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A TWO-GRID METHOD OF A MIXED STOKES–DARCY MODEL FOR COUPLING FLUID FLOW WITH POROUS MEDIA FLOW*

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Abstract. We study numerical methods for solving a coupled Stokes–Darcy problem in porous media flow applications. A two-grid method is proposed for decoupling the mixed model by a coarse grid approximation to the interface coupling conditions. Error estimates are derived for the proposed method. Both theoretical analysis and numerical experiments show the efficiency and effectiveness of the two-grid approach for solving multimodeling problems. Potential extensions and future directions are discussed.

Key words. porous media flow, Stokes equations, Darcy's law, multimodeling problems, two-grid method

AMS subject classifications. 65N15, 65N30, 76D07, 76S05

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1. Introduction.

There are many multimodeling problems in real applications of complex systems. They consist of multiple models in different regions coupled through interface conditions. The local models may be very varied in type, scale, control variable, and many other physical and mathematical properties. The corresponding numerical treatments may, of course, also vary significantly in geometric and PDE discretization, algebraic solution, and so on, in order to cope with local properties. The mixture of coupled models also leads to various mathematical and numerical difficulties. For instance, interface coupling conditions involve different control variables from different local models and may have complex, or even nonlinear, forms. Coupling different models may lead to very singular and complex structures across the interface and strong stiffness due to different scales, which would present considerable numerical difficulties. Examples of coupled multimodel applications include viscous-inviscid flows [5], compressible-incompressible fluids [17], turbulent-laminar flows [9], viscous-porous media flows [11, 16, 21, 27], and inertial confinement fusion with high ratio of density and temperature [31].

In general, there are two types of approaches to solving multimodel problems. One is to solve coupled problems directly, and the other is to first decouple mixed models and then apply appropriate local solvers individually. There are many appealing reasons to use the decoupling approach. First, it allows one to tailor algorithm components flexibly and conveniently in terms of physical, mathematical, and numerical properties for each local model and solver. Second, it is suitable for today's grid computing environment because it can efficiently and effectively exploit the existing computing resources, including both hardware and software, that are distributed

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over the Internet and that have been developed by different experts for use in various application fields [26]. As a by-product, it naturally results in parallelism in the conventional sense.

There are various decoupling techniques. Many of them are in the spirit of domain decomposition in general. For instance, Quarteroni and Valli [29] have extensively investigated heterogeneous domain decomposition methods for various coupled models. The Lagrange multiplier approach is also widely used [15, 28] for decoupling multimodel problems. The interface relaxation approach [24, 25] has also been successfully applied in multimodel simulations. We note that two-grid methods were proposed in [34, 35] for discretizing nonsymmetric and indefinite PDEs. The approach was also used for linearizing nonlinear problems [23, 36, 37], for localization and parallelization [38, 39, 40], as well as for many other applications; see, for instance, Axelssson and coworkers [2, 3, 4], Girault and Lions [13], Layton and coworkers [18, 19, 20], and Utnes [32]. In this paper, we demonstrate that the two-grid approach can also be applied successfully to solve multimodel problems.

The rest of the paper is organized as follows. A coupled Stokes–Darcy model is described in the next section as our model problem. A two-grid algorithm is proposed in section 3 for decoupling the mixed model. The basic idea is to first solve a much smaller problem on a coarse grid. The coarse grid solution is then used to interpolate the interface condition, which leads to a decoupled problem on the fine grid. Section 4 contains the error analysis for the two-grid method and discusses its computational aspects as well as potential extensions and future directions. Both theoretical analysis and numerical experiments confirm that approximation accuracy does not deteriorate under the proposed two-grid decoupling technique so that the decoupled discrete problem is of the same accuracy as the couple discrete problem for approximating the mixed Stokes–Darcy model. Concluding remarks follow in section 5.

2. Coupled Stokes–Darcy model. Let us consider a mixed model of Stokes equations and Darcy equations for coupling a fluid flow with a porous media flow. There has been very active research done recently on its applications, mathematical analysis, finite element approximation, and numerical solution; see, e.g., [1, 11, 16, 21, 27] and references therein. In particular, a subdomain iterative method is proposed to decouple the Stokes–Darcy problem by applying the preconditioned Richardson–Franklin method to the interface equation with the Steklov–Poincaré pseudo-PDE operator [11].

We consider a fluid flow in Ω_f coupled with a porous media flow in Ω_p; see Figure 1, where Ω_f and Ω_p are two- or three-dimensional bounded domains, $\Omega_f \bigcap \Omega_p = \emptyset$, and $\overline{\Omega_f} \bigcap \overline{\Omega_p} = \Gamma$. Denote by $\Omega = \Omega_f \bigcup \Omega_p$, $\mathbf{n}_f$, and $\mathbf{n}_p$ as usual the unit outward normal directions on $\partial\Omega_f$ and $\partial\Omega_p$.

