

## A TWISTED MOMENT MAP AND ITS EQUIVARIANCE

TAKASHI HASHIMOTO

(Received November 9, 2012, revised November 21, 2013)

**Abstract.** Let  $G$  be a linear connected complex reductive Lie group. The purpose of this paper is to construct a  $G$ -equivariant symplectomorphism in terms of local coordinates from a holomorphic twisted cotangent bundle of the generalized flag variety of  $G$  onto the semisimple coadjoint orbit of  $G$ . As an application, one can obtain an explicit embedding of a noncompact real coadjoint orbit into the twisted cotangent bundle.

**1. Introduction.** The main purpose of this paper is to construct an equivariant symplectomorphism from a holomorphic twisted cotangent bundle of the complex generalized flag variety onto the complex coadjoint orbit of a semisimple element concretely in terms of local coordinates. As an application, one can obtain an explicit embedding of a noncompact real coadjoint orbit into the twisted cotangent bundle.

More precisely, let  $G$  denote a linear connected complex reductive Lie group with Lie algebra  $\mathfrak{g}$ . We fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , and consider a nonzero element  $\lambda$  of  $\mathfrak{h}^*$ , the dual space of  $\mathfrak{h}$ . Under the assumption that the isotropy subalgebra of  $\lambda$  in  $\mathfrak{g}$  is distinct from  $\mathfrak{g}$ , we take a parabolic subgroup  $Q$  of  $G$  whose Levi factor is the isotropy subgroup of  $\lambda$  in  $G$ , and let  $\{U_\sigma\}_{\sigma \in W/W_\lambda}$  be the open covering of the flag variety  $G/Q$  indexed by  $W/W_\lambda$  (see (2.10) below for details). Then, based on the key observation that cotangent vectors can be written in terms of 1-form taking values in a subalgebra of  $\mathfrak{g}$  (Lemma 2.3 below), we construct a holomorphic isomorphism  $\mu_{\lambda;\sigma}$  from the cotangent bundle  $T^*U_\sigma$  into the complex coadjoint orbit  $\Omega_\lambda := G \cdot \lambda$  for each  $\sigma$ . Note that  $U_\sigma$  is homeomorphic to  $\mathbf{C}^n$  and that  $T^*U_\sigma$  is trivial, i.e.,  $T^*U_\sigma \simeq U_\sigma \times \mathbf{C}^n$  with  $n = \dim(G/Q)$  for each  $\sigma$ . We shall see that the isomorphisms  $\{\mu_{\lambda;\sigma}\}_{\sigma \in W/W_\lambda}$  are closely related to the triangular decomposition of  $G$  (or rather, of  $\mathfrak{g}$ ).

A prototype of the isomorphism  $\mu_{\lambda;\sigma}$  was obtained in the process of proving the formula for the generating function of the principal symbols of the invariant differential operators that play an essential role in the Capelli identities in the case of Hermitian symmetric spaces (see [5] and [3]). Namely, let  $(G_{\mathbf{R}}, K_{\mathbf{R}})$  be a classical Hermitian symmetric pair of noncompact type such that  $G_{\mathbf{R}}$  is a real form of  $G$ , and assume that  $\lambda$  is proportional to the fundamental weight corresponding to the unique noncompact simple root. Then the symbols of the differential operators  $\pi_\lambda(X)$ ,  $X \in \mathfrak{g}$ , naturally constitute a holomorphic isomorphism from  $T^*(G_{\mathbf{R}}/K_{\mathbf{R}})$  into  $\mathfrak{g} \simeq \mathfrak{g}^*$ , where  $\pi_\lambda$  denotes the representation of the complex Lie algebra

---

2000 *Mathematics Subject Classification.* Primary 53D20; Secondary 22F30.

*Key words and phrases.* Twisted moment map,  $G$ -equivariance, holomorphic twisted cotangent bundle, complex coadjoint orbit, symplectic isomorphism.

Partly supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

$\mathfrak{g}$  induced from that of  $G_{\mathbf{R}}$ , the so-called holomorphic discrete series representation. Note that the Hermitian symmetric case corresponds to the case where the flag variety  $G/Q$  is Grassmannian.

Furthermore, the isomorphisms  $\{\mu_{\lambda;\sigma}\}$  acquire equivariance under  $G$  if we let  $G$  act on  $T^*U_{\sigma}$  by affine transformation instead of the canonical linear one. Since each coset of  $W/W_{\lambda}$  is represented by an element  $\dot{\sigma} \in G$ , we can glue together the trivial bundles  $\{T^*U_{\sigma}\}_{\sigma}$  by transition functions induced from the affine action of  $G$  to form a holomorphic twisted cotangent bundle which we denote by  $T^*(G/Q)_{\lambda}$  in this paper. Since the twisted cotangent bundle is locally isomorphic to the (standard) holomorphic cotangent bundle  $T^*(G/Q)$  by its construction, we can define local isomorphisms from  $T^*(G/Q)_{\lambda}|_{U_{\sigma}}$  into  $\Omega_{\lambda}$  by the same formulae as  $\{\mu_{\lambda;\sigma}\}$ , which satisfy the compatibility condition

$$\mu_{\lambda;\sigma}|_{\varpi^{-1}(U_{\sigma} \cap U_{\tau})} = \mu_{\lambda;\tau}|_{\varpi^{-1}(U_{\sigma} \cap U_{\tau})}$$

for  $\sigma$  and  $\tau \in W/W_{\lambda}$ , where  $\varpi : T^*(G/Q)_{\lambda} \rightarrow G/Q$  is the projection. By patching together the maps  $\{\mu_{\lambda;\sigma}\}_{\sigma}$ , we obtain an holomorphic isomorphism  $\mu_{\lambda}$  from the holomorphic twisted cotangent bundle onto the complex coadjoint orbit.

We remark that when  $\lambda \rightarrow 0$  the transition functions of the twisted cotangent bundle  $T^*(G/Q)_{\lambda}$ , which are affine transformations of the fibers as mentioned above, reduces to the canonical transition functions of the cotangent bundle  $T^*(G/Q)$ , which are linear transformations of the fibers, and that the map  $\mu_{\lambda}$  reduces to the moment map from  $T^*(G/Q)$  into  $\mathfrak{g}^*$ . In this sense, the isomorphism  $\mu_{\lambda}$  can be regarded as a twisted moment map (see [8] and [9]).

It is well known that the coadjoint orbit possesses a canonical symplectic form called the Kostant-Kirillov-Souriau form, and it is shown that our twisted cotangent bundle possesses a (holomorphic) symplectic form expressed locally by the same formula as that of the canonical holomorphic symplectic form on the holomorphic cotangent bundle, which will be denoted by  $\omega$  below. The isomorphism  $\mu_{\lambda}$  preserves the symplectic forms. Thus, the holomorphic map  $\mu_{\lambda}$  gives a  $G$ -equivariant symplectomorphism from the holomorphic twisted cotangent bundle of the complex generalized flag variety onto the complex coadjoint orbit. Furthermore, it provides a moment map on the symplectic  $G$ -manifold  $(T^*(G/Q)_{\lambda}, \omega)$ .

The rest of this paper is organized as follows. In Section 2, we first review the Hermitian symmetric case by a basic example, then construct holomorphic local isomorphisms  $\mu_{\lambda;\sigma}$  from cotangent bundle  $T^*U_{\sigma}$  into the coadjoint orbit for  $\sigma \in W/W_{\lambda}$ . In Section 3, we define an action of  $G$  on the cotangent bundle by affine transformation, and show that  $\mu_{\lambda;\sigma}$  is  $G$ -equivariant. Replacing the canonical transition functions by the ones induced from the affine action of  $G$ , we construct the holomorphic twisted cotangent bundle mentioned above, and show that the maps  $\{\mu_{\lambda;\sigma}\}_{\sigma}$  provides the isomorphism  $\mu_{\lambda}$  from the twisted cotangent bundle onto the complex coadjoint orbit. Finally, we prove that the isomorphism  $\mu_{\lambda}$  is symplectic. As an application, we give an explicit embeddings of coadjoint  $G_{\mathbf{R}}$ -orbits into the twisted cotangent bundles when  $(G_{\mathbf{R}}, K_{\mathbf{R}})$  is a classical Hermitian symmetric space of noncompact type.

**2. Twisted moment map.** Throughout, let  $G$  be a linear connected complex reductive Lie group with Lie algebra  $\mathfrak{g}$ . We fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and denote the dual space of  $\mathfrak{h}$  by  $\mathfrak{h}^*$ . Let  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$  be the root space decomposition with  $\Delta$  a root system of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Choosing a positive root system  $\Delta^+ \subset \Delta$ , we set  $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ . We take a nonzero root vector  $E_\alpha$  from  $\mathfrak{g}_\alpha$  for each  $\alpha \in \Delta$ .

**2.1. Hermitian symmetric space.** In this subsection, let  $(G_R, K_R)$  denote a Hermitian symmetric pair of noncompact type. Let  $G$  and  $K$  be the complexifications of  $G_R$  and  $K_R$  respectively, and  $Q$  be a maximal parabolic subgroup of  $G$  whose Levi factor is  $K$ . Let  $\mathfrak{k}$  and  $\mathfrak{q}$  denote the Lie algebras of  $K$  and  $Q$  respectively. We assume that  $\mathfrak{h} \subset \mathfrak{k}$ . Denoting by  $\mathfrak{u}^-$  the nilradical of  $\mathfrak{q}$  and by  $\mathfrak{u}$  its opposite, we put  $U := \exp \mathfrak{u}$  and  $U^- := \exp \mathfrak{u}^-$ .

