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A TWO-DIMENSIONAL NON-NOETHERIAN FACTORIAL RING[1]

ROBERT GILMER

ABSTRACT. Let R be a commutative ring with identity and let G be an abelian group of torsion-free rank α. If $\{X_\lambda\}$ is a set of indeterminates over R of cardinality α, then the group ring of G over R and the polynomial ring $R[\{X_\lambda\}]$ have the same (Krull) dimension. The preceding result and a theorem due to T. Parker and the author imply that for each integer $k \geqq 2$, there is a k-dimensional non-Noetherian unique factorization domain of arbitrary characteristic.

Assume that R is an associative ring and S is a semigroup (with operation written as addition). The semigroup ring of S over R [**12**, p. 95] is the set of functions from S into R that are finitely nonzero, where addition and multiplication are defined by the rules

$$(f+g)(s) = f(s) + g(s),$$
$$(fg)(s) = \sum_{t+u=s} f(t)g(u).$$

Following D. G. Northcott [**15**, p. 128], we denote the semigroup ring of S over R by the symbol $R[X; S]$; we write the elements of $R[X; S]$ as "polynomials" $a_1X^{s_1} + \cdots + a_nX^{s_n}$, where each a_i is in R and each s_i is in S. In the case where R is an integral domain with identity and S is abelian with a zero element, the author and T. Parker [**10**] have recently determined necessary and sufficient conditions in order that the semigroup ring $R[X; S]$ be a GCD-domain, a unique factorization domain[2] (UFD), or a principal ideal domain (PID). A special case of Theorem 7.5 of [**10**] is the following result, which we label as Theorem 1 for ease of reference.

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[2] We sometimes use the term *factorial ring* instead of unique factorization domain.

THEOREM 1. *If D is an integral domain with identity and G is an abelian group, then the group ring $D[X; G]$ of G over D is a UFD if and only if D is a UFD and G is a torsion-free group with the property that each rank 1 subgroup of G is cyclic.*[3]

To say that each rank one subgroup of G is cyclic is equivalent to the condition that for each nonzero element g of G, there is a largest positive integer n_g such that the equation $n_g x = g$ is solvable in G. We are able to use Theorem 1 to give an example of a two-dimensional non-Noetherian UFD of arbitrary characteristic. The question of the existence of finite-dimensional non-Noetherian factorial rings has been investigated recently by J. David in [**4**] and [**5**]. In particular, he has proved the existence of a k-dimensional non-Noetherian UFD of characteristic 0 or 2 for each $k \geqq 3$. In [**4**, Conjecture 1.1, Chapter VIII], David conjectures that such domains exist for arbitrary characteristic, and we are able to verify this conjecture, but our examples of two-dimensional non-Noetherian factorial rings negate Conjecture 5.2, Chapter VII of [**4**]. We begin with some considerations concerning the dimension of $R[X; G]$, where R is a commutative ring with identity and G is an abelian group.

If H is a subgroup of G and if G/H is a torsion group, then $R[X; G]$ is integral over its subring $R[X; H]$. This follows since $\{X^g\}_{g \in G}$ generates $R[X; G]$ as a ring extension of $R[X; H]$ and since, for each $g \in G$, there is a positive integer k_g such that $(X^g)^{k_g} \in R[X; H]$. Thus, if G has torsion-free rank α (recall that G has torsion-free rank α if $\dim_Q(Q \otimes G) = \alpha$), then there is a free subgroup F of G of rank α such that G/F is a torsion group, and $\dim R[X; G] = \dim R[X; F]$. Moreover, $R[X; F]$ is isomorphic to the ring $R[\{X_\lambda\}, \{X_\lambda^{-1}\}]_{\lambda \in A}$, where A is a set of cardinality α. Our first result concerns the dimension of the ring $R[\{X_\lambda\}, \{X_\lambda^{-1}\}]$.

PROPOSITION 1. *Let R be a commutative ring with identity, let $\{X_\lambda\}_{\lambda \in A}$ be a set of indeterminates over R, and let $S = R[\{X_\lambda\}, \{X_\lambda^{-1}\}]$. Then* $\dim S = \dim R[\{X_\lambda\}]$.[4]

PROOF. If R is infinite-dimensional, then it is clear that both S and $R[\{X_\lambda\}]$ are infinite-dimensional. If R is finite-dimensional and if A is infinite, then again $R[\{X_\lambda\}]$ and S are infinite-dimensional, for if M is a maximal ideal of R and if $\{X_{\lambda_i}\}_{i=1}^{\infty}$ is an infinite subset of $\{X_\lambda\}$, then $(M, X_{\lambda_1} - 1) \subset (M, X_{\lambda_1} - 1, X_{\lambda_2} - 1) \subset \cdots$ is an infinite chain of prime ideals of $R[\{X_\lambda\}]$ that misses the multiplicative system generated by $\{X_\lambda\}$.

