# A two phase Stefan problem: regularity of the free boundary.

J. R. CANNON [1] and M. PRIMICERIO [2] (*) (**)

**Riassunto.** - *Con riferimento ad un problema di Stefan a due fasi in uno strato piano indefinito, viene dimostrata la infinita differenziabilità della funzione $x = s(t)$ che rappresenta ad ogni istante la ascissa del piano di separazione tra le due fasi.*

*La trattazione è valida sia per il caso in cui si assegni la temperatura sulle facce dello strato, sia per quello in cui venga assegnato il flusso.*

**Abstract.** - *We proved the infinite differentiability of the function $x = s(t)$ giving, for all $t$, the abscissa of the interface plane for a two phase Stefan problem in a plane infinite slab.*

*The proof applies in both cases of temperature or thermal fluxes prescribed on the two limiting planes.*

## 1. - Introduction.

Consider the following two-phase STEFAN problem:

$$
(1.1)\qquad \begin{cases} \mathfrak{L}_1^* u_1(x,\ t) \equiv \varkappa_1 u_{1xx} - u_{1t} = 0 \quad \text{in} \quad 0 < x < s(t), & 0 < t \leq T, \\ u_1(0,\ t) = f(t) > 0, \qquad u_1(s(t),\ t) = 0, & 0 < t \leq T, \\ u_1(x,\ 0) = 0, \qquad 0 < x \leq s(0) = b \neq 0. \end{cases}
$$

$$
(1.2)\qquad \begin{cases} \mathfrak{L}_2^* u_2(x,\ t) \equiv \varkappa_2 u_{2xx} - u_{2t} = 0 \quad \text{in} \quad s(t) < x < 1, & 0 < t \leq T, \\ u_2(1,\ t) = g(t) < 0, \qquad u_2(s(t),\ t) = 0, & 0 < t \leq T, \\ u_2(x,\ 0) = 0, \qquad b \leq x < 1. \end{cases}
$$

$$
(1.3)\qquad \dot{s}(t) = - K_1 u_{1x}(s(t),\ t) + K_2 u_{2x}(s(t),\ t), \qquad 0 < t \leq T,
$$

---

[1] Mathematics Department, University of Texas at Austin, Austin, Texas, 78712.

[2] Università di Firenze, Istituto Matematico «Ulisse Dini» Viale Morgagni, 67/A, 50134, Firenze, Italy.

(*) The research was supported in part by the National Science Foundation contract GP 15724 and the NATO Senior Fellowship program.

(**) Entrata in Redazione il 14 settembre 1970.

where $\varkappa_i = k_i \rho_i^{-1} c_i^{-1}$, $i = 1, 2$, $K_i = k_i \rho_2^{-1} L^{-1}$, $i = 1, 2$, $\rho_i$, $c_i$, $k_i$, $i = 1, 2$, and $L$ are positive constants describing physical properties of the media.

We have proved in [8] that (1.1)-(1.3) has a unique solution under the following assumptions:

(A) $f(t)$ and $g(t)$ are piecewise continuous functions such that there exist four positive constants $\alpha_i$, $\beta_i$ ($i = 1, 2$) such that

$$0 < \alpha_1 < f(t) < \beta_1 \quad \text{and} \quad -\beta_2 < g(t) < -\alpha_2 < 0. \tag{1.4}$$

(B) $\Gamma = \max\,(2K_1\varkappa_1^{-1}\beta_1,\ 2K_2\varkappa_2^{-1}\beta_2) < 1$.

Also, we demonstrated that $s \in C_1$ and that there exists a constant $\Delta$, $0 < \Delta < 1$ such that

$$\Delta \leq s(t) \leq 1 - \Delta, \qquad 0 \leq t \leq T.$$

This paper is concerned with the study of the differentiability properties of $s(t)$. We can prove the infinite differentiability of $s(t)$ by developing estimates on the successive derivatives of the functions $s_\delta(t)$ which satisfy approximating $\delta$-problems that will be specified later.

Since these estimates will be shown to be independent of $\delta$, the ASCOLI-ARZELA's Theorem will hold for each $\delta$-family $\frac{d^n}{dt^n} s_\delta(t)$ and from this will follow the infinite differentiability of $s$. The technique used here is similar to the method of [7]. The main difference is the lack of monotonicity of the boundary and the form of the barriers required to obtain the results. Some computations are tedious but the logic of the proof is direct and conceptually simple. As in [7], the method relates the size of the successive derivatives of $s_\delta(t)$ via 13 explicit recursion relations.