The fluid motion is governed by the Stokes equations for the velocity $\mathbf{V}_f$ and the pressure p_f: $\forall t > 0$,

$$\begin{cases} \dfrac{\partial \mathbf{V}_f}{\partial t} - \mathrm{div}\mathbf{T}(\mathbf{V}_f, p_f) = \mathbf{g}_f & \forall \mathbf{x} \in \Omega_f \text{ (conservation of momentum)}, \\ \mathrm{div}\mathbf{V}_f = 0, & \forall \mathbf{x} \in \Omega_f \text{ (conservation of mass)}, \end{cases} \tag{1}$$

where

$$\mathbf{T}(\mathbf{V}_f, p_f) = -p_f\mathbf{I} + 2\mu\mathbf{D}(\mathbf{V}_f)$$

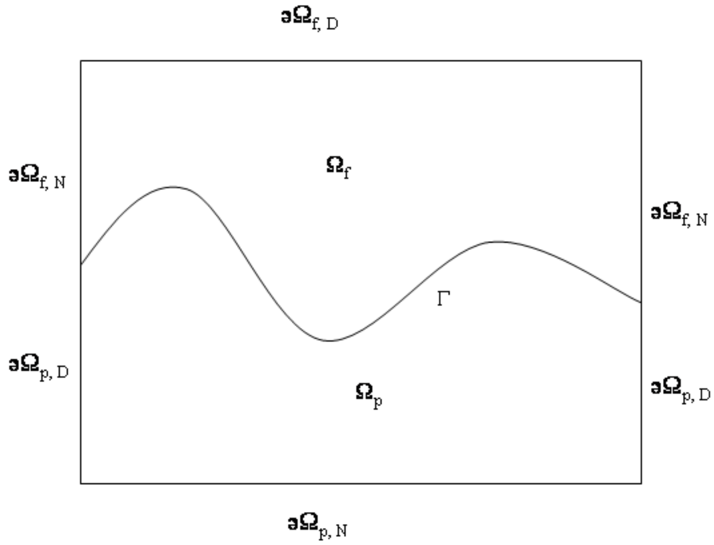

FIG. 1. *A global domain Ω consisting of a fluid region Ω_f and a porous media region Ω_p separated by an interface Γ.*

is the stress tensor, $\mu > 0$ is the kinematic viscosity, $\mathbf{g}_f$ is the external force, and

$$\mathbf{D}(\mathbf{V}_f) = \frac{1}{2}(\nabla \mathbf{V}_f + \nabla^T \mathbf{V}_f)$$

is the deformation rate tensor.

The porous media flow motion is governed by Darcy's law for the *piezometric head* ϕ and the *discharge vector* $\mathbf{q}$ that is proportional to the velocity $\mathbf{V}_p$, namely, $\mathbf{q} = n\mathbf{V}_p$ with n being the volumetric porosity: $\forall t > 0$,

$$(2) \qquad \begin{cases} S_0 \dfrac{\partial \phi}{\partial t} + \mathrm{div}\mathbf{q} = g_p & \forall \mathbf{x} \in \Omega_p \text{ (conservation of mass)}, \\ \mathbf{q} = -\mathbf{K}\nabla\phi & \forall \mathbf{x} \in \Omega_p \text{ (Darcy's law)}, \end{cases}$$

where S_0 is the mass storativity, $\mathbf{K}$ is the hydraulic conductivity tensor of the porous medium, and the source g_p satisfies the solvability condition

$$\int_{\Omega_p} g_p = 0,$$

and

$$\phi = z + \frac{p_p}{\rho_f g},$$

where z is the elevation from a reference level, p_p is the pressure in Ω_p, ρ_f is the density, and g is the gravity acceleration.

We consider the following boundary conditions. Denote $\partial\Omega_f \setminus \Gamma = \partial\Omega_{f,D} \bigcup \partial\Omega_{f,N}$ and $\partial\Omega_p \setminus \Gamma = \partial\Omega_{p,D} \bigcup \partial\Omega_{p,N}$, as shown in Figure 1. For the fluid flow, we impose

$$\begin{cases} \mathbf{V}_f = 0 & \text{on } \partial\Omega_{f,D} \text{ with } meas(\partial\Omega_{f,D}) \neq 0, \\ -(\mathbf{T}(\mathbf{V}_f, p_f)) \cdot \mathbf{n}_f = \mathbf{h} & \text{on } \partial\Omega_{f,N}, \end{cases}$$

where $\mathbf{h}$ is a given vector. For the porous medium, we assume

$$\begin{cases} \phi = \phi_p & \text{on } \partial\Omega_{p,D}, \\ \mathbf{V}_p \cdot \mathbf{n}_p = v_p & \text{on } \partial\Omega_{p,N}. \end{cases}$$