[3] In alternate terminology, this is the condition that each nonzero element of G has type $(0, 0, 0, \cdots)$; see [**6**, p. 147] or [**18**, p. 203].

[4] Compare Proposition 2 with part (2) of Exercise 7, p. 415 of [**9**].

If R is finite-dimensional and $A=\{1, 2, \cdots, k\}$ is finite, then $\dim S \leqq \dim R[\{X_\lambda\}]$ since S is a quotient ring of $R[\{X_\lambda\}]$. On the other hand, it is known that $\dim R[\{X_\lambda\}]=\text{rank}(M[\{X_\lambda\}])+k$ for some maximal ideal M of R [**2**, Corollary 2.10]. Hence if $P_0 \subset P_1 \subset \cdots \subset P_t = M[\{X_\lambda\}]$ is a chain of prime ideals of $R[\{X_\lambda\}]$ of length $t=\text{rank}(M[\{X_\lambda\}])$, then

$$P_0 \subset P_1 \subset \cdots \subset P_t \subset P_t + (X_1 - 1) \subset \cdots \subset P_t + (X_1 - 1, \cdots, X_k - 1)$$

is a chain of primes of $R[\{X_\lambda\}]$, and each of these prime ideals extends to a proper ideal of S. Consequently, $\dim S \geqq \dim R[\{X_\lambda\}]$, and equality holds in each case.[5]

Corollary 1. *Assume that α is the torsion-free rank of the group G, and let $\{X_\lambda\}_{\lambda \in A}$ be a set of indeterminates over R, where $|A|=\alpha$. Then* $\dim R[X; G]=\dim R[\{X_\lambda\}_{\lambda \in A}]$.

Corollary 2. *If R is a Prüfer domain or a commutative Noetherian ring with identity, and if G is an abelian group of finite torsion-free rank α, then* $\dim R[X; G]=\dim R+\alpha$.

Proof. If R satisfies the hypothesis of Corollary 2, then it is well known that[6] $\dim R[X_1, \cdots, X_n]=\dim R+n$ for each positive integer n.

We are now in a position to give examples of non-Noetherian factorial rings of arbitrary characteristic and of arbitrary dimension $k \geqq 2$. We begin with the result that there exists a torsion-free abelian group L of rank two such that each rank one subgroup of L is cyclic, but L is not finitely generated; the first example of such a group in the literature seems to be due to Pontryagin [**16**], but Pontryagin's original construction has been generalized extensively (see [**6**, p. 151] or [**7**, Vol. II, §88]). If r is a nonnegative integer, and if L_r is the direct sum of the group L and r copies of the infinite cyclic group, then L_r is a torsion-free group of rank $r+2$, each rank one subgroup of L_r is cyclic, and L_r is not finitely generated. If K is a field, it follows from Theorem 1 that $K[X; L_r]$ is a UFD, and Corollary 1 implies that $K[X; L_r]$ has dimension $r+2$. Finally,

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[5] In the case in which A is finite, an alternate proof of the equality $\dim S = \dim R[\{X_\lambda\}]$ is obtained from the fact that $R[X_1, \cdots, X_k, X_1^{-1}, \cdots, X_k^{-1}]$ is integral over its subring $R[X_1+X_1^{-1}, \cdots, X_k+X_k^{-1}]$.

[6] For an in-depth study of the sequence $\dim R[X_1]$, $\dim R[X_1, X_2], \cdots$, $\dim R[X_1, \cdots, X_n], \cdots$, see [**1**], [**2**]. The case of Corollary 2 in which R is Noetherian can be obtained from Theorems 2.5 and 3.7 of [**19**], together with (h) of [**17**] (also, see Corollaire 3, p. 426 of [**8**]). While the considerations of [**19**] may seem more general than those we have undertaken in regard to the dimension of $R[X; G]$, there is actually little overlap between our results and those of [**19**] since, in general, only the inequality $\dim R \leqq K\text{-}\dim R$ need hold, even for a commutative ring R [**11**, Example 2.9 and Proposition 7.8] (the notation K-dim R is that of [**19**]).

$K[X; L_r]$ is not Noetherian, for it is known that if R is an associative ring with identity and if G is an abelian group, then $R[X; G]$ is right Noetherian if and only if R is right Noetherian and G is finitely generated [**3**], [**13**, p. 154]. We have therefore proved the following theorem.