For each $\delta$, $0 < \delta < \Delta$. consider the following approximating problem:

$$\begin{cases} \mathfrak{L}_1^* u_{1\delta} = 0, \quad \text{in} \quad 0 < x < s_\delta(t), \qquad 0 < t \leq T, \\ u_{1\delta}(0, t) = f(t), \quad \text{and} \quad u_{1\delta}(s_\delta(t), t) = 0 \quad \text{for} \quad 0 < t \leq T, \\ u_{1\delta}(x, 0) = 0 \quad \text{for} \quad 0 < x \leq s_\delta(0), \qquad s_\delta(0) = b \end{cases} \tag{1.5}$$

$$\begin{cases} \mathfrak{L}_1^* u_{2\delta} = 0, \quad \text{in} \quad s_\delta(t) < x < 1, \qquad 0 < t \leq T, \\ u_{2\delta}(1, t) = g(t) \quad \text{and} \quad u_{2\delta}(s_\delta(t), t) = 0 \quad \text{for} \quad 0 < t \leq T, \\ u_{2\delta}(x, 0) = 0 \quad \text{for} \quad s_\delta(0) \leq x < 1 \qquad s_\delta(0) = b \end{cases} \tag{1.6}$$

$$\dot{s}_\delta(t) = \delta^{-1}[K_1 u_{1\delta}(s_\delta(t) - \delta, t) + K_2 u_{2\delta}(s_\delta(t) + \delta, t)] \quad \text{for} \quad 0 < t \leq T, \tag{1.7}$$

where the dot denotes differentiation with respect to $t$.

Existence (for all $T$) and uniqueness of a solution $(u_{1\delta}, u_{2\delta}, s_\delta)$ to (1.5)-(1.7) can be proved by using retarded argument method (applied to (1.7)) and the maximum principle. The proof can easily be made by means of trivial modifications of Theorems 1 and 3 of [8].

By the same technique one can prove the following results:

(i) there exists a constant $\mu$, $0 < \mu < 1/2$, independent of $\delta$ such that:

$$\mu < s_\delta(t) < 1 - \mu; \tag{1.8}$$

(ii) let $\eta$ denote an arbitrary positive constant, let $\nu = \mu \min(\varkappa_1^{-1}, \varkappa_2^{-1})$, and let

$$H = \max\{\nu^{-1} \log[(1 - \Gamma)/2],\ 2\Gamma\eta(1 - \Gamma)^{-1}\}.$$

Then,

$$\| \dot{s}_\delta \|_T = \max_{0 \le \tau \le T} | \dot{s}_\delta(\tau) | \le H; \tag{1.9}$$

(iii) with the same meaning of the constants

$$|\delta^{-1} u_{i\delta}(s_\delta(t) \pm \delta, t)| \le \beta_i \varkappa_i^{-1} \{1 - \exp[-\varkappa_i^{-1}(\|\dot{s}_\delta\|_T + \eta)\mu]\}^{-1}(\|\dot{s}_\delta\|_T + \eta) \equiv C_i, \tag{1.10}$$

for $i = 1, 2$.

## 2. - Starting estimates.

Estimates of the derivatives of $s_\delta(t)$ involve the computation of directional derivatives of $u_{1\delta}$ and $u_{2\delta}$ according to equation (1.7). Consequently the following change of variables will help to simplify the process. Set

$$\xi = x - s_\delta(t), \qquad \tau = t,$$

and define

$$w_1(\xi, \tau) = u_{1\delta}(\xi + s_\delta(\tau), \tau), \qquad w_2(\xi, \tau) = u_{2\delta}(\xi + s_\delta(\tau), \tau). \tag{2.1}$$

It is easy to see that

$$\begin{cases} \mathfrak{L}_1 w_1 \equiv \varkappa_1 w_{1\xi\xi} + \dot{s}_\delta w_{1\xi} - w_{1\tau} = 0, & -s_\delta(\tau) < \xi < 0, \quad 0 < \tau \le T, \\ w_1(-s_\delta(\tau), \tau) = f(\tau) \quad \text{and} \quad w_1(0, \tau) = 0 \quad \text{for} \quad 0 < \tau \le T, \\ w_1(\xi, 0) = 0 \quad \text{for} \quad -s_\delta(0) < \xi \le 0 \end{cases} \tag{2.2}$$

that

$$(2.3)\qquad \begin{cases} \mathcal{L}_2 w_2 \equiv \varkappa_2 w_{2\xi\xi} + \dot{s}_\delta w_{2\xi} - w_{2\tau} = 0, & 0 < \xi < 1 - s_\delta(\tau), \quad 0 < \tau \leq T, \\ w_2(1 - s_\delta(\tau), \tau) = g(\tau) \text{ and } w_2(0, \tau) = 0, & \text{for } 0 < \tau \leq T, \\ w_2(\xi, 0) = 0, \quad \text{for } 0 \leq \xi < 1 - s_\delta(0), \end{cases}$$

and that

$$(2.4)\qquad \dot{s}_\delta(\tau) = \delta^{-1}[K_1 w_1(-\delta, \tau) + K_2 w_2(\delta, \tau)], \qquad 0 < \tau \leq T.$$