A key part in a mixed model is the interface coupling conditions. The following interface conditions have been extensively used and studied in the literature [6, 21, 27]:

$$(3) \quad \begin{cases} \mathbf{V}_f \cdot \mathbf{n}_f + \mathbf{V}_p \cdot \mathbf{n}_p = 0, \\ -[(\mathbf{T}(\mathbf{V}_f, p_f)) \cdot \mathbf{n}_f] \cdot \mathbf{n}_f = \rho_f g \phi, \\ -[(\mathbf{T}(\mathbf{V}_f, p_f)) \cdot \mathbf{n}_f] \cdot \boldsymbol{\tau}_i = \dfrac{\alpha}{\sqrt{\boldsymbol{\tau}_i \cdot \mathbf{K} \cdot \boldsymbol{\tau}_i}} (\mathbf{V}_f - \mathbf{V}_p) \cdot \boldsymbol{\tau}_i, \quad i = 1, \ldots, d-1, \end{cases}$$

where $\{\boldsymbol{\tau}_i\}_{i=1}^{d-1}$ are linearly independent unit tangential vectors on Γ, d is the spacial dimension, and α is a positive parameter depending on the properties of the porous medium and must be experimentally determined. The first interface condition ensures mass conservation across Γ. The second one is a balance of normal forces across the interface. The third one states that the slip velocity along Γ is proportional to the shear stress along Γ. There have been many discussions in the literature on the slip condition along the interface. It is even unclear if the third condition in (3) leads to a well-posed problem. However, it has been observed that in practice the term $\mathbf{V}_p \cdot \boldsymbol{\tau}_i$ on the right-hand side from the porous media flow is much smaller than the other terms. The most accepted interface condition, known as the Beavers–Joseph–Saffman law, is then given by

$$(4) \qquad -[2\mu \mathbf{D}(\mathbf{V}_f) \cdot \mathbf{n}_f] \cdot \boldsymbol{\tau}_i = \frac{\alpha}{\sqrt{\boldsymbol{\tau}_i \cdot \mathbf{K} \cdot \boldsymbol{\tau}_i}} \mathbf{V}_f \cdot \boldsymbol{\tau}_i, \ i = 1, \ldots, d-1,$$

which can be justified by a statistical approach and the Brinkman approximation [30]. We note that different interface conditions have been used in numerical studies. For instance, the Beavers–Joseph–Saffman condition is used in [1, 21], while the free-slip condition with $\alpha = 0$ is assumed in [10, 11, 12]. We will assume the Beavers–Joseph–Saffman condition (4) from now on.

For simplicity, let us assume n, ρ_f, and g are constants. We also assume the homogenous boundary condition on ϕ, $\phi_p = 0$, which can be easily handled by a lifting function in the nonhomogenous case.

Denote

$$\begin{aligned} H_f &= \{\mathbf{v} \in (H^1(\Omega_f))^d \mid \mathbf{v} = 0 \text{ on } \partial\Omega_{f,D}\}, \\ H_p &= \{\phi \in H^1(\Omega_p) \mid \phi = 0 \text{ on } \partial\Omega_{p,D}\}, \\ W &= H_f \times H_p, \\ Q &= L^2(\Omega_f). \end{aligned}$$

By integration by parts as in [21], the weak formulation for the above coupled (stationary) Stokes–Darcy problem reads as follows: For $f \in W'$, find $u = (\mathbf{u}, \phi) \in W$, $p \in Q$ such that

$$(5) \qquad \begin{cases} a(u,v) + b(v,p) = f(v) & \forall v = (\mathbf{v}, \psi) \in W, \\ b(u,q) = 0 & \forall q \in Q, \end{cases}$$

where

$$a(u,v) = a_\Omega(u,v) + a_\Gamma(u,v),$$

with

$$\begin{aligned} a_\Omega(u,v) &= a_{\Omega_f}(\mathbf{u},\mathbf{v}) + a_{\Omega_p}(\phi,\psi), \\ a_{\Omega_f}(\mathbf{u},\mathbf{v}) &= \int_{\Omega_f} 2n\mu D(\mathbf{u}) \cdot D(\mathbf{v}) + \sum_{i=1}^{d-1} \frac{\alpha n}{\sqrt{\boldsymbol{\tau}_i \cdot \mathbf{K} \cdot \boldsymbol{\tau}_i}} \int_\Gamma (\mathbf{u} \cdot \boldsymbol{\tau}_i)(\mathbf{v} \cdot \boldsymbol{\tau}_i), \\ a_{\Omega_p}(\phi,\psi) &= \int_{\Omega_p} \rho_f g \nabla\psi \cdot \mathbf{K} \nabla\phi, \\ a_\Gamma(u,v) &= \int_\Gamma n\rho_f g[\phi\mathbf{v} - \psi\mathbf{u}] \cdot \mathbf{n}_f \end{aligned}$$