Consider a holomorphic character  $\lambda : Q \rightarrow \mathbb{C}^\times$  whose differential is pure-imaginary on  $\mathfrak{h} \cap \mathfrak{k}_R$  and is proportional to the fundamental weight corresponding to the unique noncompact simple root on  $\mathfrak{h}$ , where  $\mathfrak{k}_R$  denotes the Lie algebra of  $K_R$ . Let us denote by  $\mathcal{C}_\lambda$  the one-dimensional representation  $(\lambda, \mathcal{C})$  of  $Q$ . Then one can construct an irreducible unitary representation  $(\pi_\lambda, \mathcal{H}_\lambda)$  of  $G_R$ , the so-called holomorphic discrete series representation, by Borel-Weil theory as follows. Let  $\mathcal{L}_\lambda$  be the pull-back by the open embedding  $G_R/K_R \hookrightarrow G/Q$  of the holomorphic line bundle  $G \times_Q \mathcal{C}_\lambda$  associated to the principal bundle  $G \rightarrow G/Q$ . The Hilbert space  $\mathcal{H}_\lambda$  consists of square-integrable holomorphic sections for  $\mathcal{L}_\lambda$ , which we identify with the space of holomorphic functions  $f$  on the open subset  $G_R Q \subset G$  that satisfy the following conditions

$$f(xq) = \lambda(q)^{-1} f(x) \quad (x \in G_R Q, q \in Q) \quad \text{and} \quad \int_{G_R} |f(g)|^2 dg < \infty,$$

where  $dg$  denotes the Haar measure on  $G_R$ . If  $\lambda$  satisfies certain conditions,  $\mathcal{H}_\lambda$  is non trivial. Now the irreducible unitary representation  $\pi_\lambda$  of  $G_R$  is defined by

$$\pi_\lambda(g)f(x) := f(g^{-1}x) \quad \text{for } f \in \mathcal{H}_\lambda.$$

This induces a complex linear representation of  $\mathfrak{g}$ , which we also denote by  $\pi_\lambda$  (see [6] for details).

Take a basis  $\{X_i\}$  for  $\mathfrak{g}$ , and its dual basis  $\{X_i^\vee\}$ , i.e., the basis for  $\mathfrak{g}$  satisfying that

$$B(X_i, X_j^\vee) = \delta_{ij},$$

where  $B$  is a nondegenerate invariant symmetric bilinear form on  $\mathfrak{g}$ . For  $X \in \mathfrak{g}$  given, denoting by  $\sigma_\lambda(X)(x, v^*)$  the symbol of the differential operator  $\pi_\lambda(X)$  at  $x \in G_R/K_R$  with cotangent vector  $v^* \in T_x^*(G_R/K_R)$ , we define

$$\mu_{\lambda;0}(x, v^*) := \sum_i \sigma_\lambda(X_i)(x, v^*) \otimes X_i^\vee.$$

Note that  $\mu_{\lambda;0}$  is independent of the basis  $\{X_i\}$  chosen.

Recall from [7], for example, that for a Lie group  $A$  and an  $A$ -manifold  $M$ , the cotangent bundle  $T^*M$  is a symplectic  $A$ -manifold in the canonical way. Namely, the Lie group  $A$  acts

on  $T^*M$  by

$$(2.1) \quad g.(x, v^*) = (g.x, (g^{-1})^*v^*)$$

for  $x \in M$  and  $v^* \in T_x^*M$ , where  $(g^{-1})^*$  denotes the transpose map of the differential  $(g^{-1})_* : T_{g.x}M \rightarrow T_xM$  induced from the translation by  $g^{-1}$  on the base manifold  $M$ .

Then, for  $X \in \mathfrak{a} := \text{Lie}(A)$ , the moment map on the cotangent bundle  $\mu : T^*M \rightarrow \mathfrak{a}^*$  is given by

$$(2.2) \quad \langle \mu(x, v^*), X \rangle = v^*(X_M(x)) \quad (x \in M, v^* \in T_x^*M),$$

where  $\mathfrak{a}^*$  denotes the dual of  $\mathfrak{a}$ ,  $\langle \cdot, \cdot \rangle$  the canonical pairing between  $\mathfrak{a}^*$  and  $\mathfrak{a}$ , and  $X_M$  the vector field on  $M$  generated by  $X$ :

$$(2.3) \quad X_M(x) \varphi = \left. \frac{d}{dt} \right|_{t=0} \varphi(\exp(-tX).x)$$

for functions  $\varphi$  defined around  $x \in M$ .

It follows from (2.2) and (2.3) that the principal part of  $\mu_{\lambda;0}$  is identical to the moment map  $\mu : T^*(G_R/K_R) \rightarrow \mathfrak{g}^*$  composed by the isomorphism  $\mathfrak{g}^* \simeq \mathfrak{g}$  via the bilinear form  $B$ , which we also denote by  $\mu$ . Here  $T^*(G_R/K_R)$  denotes the holomorphic cotangent bundle of the Hermitian symmetric space  $G_R/K_R$ .

We showed in [3] that if  $(G_R, K_R)$  is classical, then the total symbol  $\mu_{\lambda;0}$  can be regarded as a variant of the twisted moment map  $\tilde{\mu}_\lambda : T^*(G_R/K_R) \rightarrow \mathfrak{g}^* \simeq \mathfrak{g}$  due to Rossmann (see [8], or §7 of [9]). In fact, the difference  $\tilde{\mu}_\lambda - \mu$ , which is denoted by  $\lambda_x$  with  $x \in G/Q$  therein, can be expressed as  $\tilde{\mu}_\lambda - \mu = \text{Ad}(g)\lambda^\vee$ , or

$$\tilde{\mu}_\lambda(x, v^*) = \text{Ad}(g)\lambda^\vee + \mu(x, v^*),$$

where  $\lambda^\vee \in \mathfrak{g}$  corresponds to  $\lambda \in \mathfrak{g}^*$  under the isomorphism  $\mathfrak{g}^* \simeq \mathfrak{g}$  via the bilinear form  $B$ , and  $g$  is an element of a compact real form  $G_u$  of  $G$  such that  $x = g.e_Q$  with  $e_Q$  the origin of  $G/Q$ . Now, if  $x$  is in the open subset  $G_R/K_R \subset G/Q$ , one can choose a unique element  $u_x$  from a certain open subset of  $U$  so that  $x = u_x.e_Q$ , instead of  $g$  from  $G_u$ . Then, one can immediately verify that

$$\mu_{\lambda;0}(x, v^*) = \text{Ad}(u_x)\lambda^\vee + \mu(x, v^*).$$

Moreover, the following relation holds:

$$(2.4) \quad \begin{aligned} \text{Ad}(u_x^{-1})\mu_{\lambda;0}(x, v^*) &= \mu_{\lambda;0}(e_Q, u_x^*v^*) \\ &= \mu_{\lambda;0}(u_x^{-1}.(x, v^*)). \end{aligned}$$

EXAMPLE 2.1. Let  $(G_R, K_R) = (\text{SU}(p, q), \text{S}(\text{U}(p) \times \text{U}(q)))$  ( $p \geq q$ ), where we realize  $\text{SU}(p, q)$  as

$$\text{SU}(p, q) = \{g \in \text{SL}_{p+q}(\mathbf{C}); {}^t\bar{g}I_{p,q}g = I_{p,q}\}$$

with  $I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$ . Then we take  $K, Q$  to be given by

$$K = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \mathrm{SL}_{p+q}(\mathbf{C}); a \in \mathrm{GL}_p(\mathbf{C}), d \in \mathrm{GL}_q(\mathbf{C}) \right\},$$

$$Q = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in \mathrm{SL}_{p+q}(\mathbf{C}); a \in \mathrm{GL}_p(\mathbf{C}), d \in \mathrm{GL}_q(\mathbf{C}), c \in \mathrm{Mat}_{q,p}(\mathbf{C}) \right\},$$

respectively. We can assume that the holomorphic character  $\lambda : Q \rightarrow \mathbf{C}^\times$  in this case is given by

$$(2.5) \quad \lambda \left( \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \right) = (\det d)^{-s}$$

for some integer  $s$ .

Note that  $G_{\mathbf{R}}/K_{\mathbf{R}}$  is isomorphic to the bounded symmetric domain given by

$$\left\{ z = (z^{ij}) \in \mathrm{Mat}_{p,q}(\mathbf{C}); 1_q - {}^t \bar{z} z \text{ is positive definite} \right\}.$$

Therefore, we can take holomorphic coordinates  $(z^{ij}, \xi_{ij})_{i=1,\dots,p; j=1,\dots,q}$  around an arbitrary point  $(x, v^*)$  on the whole  $T^*(G_{\mathbf{R}}/K_{\mathbf{R}})$  such that

$$v^* = \sum_{i,j} \xi_{ij} dz^{ij}.$$

Using the fact that  $\mathfrak{u}$  is abelian, it is easy to show that the right-hand side of (2.4) equals

$$(2.6) \quad \begin{bmatrix} \frac{q}{p+q}s1_p & 0 \\ -\xi & -\frac{p}{p+q}s1_q \end{bmatrix}$$

(Theorem 4.9, [3]), where we denote the complex  $q \times p$ -matrix  ${}^t(\xi_{ij})$  by  $\xi$ . Thus, if  $s \neq 0$  and if we put

$$(2.7) \quad w := -s^{-1}\xi \quad \text{and} \quad u_w^- := \begin{bmatrix} 1_p & 0 \\ w & 1_q \end{bmatrix},$$

then it is immediate to show that (2.6) is equal to  $\mathrm{Ad}(u_w^-)\lambda^\vee$ . Hence

$$\mu_{\lambda;0}(x, v^*) = \mathrm{Ad}(u_x) \mathrm{Ad}(u_w^-)\lambda^\vee.$$

This yields an injective holomorphic map  $\mu_{\lambda;0} : T^*(G_{\mathbf{R}}/K_{\mathbf{R}}) \rightarrow \mathfrak{g}$ , which is a prototype of our main object.

Observe that there is no need to restrict the domain of  $\mu_{\lambda;0}$  to  $T^*(G_{\mathbf{R}}/K_{\mathbf{R}})$ . Indeed, it naturally extends to the holomorphic cotangent bundle of the open subset  $UQ/Q \subset G/Q$  if we do not take the real form  $G_{\mathbf{R}}$  into account. Furthermore, we can take an arbitrary  $\lambda \in \mathfrak{h}^*$ ; we shall carry out this extended case in the next subsection.