Let us introduce the following notation:

$$(2.5)\qquad v_1^{(j)} = \frac{\partial^j}{\partial \tau^j} w_1, \qquad v_2^{(j)} = \frac{\partial^j}{\partial \tau^j} w_2$$

$$(2.6)\qquad s_\delta^{(j)} = \frac{d^{j+1}}{d\tau^{j+1}} s_\delta(\tau).$$

Remark that notation (2.6) — which is required by symmetry and convenience — implies in particular $s_\delta^{(0)}(\tau) = \dot{s}_\delta(\tau)$. According to (2.5), (2.6) and (2.2), (2.3), (2.4) we have

$$(2.7)\qquad \mathcal{L}_i v_i^{(0)} \equiv \varkappa_i v_{i\xi\xi}^{(0)} + s_\delta^{(0)} v_{i\xi}^{(0)} - v_{i\tau}^{(0)} = 0 \qquad (i = 1, 2),$$

and

$$(2.8)\qquad s_\delta^{(0)} = \delta^{-1}[K_1 v_1^{(0)}(-\delta, \tau) + K_2 v_2^{(0)}(\delta, \tau)].$$

Hence estimates of $s_\delta^{(j)}$ ($j = 1, 2, \ldots$) can be obtained in terms of estimates of $v_i^{(j)}$ ($i = 1, 2$ and $j = 1, 2, \ldots$), which in turn, will imply estimates of $v_{i\xi}^{(j)}$. In what follows it will be useful to estimate functions $v_i^{(j)}$ and $v_{i\xi}^{(j)}$ in a sufficiently narrow strip around $\xi = 0$ (i.e. we confine our attention to the temperatures — and their directional derivatives — in the vicinity of the interface $x = s(t)$) where $v_i^{(0)}$ are $C^\infty$. More precisely, let $\alpha$ be a positive number less than $\mu$ which will be specified later. We consider (2.7) in the region limited by planes $\xi = 0$ and $\xi = (-1)^i \alpha$. In addition assume $\delta < \alpha$. Estimates for $v_i^{(0)}$ are directly given by the maximum principle

$$(2.9)\qquad v_i^{(0)}(\xi, \tau) \leq \beta_1, \qquad -\alpha \leq \xi \leq 0,$$

$$(2.10)\qquad |v_2^{(0)}(\xi, \tau)| \leq \beta_2, \qquad 0 \leq \xi \leq \alpha.$$

To estimate $v_{i\xi}^{(0)}$ we need the following Lemma which is based upon the maximum principle.

LEMMA 1. - *Let* $r_i = r_i(\xi)$, $i = 1, 2$, *be defined as follows*:

$$r_i(\xi) = L_i \frac{-\xi^2 + 2(-1)^i\alpha\xi}{2} + M_i \{ 1 - \exp[-\varkappa_i^{-1}(\| s_\delta^{(0)} \| + \bar{\eta})\alpha] \}^{-1} \cdot \tag{2.11}$$

$$\cdot \{ 1 - \exp[-(-1)^i\varkappa_i^{-1}(\| s_\delta^{(0)} \| + \bar{\eta})\xi] \} + N_i,$$

$L_i$, $M_i$, $N_i$, $\bar{\eta}$ being positive constants, and let $\alpha$ be such that

$$\alpha \leq \min \{ \varkappa_1/2 \| s_\delta^{(0)} \|, \varkappa_2/2 \| s_\delta^{(0)} \| \}. \tag{2.12}$$

*Then*

$$\mathfrak{L}_1 r_1(\xi) \leq -\frac{\varkappa_1}{2} L_1 \quad \text{for} \quad -\alpha \leq \xi < 0, \qquad 0 \leq \tau \leq T, \tag{2.13}$$

$$\mathfrak{L}_2 r_2(\xi) \leq -\frac{\varkappa_2}{2} L_2 \quad \text{for} \quad 0 < \xi \leq \alpha, \qquad 0 \leq \tau \leq T. \tag{2.14}$$

*Moreover*

$$r_i(\xi) \leq r_i[(-1)^i\alpha], \qquad r_i[(-1)^i\alpha] \geq M_i, \tag{2.15}$$

$$r_i(0) = N_i, \tag{2.16}$$

*and, if* $N_i = 0$, *for any* $\delta$, $0 < \delta < \alpha$,

$$\delta^{-1} r_i[(-1)^i\delta] \leq \alpha L_i + G_i M_i, \tag{2.17}$$

$G_1$ *and* $G_2$ *being defined by* (2.18) *below.*

PROOF. - Direct computations show that

$$\mathfrak{L}_i r_i(\xi) \equiv \varkappa_i r_{i\xi\xi} + s_\delta^{(0)} r_{i\xi} = -L_i[\varkappa_i - s_\delta^{(0)} \{ -\xi + (-1)^i\alpha \}] -$$

$$- M_i \{ 1 - \exp[-\varkappa_i^{-1}(\| s_\delta^{(0)} \| + \bar{\eta})\alpha] \}^{-1} \varkappa_i^{-1}(\| s_\delta^{(0)} \| + \bar{\eta}) \exp[-(-1)^i\varkappa_i^{-1}(\| s_\delta^{(0)} \| + \bar{\eta})\xi] \cdot$$

$$\cdot \{ \| s_\delta^{(0)} \| + \bar{\eta} - (-1)^i s_\delta^{(0)} \}.$$

The second term is clearly negative because $\{ \| s_\delta^{(0)} \| + \bar{\eta} - (-1)^i s_\delta^{(0)} \} > 0$ independently of $i$.