THEOREM 2. *If K is a field and r is a nonnegative integer, then the group ring $K[X; L_{r+2}]$ is a non-Noetherian unique factorization domain of dimension $r+2$.*

The domains $K[X; L_{r+2}]$ of Theorem 2 are not quasi-local, and hence the following question arises. Does there exist a non-Noetherian quasi-local UFD of dimension r for each $r \geqq 2$? We show presently that the answer to this question is affirmative, and in fact, we show that the characteristic of such a UFD may be an arbitrary prime integer. Our approach to this problem is to consider localizations of the domains $K[X; L_{r+2}]$; in fact, we restrict to localizations at the augmentation ideal $(\{1-X^g | g \in L_{r+2}\})$. Our proof uses the following result, which follows immediately from Proposition 6 of [**3**].

PROPOSITION 2. *Assume that R is a commutative ring with identity, S is a cancellative additive abelian semigroup with zero, and $s \in S-\{0\}$. The element $1-X^s$ is a zero divisor of the semigroup ring $R[X; S]$ if and only if $ns=0$ for some positive integer n. If $1-X^s$ is a zero divisor in $R[X; S]$ and if k is the smallest positive integer such that $ks=0$, then the annihilator of $1-X^s$ is the principal ideal of $R[X; S]$ generated by $1+X^s+X^{2s}+\cdots+X^{(k-1)s}$.*

If H is a normal subgroup of the group G and if R is an associative ring with identity, then the natural homomorphism $\phi: G \to G/H$ of G onto G/H induces a unique R-homomorphism ϕ^* of $R[X; G]$ onto $R[X; G/H]$ such that $\phi^*(rX^g)=rX^{\phi(g)}$ for each r in R and each g in G. Moreover, if $\{g_\alpha\}$ is a subset of H that generates H, then $\{1-X^{g_\alpha}\}$ generates the kernel of ϕ^*; see [**3**], [**13**, p. 154]. We use these results to obtain the next theorem.

THEOREM 3. *Let F be a field and let G be a nonfinitely generated torsion-free abelian group with a finitely generated subgroup H such that G/H is a p-group. If M is the maximal ideal of $D=F[X; G]$ generated by $\{1-X^g | g \in G\}$, then the ideal MD_M of D_M is finitely generated if and only if the characteristic of F is distinct from p.*

PROOF. The ideal M consists of all elements $\sum_1^n a_i X^{g_i}$ such that $\sum_1^n a_i=0$. We let $\{h_i\}_{i=1}^m$ be a finite set of generators of the subgroup H of G. If the characteristic of F is different from p, then $\{1-X^{h_i}\}_1^m$ generates MD_M, for if $g \in G-H$, then $g+H$ has order p^k in G/H for some positive integer k. Since $1+X^g+\cdots+X^{(p^k-1)g}\notin M$ because $p^k\neq 0$, it follows that

$$1 - X^g = (1 - X^{p^k g})/(1 + X^g + \cdots + X^{(p^k-1)g})$$

is in the ideal of D_M generated by $\{1-X^{h_i}\}_{i=1}^m$.

On the other hand, we prove that if F has characteristic p, then MD_M is not finitely generated. We assume, on the contrary, that $\{1-X^{k_i}\}_{i=1}^r$ is a finite set of generators for MD_M. Without loss of generality, assume that $\{h_i\}_1^m\subseteq\{k_i\}_1^r$, and hence G/K is a p-group, where K is the subgroup of G generated by $\{k_i\}_1^r$. Since G is not finitely generated, there is an element g in $G-K$. By assumption, $(1-X^g)f\in(\{1-X^{k_i}\}_1^r)$ for some f in $D-M$. If ϕ is the natural homomorphism of G onto G/K and if ϕ^* is the induced homomorphism of D onto $F[X; G/K]$, then $\phi^*(1-X^g)\phi^*(f)=0$. But this contradicts Proposition 2—since the order of $g+K$ is a positive power of p, the annihilator of $\phi^*(1-X^g)=1-X^{g+K}$ is contained in $(\{1-X^{a+K}|a\in G\})$, and f is in $(\{1-X^a|a\in G\})=M$. This contradiction establishes the assertion of Theorem 3 that MD_M is not finitely generated if F is of characteristic p.