**2.2. Generalized flag variety.** Let  $G$  be a linear connected complex reductive Lie group with Lie algebra  $\mathfrak{g}$ , as above. Consider a nonzero  $\lambda \in \mathfrak{h}^*$ , which is not necessarily the same as in the previous subsection. Put

$$\mathfrak{l} := \mathfrak{g}(\lambda) = \{X \in \mathfrak{g}; \mathrm{ad}^*(X)\lambda = 0\};$$

we assume that  $\mathfrak{l}$  is distinct from  $\mathfrak{g}$  throughout. Let  $\mathfrak{q}$  be a parabolic subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{b}$  whose Levi part is  $\mathfrak{l}$ . We assume that  $\mathfrak{q}$  is not necessarily maximal. Let  $\mathfrak{u}^-$  be the nilradical of  $\mathfrak{q}$ , and  $\mathfrak{u}$  the opposite of  $\mathfrak{u}^-$ . Our assumption on  $\mathfrak{q}$  implies that the subalgebras  $\mathfrak{u}^-$  and  $\mathfrak{u}$  need not be abelian. At any rate, we have the following decompositions:

$$(2.8) \quad \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}^-, \quad \text{and} \quad \mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{u}^-.$$

Denote by  $\Delta(\mathfrak{u})$  the subset of  $\Delta^+$  such that  $\mathfrak{u} = \bigoplus_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_\alpha$  and  $\mathfrak{u}^- = \bigoplus_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_{-\alpha}$ .

Let  $L := G(\lambda) = \{g \in G; \text{Ad}^*(g)\lambda = \lambda\}$ , the isotropy subgroup of  $\lambda$  in  $G$ . Denoting the analytic subgroup of  $\mathfrak{u}^-$  (resp.  $\mathfrak{u}$ ) by  $U^-$  (resp.  $U$ ), let us introduce holomorphic coordinates  $z = (z^\alpha)_{\alpha \in \Delta(\mathfrak{u})}$  on  $U$  and  $w = (w_\alpha)_{\alpha \in \Delta(\mathfrak{u})}$  on  $U^-$  by parametrizing elements  $u \in U$  and  $u^- \in U^-$  as

$$(2.9) \quad u = \exp \sum_{\alpha \in \Delta(\mathfrak{u})} z^\alpha E_\alpha \quad \text{and} \quad u^- = \exp \sum_{\alpha \in \Delta(\mathfrak{u})} w_\alpha E_{-\alpha},$$

which we denote by  $u_z$  and  $u_w^-$  respectively.

Put  $Q = LU^- = U^-L$ , the parabolic subgroup of  $G$  whose Lie algebra is  $\mathfrak{q}$ . Note that  $L$  is connected since  $G$  is connected. Let  $T^*(G/Q)$  denote the holomorphic cotangent bundle of the flag variety  $G/Q$  and

$$\begin{aligned} p &: G \rightarrow G/Q, \\ \pi &: T^*(G/Q) \rightarrow G/Q, \end{aligned}$$

the canonical projections. Fixing a representative  $\dot{\sigma} \in G$  of each  $\sigma \in W/W_\lambda$  once and for all, let us identify  $\sigma$  with  $\dot{\sigma}$ , where  $W_\lambda$  denotes the isotropy subgroup of  $\lambda$  in the Weyl group  $W$ . Take the open covering  $\{U_\sigma\}$  of  $G/Q$ :

$$(2.10) \quad G/Q = \bigcup_{\sigma \in W/W_\lambda} U_\sigma \quad \text{with} \quad U_\sigma := \sigma U Q/Q.$$

Since any element  $x$  of  $U_\sigma$  is expressed as

$$(2.11) \quad x = \sigma u.e_Q$$

for a unique  $u \in U$ , one can introduce holomorphic local coordinates  $z_\sigma = (z_\sigma^\alpha)_{\alpha \in \Delta(\mathfrak{u})}$  on  $U_\sigma$  by

$$(2.12) \quad u = \exp \sum_{\alpha \in \Delta(\mathfrak{u})} z_\sigma^\alpha E_\alpha,$$

which we denote by  $u_{z_\sigma}$  following the notation (2.9). Since every cotangent vector  $v^* \in T_x^*(G/Q)$  is written as

$$v^* = \sum_{\alpha \in \Delta(\mathfrak{u})} \xi_{\sigma\alpha} dz_\sigma^\alpha,$$

it provides holomorphic coordinates  $(z_\sigma, \xi_\sigma)$  on  $\pi^{-1}(U_\sigma)$  with  $\xi_\sigma = (\xi_{\sigma\alpha})_{\alpha \in \Delta(\mathfrak{u})}$ . In other words, one obtains a local triviality

$$(2.13) \quad \phi_\sigma : \pi^{-1}(U_\sigma) \xrightarrow{\sim} U_\sigma \times \mathbb{C}^n, \quad (x, v^*) \mapsto (z_\sigma, \xi_\sigma),$$

with  $n = \#\Delta(\mathfrak{u}) = \dim(G/Q)$  for each  $\sigma \in W/W_\lambda$ . In the sequel, however, if  $(z_e, \xi_e)$  is in  $\pi^{-1}(U_e)$  i.e., if  $\sigma$  happens to equal the identity element  $e$ , we suppress the subscripts and just write  $(z, \xi)$  for brevity.

REMARK 2.2. For  $\sigma \in W/W_\lambda$  given, we have a unique decomposition  $p^{-1}(U_\sigma) = \sigma U Q = \sigma U U^- L$ . Namely, any  $g \in p^{-1}(U_\sigma)$  uniquely factorizes into a product

$$(2.14) \quad g = \sigma u u^- t \quad (u \in U, u^- \in U^-, t \in L),$$

which plays a role throughout the paper, as we shall see.

Now, let us fix an element  $\sigma \in W/W_\lambda$ , and discuss inside the product bundle  $T^*U_\sigma = \pi^{-1}(U_\sigma)$  until the end of this section.

LEMMA 2.3. For  $(z_\sigma, \xi_\sigma) \in T^*U_\sigma$  given, there correspond unique  $u_{z_\sigma} \in U$  and  $u_{w_\sigma}^- \in U^-$  such that

$$(2.15) \quad z_\sigma = \sigma u_{z_\sigma} \cdot e_Q \quad \text{and} \quad \xi_\sigma = -\langle \text{Ad}^*(u_{w_\sigma}^-)\lambda, u_{z_\sigma}^{-1} du_{z_\sigma} \rangle,$$

where we identify<sup>1</sup>  $\xi_\sigma = (\xi_{\sigma\alpha})_{\alpha \in \Delta(\mathfrak{u})}$  with  $\sum_{\alpha \in \Delta(\mathfrak{u})} \xi_{\sigma\alpha} dz_{\sigma\alpha}$ , which we abbreviate  $\xi_\sigma dz_\sigma$ .

PROOF. It is trivial that such  $u_{z_\sigma} \in U$  uniquely exists. If we identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$  via the nondegenerate invariant symmetric bilinear form  $B$  on  $\mathfrak{g}$ , the second formula of (2.15) can be rewritten as

$$B(\text{Ad}(u_{w_\sigma}^-)\lambda^\vee, u_{z_\sigma}^{-1} du_{z_\sigma}) = -\xi_\sigma,$$

where  $\lambda^\vee \in \mathfrak{h}$  corresponds to  $\lambda \in \mathfrak{h}^*$  under the identification.

Now, since  $u_{z_\sigma}^{-1} du_{z_\sigma}$  is a 1-form taking values in  $\mathfrak{u}$ , the nondegeneracy of  $B|_{\mathfrak{u} \times \mathfrak{u}}$  implies that there exists a unique  $Y \in \mathfrak{u}^-$  satisfying

$$B(Y, u_{z_\sigma}^{-1} du_{z_\sigma}) = -\xi_\sigma.$$

Thus it suffices to show that there exists a unique  $u_{w_\sigma}^- \in U^-$  such that

$$(2.16) \quad \text{Ad}(u_{w_\sigma}^-)\lambda^\vee = \lambda^\vee + Y$$

since  $\text{Ad}(u_{w_\sigma}^-)\lambda^\vee$  is in  $\mathfrak{l} \oplus \mathfrak{u}^-$  with its  $\mathfrak{l}$ -component equal to  $\lambda^\vee$ . Parametrizing  $u_{w_\sigma}^-$  as

$$(2.17) \quad u_{w_\sigma}^- = \exp \sum_{\alpha \in \Delta(\mathfrak{u})} w_{\sigma\alpha} E_{-\alpha},$$

one can determine the coefficients  $w_{\sigma\alpha}$  inductively from (2.16) with respect to the height of  $\alpha$ . This completes the proof. □

EXAMPLE 2.4. Let us consider the case where  $G = \text{GL}_3(\mathbb{C})$  and  $\lambda = \sum_{i=1}^3 \lambda_i \varepsilon_i \in \mathfrak{h}^*$  is regular semisimple, i.e., it satisfies  $\lambda_i \neq \lambda_j$  if  $i \neq j$ . Then  $\mathfrak{l} = \mathfrak{g}(\lambda)$  is equal to  $\mathfrak{h}$ , the Cartan subalgebra consisting of all diagonal matrices in  $\mathfrak{g} = \mathfrak{gl}_3(\mathbb{C})$ ,  $\mathfrak{q}$  the Borel subalgebra  $\mathfrak{b}$  of all lower triangular matrices in  $\mathfrak{g}$ , and  $\mathfrak{u}^-$  (resp.  $\mathfrak{u}$ ) the nilpotent subalgebra of all strictly lower (resp. upper) matrices in  $\mathfrak{g}$ .

---

<sup>1</sup>We shall sometimes use this convention throughout the paper.