Hence

$$\mathfrak{L}_i r_i(\xi) \leq -L_i[\varkappa_i - s_\delta^{(0)} \{ -\xi + (-1)^i\alpha \}] \leq -L_i[\varkappa_i - \alpha \| s_\delta^{(0)} \| \},$$

the last inequality being a consequence of the interval of variation for $\xi$. Recalling (2.12), the proof of (2.13) and (2.14) is immediate.

The first inequality in (2.15) follows from the sign of $r_{i\xi}$ and the second one is immediate, as well as the proof of (2.16).

Finally (2.17) follows from the following inequality (recall that $1-e^{-z}\leq z$)

$$\delta^{-1}r_i[(-1)^i\delta]=\delta^{-1}L_i\,\frac{-\delta^2+2\alpha\delta}{2}+$$
$$+M_i\delta^{-1}\{1-\exp[-\varkappa_i^{-1}(\|s_\delta^{(0)}\|+\bar\eta)\alpha\}^{-1}\{1-\exp[-\varkappa_i^{-1}(\|s_\delta^{(0)}\|+\bar\eta)\delta]\}\leq$$
$$\leq L_i\alpha+M_i\varkappa_i^{-1}(\|s_\delta^{(0)}\|+\bar\eta)\{1-\exp[-\varkappa_i^{-1}(\|s_\delta^{(0)}\|+\bar\eta)\alpha]\}^{-1},$$

and (2.17) holds with constants $G_i$ given by

$$G_i=\varkappa_i^{-1}(\|s_\delta^{(0)}\|+\bar\eta)\{1-\exp[-\varkappa_i^{-1}(\|s_\delta^{(0)}\|+\bar\eta)\alpha]\}^{-1} \tag{2.18}$$

Note that $G_i$ are independent of $\delta$ (and that $G_i\to\infty$ like $\alpha^{-1}$ for $\alpha$ tending to zero).

The barrier (2.11) and some results on the interior estimates of $v_{i\xi}^{(0)}$ enable us to prove the following Lemma.

LEMMA 2. - *Choose* $\bar\mu$ *such that* $0<\bar\mu<2\min(\varkappa_i/2\|s_\delta^{(0)}\|,\ \mu/2)$.
*Then,*

$$|v_{i\xi}^{(0)}(\xi,\tau)|\leq D_i,\qquad -\frac{\bar\mu}{2}\leq\xi\leq 0,\quad i=1,\ 0\leq\xi\leq\frac{\bar\mu}{2},\quad i=2,\ 0<t\leq T, \tag{2.19}$$

where $D_i$ depends upon $\bar\mu$.

PROOF. - Simple modifications of Lemma 3 and 4 of [2] give the following results. For each $\sigma$, $0<\sigma<\bar\mu/2$

$$\left|\frac{\partial^n}{\partial x^n}u_{1\delta}(x,t)\right|\leq\gamma_1Q_1^n n! \tag{2.20}$$

$$\text{in}\quad s_\delta(t)+\sigma-\bar\mu\leq x\leq s_\delta-\sigma\quad\text{and}\quad 0<t\leq T,$$

and

$$\left|\frac{\partial^n}{\partial x^n}u_2(x,t)\right|\leq\gamma_2Q_2^n n! \tag{2.21}$$

$$\text{in}\quad s_\delta(t)+\sigma\leq x\leq s_\delta+\bar\mu-\sigma\quad\text{and}\quad 0<t\leq T,$$

where $Q_i$ and $\gamma_i$ are constants depending upon $\bar\mu$, $\sigma$, $\beta_i$, $T$.

Set $\sigma \leq \bar{\mu}/4$ and consider $v_{i\xi}^{(0)}$ in the region bounded by the planes $\xi = 0$, $\xi = (-1)^i \bar{\mu}/2$, for $0 < t \leq T$. Then,

$$\mathcal{L}_i v_{i\xi}^{(0)} = 0,$$

$$|v_{i\xi}^{(0)}(0, \tau)| \leq C_i, \quad \text{(see (1.10))},$$

$$v_{i\xi}^{(0)}(\xi, 0) = 0,$$

$$\left| v_{i\xi}^{(0)}\left[(-1)^i \frac{\bar{\mu}}{2}, \tau\right] \right| \leq \gamma_i Q_i \quad \text{(see (2.20))}.$$

Consequently, if we define

$$\gamma_{i\pm}(\xi, \tau) = r_i(\xi) \pm v_{i\xi}^{(0)}, \qquad i = 1, 2.$$

where in the definition of $r_i(\xi)$ according to (2.11) we set $L_i = 0$, $M_i = \gamma_i Q_i$ and $N_i = C_i$, and if $\alpha = \bar{\mu}/2$, (2.19) follows directly from Lemma 1.

