## A Two-Parameter Trigonometric Series

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Let us consider the following dual questions.
A. Given a function $f(x)$, find its Fourier series

$$
\frac{\alpha_{0}}{2}+\sum_{k=1}^{\infty}\left(\alpha_{k} \cos k x+\beta_{k} \sin k x\right)
$$

B. Given a trigonometric series $\sum_{k=1}^{\infty}\left(A_{k} \cos k x+B_{k} \sin k x\right)$, find a function $f(x)$ such that the given trigonometric series is its Fourier series (then we can find the sum of the given trigonometric series).

Upon learning the theory of Fourier series, we know questions of type A are straightforward but questions of type B are not, for a given trigonometric series may not even be a Fourier series, as

$$
\sum_{k=2}^{\infty} \frac{1}{\ln k} \sin k x
$$

shows, let alone to find a such function. Nevertheless, our experiences in doing many type A exercises are crucially helpful when tackling type B problems. For instance, if you are asked to find the sum of the trigonometric series

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}} \cos (k x) \quad(x \in[0, \pi]),
$$

your experiences of calculating the Fourier series of $f(x)=x^{2}$ on $[-\pi, \pi]$, which gives you

$$
x^{2}=\frac{\pi^{2}}{3}+4 \sum_{k=1}^{\infty}(-1)^{k} \frac{\cos k x}{k^{2}}
$$

and of $f(x)=x$ on $[0, \pi]$, which gives you

$$
x=\frac{\pi}{2}-\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos (2 k-1) x}{(2 k-1)^{2}}
$$

will help you to figure out that for $x \in[0, \pi]$,

$$
\begin{equation*}
\sum_{k=1}^{\infty} \frac{1}{k^{2}} \cos (k x)=\frac{1}{12}\left(3 x^{2}-6 \pi x+2 \pi^{2}\right) \tag{1}
\end{equation*}
$$

However, in most such problems, there is only one parameter involved. In this short note, we are going to obtain the sum of a two-parameter trigonometric series, namely the sum of

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(1-\cos \left(\frac{2 k \pi}{l}\right)\right) \cos \left(\frac{2 k \pi}{l} n\right)
$$

for integer parameters $l$ and $n$, by carefully locating a function $f(x)$ and calculating its Fourier series. How did we find this function? By trying many type A exercises.

First, let us consider the general term $a_{n}$ for the periodic sequence

$$
\begin{equation*}
1, \underbrace{0,0, \ldots, 0}_{l-1}, 1, \underbrace{0,0, \ldots, 0}_{l-1}, 1,0, \ldots \tag{2}
\end{equation*}
$$

If $l=2$, we know $a_{n}=\frac{1}{2}\left((-1)^{n}+1\right), n=0,1,2 \ldots$ For the general case, we show

$$
\begin{equation*}
a_{n}=\frac{1}{l} \sum_{k=0}^{l-1} \exp \left(2 \pi i \frac{n}{l} k\right) . \tag{3}
\end{equation*}
$$

In fact, it is a direct corollary of the following simple lemma.

## Lemma.

$$
\sum_{k=0}^{l-1} \exp \left(2 \pi i \frac{n}{l} k\right)= \begin{cases}l, & \text { if } l \mid n \\ 0, & \text { otherwise }\end{cases}
$$

Proof. If $l \mid n, \exp \left(2 \pi i \frac{n}{l} k\right)=1$, and we get

$$
\sum_{k=0}^{l-1} \exp \left(2 \pi i \frac{n}{l} k\right)=l
$$

If $l \nmid n, \exp \left(\left(2 \pi i \frac{n}{l} k\right) \neq 1\right.$, and we have

$$
\sum_{k=0}^{l-1} \exp \left(2 \pi i \frac{n}{l} k\right)=\sum_{k=0}^{l-1}\left(\exp \left(2 \pi i \frac{n}{l}\right)\right)^{k}=\frac{1-\left(\exp \left(2 \pi i \frac{n}{l}\right)\right)^{l}}{1-\exp \left(2 \pi i \frac{n}{l}\right)}=0 .
$$

Next, let us consider $a_{n}$ from a different angle. Define a continuous function $f(x)$ on $[0, l]$ as

$$
f(x)= \begin{cases}-x+1, & \text { for } 0 \leq x \leq 1 \\ 0, & \text { for } 1 \leq x \leq l-1 \\ x+1-l, & \text { for } l-1 \leq x \leq l\end{cases}
$$

It is easy to see that $f(x)$ is the linear interpolation of the sequence in (2). Now, extend this $f(x)$, first evenly to $[-l, l]$, then periodically to the whole real line $R$. Denote the extended function as $\tilde{f}(x)$. The famous Dirichlet-Jordan theorem (see [1]) assures the convergence of its Fourier series to $\tilde{f}(x)$; hence $a_{n}=\tilde{f}(n)$.