and with

$$b(v,p) \equiv b(\mathbf{v},p) = -\int_{\Omega_f} np \,\mathrm{div}\, \mathbf{v}.$$

Similarly to [10], it is easy to verify that (i) $a(\cdot,\cdot)$ is continuous and coercive on W, and that (ii) $b(\cdot,\cdot)$ is continuous on $W \times Q$ and satisfies the well-known Brezzi–Babuska condition as follows: There exists a positive constant $\beta > 0$ such that $\forall q \in Q, \exists w \in W$ such that

$$(6) \qquad b(w,q) \geq \beta ||w||_W ||q||_Q.$$

The well-posedness of the model problem (5) then follows from Brezzi's theory for saddle-point problems [7]. The only difference from [10] is that the extension from the free-slip interface condition to the case of nonzero α results in the inclusion of an extra term $\sum_{i=1}^{d-1} \frac{\alpha n}{\sqrt{\boldsymbol{\tau}_i \cdot \mathbf{K} \cdot \boldsymbol{\tau}_i}} \int_\Gamma (\mathbf{u} \cdot \boldsymbol{\tau}_i)(\mathbf{v} \cdot \boldsymbol{\tau}_i)$ in the bilinear form $a_{\Omega_f}(\mathbf{u},\mathbf{v})$. Note that this extension does not affect property (i) for the bilinear form $a(\cdot,\cdot)$. The continuity is obvious, while the coercivity is still a consequence of the well-known Poincaré inequality and Korn inequality as in the free-slip case because α is positive and the corresponding term can thus be ignored in the estimation.

3. A two-grid algorithm. Let $W_h = H_{f,h} \times H_{p,h} \subset W$ and $Q_h \subset Q$ be two finite element spaces. The finite element discretization applied to the model problem (5) leads to a coupled discrete problem as follows: Find $u_h = (\mathbf{u_h}, \phi_h) \in W_h$, $p_h \in Q_h$ such that

$$(7) \qquad \begin{cases} a(u_h,v_h) + b(v_h,p_h) = f(v_h) & \forall v_h = (\mathbf{v_h}, \psi_h) \in W_h, \\ b(u_h,q_h) = 0 & \forall q_h \in Q_h. \end{cases}$$

The construction of the finite element spaces W_h and Q_h is described more specifically as follows. Let the triangulation of the global domain be regular, as well as compatible and quasi-uniform on Γ as described in [11]. Furthermore, the finite element spaces $H_{f,h}$ and Q_h approximating the velocity and pressure fields in the fluid region are assumed to satisfy the discrete inf-sup condition as follows: There exists a positive constant $\beta^* > 0$, independent of h, such that $\forall \mathbf{v_h} \in H_{f,h}, q_h \in Q_h$,

$$b(\mathbf{v_h}, q_h) \geq \beta^* ||\mathbf{v_h}||_{H_f} ||q_h||_Q. \tag{8}$$

Several families of finite element spaces designed for the Stokes problem are provided in IV.2 and Chapter VI in [7]. They all satisfy the discrete inf-sup condition (8) and can thus be applied for $H_{f,h}$ and Q_h. Finally, standard finite element approximations of $H^m(\Omega_p)$, such as piecewise linear elements for $m = 1$, can be applied for $H_{p,h}$ in the porous media region. The well-posedness and error analysis of the coupled discrete model (7) can be found in [11].

We now propose a *two-grid algorithm* consisting of the following two steps.

Algorithm.