### 3. - The recursion relations.

Throughout this section we shall use the following notation:

$$S^{(j)} = \| s_\delta^{(0)} \|_T, \tag{3.1}$$

$$U_i^{(j)} = \| v_i^{(j)} \|_{T, \alpha}, \tag{3.2}$$

$$V_i^{(j)} = \| v_{i\xi}^{(j)} \|_{T, \alpha}, \tag{3.3}$$

$$W_i^{(j)} = \| \mathcal{L}_i v_i^{(j)} \|_{T, \alpha}, \tag{3.4}$$

$$X_i^{(j)} = \| v_i^{(j)}[(-1)^i\alpha, \tau] \|_T, \tag{3.5}$$

$$Y_i^{(j)} = \| v_{i\xi}^{(j)}[0, \tau] \|_T, \tag{3.6}$$

$$Z_i^{(j)} = \| v_{i\xi}^{(j)}[(-1)^i\alpha, \tau] \|_T, \tag{3.7}$$

where $i = 1, 2$, $j = 1, 2, \ldots$, and for any function $g_i = g_i(\xi, \tau)$, $i = 1, 2$,

$$\| g_1 \|_{T, \alpha} = \max_{\substack{0 \leq \tau \leq T \\ -\alpha \leq \xi \leq 0}} | g_1(\xi, \tau) |$$

and

$$\| g_2 \|_{T, \alpha} = \max_{\substack{0 \leq \tau \leq T \\ 0 \leq \xi \leq \alpha}} | g_2(\xi, \tau) | .$$

It is clear that results of section 1 and 2 provide estimates for $S^{(0)}$ (see (1.9)), $U_i^{(0)}$ (see (2.9) and (2.10)), $V_i^{(0)}$ (see (2.19)), $W_i^{(0)}$ (which is zero according to (2.7)), $X_i^{(0)}$ (same estimate as $U_i^{(0)}$), $Y_i^{(0)}$ (see (1.10), and $Z_i^{(0)}$ (same estimate as $V_i^{(0)}$). Note that these estimates are independent of $\delta$. The aim of this section is to provide estimates of quantities defined by (3.1)-(3.7) by means of ***recursion relations***, i.e., let us assume that $S^{(r)}$, $U_i^{(r)} \dots Z_i^{(r)}$, $i = 1, 2$, and $r = 1, \dots, j-1$ are known, and estimate $S^{(j)}$, $U_i^{(j)} \dots Z_i^{(j)}$.

First, we shall deal with estimates of $U_i^{(j)}$. Consider the following equality:

$$\mathcal{L}_i v_i^{(j)} = \frac{\partial^j}{\partial \tau^j} \mathcal{L}_i v_i^{(0)} + \left[\mathcal{L}_i \frac{\partial^j}{\partial \tau^j} - \frac{\partial^j}{\partial \tau^j} \mathcal{L}_i\right] v_i^{(0)} = \left[\mathcal{L}_i \frac{\partial^j}{\partial \tau^j} - \frac{\partial^j}{\partial \tau^j} \mathcal{L}_i\right] v_i^{(0)} .$$

Since operator $\frac{\partial^j}{\partial \tau^j}$ commutes with $\varkappa_i \frac{\partial^2}{\partial \xi^2}$ and $\frac{\partial}{\partial \tau}$, we have

$$\mathcal{L}_i v_i^{(j)} = \left\{ \left[ s_\delta^{(0)} \frac{\partial^j}{\partial \tau^j} - \frac{\partial^j}{\partial \tau^j} s_\delta^{(0)} \right] \frac{\partial}{\partial \xi} \right\} v_i^{(0)} = - \sum_{h=1}^{j} \binom{j}{h} s_\delta^{(h)} v_{i\xi}^{(j-h)} . \tag{3.8}$$

Next, define

$$\gamma_{i\pm}(\xi, \tau) = r_i(\xi) \pm v_i^{(j)}(\xi, \tau),$$

with $r_i(\xi)$ as in (2.11) with

$$L_i = 2\varkappa_i^{-1} \sum_{h=1}^{j} \binom{j}{h} S^{(h)} V_i^{(j-h)}, \qquad M_i = X_i^{(j)}, \qquad N_i = 0.$$

From Lemma 1 we have in the regions limited by planes $\xi = 0$, $\xi = (-1)^i \alpha$:

$\mathcal{L}_i \gamma_{i\pm}(\xi, \tau) \leq 0$, from (2.13) and (2.14),

$\gamma_{i\pm}((-1)^i \alpha, \tau) \geq X_i^{(j)} \pm v_i^{(j)}((-1)^i \alpha, \tau) \geq 0$, by (2.15) and (3.5),

$\gamma_{i\pm}(0, \tau) = 0$,

$\gamma_{i\pm}(\xi, 0) = r_i(\xi) \geq 0$, by (2.15).