Now, let us calculate the Fourier series of the even function $\tilde{f}(x)$. By the EulerFourier formulas, we have

$$
\begin{aligned}
\alpha_{0} & =\frac{2}{l} \int_{0}^{l} \tilde{f}(x) d x=\frac{2}{l}\left(\int_{0}^{l}(1-x) d x+\int_{0}^{l}(x+1-l) d x\right) \\
& =\frac{2}{l}\left(\frac{1}{2}+\frac{1}{2}\right)=\frac{2}{l},
\end{aligned}
$$

and

$$
\begin{align*}
\alpha_{m} & =\frac{2}{l} \int_{0}^{l} \tilde{f}(x) \cos \left(\frac{m \pi}{l} x\right) d x \\
& =\frac{2}{l}\left\{\int_{0}^{1}(-x+1) \cos \left(\frac{m \pi}{l} x\right) d x+\int_{l-1}^{l}(x+1-l) \cos \left(\frac{m \pi}{l} x\right) d x\right\} \tag{4}
\end{align*}
$$

It is easy to check that

$$
\int_{0}^{1} \cos \left(\frac{m \pi}{l} x\right) d x=\frac{l}{m \pi} \sin \left(\frac{m \pi}{l}\right)
$$

and

$$
\int_{l-1}^{l} \cos \left(\frac{m \pi}{l} x\right)=\frac{l}{m \pi}(-1)^{m+2} \sin \left(\frac{m \pi}{l}\right)
$$

and by integrating by parts, we also have the following:

$$
\begin{aligned}
& \int_{0}^{1} x \cos \left(\frac{m \pi}{l} x\right) d x=\frac{l}{m \pi}\left(\sin \left(\frac{m \pi}{l}\right)+\frac{l}{m \pi} \cos \left(\frac{m \pi}{l}\right)-\frac{l}{m \pi}\right) \\
& \int_{l-1}^{l} x \cos \left(\frac{m \pi}{l} x\right) d x= \\
& \quad \frac{l}{m \pi}\left\{(-1)^{m+1}(l-1) \sin \left(\frac{m \pi}{l}\right)+\frac{l}{m \pi}(-1)^{m}\left(1-\cos \left(\frac{m \pi}{l}\right)\right)\right\}
\end{aligned}
$$

Now putting all these into (4), we get that

$$
\alpha_{m}= \begin{cases}0, & \text { for } m \text { odd } \\ \frac{l}{k^{2} \pi^{2}}\left(1-\cos \left(\frac{2 k \pi}{l}\right)\right), & \text { for } m=2 k\end{cases}
$$

Therefore,

$$
\begin{aligned}
\tilde{f}(x) & =\frac{\alpha_{0}}{2}+\sum_{m=1}^{\infty} \alpha_{m} \cos \left(\frac{m \pi}{l} x\right) \\
& =\frac{1}{l}+\frac{l}{\pi^{2}} \sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(1-\cos \left(\frac{2 k \pi}{l}\right)\right) \cos \left(\frac{2 k \pi}{l} x\right) .
\end{aligned}
$$

Thus,

$$
a_{n}=\tilde{f}(n)=\frac{1}{l}+\frac{l}{\pi^{2}} \sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(1-\cos \left(\frac{2 k \pi}{l}\right)\right) \cos \left(\frac{2 k \pi}{l} n\right) .
$$

Comparing the above with (3), we have the following result.
Theorem.

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(1-\cos \left(\frac{2 k \pi}{l}\right)\right) \cos \left(\frac{2 k \pi}{l} n\right)=\frac{\pi^{2}}{l^{2}} \sum_{k=1}^{l-1} \exp \left(2 \pi i \frac{n}{l} k\right),
$$

or equivalently,

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}}\left(1-\cos \left(\frac{2 k \pi}{l}\right)\right) \cos \left(\frac{2 k \pi}{l} n\right)= \begin{cases}\frac{l-1}{l^{2}} \pi^{2}, & \text { for } l \mid n  \tag{5}\\ -\frac{1}{l^{2}} \pi^{2}, & \text { otherwise }\end{cases}
$$

## Remarks.

1. Letting $n=0$ and $l=2$, we get the well-known identity

$$
\sum_{k=1}^{\infty}(-1)^{k-1} \frac{1}{k^{2}}=\frac{\pi^{2}}{12}
$$

which in turn implies the other well-known identity

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}}=\frac{\pi^{2}}{6}
$$

2. Letting $n=0$ in the theorem, and using the fact

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}}=\frac{\pi^{2}}{6},
$$

we can easily get

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2}} \cos \left(\frac{2 k \pi}{l}\right)=\frac{\pi^{2}}{6}-\frac{l-1}{l} \pi^{2},
$$

which, in essence, is a different version of what we got in (1). In contrast to there, where we need two Fourier expansions, here only one function $f(x)$ is needed to derive a much more general result (5).
3. The lemma, although simple, is useful in classical number theory. For instance, by this very lemma, the number of solutions of $f\left(x_{1}, \ldots, x_{n}\right) \equiv N(\bmod l), 0 \leq x_{j} \leq$ $l-1$, can be expressed as

$$
\frac{1}{l} \sum_{x_{1}=0}^{l-1} \ldots \sum_{x_{n}=0}^{l-1} \sum_{k=0}^{l-1} \exp \left(2 \pi i \frac{f\left(x_{1}, \ldots, x_{n}\right)-N}{l} k\right) .
$$

For this and related rich discussions in classical number theory, see [2].

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## References

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2. L. G. Hua, Introduction to Number Theory, Springer-Verlag, 1982.

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