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ABSTRACT

Thompson sampling is an algorithm for online decision problems where actions are taken sequentially in a manner that must balance between exploiting what is known to maximize immediate performance and investing to accumulate new information that may improve future performance. The algorithm addresses a broad range of problems in a computationally efficient manner and is therefore enjoying wide use. This tutorial covers the algorithm and its application, illustrating concepts through a range of examples, including Bernoulli bandit problems, shortest path problems, product recommendation, assortment, active learning with neural networks, and reinforcement learning in Markov decision processes. Most of these problems involve complex information structures, where information revealed by taking an action informs beliefs about other actions. We will also discuss when and why Thompson sampling is or is not effective and relations to alternative algorithms.

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In memory of Arthur F. Veinott, Jr.

1

Introduction

The multi-armed bandit problem has been the subject of decades of intense study in statistics, operations research, electrical engineering, computer science, and economics. A "one-armed bandit" is a somewhat antiquated term for a slot machine, which tends to "rob" players of their money. The colorful name for our problem comes from a motivating story in which a gambler enters a casino and sits down at a slot machine with multiple levers, or arms, that can be pulled. When pulled, an arm produces a random payout drawn independently of the past. Because the distribution of payouts corresponding to each arm is not listed, the player can learn it only by experimenting. As the gambler learns about the arms' payouts, she faces a dilemma: in the immediate future she expects to earn more by *exploiting* arms that yielded high payouts in the past, but by continuing to *explore* alternative arms she may learn how to earn higher payouts in the future. Can she develop a sequential strategy for pulling arms that balances this tradeoff and maximizes the cumulative payout earned? The following Bernoulli bandit problem is a canonical example.

Example 1.1. (*Bernoulli Bandit*) Suppose there are K actions, and when played, any action yields either a success or a failure. Action

$k \in \{1, ..., K\}$ produces a success with probability $\theta_k \in [0, 1]$. The success probabilities $(\theta_1, .., \theta_K)$ are unknown to the agent, but are fixed over time, and therefore can be learned by experimentation. The objective, roughly speaking, is to maximize the cumulative number of successes over T periods, where T is relatively large compared to the number of arms K.

The "arms" in this problem might represent different banner ads that can be displayed on a website. Users arriving at the site are shown versions of the website with different banner ads. A success is associated either with a click on the ad, or with a conversion (a sale of the item being advertised). The parameters θ_k represent either the click-through-rate or conversion-rate among the population of users who frequent the site. The website hopes to balance exploration and exploitation in order to maximize the total number of successes.

A naive approach to this problem involves allocating some fixed fraction of time periods to exploration and in each such period sampling an arm uniformly at random, while aiming to select successful actions in other time periods. We will observe that such an approach can be quite wasteful even for the simple Bernoulli bandit problem described above and can fail completely for more complicated problems.

Problems like the Bernoulli bandit described above have been studied in the decision sciences since the second world war, as they crystallize the fundamental trade-off between exploration and exploitation in sequential decision making. But the information revolution has created significant new opportunities and challenges, which have spurred a particularly intense interest in this problem in recent years. To understand this, let us contrast the Internet advertising example given above with the problem of choosing a banner ad to display on a highway. A physical banner ad might be changed only once every few months, and once posted will be seen by every individual who drives on the road. There is value to experimentation, but data is limited, and the cost of of trying a potentially ineffective ad is enormous. Online, a different banner ad can be shown to each individual out of a large pool of users, and data from each such interaction is stored. Small-scale experiments are now a core tool at most leading Internet companies.

Our interest in this problem is motivated by this broad phenomenon. Machine learning is increasingly used to make rapid data-driven decisions. While standard algorithms in supervised machine learning learn passively from historical data, these systems often drive the generation of their own training data through interacting with users. An online recommendation system, for example, uses historical data to optimize current recommendations, but the outcomes of these recommendations are then fed back into the system and used to improve future recommendations. As a result, there is enormous potential benefit in the design of algorithms that not only learn from past data, but also explore systemically to generate useful data that improves future performance. There are significant challenges in extending algorithms designed to address Example 1.1 to treat more realistic and complicated decision problems. To understand some of these challenges, consider the problem of learning by experimentation to solve a shortest path problem.

Example 1.2. (Online Shortest Path) An agent commutes from home to work every morning. She would like to commute along the path that requires the least average travel time, but she is uncertain of the travel time along different routes. How can she learn efficiently and minimize the total travel time over a large number of trips?

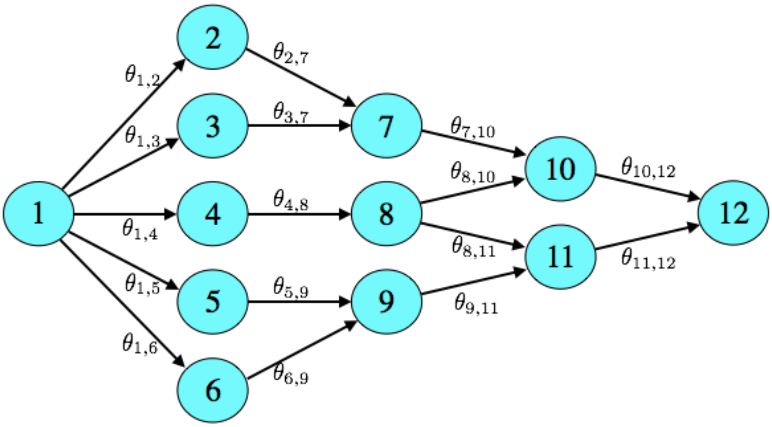

Figure 1.1: Shortest path problem.