Consequently,

$$|v_i^{(j)}(\xi, \tau)| \leq r_i(\xi) \quad \text{and} \tag{3.9}$$

$$U_i^{(j)} \leq \alpha^2 \varkappa_i^{-1} \left[ S^{(j)} V_i^{(0)} + \sum_{h=1}^{j-1} \binom{j}{h} S^{(h)} V_i^{(j-h)} \right] + X_i^{(j)} . \tag{3.10}$$

From (2.8) and (3.9) we have

$$S^{(j)} \leq \delta^{-1} \{ K_1 r_1(-\delta) + K_2 r_2(\delta) \}.$$

and from (2.17)

$$S^{(j)} \leq \alpha(K_1 L_1 + K_2 L_2) + K_1 G_1 M_1 + K_2 G_2 M_2 ,$$

whence

$$(3.11) \qquad S^{(j)} \leq [2\varkappa_1^{-1} K_1 V_1^{(0)} \alpha + 2\varkappa_2^{-1} K_2 V_2^{(0)} \alpha] S^{(j)} +$$
$$+ 2\alpha \left[ K_1 \varkappa_1^{-1} \sum_{h=1}^{j-1} \binom{j}{h} S^{(h)} V_1^{(j-h)} + K_2 \varkappa_2^{-1} \sum_{h=1}^{j-1} \binom{j}{h} S^{(h)} V_2^{(j-h)} \right] + K_1 G_1 X_1^{(j)} + K_2 G_2 X_2^{(j)} .$$

Recall (2.19) and choose $\alpha > 0$ now such that $\alpha \leq \bar{\mu}/2$ and

$$B \equiv 1 - 2\alpha[K_1 \varkappa_1^{-1} D_1 + K_2 \varkappa_2^{-1} D_2] > 0.$$

The parameter $\alpha$ will now remain fixed for the remainder of the paper. Hence, (3.11) gives

$$(3.12) \qquad S^{(j)} \leq 2\alpha B^{-1} \sum_{h=1}^{j-1} \binom{j}{h} S^{(h)} [K_1 \varkappa_1^{-1} V_1^{(j-h)} + K_2 \varkappa_2^{-1} V_2^{(j-h)}] +$$
$$+ B^{-1}[K_1 G_1 X_1^{(j)} + K_2 G_2 X_2^{(j)}].$$

Substituting in (3.10) one gets

$$(3.13) \qquad U_i^{(j)} \leq \alpha^2 \varkappa_2^{-1} \Big\{ 2 V_i^{(0)} \alpha B^{-1} \sum_{h=1}^{j-1} \binom{j}{h} S^{(h)} [K_1 \varkappa_1^{-1} V_1^{(j-h)} + K_2 \varkappa_2^{-1} V_2^{(j-h)}] +$$
$$+ B^{-1} V_i^{(0)} [G_1 K_1 X_1^{(j)} + G_2 K_2 X_2^{(j)}] +$$
$$+ \sum_{h=1}^{j-1} \binom{j}{h} S^{(h)} V_i^{(j-h)} \Big\} + X_i^{(j)}, \qquad i = 1,\ 2 \quad \text{and} \quad j = 1,\ 2,\ \dots .$$

Next, we shall estimate $X_i^{(j)}$ and consequently inequalities (3.12) and (3.13) shall contain only known quantities (i.e., $V_i^{(l)}$, $X_i^{(l)}$, $S^{(l)}$ with $l < j$).

From the transformation of variables $\xi = x - s_\delta(t)$, $\tau = t$ it is clear that

$$(3.14) \qquad v_i^{(j)}[(-1)^i \alpha,\ \tau] = \left[ \left( \frac{\partial}{\partial t} + s_\delta^{(0)} \frac{\partial}{\partial x} \right)^j u_{i\delta}(x,\ t) \right]_{\substack{x = s_\delta(t) + (-1)^i \alpha \\ t=\tau}} =$$

$$= \left[ \sum_{h=0}^{j} \binom{j}{h} \left\{ \sum_{l=0}^{h} \binom{h}{l} \frac{\partial^{j+h-2l}}{\partial x^{j+h-2l}} u_{i\delta}(x,\ t) \sum_{a_1 + a_2 + \dots + a_{j-h} = l} \frac{l!}{a_1!\, a_2! \dots a_{j-h}!} s_\delta^{(a_1)} \dots s_\delta^{(a_{j-h})} \right]_{\substack{x = s_\delta + (-1)^i \alpha \\ t=\tau}}$$

