A Two-Parameter Trigonometric Series

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Let us consider the following dual questions.

A. *Given a function $f(x)$, find its Fourier series*

$$\frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx).$$

B. *Given a trigonometric series $\sum_{k=1}^{\infty}(A_k \cos kx + B_k \sin kx)$, find a function $f(x)$ such that the given trigonometric series is its Fourier series (then we can find the sum of the given trigonometric series).*

Upon learning the theory of Fourier series, we know questions of type A are straightforward but questions of type B are not, for a given trigonometric series may not even be a Fourier series, as

$$\sum_{k=2}^{\infty} \frac{1}{\ln k} \sin kx$$

shows, let alone to find a such function. Nevertheless, our experiences in doing many type A exercises are crucially helpful when tackling type B problems. For instance, if you are asked to find the sum of the trigonometric series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \cos(kx) \quad \big(x \in [0, \pi]\big),$$

your experiences of calculating the Fourier series of $f(x) = x^2$ on $[-\pi,\ \pi]$, which gives you

$$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}(-1)^k \frac{\cos kx}{k^2},$$

and of $f(x) = x$ on $[0, \pi]$, which gives you

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2},$$

will help you to figure out that for $x \in [0, \pi]$,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \cos(kx) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2). \tag{1}$$

However, in most such problems, there is only one parameter involved. In this short note, we are going to obtain the sum of a two-parameter trigonometric series, namely the sum of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}\left(1-\cos\left(\frac{2k\pi}{l}\right)\right)\cos\left(\frac{2k\pi}{l}n\right)$$

for integer parameters l and n, by carefully locating a function $f(x)$ and calculating its Fourier series. How did we find this function? By trying many type A exercises.

First, let us consider the general term a_n for the periodic sequence

$$1, \underbrace{0, 0, \ldots, 0}_{l-1}, 1, \underbrace{0, 0, \ldots, 0}_{l-1}, 1, 0, \ldots. \tag{2}$$

If $l = 2$, we know $a_n = \frac{1}{2}((-1)^n + 1)$, $n = 0, 1, 2\ldots$ For the general case, we show

$$a_n = \frac{1}{l}\sum_{k=0}^{l-1} \exp\left(2\pi i \frac{n}{l} k\right). \tag{3}$$

In fact, it is a direct corollary of the following simple lemma.

Lemma.

$$\sum_{k=0}^{l-1} \exp\left(2\pi i \frac{n}{l} k\right) = \begin{cases} l, & \textit{if } l \mid n; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $l \mid n$, $\exp(2\pi i \frac{n}{l} k) = 1$, and we get

$$\sum_{k=0}^{l-1} \exp\left(2\pi i \frac{n}{l} k\right) = l.$$

If $l \nmid n$, $\exp((2\pi i \frac{n}{l} k) \neq 1$, and we have

$$\sum_{k=0}^{l-1} \exp\left(2\pi i \frac{n}{l} k\right) = \sum_{k=0}^{l-1}\left(\exp\left(2\pi i \frac{n}{l}\right)\right)^k = \frac{1-(\exp(2\pi i \frac{n}{l}))^l}{1-\exp(2\pi i \frac{n}{l})} = 0. \qquad \blacksquare$$

Next, let us consider a_n from a different angle. Define a continuous function $f(x)$ on $[0, l]$ as

$$f(x) = \begin{cases} -x+1, & \text{for } 0 \le x \le 1; \\ 0, & \text{for } 1 \le x \le l-1; \\ x+1-l, & \text{for } l-1 \le x \le l. \end{cases}$$

It is easy to see that $f(x)$ is the linear interpolation of the sequence in (2). Now, extend this $f(x)$, first evenly to $[-l, l]$, then periodically to the whole real line R. Denote the extended function as $\tilde{f}(x)$. The famous Dirichlet-Jordan theorem (see [**1**]) assures the convergence of its Fourier series to $\tilde{f}(x)$; hence $a_n = \tilde{f}(n)$.