We restrict ourselves to the case where  $\sigma = e$  since the other cases are similar. For  $(z, \xi) \in T^*U_e$  with  $z = (z^{i,j})_{1 \leq i < j \leq 3}$  and  $\xi dz = \sum_{1 \leq i < j \leq 3} \xi_{i,j} dz^{i,j}$ , if one writes

$$u_z = \begin{bmatrix} 1 & z^{1,2} & z^{1,3} + \frac{1}{2}z^{1,2}z^{2,3} \\ & 1 & z^{2,3} \\ & & 1 \end{bmatrix}, \quad u_w^- = \begin{bmatrix} & 1 & & \\ & w_{1,2} & & 1 \\ w_{1,3} + \frac{1}{2}w_{1,2}w_{2,3} & & w_{2,3} & 1 \end{bmatrix}$$

as in (2.12) and (2.17), the second formula of (2.15) is equivalent to

$$\begin{aligned} \lambda_{1,2}w_{1,2} + \frac{1}{2}z^{2,3} \left( \lambda_{1,3}w_{1,3} + \frac{1}{2}(\lambda_{1,2} - \lambda_{2,3})w_{1,2}w_{2,3} \right) &= -\xi_{1,2}, \\ \lambda_{2,3}w_{2,3} - \frac{1}{2}z^{1,2} \left( \lambda_{1,3}w_{1,3} + \frac{1}{2}(\lambda_{1,2} - \lambda_{2,3})w_{1,2}w_{2,3} \right) &= -\xi_{2,3}, \\ \lambda_{1,3}w_{1,3} + \frac{1}{2}(\lambda_{1,2} - \lambda_{2,3})w_{1,2}w_{2,3} &= -\xi_{1,3}, \end{aligned}$$

from which it immediately follows that

$$\begin{aligned} w_{1,2} &= \frac{1}{\lambda_{1,2}} \left( -\xi_{1,2} + \frac{1}{2}\xi_{1,3}z^{2,3} \right), \\ w_{2,3} &= \frac{1}{\lambda_{2,3}} \left( -\xi_{2,3} - \frac{1}{2}\xi_{1,3}z^{1,2} \right), \\ w_{1,3} &= \frac{1}{\lambda_{1,3}} \left( -\xi_{1,3} - \frac{1}{2} \frac{\lambda_{1,2} - \lambda_{2,3}}{\lambda_{1,2}\lambda_{2,3}} \left( -\xi_{1,2} + \frac{1}{2}\xi_{1,3}z^{2,3} \right) \left( -\xi_{2,3} - \frac{1}{2}\xi_{1,3}z^{1,2} \right) \right), \end{aligned}$$

where we put  $\lambda_{i,j} := \lambda_i - \lambda_j$  for  $i \neq j$ .

We remark that one can verify that the relation (2.4) holds if one constructs an irreducible representation of  $GL_3(\mathbb{C})$  that is induced from the character  $\lambda : Q \rightarrow \mathbb{C}^\times$  by Borel-Weil theory as in the previous subsection.

Put  $\Omega_\lambda := G \cdot \lambda = \{Ad^*(g)\lambda \in \mathfrak{g}^*; g \in G\}$ , the coadjoint orbit of  $\lambda$  under the complex Lie group  $G$ . It is canonically isomorphic to  $G/L$ , and we denote by  $p_\lambda$  the canonical surjection

$$p_\lambda : G \rightarrow \Omega_\lambda, \quad g \mapsto Ad^*(g)\lambda.$$

DEFINITION 2.5. By Lemma 2.3 above, one can define a holomorphic map

$$(2.18) \quad \mu_{\lambda;\sigma} : T^*U_\sigma \rightarrow \Omega_\lambda \quad \text{by} \quad \mu_{\lambda;\sigma}(z_\sigma, \xi_\sigma) := Ad^*(\sigma u_{z_\sigma} u_{w_\sigma}^-)\lambda,$$

where  $u_{z_\sigma} \in U$  and  $u_{w_\sigma}^- \in U^-$  are the unique elements corresponding to  $(z_\sigma, \xi_\sigma) \in T^*U_\sigma$  determined by the relation (2.15). Note in particular that  $\mu_{\lambda;\sigma}$  is injective.

REMARK 2.6. If  $g \in G$  satisfies that

$$\mu_{\lambda;\sigma}(z_\sigma, \xi_\sigma) = Ad^*(g)\lambda$$

for  $(z_\sigma, \xi_\sigma) \in T^*U_\sigma$ , then there exists an element  $t \in L$  such that  $g = \sigma u_{z_\sigma} u_{w_\sigma}^- t$ . The correspondence  $(z_\sigma, \xi_\sigma) \mapsto g = \sigma u_{z_\sigma} u_{w_\sigma}^- t$  can be regarded as a section for the fibration  $p^{-1}(U_\sigma) \rightarrow T^*U_\sigma$ :



$$\begin{array}{ccc}
 & & p^{-1}(U_\sigma) \\
 & \nearrow g & \downarrow p_\lambda \\
 T^*U_\sigma & \xrightarrow{\tilde{\mu}_{\lambda;\sigma}} & \mu_{\lambda;\sigma}(T^*U_\sigma)
 \end{array}$$

Now, let us define a  $\mathfrak{g}$ -valued 1-form  $\theta$  on  $G$  by

$$(2.19) \quad \theta_g := g^{-1} dg \quad (g \in G).$$

By abuse of notation, we use the same symbol  $\theta_g$  to denote the pull-back of the 1-form given in (2.19) by the local section  $g : T^*U_\sigma \rightarrow p^{-1}(U_\sigma)$ . Then the second formula of (2.15) can be written as

$$(2.20) \quad \begin{aligned} \xi_\sigma &= -\langle \lambda, \theta_\sigma a \rangle \\ &= -\langle \lambda, \theta_a \rangle, \end{aligned}$$

where we set  $a := u_{z_\sigma} u_{w_\sigma}^-$  for brevity. In fact, since

$$(\sigma a)^{-1} d(\sigma a) = a^{-1} da = \text{Ad}(u_{w_\sigma}^-)^{-1}(u_{z_\sigma}^{-1} du_{z_\sigma}) + u_{w_\sigma}^{-1} du_{w_\sigma}^- ,$$

the relation (2.20) follows from the fact that the second term is a 1-form taking values in  $\mathfrak{u}^-$ .

### 3. G-equivariance.

**3.1. Local G-action.** First let us consider such elements  $g \in G$  that map  $U_e$  onto itself.

**DEFINITION 3.1.** For  $(z, \xi) \in T^*U_e$ , let  $u_z \in U$  and  $u_w^- \in U^-$  be the unique elements determined by (2.15). If  $g \in G$  satisfies that  $g.z \in U_e$ , or equivalently, that  $gu_z \in p^{-1}(U_e)$ , then by Remark 2.2, one can write

$$gu_z = u_{g.z} u_{g;z}^- t_{g;z} \quad \text{with} \quad u_{g.z} \in U, u_{g;z}^- \in U^-, t_{g;z} \in L ,$$

from which it follows that

$$(3.1) \quad \begin{aligned} gu_z u_w^- &= u_{g.z} u_{g;z}^- t_{g;z} \cdot u_w^- \\ &= u_{g.z} \cdot u_{g;z}^- (t_{g;z} u_w^- t_{g;z}^{-1}) \cdot t_{g;z} . \end{aligned}$$

In particular, we see that  $gu_z u_w^-$  lies in  $UU^-L$ , and that its  $U$ - and  $L$ -components are identical to those of  $gu_z$  respectively since  $t_{g;z} u_w^- t_{g;z}^{-1}$  is in the subgroup  $U^-$ .

In view of (2.15) and (3.1), it is natural to define a cotangent vector  $\psi_{\lambda;e}(g)\xi$  by

$$(3.2) \quad \psi_{\lambda;e}(g)\xi = -\langle \text{Ad}^*(u_{g;z;w}^-)\lambda, u_{g;z}^{-1} du_{g.z} \rangle ,$$

where we set  $u_{g;z;w}^- := u_{g;z}^- (t_{g;z} u_w^- t_{g;z}^{-1})$  for brevity. If we put  $a := u_z u_w^-$ , then, as we noted in Remark 2.6, the right-hand side of (3.2) can be written as

$$(3.3) \quad \begin{aligned} \psi_{\lambda;e}(g)\xi &= -\langle \lambda, \theta_{ga t_{g;z}^{-1}} \rangle \\ &= -\langle \lambda, \theta_{ga} \rangle + \langle \lambda, dt_{g;z} t_{g;z}^{-1} \rangle \end{aligned}$$

$$(3.4) \quad = (g^{-1})^*\xi + \langle \lambda, dt_{g;z}t_{g;z}^{-1} \rangle$$

since  $\theta_{ga} = (g^{-1})^*\theta_a$ , where  $(g^{-1})^*$  denotes the transpose map of the differential  $(g^{-1})_* : T_{g,z}U_e \rightarrow T_zU_e$  induced from the translation by  $g^{-1} \in G$  on  $U_e$ . In particular, if  $g$  is an element of  $U$ , then  $\psi_{\lambda;e}(g)$  is identical to the canonical action (2.1) on  $T^*U_e$ , because the second term of (3.4) vanishes since  $t_{g;z} = e$  for all  $z$ .

Note that (3.2) implies  $\psi_{\lambda;e}(g)\xi$  belongs to  $T_{g,z}^*(G/Q)$  since the decomposition (3.1) is holomorphic, and that (3.4) reduces to the canonical  $G$ -action on the cotangent bundle given by (2.1) when  $\lambda \rightarrow 0$ . Furthermore, it follows that the second term in (3.4) is an exact 1-form since  $t_{g;z}$  is an element of  $G(\lambda)$ .