where the last summation, when $h = j$, gives zero unless $l = 0$ in which case the sum is one. Consequently the highest derivative of $s_\delta$ appearing in (3.14) is $s_\delta^{(j-1)}$. Hence, modifying (2.20) by replacing $\bar{\mu}/2$ with $\alpha$, it follows that

$$X_i^{(j)} \leq \gamma_i \sum_{h=0}^{j} \binom{j}{h} \Big\{ \sum_{l=0}^{h} \binom{h}{l} \bar{Q}_i^{j+h-2l} (j+h-2l)! \sum_{a_1+a_2+\dots+a_{j-h}=l} \frac{l!}{a_1! a_2! \dots a_{j-h}!} S^{(a_1)} S^{(a_2)} \dots S^{(a_{j-h})} \Big\}. \tag{3.15}$$

Applying the same reasoning, it is also clear that one gets

$$Z_i^{(j)} \leq \bar{\gamma}_i \sum_{h=0}^{j} \binom{j}{h} \Big\{ \sum_{l=0}^{h} \binom{h}{l} \bar{Q}_i^{j+h-2l+1} (j+h-2l+1)! \sum_{a_1+a_2+\dots+a_{j-h}=l} \frac{l!}{a_1! \, a_2! \dots a_{j-h}!} S^{(a_1)} S^{(a_2)} \dots S^{(a_{j-h})} \Big\}. \tag{3.16}$$

It is important to emphasize again that the $S^{(r)}$ with the highest index appearing in (3.15) and (3.18) is $S^{(j-h)}$.

Next, we shall estimate $V^{(j)}$. Applying the same reasoning we used to derive (3.8) we find

$$\| \mathcal{L}_i v_{i\xi}^{(j)} \|_{T,\alpha} = \Big\| - \sum_{h=1}^{j} \binom{j}{h} s_\delta^{(h)} [\mathcal{L}_i v_i^{(j-h)} - s_\delta^{(0)} v_{i\xi}^{(j-h)} + v_i^{(j-h+1)}] \Big\|_{T,\alpha} \leq$$

$$\leq \sum_{h=1}^{j} \binom{j}{h} S^{(h)} \{ W_i^{(j-h)} + S^{(0)} V_i^{(j-h)} + U_i^{(j-h+1)} \}.$$

Since

$$v_{i\xi}^{(j)}(0, \tau) \leq Y_i^{(j)},$$

$$v_{i\xi}^{(j)}(\xi, 0) = 0,$$

$$v_{i\xi}^{(j)}[(-1)^i \alpha, \tau] \leq Z_i^{(j)},$$

we can apply Lemma 1 to the function

$$\gamma_{i\pm}(\xi, \tau) = r_i(\xi) \pm v_{i\xi}^{(j)}(\xi, \tau)$$

with

$$L_i = 2\varkappa_i^{-1} \sum_{h=1}^{j} \binom{j}{h} S^{(h)} \{ W_i^{(j-h)} + S^{(0)} V_i^{(j-h)} + U_i^{(j-h+1)} \},$$

$$M_i = Z_i^{(j)},$$

$$N_i = Y_i^{(j)}.$$

Hence,

$$V_i^{(j)} \le \alpha^2 \varkappa_i^{-1} \sum_{h=1}^{j} \binom{j}{h} S^{(h)} \{ W_i^{(j-h)} + S^{(0)} V_i^{(j-h)} + U_i^{(j-h+1)} \} + Z_i^{(j)} + Y_i^{(j)} . \tag{3.17}$$

To complete the recursion relations we need finally estimates for $W_i^{(j)}$ and $Y_i^{(j)}$. For the former we have directly from (3.8)

$$W_i^{(j)} \le \sum_{h=1}^{j} \binom{j}{h} S^{(j)} V_i^{(j-h)} . \tag{3.18}$$

From the fact that

$$| v_{i\xi}^{(j)}(0, \tau) | \le \limsup_{\delta \to 0} \delta^{-1} r_i[(-1)^i \delta]$$

with $r_i(\xi)$ defined as in (3.9), we can apply (2.17) to yield

$$Y_i^{(j)} \le 2\alpha \varkappa_i^{-1} \sum_{h=1}^{j} \binom{j}{h} S^{(h)} V_i^{(j-h)} + G_i X_i^{(j)} . \tag{3.19}$$

The set of the thirteen recursion relations is then completely specified by: (3.12), (3.13), (3.17), (3.18), (3.15), (3.19) and (3.16). These relations are solved for $X_i^{(j)}$, $Z_i^{(j)}$, $S^{(j)}$, $U_i^{(j)}$, $Y_i^{(j)}$, $V_i^{(j)}$, and $W_i^{(j)}$, $i = 1, 2$, in this order.