We can formalize this as a shortest path problem on a graph $G = (V, E)$ with vertices $V = \{1, ..., N\}$ and edges E. An example is illustrated in Figure 1.1. Vertex 1 is the source (home) and vertex N is the destination (work). Each vertex can be thought of as an intersection, and for two vertices $i, j \in V$, an edge $(i, j) \in E$ is present if there is a direct road connecting the two intersections. Suppose that traveling along an edge $e \in E$ requires time θ_e on average. If these parameters were known, the agent would select a path $(e_1, .., e_n)$, consisting of a sequence of adjacent edges connecting vertices 1 and N, such that the expected total time $\theta_{e_1} + ... + \theta_{e_n}$ is minimized. Instead, she chooses paths in a sequence of periods. In period t, the realized time $y_{t,e}$ to traverse edge e is drawn independently from a distribution with mean θ_e. The agent sequentially chooses a path x_t, observes the realized travel time $(y_{t,e})_{e \in x_t}$ along each edge in the path, and incurs cost $c_t = \sum_{e \in x_t} y_{t,e}$ equal to the total travel time. By exploring intelligently, she hopes to minimize cumulative travel time $\sum_{t=1}^{T} c_t$ over a large number of periods T.

This problem is conceptually similar to the Bernoulli bandit in Example 1.1, but here the number of actions is the number of paths in the graph, which generally scales exponentially in the number of edges. This raises substantial challenges. For moderate sized graphs, trying each possible path would require a prohibitive number of samples, and algorithms that require enumerating and searching through the set of all paths to reach a decision will be computationally intractable. An efficient approach therefore needs to leverage the statistical and computational structure of problem.

In this model, the agent observes the travel time along each edge traversed in a given period. Other feedback models are also natural: the agent might start a timer as she leaves home and checks it once she arrives, effectively only tracking the total travel time of the chosen path. This is closer to the Bernoulli bandit model, where only the realized reward (or cost) of the chosen arm was observed. We have also taken the random edge-delays $y_{t,e}$ to be independent, conditioned on θ_e. A more realistic model might treat these as correlated random variables, reflecting that neighboring roads are likely to be congested at the same time. Rather than design a specialized algorithm for each possible statistical model, we seek a general approach to exploration that accommodates flexible modeling and works for a broad array of problems. We will see that Thompson sampling accommodates such flexible modeling, and offers an elegant and efficient approach to exploration in a wide range of structured decision problems, including the shortest path problem described here.

Thompson sampling – also known as *posterior sampling* and *probability matching* – was first proposed in 1933 (Thompson, 1933; Thompson, 1935) for allocating experimental effort in two-armed bandit problems arising in clinical trials. The algorithm was largely ignored in the academic literature until recently, although it was independently rediscovered several times in the interim (Wyatt, 1997; Strens, 2000) as an effective heuristic. Now, more than eight decades after it was introduced, Thompson sampling has seen a surge of interest among industry practitioners and academics. This was spurred partly by two influential articles that displayed the algorithm's strong empirical performance (Chapelle and Li, 2011; Scott, 2010). In the subsequent five years, the literature on Thompson sampling has grown rapidly. Adaptations of Thompson sampling have now been successfully applied in a wide variety of domains, including revenue management (Ferreira *et al.*, 2015), marketing (Schwartz *et al.*, 2017), web site optimization (Hill *et al.*, 2017), Monte Carlo tree search (Bai *et al.*, 2013), A/B testing (Graepel *et al.*, 2010), Internet advertising (Graepel *et al.*, 2010; Agarwal, 2013; Agarwal *et al.*, 2014), recommendation systems (Kawale *et al.*, 2015), hyperparameter tuning (Kandasamy *et al.*, 2018), and arcade games (Osband *et al.*, 2016a); and have been used at several companies, including Adobe, Amazon (Hill *et al.*, 2017), Facebook, Google (Scott, 2010; Scott, 2015), LinkedIn (Agarwal, 2013; Agarwal *et al.*, 2014), Microsoft (Graepel *et al.*, 2010), Netflix, and Twitter.

The objective of this tutorial is to explain when, why, and how to apply Thompson sampling. A range of examples are used to demonstrate how the algorithm can be used to solve a variety of problems and provide clear insight into why it works and when it offers substantial benefit over naive alternatives. The tutorial also provides guidance on approximations to Thompson sampling that can simplify computation as well as practical considerations like prior distribution specification, safety constraints and nonstationarity. Accompanying this tutorial we also release a Python package[1] that reproduces all experiments and figures presented. This resource is valuable not only for reproducible research, but also as a reference implementation that may help practioners build intuition for how to practically implement some of the ideas and algorithms we discuss in this tutorial. A concluding section discusses theoretical results that aim to develop an understanding of why Thompson sampling works, highlights settings where Thompson sampling performs poorly, and discusses alternative approaches studied in recent literature. As a baseline and backdrop for our discussion of Thompson sampling, we begin with an alternative approach that does not actively explore.

[1] Python code and documentation is available at https://github.com/iosband/ts_tutorial.

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