1. Solve a coarse grid problem (7) with spacing H as follows: Find $u_H = (\mathbf{u_H}, \phi_H) \in W_H \subset W_h$, $p_H \in Q_H \subset Q_h$ such that

$$\begin{cases} a(u_H, v_H) + b(v_H, p_H) = f(v_H) & \forall v_H = (\mathbf{v_H}, \psi_H) \in W_H, \\ b(u_H, q_H) = 0 & \forall q_H \in Q_H. \end{cases} \tag{9}$$

2. Solve a modified fine grid problem as follows: Find $u^h = (\mathbf{u^h}, \phi^h) \in W_h$, $p^h \in Q_h$ such that

$$\begin{cases} a_\Omega(u^h, v_h) + b(v_h, p^h) = f(v_h) - a_\Gamma(u_H, v_h) & \forall v_h \in W_h, \\ b(u^h, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{10}$$

It is easy to see that the modified fine grid problem (10) is also well-posed. More important, the discrete model (10) is in fact equivalent to two decoupled problems that correspond to the Stokes problem on Ω_f and the Darcy problem on Ω_p, respectively, with the boundary conditions defined by u_H on Γ. More specifically, the discrete Stokes problem on the fluid region reads as follows: Find $\mathbf{u^h} \in H_{f,h}$, $p^h \in Q_h$ such that

$$\begin{cases} a_{\Omega_f}(\mathbf{u}^h, \mathbf{v}_h) + b(\mathbf{v}_h, p^h) = (n\mathbf{g}_f, \mathbf{v}_h) - \displaystyle\int_\Gamma n\rho_f g \phi_H \mathbf{v}_h \cdot \mathbf{n}_f & \forall \mathbf{v}_h \in H_{f,h}, \\ b(\mathbf{u}^h, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{11}$$

Similarly, the discrete Darcy problem on the porous media region reads as follows: Find $\phi^h \in H_{p,h}$ such that

$$a_{\Omega_p}(\phi^h, \psi_h) = (\rho_f g g_p, \psi_h) + \int_\Gamma n\rho_f g \psi_h \mathbf{u}_H \cdot \mathbf{n}_f \quad \forall \psi_h \in H_{p,h}. \tag{12}$$

4. Error analysis. For convenience, from now on we will use $x \lesssim y$ to denote that there exists a constant C, such that $x \leq Cy$. Let W_h and Q_h be any finite element spaces as described in the previous section. In addition, for illustration assume the regularity $u \in (H^2(\Omega_f))^d \times H^2(\Omega_p)$ and $p \in H^1(\Omega_f)$, and thus finite element spaces as described above of first order approximation $O(h)$ are used for the fluid and porous media regions. Then the error analysis for the coupled model in [11] yields the estimates

$$\begin{cases} ||u - u_h||_W \lesssim h, \\ ||p - p_h||_Q \lesssim h. \end{cases} \tag{13}$$

Note that estimates (13) apply to the coupled problem (7) but not to the decoupled problem (10). Furthermore, the extended framework of the Aubin–Nitsche duality technique [7] gives the following L^2-norm estimate.

Lemma 1. *Let $W_- = (L^2(\Omega_f))^d \times L^2(\Omega_p)$. Then under the same assumptions as above, we have*

$$||u - u_h||_{W_-} \lesssim h^2. \tag{14}$$

Proof. As in the Aubin–Nitsche duality technique for the general framework of mixed problems in [7], consider the dual problem defined by the error pair $(u - u_h, p - p_h)$ to be (2.90) and (2.93) from [7]. For the solution (w, s) of the dual problem, from the regularity of the dual problem we have $(w, s) \in W_{++} \times Q_{++} = ((H^2(\Omega_f))^d \times H^2(\Omega_p)) \times H^1(\Omega_f)$ in the particular setting of our problem. Then, Theorem 2.2 (in particular the estimate of (2.100)) in [7] gives

$$||u - u_h||_{W_-} \lesssim m(h)(\|u - u_h\|_W + \|p - p_h\|_Q) + n(h)\|u - u_h\|_W,$$

where

$$\inf_{w_h \in W_h} \|w - w_h\|_W \le m(h)\|w\|_{W_{++}},$$

and

$$\inf_{q_h \in Q_h} \|s - q_h\|_Q \le n(h)\|s\|_{Q_{++}}.$$

Note that both $m(h)$ and $n(h)$ are of the order of $O(h)$ as shown in [7]. Estimate (14) then follows immediately from (13), which completes the proof. ∎

As a consequence, the following estimates, which will be used in the proof of the next theorem, follow immediately from (13) and (14):

$$\begin{cases} ||\mathbf{u}_h - \mathbf{u}_H||_{H_f} \lesssim H, \quad ||\mathbf{u}_h - \mathbf{u}_H||_{(L^2(\Omega_f))^d} \lesssim H^2, \\ \|\phi_h - \phi_H\|_{H_p} \lesssim H, \quad \|\phi_h - \phi_H\|_{L^2(\Omega_p)} \lesssim H^2. \end{cases} \tag{15}$$