PROPOSITION 3.2. *Let  $(z, \xi) \in T^*U_e$ . For  $g, h \in G$  such that both  $h.z$  and  $gh.z$  are in  $U_e$ , we have*

$$(3.5) \quad \psi_{\lambda;e}(g)(\psi_{\lambda;e}(h)\xi) = \psi_{\lambda;e}(gh)\xi .$$

PROOF. We use (3.3) to prove the proposition. Let  $u_z \in U$  and  $u_w^- \in U^-$  be the unique elements determined by (2.15) and let  $a = u_zu_w^-$ . Since both  $hu_z$  and  $gu_{h.z}$  are in  $p^{-1}(U_e)$  by assumption, they decompose as

$$(3.6) \quad hu_z = u_{h.z}u_{h;z}^-t_{h;z} \in UU^-L ,$$

$$(3.7) \quad gu_{h.z} = u_{gh.z}u_{g;h.z}^-t_{g;h.z} \in UU^-L .$$

Then, we see that

$$(3.8) \quad \begin{aligned} g(hu_z) &= g(u_{h.z}u_{h;z}^-t_{h;z}) \\ &= u_{gh.z}u_{g;h.z}^-t_{g;h.z} \cdot u_{h;z}^-t_{h;z} \\ &= u_{gh.z} \cdot u_{g;h.z}^- (t_{g;h.z}u_{h;z}^-t_{g;h.z}^{-1}) \cdot t_{g;h.z}t_{h;z} . \end{aligned}$$

Namely, the  $L$ -component of  $g(hu_z)$  equals  $t_{g;h.z}t_{h;z}$ .

Now, it follows from (3.6) and (3.7) that

$$\begin{aligned} \psi_{\lambda;e}(g)(\psi_{\lambda;e}(h)\xi) &= \psi_{\lambda;e}(g)(\psi_{\lambda;e}(h)\langle -\lambda, \theta_a \rangle) \\ &= \psi_{\lambda;e}(g)\langle -\lambda, \theta_{ha t_{h;z}^{-1}} \rangle \\ &= \langle -\lambda, \theta_{gha t_{h;z}^{-1} t_{g;h.z}^{-1}} \rangle . \end{aligned}$$

On the other hand, it follows from (3.8) that

$$\begin{aligned} \psi_{\lambda;e}(gh)\xi &= \langle -\lambda, \theta_{(gh)a(t_{g;h.z}t_{h.z})^{-1}} \rangle \\ &= \langle -\lambda, \theta_{gha t_{h;z}^{-1} t_{g;h.z}^{-1}} \rangle . \end{aligned}$$

This completes the proof. □

For  $(z, \xi) \in T^*U_e$  and  $g \in G$  such that  $g.z \in U_e$ , we define

$$(3.9) \quad \Psi_{\lambda;e}(g) : T^*U_e \rightarrow T^*U_e \quad \text{by} \quad \Psi_{\lambda;e}(g)(z, \xi) := (g.z, \psi_{\lambda;e}(g)\xi) .$$

Note that  $\Psi_{\lambda;e}(g)|_{T_z^*(G/Q)} = \psi_{\lambda;e}(g)$  is a bi-holomorphic map from  $T_z^*(G/Q)$  onto  $T_{g.z}^*(G/Q)$  for all  $z \in U_e$ .

PROPOSITION 3.3. For  $(z, \xi) \in T^*U_e$  and  $g \in G$  such that  $g.z \in U_e$ , we have

$$(3.10) \quad \mu_{\lambda;e}(\Psi_{\lambda;e}(g)(z, \xi)) = \text{Ad}^*(g)\mu_{\lambda;e}(z, \xi).$$

PROOF. This is equivalent to the definition of  $\psi_{\lambda;e}(g)$ , with  $g \in G$ , as we shall see soon. In fact, the elements of  $U$  and  $U^-$  corresponding to  $\psi_{\lambda;e}(g)\xi$  by  $\mu_{\lambda;e}$  are  $u_{g.z}$  and  $u_{g.z;w}^-$  respectively in the notation of Definition 3.1. Therefore, the left-hand side of (3.10) equals

$$\begin{aligned} \text{Ad}^*(u_{g.z}u_{g.z;w}^-)\lambda &= \text{Ad}^*(gu_zu_w^-)\lambda = \text{Ad}^*(g)\text{Ad}^*(u_zu_w^-)\lambda \\ &= \text{Ad}^*(g)\mu_{\lambda;e}(z, \xi) \end{aligned}$$

by (3.1) and (2.18). □

EXAMPLE 3.4 (Example 2.1 continued). Let  $G = \text{SL}_{p+q}(\mathbb{C})$ ,  $Q$  and  $\lambda$  be as in Example 2.1. Then, for  $(z, \xi) \in T^*U_e$ , let  $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be an element of  $G$  such that  $g.z \in U_e$ . If one writes  $\begin{bmatrix} 1 & \hat{z} \\ 0 & 1 \end{bmatrix} := u_{g.z}$ ,  $\begin{bmatrix} 1 & 0 \\ \hat{w} & 1 \end{bmatrix} := u_{g.z;w}^-$  and  $\begin{bmatrix} \hat{a} & 0 \\ 0 & \hat{d} \end{bmatrix} := t_{g;z}$  in the decomposition (3.1), an elementary matrix calculation shows that

$$\begin{aligned} \hat{z} &= (az + b)(cz + d)^{-1} = g.z, \\ \hat{w} &= (c + (cz + d)w)(a - (az + b)(cz + d)^{-1}c)^{-1}, \\ \hat{a} &= a - (az + b)(cz + d)^{-1}c, \\ \hat{d} &= cz + d. \end{aligned}$$

In particular, one sees

$$\begin{aligned} d\hat{z} &= d((az + b)(cz + d)^{-1}) \\ &= (a - (az + b)(cz + d)^{-1}c) dz(cz + d)^{-1}. \end{aligned}$$

Using the relation (2.15), i.e.,  $\hat{\xi} = -s\hat{w}$ , one has

$$\hat{\xi} d\hat{z} = (cz + d)\xi dz(cz + d)^{-1} - sc dz(cz + d)^{-1}.$$

Taking the trace of the both sides, one obtains (3.4) in this case; in particular, the second term of (3.4) is given by

$$\begin{aligned} \langle \lambda, dt_{g;z}t_{g;z}^{-1} \rangle &= -s \text{tr}(c dz(cz + d)^{-1}) \\ &= -s d \log \det(cz + d), \end{aligned}$$

and hence (3.5) corresponds to the cocycle condition of the automorphy factor.

**3.2. Global construction.** Recall that the flag variety  $G/Q$  has the open covering  $\{U_\sigma\}_{\sigma \in W/W_\lambda}$ , and that each  $\pi^{-1}(U_\sigma)$  is bi-holomorphic to  $U_\sigma \times \mathbb{C}^n$  with  $n = \dim(G/Q)$ . If a point  $x \in G/Q$  is in  $U_\sigma \cap U_\tau$ , then it can be written

$$x = \sigma u_{z_\sigma}.e_Q = \tau u_{z_\tau}.e_Q.$$

Therefore, if we take into account (2.15), (2.20) and Proposition 3.3, it is natural from the group-theoretic point of view to glue together the product bundles  $\{\pi^{-1}(U_\sigma)\}$ , using the transition functions given by  $\{\psi_{\lambda;e}(\tau^{-1}\sigma)\}$ , as follows.

For two points in the disjoint union  $\bigsqcup_{\sigma \in W/W_\lambda} (U_\sigma \times \mathbf{C}^n)$ , say,  $(z_\sigma, \xi_\sigma) \in U_\sigma \times \mathbf{C}^n$  and  $(z_\tau, \xi_\tau) \in U_\tau \times \mathbf{C}^n$  for some  $\sigma$  and  $\tau \in W/W_\lambda$ , then  $(z_\tau, \xi_\tau)$  is defined to be equivalent to  $(z_\sigma, \xi_\sigma)$  if and only if

$$(3.11) \quad \tau u_{z_\tau} \cdot e_Q = \sigma u_{z_\sigma} \cdot e_Q \quad \text{and} \quad \xi_\tau = \psi_{\lambda;e}(\tau^{-1}\sigma)\xi_\sigma;$$

we write  $(z_\tau, \xi_\tau) \sim (z_\sigma, \xi_\sigma)$  in this case. Then we define our twisted cotangent bundle to be the quotient space by this equivalence relation:

$$(3.12) \quad T^*(G/Q)_\lambda := \bigsqcup_{\sigma \in W/W_\lambda} (U_\sigma \times \mathbf{C}^n) / \sim .$$

We denote by  $[z_\sigma, \xi_\sigma]$  the equivalence class of  $(z_\sigma, \xi_\sigma) \in U_\sigma \times \mathbf{C}^n$  and by  $\varpi$  the projection

$$T^*(G/Q)_\lambda \rightarrow G/Q, \quad [z_\sigma, \xi_\sigma] \mapsto \sigma u_{z_\sigma} \cdot e_Q .$$

Our twisted cotangent bundle  $T^*(G/Q)_\lambda$  is the affine bundle associated to the holomorphic cotangent bundle  $T^*(G/Q)$ . Namely, it is identical set-theoretically to the cotangent bundle

$$T^*(G/Q)_\lambda = \bigcup_{x \in G/Q} T_x^*(G/Q),$$

and its local triviality on  $\varpi^{-1}(U_\sigma)$  is given by

$$\varpi^{-1}(U_\sigma) \simeq U_\sigma \times \mathbf{C}^n, \quad [z_\sigma, \xi_\sigma] \mapsto (z_\sigma, \xi_\sigma)$$

for each  $\sigma$ ; however, each fiber of  $T^*(G/Q)_\lambda$  is considered to be an affine space and its transition functions are affine transformations  $\{\psi_{\lambda;e}(\tau^{-1}\sigma)\}_{\sigma,\tau}$  given in (3.11). Note that the transition functions of our twisted cotangent bundle reduce to those of the cotangent bundle when  $\lambda \rightarrow 0$ .

REMARK 3.5. Since the second term of (3.4) is exact, one obtains that  $d(\xi_\sigma dz_\sigma) = d(\xi_\tau dz_\tau)$  on  $\varpi^{-1}(U_\sigma \cap U_\tau)$ . Therefore, our twisted cotangent bundle possesses a holomorphic symplectic form that is locally identical to the canonical one on the cotangent bundle  $T^*(G/Q)$ , which we shall denote by  $\omega$ .