## 4. - Conclusion.

In sec. 3 we have found recursion relations relating the quantities (3.1)-(3.7). Since these relations are independent of $\delta$ and since also the estimates of $S^{(0)}$, $U_i^{(0)}$, ..., $Z_i^{(0)}$ we derived in sec. 2 are independent of $\delta$, we can claim that $\delta$-families $s_\delta^{(j)}$ are families of equibounded equicontinuous functions. Consequently, a repeated application of the ASCOLI-ARZELA theorem shows that $s$ is infinitely differentiable and that the limit functions of the $s_\delta^{(j)}$ families, $\dot{s}$, $\ddot{s}$, ..., $s^{(n)}$ ... satisfy the same limitations of $s_\delta^{(0)}$, ..., $s_\delta^{(n-1)}$ ... .

Let $u_1(x, t)$, $u_2(x, t)$ denote the solution of (1.1), (1.2) with $s(t)$ given by $\lim_{\delta \to 0} s_\delta(t)$. It is easily seen that $u_{1\delta}$ and $u_{2\delta}$ tend uniformly to $u_1$ and $u_2$ as $\delta \to 0$ and, using the mean value theorem one can also show that $(s, u_1, u_2)$ satisfy (1.3) in the sense of its integral equivalence expressing the heat balance across the interface. Hence $(s, u_1, u_2)$ form the solution to the given STEFAN problem. See [8].

In conclusion in (1.1) we can replace $u_1(0, t)$ by $k_1 u_{1x}(0, t)$, and in (1.2) we can replace $u_2(1, t)$ by $k_2 u_{2x}(1, t)$ and appeal to the results of [9] for the necessary preliminary estimates from which the infinite differentiability of $s$ follows.

## REFERENCES

[1] B. M. BUDAK and M. Z. MOSKAL, *Classical solution of the multidimensional multifront Stefan problem*, Soviet Math. Dokl., Vol. 10 (1969), # 5, pp. 1043-1046.

[2] J. R. CANNON, *A priori estimate for continuation of the solution of the heat equation in the space variable*, Ann. Mat. Pura Appl. 65 (1964), pp. 377-388.

[3] J. R. CANNON and J. DOUGLAS Jr., *The stability of the boundary in a Stefan problem*, Ann. della Scuola Normale Superiore di Pisa, Vol. XXI, Fasc. I, (1967), pp. 83-91.

[4] J. R. CANNON and C. D. HILL, *Existence, uniqueness, stability and monotone dependance in a Stefan problem for the heat equation*, J. of Math. and Mech., Vol. 17, (1967), pp. 1-20.

[5] J. R. CANNON, J. DOUGLAS Jr. and C. D. HILL, *A multi-boundary Stefan problem and the disappearance of phases*, J. of Math. and Mech., Vol. 17, (1967), pp. 21-34.

[6] J. R. CANNON and C. D. HILL, *Remarks on a Stefan problem*, J. of Math. and Mech., Vol. 17, (1967), pp. 433-442.

[7] — — and — —, *On the infinite differentiability of the free boundary in a Stefan problem*, J. of Math. Anal. and Appl., Vol. 22, (1968), pp. 385-397.

[8] J. R. CANNON and M. PRIMICERIO, *A two phase Stefan problem with temperature boundary conditions*, Ann. Mat. Pura Appl. 88 (1971), pp 177-192.

[9] — — and — —, *A two phase Stefan problem with flux boundary conditions*, Ann. Mat. Pura Appl. 88 (1971), pp. 193-206.

[10] A. FRIEDMAN, *The Stefan problem in several space variables*, Transactions of the A.M.S., Vol. 133, (1968), pp. 51-87.

[11] — —, *One dimensional Stefan problems with nonmonotone free boundary*, Transactions of the A.M.S., Vol. 133, (1968), pp. 89-114.

[12] — —, Correction to *the Stefan problem in several space variables*, Transactions of the A.M.S., Vol. 142, (1969), p. 557.

[13] M. GEVREY, *Sur les équations anx derivées partielles du type paraboliques*, J. Math. (ser. 6), 9 (1913), pp. 305-471.

[14] S L. KAMENOMOSTSKAJA, *On Stefan's problem*, Mat. Sb. 53 (95), (1965), pp. 485-514.

[15] JIANG LI-SHANG, *Existence and differentiability of the solution of a two-phase Stefan problem for quasi-linear parabolic equations*, Chinese Math. 7, (1965), pp. 481-496.

[16] D. QUILGHINI, *Una analisi fisico-matematica del processo del cambiamento di fase*, Ann. di Mat. pura ed applicata, (IV), Vol. LXVII (1965), pp. 33-74.

[17] L. I. RUBINSTEIN, *Two-phase Stefan problem on a segment with one-phase initial state of thermoconductive medium*, Ucen. Zap. Lat. Gos. Univ. Stucki 58 (1964), pp. 111-148.

[18] G. SESTINI, *Esistenza ed unicità nel problema di Stefan relativo a campi dotati di simmetria*, Rivista Mat. Univ. Parma, 3 (1952), pp. 103-113.