Theorem 3 can be generalized, but the statement of Theorem 3 given serves our purposes well. Thus if p is prime, then there is a nonfinitely generated abelian group L of rank two such that each rank one subgroup of L is cyclic and such that L/H is a p-group for some finitely generated subgroup H of L (see [**6**, p. 151] or [**7**, Vol. II, §88]). If F is a field of characteristic p and if $D=F[X; L]$, then as we have already observed, D is a two-dimensional non-Noetherian UFD. If M is the maximal ideal of D generated by $\{1-X^g|g\in L\}$, then Theorem 3 shows that D_M is a non-Noetherian quasi-local UFD of dimension two and characteristic p. More generally, if $J=D[X_1^{\pm 1},\cdots,X_r^{\pm 1}]$ and if $M_1=M+(X_1-1,\cdots,X_r-1)$, then J_{M_1} is a non-Noetherian quasi-local UFD of characteristic p and dimension $r+2$. That J_{M_1} is a quasi-local UFD of characteristic p is clear; J_{M_1} is not Noetherian because $J_{M_1}/\{X_1-1,\cdots,X_r-1\}J_{M_1}$ is isomorphic to D_M and D_M is not Noetherian. The domain J is isomorphic to $F[X; L\oplus G]$, where G is a direct sum of r copies of Z, and hence J has dimension $r+2$ by Corollary 2. Moreover, the proof of Proposition 1 shows that the ideal M_1 has height $r+2$ so that $\dim J_{M_1}=r+2$. In summary, we have established the following result, Theorem 4.

THEOREM 4. *If p is a prime integer and r is an integer greater than or equal to two, then there exists a quasi-local non-Noetherian UFD of dimension r and characteristic p.*

We remark that J. Brewer, D. Costa, and L. Lady have recently considered the prime ideal structure of $D[X; G]$, where G is an abelian group of finite torsion-free rank. In particular, they have shown that if R is a field of characteristic 0, then $R[X; G]_P$ is a regular local ring for each proper prime ideal P of $R[X; G]$. This means that the technique used to obtain Theorem 4 fails in the case of characteristic 0, but they have shown that an appropriate localization of $Z[G]$, where G is the direct sum of three copies of the additive group of rationals whose denominators are powers of the prime integer p, is a two-dimensional non-Noetherian quasi-local UFD of characteristic 0.

References

1. J. T. Arnold and R. Gilmer, *The dimension sequence of a commutative ring*, Bull. Amer. Math. Soc. **79** (1973), 407–409.

2. ———, *The dimension sequence of a commutative ring*, Amer. J. Math. (to appear).

3. Ian G. Connell, *On the group ring*, Canad. J. Math. **15** (1963), 650–685. MR **27** #3666.

4. John E. David, *Some non-Noetherian factorial rings*, Dissertation, University of Rochester, Rochester, N.Y., 1972.

5. ———, *A non-Noetherian factorial ring*, Trans. Amer. Math. Soc. **169** (1972), 495–502.

6. L. Fuchs, *Abelian groups*, Pergamon Press, London, 1967.

7. ———, *Infinite abelian groups*. Vol. I, Pure and Appl. Math., vol. 36, Academic Press, New York, 1970; Vol. II, Academic Press, New York, 1973. MR **41** #333.

8. P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448. MR **38** #1144.

9. R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972.

10. R. Gilmer and T. Parker, *Divisibility properties of semigroup rings*, Michigan Math. J. (to appear).

11. R. Gordon and J. C. Robson, *Krull dimension*, Mem. Amer. Math. Soc. No. 133 (1973).

12. N. Jacobson, *Lectures in abstract algebra*, Vol. I, *Basic concepts*, Van Nostrand, Princeton, N.J., 1951. MR **12,** 794.

13. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

14. M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR **27** #5790.

15. D. G. Northcott, *Lessons on rings, modules, and multiplicities*, Cambridge Univ. Press, London, 1968. MR **38** #144.

16. L. Pontryagin, *The theory of topological commutative groups*, Ann. of Math. **35** (1934), 361–388.

17. R. Rentschler and P. Gabriel, *Sur la dimension des anneaux et ensembles ordonnés*, C.R. Acad. Sci. Paris, Sér. A–B **265** (1967), A712–A715. MR **37** #243.

18. J. Rotman, *The theory of groups*: *An introduction*, Allyn and Bacon, Boston, Mass., 1965. MR **34** #4338.

19. P. F. Smith, *On the dimension of group rings*, Proc. London Math. Soc. (3) **25** (1972), 288–302.

Department of Mathematics, Florida State University, Tallahassee, Florida 32306

Current address: Department of Mathematics, Latrobe University, Bundoora, Victoria 3083, Australia