DEFINITION 3.6. For given  $g \in G$  and  $[z_\sigma, \xi_\sigma] \in \varpi^{-1}(U_\sigma) \subset T^*(G/Q)_\lambda$ , take any  $\tau \in W/W_\lambda$  such that  $g \cdot z_\sigma \in U_\tau$ . Suppose that  $\xi_\sigma$  is written as  $\xi_\sigma = \langle -\lambda, \theta_{u_{z_\sigma} u_{w_\sigma}^-} \rangle$  with  $u_{z_\sigma}$  and  $u_{w_\sigma}^-$  being the unique elements of  $U$  and  $U^-$  determined by (2.20). Then we define a cotangent vector  $\psi_\lambda(g)\xi_\sigma \in T_{g \cdot z_\sigma}^*(G/Q)$  by

$$(3.13) \quad \psi_\lambda(g)\xi_\sigma := \psi_{\lambda;e}(\tau^{-1}g\sigma)\langle -\lambda, \theta_{u_{z_\sigma} u_{w_\sigma}^-} \rangle$$

and a holomorphic map  $\Psi_\lambda(g)$  from  $T^*(G/Q)_\lambda$  to itself by

$$(3.14) \quad \Psi_\lambda(g)[z_\sigma, \xi_\sigma] := [g \cdot z_\sigma, \psi_\lambda(g)\xi_\sigma] .$$

LEMMA 3.7. *The map  $\Psi_\lambda$  is well defined, i.e., is independent of the choices of  $\sigma$  and  $\tau$  in (3.13) above. Furthermore, we have*

$$(3.15) \quad \Psi_\lambda(g)\Psi_\lambda(h) = \Psi_\lambda(gh)$$

for all  $g, h \in G$ . Namely,  $G$  acts on the twisted cotangent bundle  $T^*(G/Q)_\lambda$  through  $\Psi_\lambda$ .

PROOF. Suppose that  $[z_\sigma, \xi_\sigma] = [z_{\hat{\sigma}}, \xi_{\hat{\sigma}}]$  and take another  $\hat{\tau}$  such that  $g.z_{\hat{\sigma}} \in U_{\hat{\tau}}$ . Then, by definition, one has

$$\xi_{\hat{\sigma}} = \psi_{\lambda;e}(\hat{\sigma}^{-1}\sigma)\xi_\sigma.$$

It suffices to show that

$$\psi_{\lambda;e}(\hat{\tau}^{-1}g\hat{\sigma})\xi_{\hat{\sigma}} = \psi_{\lambda;e}(\hat{\tau}^{-1}\tau)\psi_{\lambda;e}(\tau^{-1}g\sigma)\xi_\sigma.$$

Now one sees

$$\begin{aligned} \psi_{\lambda;e}(\hat{\tau}^{-1}g\hat{\sigma})\xi_{\hat{\sigma}} &= \psi_{\lambda;e}(\hat{\tau}^{-1}g\hat{\sigma})\psi_{\lambda;e}(\hat{\sigma}^{-1}\sigma)\xi_\sigma \\ &= \psi_{\lambda;e}(\hat{\tau}^{-1}g\sigma)\xi_\sigma \\ &= \psi_{\lambda;e}(\hat{\tau}^{-1}\tau)\psi_{\lambda;e}(\tau^{-1}g\sigma)\xi_\sigma \end{aligned}$$

by Proposition 3.2. The second assertion also follows from Proposition 3.2.  $\square$

Since  $\varpi^{-1}(U_\sigma) = \pi^{-1}(U_\sigma)$ , one can define  $\mu_{\lambda;\sigma} : \varpi^{-1}(U_\sigma) \rightarrow \Omega_\lambda$  by the same formula as (2.18) for each  $\sigma \in W/W_\lambda$ :

$$(3.16) \quad \mu_{\lambda;\sigma} : \varpi^{-1}(U_\sigma) \rightarrow \Omega_\lambda, \quad [z_\sigma, \xi_\sigma] \mapsto \text{Ad}^*(\sigma u_{z_\sigma} u_{w_\sigma}^-)\lambda,$$

where  $u_{z_\sigma} \in U$  and  $u_{w_\sigma}^- \in U^-$  are determined by  $(z_\sigma, \xi_\sigma)$  as in (2.15).

PROPOSITION 3.8. *The local isomorphisms  $\{\mu_{\lambda;\sigma}\}_{\sigma \in W/W_\lambda}$  satisfy the compatibility condition*

$$(3.17) \quad \mu_{\lambda;\sigma}|_{\varpi^{-1}(U_\sigma \cap U_\tau)} = \mu_{\lambda;\tau}|_{\varpi^{-1}(U_\sigma \cap U_\tau)} \quad (\sigma, \tau \in W/W_\lambda).$$

Thus we can define a globally defined bi-holomorphic map

$$(3.18) \quad \mu_\lambda : T^*(G/Q)_\lambda \rightarrow \Omega_\lambda \quad \text{by} \quad \mu_\lambda|_{\varpi^{-1}(U_\sigma)} := \mu_{\lambda;\sigma}.$$

Furthermore this map is  $G$ -equivariant, i.e., we have

$$(3.19) \quad \mu_\lambda \circ \Psi_\lambda(g) = \text{Ad}^*(g) \circ \mu_\lambda$$

for all  $g \in G$ .

PROOF. Suppose that a point of  $\varpi^{-1}(U_\sigma \cap U_\tau)$  is expressed in two ways:

$$[z_\sigma, \xi_\sigma] = [z_\tau, \xi_\tau] \in \varpi^{-1}(U_\sigma \cap U_\tau),$$

where we regard  $[z_\sigma, \xi_\sigma] \in \varpi^{-1}(U_\sigma)$  and  $[z_\tau, \xi_\tau] \in \varpi^{-1}(U_\tau)$ . Let  $u_{z_\sigma}, u_{z_\tau} \in U$  and  $u_{w_\sigma}^-, u_{w_\tau}^- \in U^-$  satisfy

$$\xi_\sigma = \langle -\lambda, \theta_{u_\sigma u_\sigma^-} \rangle \quad \text{and} \quad \xi_\tau = \langle -\lambda, \theta_{u_\tau u_\tau^-} \rangle$$

as in (2.20) (we shall abbreviate  $u_\sigma := u_{z_\sigma}, u_\sigma^- := u_{w_\sigma}^-$  etc. until the end of the proof). Then by the definition of the equivalence relation (3.11), we have

$$\sigma^{-1} \tau u_\tau u_\tau^- = u_\sigma u_\sigma^- t$$

for some  $t \in L = G(\lambda)$ . Therefore, we see that

$$\begin{aligned} \mu_{\lambda; \tau}([z_\tau, \xi_\tau]) &= \text{Ad}^*(\tau u_\tau u_\tau^-) \lambda = \text{Ad}^*(\sigma u_\sigma u_\sigma^- t) \lambda \\ &= \text{Ad}^*(\sigma u_\sigma u_\sigma^-) \lambda \\ &= \mu_{\lambda; \sigma}([z_\sigma, \xi_\sigma]). \end{aligned}$$

Next, for  $g \in G$  and  $[z_\sigma, \xi_\sigma] \in \varpi^{-1}(U_\sigma)$ , take  $\hat{\sigma} \in W/W_\lambda$  satisfying  $g \cdot z_\sigma \in U_{\hat{\sigma}}$ . Since  $\hat{\sigma}^{-1} g \sigma u_\sigma u_\sigma^-$  is in  $UU^-L$ , it decomposes as

$$\hat{\sigma}^{-1} g \sigma u_\sigma u_\sigma^- = u_1 u_1^- t_1 \quad \text{with} \quad u_1 \in U, u_1^- \in U^-, t_1 \in L.$$

Then

$$\begin{aligned} \psi_\lambda(g) \xi_\sigma &= \psi_{\lambda; e}(\hat{\sigma}^{-1} g \sigma) \xi_\sigma \\ &= \psi_{\lambda; e}(\hat{\sigma}^{-1} g \sigma) \langle -\lambda, \theta_{u_\sigma u_\sigma^-} \rangle \\ &= \langle -\lambda, \theta_{\hat{\sigma}^{-1} g \sigma u_\sigma u_\sigma^- t_1^{-1}} \rangle. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} \mu_\lambda(\Psi_\lambda(g)[z_\sigma, \xi_\sigma]) &= \mu_{\lambda; \hat{\sigma}}([g \cdot z_\sigma, \psi_\lambda(g) \xi_\sigma]) \\ &= \text{Ad}^*(\hat{\sigma}) \text{Ad}^*(\hat{\sigma}^{-1} g \sigma u_\sigma u_\sigma^- t_1^{-1}) \lambda \\ &= \text{Ad}^*(g) \mu_\lambda([z_\sigma, \xi_\sigma]). \end{aligned}$$

This completes the proof.  $\square$

EXAMPLE 3.9. Let us consider the case where  $p = q = 1$  in Example 2.1, i.e.,  $G = \text{SL}_2(\mathbf{C})$ ,  $Q = \left\{ \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \in G \right\}$ , the Borel subgroup of  $G$ , and

$$\lambda : Q \rightarrow \mathbf{C}^\times, \quad \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \mapsto a^s.$$

The flag variety  $G/Q$  is identified with the complex projective line  $\mathbf{C}P^1$ . Under this identification, the open covering  $\{U_e, U_\sigma\}$  (with  $\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ) is given by

$$U_e = \{(z : 1) \in \mathbf{C}P^1; z \in \mathbf{C}\} \simeq \mathbf{C}, \quad U_\sigma = \{(1 : z_\sigma) \in \mathbf{C}P^1; z_\sigma \in \mathbf{C}\} \simeq \mathbf{C}.$$

For  $[z, \xi] \in \varpi^{-1}(U_e)$  and  $[z_\sigma, \xi_\sigma] \in \varpi^{-1}(U_\sigma)$ , let  $u_z, u_{z_\sigma} \in U$  and  $u_w^-, u_{w_\sigma}^- \in U^-$  satisfy  $\xi = -s w$ ,  $\xi_\sigma = -s w_\sigma$  as in (2.15). If  $[z_\sigma, \xi_\sigma] = [z, \xi]$ , then one sees

$$(3.20) \quad z_\sigma = -\frac{1}{z}, \quad w_\sigma = z^2 w + z$$

since  $\xi_\sigma = \psi_{\lambda; e}(\sigma^{-1}) \xi = z^2 \xi - s z$ .