Theorem 2. *Let u_h, p_h and u^h, p^h be defined by the two discrete models* (7) *and* (10) *on the fine grid. The following error estimates hold:*

$$||\phi_h - \phi^h||_{H_p} \lesssim H^2, \tag{16}$$

$$||\mathbf{u}_h - \mathbf{u}^h||_{H_f} \lesssim H^{3/2}, \tag{17}$$

$$||p_h - p^h||_Q \lesssim H^{3/2}. \tag{18}$$

Proof. Note that by comparing the two discrete models (7) and (10) on the fine grid, we have

$$\begin{cases} a_\Omega(u_h - u^h, v_h) + a_\Gamma(u_h - u_H, v_h) + b(v_h, p_h - p^h) = 0 & \forall v_h \in W_h, \\ b(u_h - u^h, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{19}$$

First, taking $v_h = (\mathbf{0}, \psi_h) \in W_h$ in (19), we obtain

$$a_{\Omega_p}(\phi_h - \phi^h, \psi_h) + a_\Gamma(u_h - u_H, v_h) = 0.$$

In particular, when $\psi_h = \phi_h - \phi^h$, it is further reduced to

$$a_{\Omega_p}(\phi_h - \phi^h, \phi_h - \phi^h) = \int_\Gamma n\rho_f g(\phi_h - \phi^h)(\mathbf{u}_h - \mathbf{u}_H) \cdot \mathbf{n}_f.$$

Let $\theta \in H^1(\Omega_f)$ be a harmonic extension of $\phi_h - \phi^h$ to the fluid flow region, satisfying

$$\begin{cases} -\Delta\theta = 0 & \text{in } \Omega_f, \\ \theta = \phi_h - \phi^h & \text{on } \Gamma, \\ \theta = 0 & \text{on } \partial\Omega_f/\Gamma. \end{cases}$$

Let $H_{00}^{1/2}(\Gamma)$ denote the interpolation space [22]

$$H_{00}^{1/2}(\Gamma) = [L^2(\Gamma), H_0^1(\Gamma)]_{1/2}.$$

Apparently,

$$||\theta||_{H^1(\Omega_f)} \lesssim ||\phi_h - \phi^h||_{H_{00}^{1/2}(\Gamma)} \lesssim ||\phi_h - \phi^h||_{H_p}.$$

Note that $\forall q_H \in Q_H$,

$$\begin{aligned} &\int_\Gamma n\rho_f g(\phi_h - \phi^h)(\mathbf{u}_h - \mathbf{u}_H) \cdot \mathbf{n}_f \\ &\quad = \int_{\partial\Omega_f} n\rho_f g\theta(\mathbf{u}_h - \mathbf{u}_H) \cdot \mathbf{n}_f \\ &\quad = \int_{\Omega_f} \mathrm{div}(\mathbf{u}_h - \mathbf{u}_H)(n\rho_f g\theta) + \int_{\Omega_f} (\mathbf{u}_h - \mathbf{u}_H) \cdot \nabla(n\rho_f g\theta) \\ &\quad = n\rho_f g \left(\int_{\Omega_f} (\theta - q_H)\mathrm{div}(\mathbf{u}_h - \mathbf{u}_H) + \int_{\Omega_f} (\mathbf{u}_h - \mathbf{u}_H) \cdot \nabla\theta \right), \end{aligned}$$

where in the last equality we use the discrete divergence-free property for $\mathbf{u}_h$ and $\mathbf{u}_H$,

$$b(u_h - u_H, q_H) = \int_{\Omega_f} nq_H \mathrm{div}(\mathbf{u}_h - \mathbf{u}_H) = 0 \ \forall q_H \in Q_H.$$

Therefore, we have

$$\begin{aligned} &||\phi_h - \phi^h||_{H_p}^2 \\ &\quad \lesssim a_{\Omega_p}(\phi_h - \phi^h, \phi_h - \phi^h) \\ &\quad \lesssim \inf_{\forall q_H \in Q_H} \left| \int_{\Omega_f} (\theta - q_H)\mathrm{div}(\mathbf{u}_h - \mathbf{u}_H) \right| + \left| \int_{\Omega_f} (\mathbf{u}_h - \mathbf{u}_H) \cdot \nabla\theta \right| \\ &\quad \lesssim ||\mathbf{u}_h - \mathbf{u}_H||_{H_f} \inf_{\forall q_H \in Q_H} ||\theta - q_H||_{L^2(\Omega_f)} + ||\mathbf{u}_h - \mathbf{u}_H||_{(L^2(\Omega_f))^d} ||\theta||_{H^1(\Omega_f)} \\ &\quad \lesssim (H||\mathbf{u}_h - \mathbf{u}_H||_{H_f} + ||\mathbf{u}_h - \mathbf{u}_H||_{(L^2(\Omega_f))^d})||\theta||_{H^1(\Omega_f)} \\ &\quad \lesssim H^2||\phi_h - \phi^h||_{H_{00}^{1/2}(\Gamma)} \\ &\quad \lesssim H^2||\phi_h - \phi^h||_{H_p}, \end{aligned}$$

which leads to estimate (16).

