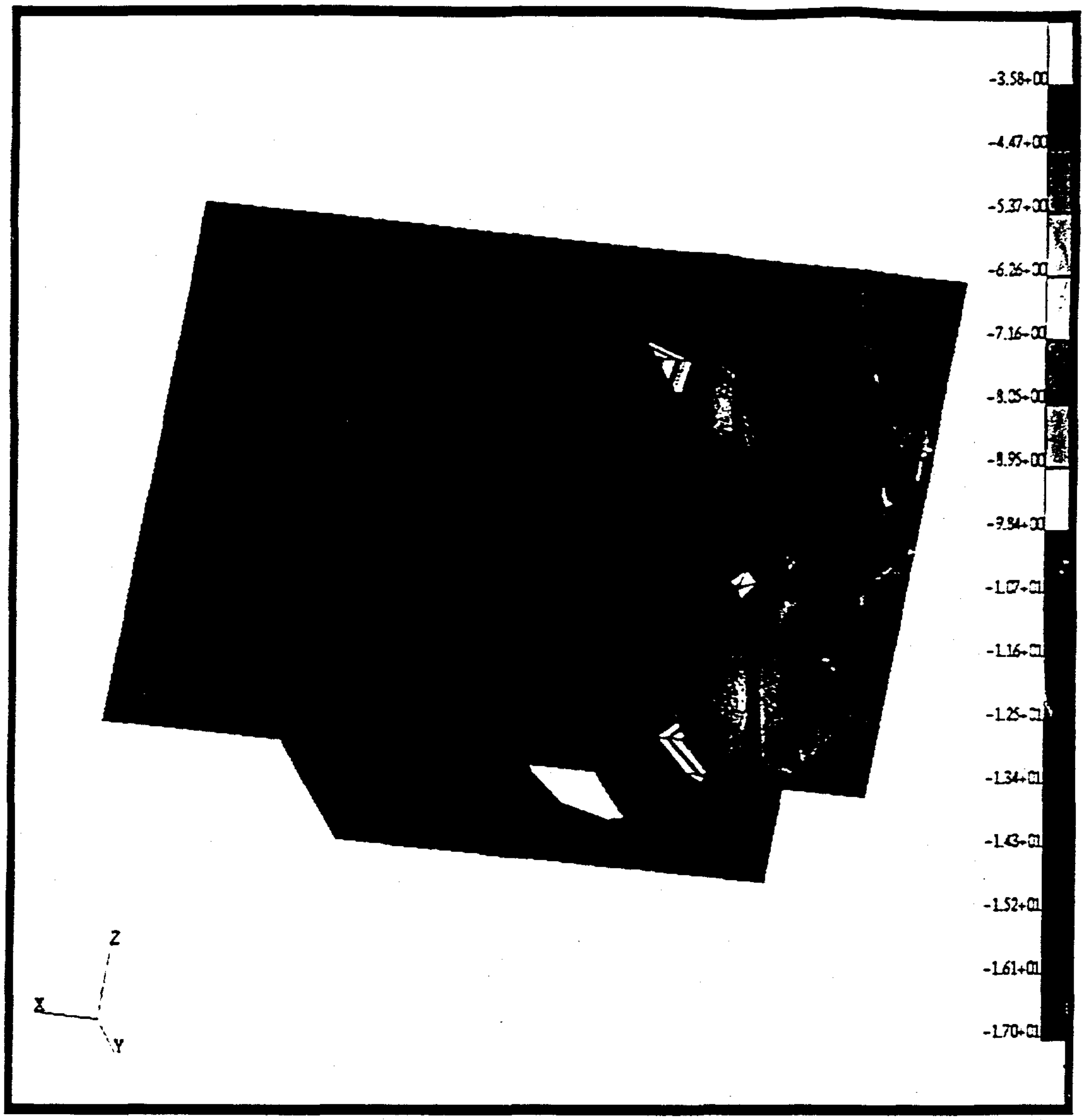

Figure 10: Log of the probability of exceeding the yield stress for the re-design.

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