Denoting the maps  $\mu_{\lambda;e}$  and  $\mu_{\lambda;\sigma}$  followed by the isomorphism  $\mathfrak{g}^* \simeq \mathfrak{g}$  via the trace form by the same notations, one obtains

$$(3.21) \quad \mu_{\lambda;e}([z, \xi]) = \frac{s}{2} \begin{bmatrix} 1 + 2zw & -2z(1 + zw) \\ 2w & -(1 + 2zw) \end{bmatrix}$$

and

$$\mu_{\lambda;\sigma}([z_\sigma, \xi_\sigma]) = \frac{s}{2} \begin{bmatrix} -(1 + 2z_\sigma w_\sigma) & -2w_\sigma \\ 2z_\sigma(1 + z_\sigma w_\sigma) & 1 + 2z_\sigma w_\sigma \end{bmatrix},$$

which, under the relation (3.20), coincide with each other.

**3.3. Symplectomorphism.** We next prove that the map  $\mu_\lambda : T^*(G/Q)_\lambda \rightarrow \Omega_\lambda$  is symplectic. Let  $\omega$  and  $\omega_\lambda$  denote the canonical  $G$ -invariant holomorphic symplectic forms on  $T^*(G/Q)_\lambda$  and  $\Omega_\lambda$  respectively. Recall that  $\omega$  is defined by

$$(3.22) \quad \omega_{[z_\sigma, \xi_\sigma]} = -d(\xi_\sigma dz_\sigma) = \sum_{\alpha \in \Delta(u)} dz_\sigma^\alpha \wedge d\xi_{\sigma\alpha}$$

if  $[z_\sigma, \xi_\sigma] \in \varpi^{-1}(U_\sigma) \subset T^*(G/Q)_\lambda$  (cf. Remark 3.5), and that  $\omega_\lambda$  is defined by

$$(3.23) \quad (\omega_\lambda)_f(X_{\Omega_\lambda}, Y_{\Omega_\lambda}) = -\langle f, [X, Y] \rangle \quad (f \in \Omega_\lambda; X, Y \in \mathfrak{g}),$$

where  $X_{\Omega_\lambda}, Y_{\Omega_\lambda}$  are the vector fields on  $\Omega_\lambda$  generated by  $X, Y \in \mathfrak{g}$  respectively that are defined by (2.3).

**PROPOSITION 3.10.** *Let  $\omega$  and  $\omega_\lambda$  be the canonical symplectic forms on  $T^*(G/Q)_\lambda$  and  $\Omega_\lambda$  respectively. Then  $\mu_\lambda$  preserves the symplectic forms:*

$$(3.24) \quad \mu_\lambda^* \omega_\lambda = \omega.$$

**PROOF.** It suffices to show the equality (3.24) on the dense open subset  $\varpi^{-1}(U_e)$ . For  $[z, \xi] \in \varpi^{-1}(U_e)$ , we put  $g := u_z u_w^-$ , where  $u_z \in U$  and  $u_w^- \in U^-$  are determined by (2.15). Then it follows from (2.20) that

$$(3.25) \quad \omega = \langle \lambda, d\theta_g \rangle = \langle -\lambda, \theta_g \wedge \theta_g \rangle$$

since  $d\theta_g = d(g^{-1} dg) = -g^{-1} dg \wedge g^{-1} dg = -\theta_g \wedge \theta_g$ . Thus, if we can show that

$$(\omega_\lambda)_{g.\lambda}(X_{\Omega_\lambda}, Y_{\Omega_\lambda}) = \langle -\lambda, (\theta_{g.\lambda} \wedge \theta_{g.\lambda})(X_{\Omega_\lambda}, Y_{\Omega_\lambda}) \rangle$$

for all  $X, Y \in \mathfrak{g}$ , then we are done. Here, we set  $g.\lambda = \text{Ad}^*(g)\lambda$  for brevity, and denote by  $\theta_{g.\lambda}$  the pull-back of the 1-form  $\theta$  by the local section

$$g : \Omega_\lambda|_{\mu_\lambda(\varpi^{-1}(U_e))} \rightarrow G, \quad g.\lambda \mapsto g$$

with  $g = u_z u_w^-$ .

Now, for  $X \in \mathfrak{g}$ , we see that

$$\begin{aligned} X_{\Omega_\lambda}(g.\lambda) &= \left. \frac{d}{dt} \right|_{t=0} \exp(-tX).(g.\lambda) \\ &= \left. \frac{d}{dt} \right|_{t=0} g \exp(-t \operatorname{Ad}(g^{-1})X).\lambda \\ &= g_*(\operatorname{Ad}(g^{-1})X)_{\Omega_\lambda}(\lambda). \end{aligned}$$

Hence we obtain that

$$\begin{aligned} \theta_{g.\lambda}(X_{\Omega_\lambda}) &= \theta_{g.\lambda}(g_*(\operatorname{Ad}(g^{-1})X)_{\Omega_\lambda}) = (g^*\theta)_\lambda((\operatorname{Ad}(g^{-1})X)_{\Omega_\lambda}) \\ &= \theta_\lambda((\operatorname{Ad}(g^{-1})X)_{\Omega_\lambda}) \\ &= -\operatorname{Ad}(g^{-1})X \end{aligned}$$

since  $g^*\theta = \theta$  and  $\theta_\lambda(Z_{\Omega_\lambda}) = -Z$  for  $Z \in \mathfrak{g}$ . Therefore, we have

$$\begin{aligned} \langle -\lambda, (\theta_{g.\lambda} \wedge \theta_{g.\lambda})(X_{\Omega_\lambda}, Y_{\Omega_\lambda}) \rangle &= \langle -\lambda, [\theta_{g.\lambda}(X_{\Omega_\lambda}), \theta_{g.\lambda}(Y_{\Omega_\lambda})] \rangle \\ &= \langle -\lambda, [\operatorname{Ad}(g^{-1})X, \operatorname{Ad}(g^{-1})Y] \rangle \\ &= \langle -\operatorname{Ad}^*(g)\lambda, [X, Y] \rangle, \end{aligned}$$

which equals  $(\omega_\lambda)_{g.\lambda}(X_{\Omega_\lambda}, Y_{\Omega_\lambda})$  by definition.  $\square$

EXAMPLE 3.11. We let  $G = \operatorname{SL}_2(\mathbf{C})$ ,  $Q$  and  $\lambda \in \mathfrak{g}^*$  be as in Example 3.9, and still identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$  by the trace form. If we parametrize an element  $f \in \Omega_\lambda \subset \mathfrak{g}$  as  $f = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$  with  $a, b$  and  $c \in \mathbf{C}$ , then it is easy to show that the canonical symplectic form  $\omega_\lambda$  on  $\Omega_\lambda$  is given by

$$\omega_\lambda = \frac{2}{s^2} (a db \wedge dc - b da \wedge dc + c da \wedge db).$$

If  $f = \operatorname{Ad}(g)\lambda^\vee$  with  $g = u_z u_w^{-1} \in p^{-1}(U_e) = UU^{-1}L$  and if we write  $u_z$  and  $u_w^{-1}$  as

$$u_z = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, \quad u_w^{-1} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \quad (z, w \in \mathbf{C}),$$

then we find that

$$\omega_\lambda = -s dz \wedge dw = \langle -\lambda, \theta_g \wedge \theta_g \rangle.$$

Now, for  $[z, \xi] \in \varpi^{-1}(U_e)$ , if  $f = \mu_\lambda([z, \xi]) = \operatorname{Ad}(u_z u_w^{-1})\lambda^\vee$ , i.e.,  $\xi$  and  $w$  are related by  $\xi = -sw$  as in Lemma 2.3 or (2.7), then it is immediate to see that  $\mu_\lambda$  preserves the symplectic forms.

Summarizing the results, we have shown that the following theorem holds (cf. [1, 2]):

THEOREM 3.12. *The holomorphic map  $\mu_\lambda : T^*(G/Q)_\lambda \rightarrow \Omega_\lambda$  given by (3.16) and (3.18) is a  $G$ -equivariant symplectic isomorphism.*

Furthermore, the map  $\mu_\lambda$  provides a moment map  $T^*(G/Q)_\lambda \rightarrow \mathfrak{g}^*$  with respect to the symplectic form  $\omega$ . Namely, we have the following.



COROLLARY 3.13. *The action  $\Psi_\lambda$  of  $G$  on the symplectic manifold  $(T^*(G/Q)_\lambda, \omega)$  is Hamiltonian with moment map  $\mu_\lambda$ .*

PROOF. Setting  $M := T^*(G/Q)_\lambda$  for brevity, we must show that

$$d\langle \mu_\lambda, X \rangle = \iota_{X_M} \omega$$

for  $X \in \mathfrak{g}$  with  $\iota$  denoting the contraction. Since  $\{Y_M; Y \in \mathfrak{g}\}$  spans the holomorphic tangent space  $T_\alpha M$ , it suffices to show that

$$(3.26) \quad d\langle \mu_\lambda(\alpha), X \rangle(Y_M) = (\iota_{X_M} \omega)_\alpha(Y_M)$$

for  $X, Y \in \mathfrak{g}$ , where we simply denote a point in  $M = T^*(G/Q)_\lambda$  by  $\alpha$ .