To show (17), taking $v_h = (\mathbf{v}_h, 0) \in W_h$ in (19), we obtain

$$a_{\Omega_f}(\mathbf{u}_h - \mathbf{u}^h, \mathbf{v}_h) + a_\Gamma(u_h - u_H, v_h) + b(v_h, p_h - p^h) = 0.$$

In particular, when $\mathbf{v}_h = \mathbf{u}_h - \mathbf{u}^h$, due to the discrete divergence-free property of $\mathbf{u}_h$ and $\mathbf{u}^h$ so that $b(\mathbf{u}_h - \mathbf{u}^h, p_h - p^h) = 0$, we further have

$$a_{\Omega_f}(\mathbf{u}_h - \mathbf{u}^h, \mathbf{u}_h - \mathbf{u}^h) = \int_\Gamma n\rho_f g(\phi_h - \phi_H)(\mathbf{u}_h - \mathbf{u}^h) \cdot \mathbf{n}_f.$$

Hence,

$$\begin{aligned} ||\mathbf{u}_h - \mathbf{u}^h||_{H_f}^2 &\lesssim a_{\Omega_f}(\mathbf{u}_h - \mathbf{u}^h, \mathbf{u}_h - \mathbf{u}^h) \\ &= \int_\Gamma n\rho_f g(\phi_h - \phi_H)(\mathbf{u}_h - \mathbf{u}^h) \cdot \mathbf{n}_f \\ &\lesssim \|\phi_h - \phi_H\|_{L^2(\Gamma)} \|\mathbf{u}_h - \mathbf{u}^h\|_{(L^2(\Gamma))^d} \\ &\lesssim \|\phi_h - \phi_H\|_{L^2(\Gamma)} \|\mathbf{u}_h - \mathbf{u}^h\|_{H_f}. \end{aligned} \tag{20}$$

Using a refined trace result (see [33, p. 27], with $\epsilon = H^{1/2}$), we get

$$\|\phi_h - \phi_H\|_{L^2(\Gamma)} \lesssim H^{-1/2}\|\phi_h - \phi_H\|_{L^2(\Omega_p)} + H^{1/2}\|\phi_h - \phi_H\|_{H^1(\Omega_p)} \lesssim H^{3/2}. \tag{21}$$

Applying (21) to (20) then yields estimate (17).

Finally, let us show (18). From the discrete Brezzi–Babuska condition on Ω_f, for $q_h = p_h - p^h \in Q_h, \exists \mathbf{v}_h \in H_{f,h}$ such that

$$||p_h - p^h||_{L^2(\Omega_f)} \lesssim \frac{-\int_{\Omega_f} n(p_h - p^h)\mathrm{div}\mathbf{v}_h}{||\mathbf{v}_h||_{H_f}}.$$

Recall that for $v_h = (\mathbf{v}_h, 0) \in W_h$ in (19), we have

$$a_{\Omega_f}(\mathbf{u}_h - \mathbf{u}^h, \mathbf{v}_h) + a_\Gamma(u_h - u_H, v_h) + b(v_h, p_h - p^h) = 0.$$

The first term above is easy to handle by

$$|a_{\Omega_f}(\mathbf{u}_h - \mathbf{u}^h, \mathbf{v}_h)| \lesssim ||\mathbf{u}_h - \mathbf{u}^h||_{H_f}||\mathbf{v}_h||_{H_f}.$$

For the second term, we have

$$\begin{aligned} |a_\Gamma(u_h - u_H, v_h)| &= \left| \int_\Gamma n\rho_f g(\phi_h - \phi_H)\mathbf{v}_h \cdot \mathbf{n}_f \right| \\ &\lesssim \|\phi_h - \phi_H\|_{L^2(\Gamma)}\|\mathbf{v}_h\|_{(L^2(\Gamma))^d} \\ &\lesssim \|\phi_h - \phi_H\|_{L^2(\Gamma)}\|\mathbf{v}_h\|_{H_f}. \end{aligned}$$