Now, let us calculate the Fourier series of the even function $\tilde{f}(x)$. By the Euler-Fourier formulas, we have

$$\begin{aligned}\alpha_0 &= \frac{2}{l}\int_0^l \tilde{f}(x)\,dx = \frac{2}{l}\left(\int_0^l (1-x)\,dx + \int_0^l (x+1-l)\,dx\right)\\ &= \frac{2}{l}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{2}{l},\end{aligned}$$

and

$$\begin{aligned}\alpha_m &= \frac{2}{l}\int_0^l \tilde{f}(x)\cos\left(\frac{m\pi}{l}x\right)dx\\ &= \frac{2}{l}\left\{\int_0^1 (-x+1)\cos\left(\frac{m\pi}{l}x\right)dx + \int_{l-1}^l (x+1-l)\cos\left(\frac{m\pi}{l}x\right)dx\right\}. \quad (4)\end{aligned}$$

It is easy to check that

$$\int_0^1 \cos\left(\frac{m\pi}{l}x\right)dx = \frac{l}{m\pi}\sin\left(\frac{m\pi}{l}\right)$$

and

$$\int_{l-1}^l \cos\left(\frac{m\pi}{l}x\right) = \frac{l}{m\pi}(-1)^{m+2}\sin\left(\frac{m\pi}{l}\right),$$

and by integrating by parts, we also have the following:

$$\begin{aligned}&\int_0^1 x\cos\left(\frac{m\pi}{l}x\right)dx = \frac{l}{m\pi}\left(\sin\left(\frac{m\pi}{l}\right)+\frac{l}{m\pi}\cos\left(\frac{m\pi}{l}\right)-\frac{l}{m\pi}\right),\\ &\int_{l-1}^l x\cos\left(\frac{m\pi}{l}x\right)dx =\\ &\qquad \frac{l}{m\pi}\left\{(-1)^{m+1}(l-1)\sin\left(\frac{m\pi}{l}\right)+\frac{l}{m\pi}(-1)^m\left(1-\cos\left(\frac{m\pi}{l}\right)\right)\right\}.\end{aligned}$$

Now putting all these into (4), we get that

$$\alpha_m = \begin{cases} 0, & \text{for } m \text{ odd};\\ \dfrac{l}{k^2\pi^2}\left(1-\cos\left(\dfrac{2k\pi}{l}\right)\right), & \text{for } m = 2k.\end{cases}$$

Therefore,

$$\begin{aligned}\tilde{f}(x) &= \frac{\alpha_0}{2} + \sum_{m=1}^{\infty}\alpha_m\cos\left(\frac{m\pi}{l}x\right)\\ &= \frac{1}{l} + \frac{l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(1-\cos\left(\frac{2k\pi}{l}\right)\right)\cos\left(\frac{2k\pi}{l}x\right).\end{aligned}$$

Thus,

$$a_n = \tilde{f}(n) = \frac{1}{l} + \frac{l}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \cos\left(\frac{2k\pi}{l}\right)\right) \cos\left(\frac{2k\pi}{l} n\right).$$

Comparing the above with (3), we have the following result.

Theorem.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \cos\left(\frac{2k\pi}{l}\right)\right) \cos\left(\frac{2k\pi}{l} n\right) = \frac{\pi^2}{l^2} \sum_{k=1}^{l-1} \exp\left(2\pi i \frac{n}{l} k\right),$$

or equivalently,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \cos\left(\frac{2k\pi}{l}\right)\right) \cos\left(\frac{2k\pi}{l} n\right) = \begin{cases} \dfrac{l-1}{l^2}\pi^2, & \textit{for } l \mid n; \\ -\dfrac{1}{l^2}\pi^2, & \textit{otherwise.} \end{cases} \tag{5}$$

Remarks.

1. Letting $n = 0$ and $l = 2$, we get the well-known identity

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2} = \frac{\pi^2}{12},$$

which in turn implies the other well-known identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

2. Letting $n = 0$ in the theorem, and using the fact

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

we can easily get

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{2k\pi}{l}\right) = \frac{\pi^2}{6} - \frac{l-1}{l}\pi^2,$$

which, in essence, is a different version of what we got in (1). In contrast to there, where we need two Fourier expansions, here only one function $f(x)$ is needed to derive a much more general result (5).

3. The lemma, although simple, is useful in classical number theory. For instance, by this very lemma, the number of solutions of $f(x_1, \ldots, x_n) \equiv N (\bmod\ l)$, $0 \leq x_j \leq l - 1$, can be expressed as

$$\frac{1}{l} \sum_{x_1=0}^{l-1} \cdots \sum_{x_n=0}^{l-1} \sum_{k=0}^{l-1} \exp\left(2\pi i \frac{f(x_1, \ldots, x_n) - N}{l} k\right).$$

For this and related rich discussions in classical number theory, see [**2**].

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References

1. G. Klambauer, *Mathematical Analysis*, Marcel Dekker, 1975.
2. L. G. Hua, *Introduction to Number Theory*, Springer-Verlag, 1982.