Now, since  $\mu_\lambda$  is  $G$ -equivariant, one has

$$\begin{aligned} d\langle \mu_\lambda(\alpha), X \rangle(Y_M) &= \left. \frac{d}{dt} \right|_{t=0} \langle \mu_\lambda(\Psi_\lambda(\exp(-tY))\alpha), X \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle \text{Ad}^*(\exp(-tY))\mu_\lambda(\alpha), X \rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} \langle \mu_\lambda(\alpha), \text{Ad}(\exp tY)X \rangle = \langle \mu_\lambda(\alpha), [Y, X] \rangle \\ &= -\langle \mu_\lambda(\alpha), [X, Y] \rangle. \end{aligned}$$

On the other hand, since  $\mu_\lambda$  is symplectic, the right-hand side of (3.26) equals

$$\omega_\alpha(X_M, Y_M) = (\mu_{\lambda*}^* \omega_\lambda)_\alpha(X_M, Y_M) = (\omega_\lambda)_{\mu_\lambda(\alpha)}(\mu_{\lambda*} X_M, \mu_{\lambda*} Y_M).$$

Here, using the  $G$ -equivariance of  $\mu_\lambda$  again, one has

$$\begin{aligned} (\mu_{\lambda*} X_M)(\mu_\lambda(\alpha)) &= \left. \frac{d}{dt} \right|_{t=0} \mu_\lambda(\Psi_\lambda(\exp(-tX))\alpha) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}^*(\exp(-tX))\mu_\lambda(\alpha) \\ &= X_{\Omega_\lambda}(\mu_\lambda(\alpha)), \end{aligned}$$

and hence

$$(\omega_\lambda)_{\mu_\lambda(\alpha)}(\mu_{\lambda*} X_M, \mu_{\lambda*} Y_M) = (\omega_\lambda)_{\mu_\lambda(\alpha)}(X_{\Omega_\lambda}, Y_{\Omega_\lambda}).$$

This implies the equation (3.26) by the definition (3.23) of  $\omega_\lambda$ . □

As an application, one can obtain an explicit embedding of  $G_{\mathbf{R}}$ -orbit of  $\lambda$  into the twisted cotangent bundle, where  $G_{\mathbf{R}}$  is a noncompact real form of  $G$  and  $\lambda$  is as given in Subsection 2.1. Namely, let  $(G_{\mathbf{R}}, K_{\mathbf{R}})$  be a classical Hermitian symmetric pair of noncompact type whose complexifications are equal to  $G$  and  $L$  respectively, and  $\lambda : \mathcal{Q} \rightarrow \mathbf{C}^\times$  the holomorphic character whose differential, which we also denote by  $\lambda$ , is pure-imaginary on  $\mathfrak{h} \cap \mathfrak{k}_{\mathbf{R}}$  and is proportional to the fundamental weight corresponding to the unique noncompact simple root on  $\mathfrak{h}$ . If  $\lambda$  satisfies the condition that we referred to [6] in §2.1, then the holomorphic discrete series representation  $(\pi_\lambda, \mathcal{H}_\lambda)$  possesses a unique element (highest weight vector)  $\varphi_\lambda$  satisfying

$$(3.27) \quad \begin{aligned} \pi_\lambda(t)\varphi_\lambda &= \lambda(t)\varphi_\lambda & (t \in T_{\mathbf{R}}); \\ \pi_\lambda(X)\varphi_\lambda &= 0 & (X \in \mathfrak{u}); \\ \varphi_\lambda(e) &= 1, \end{aligned}$$

where  $T_{\mathbf{R}}$  denotes the torus corresponding to  $\mathfrak{h} \cap \mathfrak{k}_{\mathbf{R}}$ . Note that  $1/\varphi_{\lambda}$  naturally extends to a holomorphic function defined on the dense open set  $p^{-1}(U_e)$ , which we still denote by the same notation.

Let us define a real-analytic function  $f_{\lambda} : U_e \rightarrow \mathbf{R}$  by

$$f_{\lambda}(z) := \frac{1}{\varphi_{\lambda}(u_z^{\dagger}u_z)},$$

where we write  $g^{\dagger} = \tau(g)^{-1}$  for  $g \in G$  with  $\tau$  the involution that characterizes the real form  $G_{\mathbf{R}}$  in  $G$ . Note that  $f_{\lambda}$  is positive on  $G_{\mathbf{R}}/K_{\mathbf{R}}$ .

PROPOSITION 3.14. *If one denotes by  $\Omega_{\lambda}^{\mathbf{R}}$  the coadjoint orbit of  $\lambda$  under  $G_{\mathbf{R}}$ , then one finds that*

$$(3.28) \quad \mu_{\lambda} : \{-d' \log f_{\lambda}(z); z \in G_{\mathbf{R}}/K_{\mathbf{R}}\} \xrightarrow{\sim} \Omega_{\lambda}^{\mathbf{R}},$$

where  $d'$  denotes the holomorphic part of the exterior derivative  $d$ .

For the sake of completeness we provide the proof (cf. [4, Proposition 3.3]).

PROOF. An element  $g$  of  $G_{\mathbf{R}}$  decomposes as  $g = u_z u_w^{-1} t$  with  $u_z \in U$ ,  $u_w^{-1} \in U^{-}$  and  $t \in L$ , since  $G_{\mathbf{R}} \subset UQ$ . It follows from  $g^{\dagger}g = e$  that

$$(3.29) \quad (u_w^{-1})^{\dagger} u_z^{\dagger} u_z u_w^{-1} = (tt^{\dagger})^{-1}.$$

Applying  $\varphi_{\lambda}$  to the both sides of (3.29), one finds

$$(3.30) \quad \varphi_{\lambda}(u_z^{\dagger}u_z) = \lambda(tt^{\dagger})$$

by (3.27). On the other hand, it follows from (3.29) that

$$d'(tt^{\dagger})(tt^{\dagger})^{-1} = -u_w^{-1} d'u_w^{-1} - \text{Ad}(u_w^{-1})(u_z^{-1}d'u_z) - \text{Ad}(tt^{\dagger})(d'u_w^{-1}u_w^{-1})^{-1}$$

since  $d'u_z^{\dagger} = 0$ . Therefore, one obtains

$$\begin{aligned} \langle \lambda, d'(tt^{\dagger})(tt^{\dagger})^{-1} \rangle &= -\langle \lambda, u_w^{-1} d'u_w^{-1} \rangle - \langle \lambda, \text{Ad}(u_w^{-1})^{-1}(u_z^{-1}d'u_z) \rangle - \langle \lambda, \text{Ad}(tt^{\dagger})(d'u_w^{-1}u_w^{-1})^{-1} \rangle \\ &= -\langle \lambda, \text{Ad}(u_w^{-1})^{-1}(u_z^{-1}d'u_z) \rangle \end{aligned}$$

since  $\text{Ad}^*(tt^{\dagger})\lambda = \lambda$ , and  $u_w^{-1} d'u_w^{-1}$  and  $d'u_w^{-1}u_w^{-1}$  are  $u^{-}$ - and  $u$ -valued 1-form respectively. Now the proposition follows from the formulae  $\langle \lambda, d'(tt^{\dagger})(tt^{\dagger})^{-1} \rangle = d' \log \lambda(tt^{\dagger})$  and (3.30). □

If we denote by  $a^{\mathbf{R}}$  the canonical isomorphism from the real-analytic cotangent bundle  $T^*(U_e)^{\mathbf{R}}$  onto the holomorphic cotangent bundle  $T^*U_e$  given in terms of coordinates by

$$dx^{\alpha} \longleftrightarrow \frac{1}{2} dz^{\alpha} \quad \text{and} \quad dy^{\alpha} \longleftrightarrow -\frac{\sqrt{-1}}{2} dz^{\alpha}$$

with  $z^{\alpha} = x^{\alpha} + \sqrt{-1}y^{\alpha}$  for each  $\alpha \in \Delta(\mathfrak{u})$ , then the equation (3.28) reads

$$\mu_{\lambda}^{\mathbf{R}} : \{-d \log f_{\lambda}(z); z \in G_{\mathbf{R}}/K_{\mathbf{R}}\} \xrightarrow{\sim} \Omega_{\lambda}^{\mathbf{R}},$$

where we put  $\mu_\lambda^R := \mu_\lambda \circ a^R$ . Thus, this is our version of [10, Lemma 7.17], which plays a prominent role in establishing the character formula therein.

## REFERENCES

- [1] N. CHRISS AND V. GINZBURG, Representation theory and complex geometry, Birkhäuser, 1997.
- [2] W. GRAHAM AND D. VOGAN, Geometric quantization for nilpotent coadjoint orbits, Geometry and representation theory of real and  $p$ -adic groups (Ed. J. Tirao et al.), PM, vol. 158, pp. 69–137, Birkhäuser, 1997.
- [3] T. HASHIMOTO, On the principal symbols of  $K_C$ -invariant differential operators on Hermitian symmetric spaces, J. Math. Soc. Japan 63 (2011), 837–869.
- [4] T. HASHIMOTO, K. OGURA, K. OKAMOTO AND R. SAWAE, Borel-Weil theory and Feynman path integrals on flag manifolds, Hiroshima Math. J. 23 (1993), 231–247.
- [5] R. HOWE AND T. UMEDA, The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann. 290 (1991), 565–619.
- [6] A. W. KNAPP, Representation theory of semisimple groups: An overview based on examples, Princeton Mathematical Series, vol. 36, Princeton Univ. Press, 1986.
- [7] D. MCDUFF AND D. SALAMON, Introduction to symplectic topology, Oxford Univ. Press, 1995.
- [8] W. ROSSMANN, Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety I, II, J. Funct. Anal. 96 (1991), 130–154, 155–193.
- [9] W. SCHMID AND K. VILONEN, Characteristic cycles of constructible sheaves, Invent. Math. 124 (1996), 451–502.
- [10] W. SCHMID AND K. VILONEN, Two geometric character formulas for reductive Lie groups, J. of Amer. Math. Soc. 11 (1998), 799–867.

UNIVERSITY EDUCATION CENTER  
TOTTORI UNIVERSITY  
4-101, KOYAMA-MINAMI  
TOTTORI, 680-8550  
JAPAN

*E-mail address:* thashi@uec.tottori-u.ac.jp