Using (21) and (17), we have

$$\begin{aligned} ||p_h - p^h||_{L^2(\Omega_f)} &\lesssim \frac{|a_{\Omega_f}(\mathbf{u}_h - \mathbf{u}^h, \mathbf{v}_h)| + |a_\Gamma(u_h - u_H, v_h)|}{||\mathbf{v}_h||_{H_f}} \\ &\lesssim ||\mathbf{u}_h - \mathbf{u}^h||_{H_f} + \|\phi_h - \phi_H\|_{L^2(\Gamma)} \\ &\lesssim H^{3/2}, \end{aligned}$$

which leads to estimate (18). This completes the proof. ∎

COROLLARY 3. *Let* $(u^h, p^h) \in W_h \times Q_h$ *be the solution of the two-grid algorithm with* $H = \sqrt{h}$. *We have*

$$||\phi - \phi^h||_{H_p} \lesssim h \tag{22}$$

and

$$||\mathbf{u} - \mathbf{u}^h||_{H_f} + ||p - p^h||_Q \lesssim h^{3/4}. \tag{23}$$

If $H = h^{2/3}$, *estimate* (23) *is further improved to the optimal order as follows:*

$$||\mathbf{u} - \mathbf{u}^h||_{H_f} + ||p - p^h||_Q \lesssim h. \tag{24}$$

We remark that error estimates (17) and (18) for $\mathbf{u}_h - \mathbf{u}^h$ and $p_h - p^h$ may not be optimal due to technical reasons. These two estimates might be further improved to $O(H^2)$ by a finer analysis, as suggested by numerical experiments in [8], which could then lead to an improvement of (23) to an optimal estimate of the order of $O(h)$ for $\mathbf{u} - \mathbf{u}^h$ and $p - p^h$, yet still with $H = \sqrt{h}$. Furthermore, the error analysis may be extended to finite element spaces with higher order approximation $O(h^m)$, provided that the solution is locally smooth enough within each subdomain. Specifically, if $W_h \subset W$ and $Q_h \subset Q$ are finite element spaces with the approximation order $O(h^m)$, and the solution (u, p) is locally smooth enough within each subdomain, we expect the following estimates to hold:

$$||\phi_h - \phi^h||_{H_p} \lesssim H^{m+1} \tag{25}$$

and

$$||\mathbf{u_h} - \mathbf{u}^h||_{H_f} + ||p_h - p^h||_Q \lesssim H^{m+1}, \tag{26}$$

which implies the optimal error estimates if we take $H = h^{\frac{m}{m+1}}$:

$$||\phi - \phi^h||_{H_p} \lesssim h^m \tag{27}$$

and

$$||\mathbf{u} - \mathbf{u}^h||_{H_f} + ||p - p^h||_Q \lesssim h^m. \tag{28}$$

We refer readers to [8] for more details on this extension.

Comprehensive numerical experiments on various aspects of the proposed theoretical framework are under investigation and will be reported in [8]. For instance, if the well-known Taylor–Hood elements [7], also known as the P2-P1 elements, are applied to the Stokes model, and the P2 elements are applied to the Darcy model, and for convenience we simply take $H = \sqrt{h}$, the numerical approximations of the two-grid algorithm to a locally very smooth solution clearly demonstrate an optimal convergence rate of $O(h^2)$, which confirms our theoretical expectation. For more details, see [8].

Most important, the presented theory suggests that one can effectively and efficiently decouple a coupled multimodel problem by proper multigrid techniques. This allows for different submodel problems to be solved independently by applying the most appropriate numerical techniques individually. In addition, these decoupled local problems can be solved by different processors on a parallel multiprocessor or by different computing nodes on a traditional cluster or even a remotely distributed computational grid. Furthermore, in a grid computing environment, powerful and efficient local solvers are usually available which were developed at different sites by different experts for various single models. Therefore, substantial coding tasks can also be reduced thanks to resource sharing in grid computing.

We also remark that the proposed two-grid algorithm still requires a coarse grid solver for the coupling purpose. The coarse grid problem usually has a much smaller size, say $H = \sqrt{h}$, and can thus be solved on a front end machine or a client machine. It is also numerically easier to solve than a fine grid problem in various aspects such as approximation accuracy, stiffness, and so on.

In addition, iterative strategies such as preconditioned error correction can be applied for the coarse grid solver by restricting the computed fine grid approximation to the coarse grid so that the coarse grid problem is also similarly decoupled. This then leads to a fully decoupled iterative two-grid algorithm. Finally, we remark that the same strategy can be applied recursively to the coarse grid problem, if necessary, which then leads to a multigrid algorithm.

5. Conclusions. We have proposed a two-grid method for solving the coupled Stokes–Darcy problem. Error estimates are obtained, which suggests that multigrid can provide a general framework for solving multimodeling problems. It is promising to extend this approach to more general settings, such as other boundary and interface conditions, Navier–Stokes/Darcy coupling, time-dependent problems, as well as other coupling applications. It is also possible to generalize the framework to other versions, including iterative two-grid methods and multilevel methods.

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