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A TWO POINT CALIBRATION ON AN SP(1) BUNDLE OVER THE THREE-SPHERE

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Abstract

Gluck and Ziller proved that Hopf vector fields on S^3 have minimum volume among all unit vector fields. Thinking of S^3 as a Lie group, Hopf vector fields are exactly those with unit length which are left or right invariant, and TS^3 is a trivial vector bundle with a connection induced by the adjoint representation. We prove the analogue of the stated result of Gluck and Ziller for the representation given by quaternionic multiplication. The resulting vector bundle over S^3, with the Sasaki metric, has as well no parallel unit sections. We provide an application of a double point calibration, proving that the submanifolds determined by the left and right invariant sections minimize volume in their homology classes.

Introduction

Gluck and Ziller proved in [2] that Hopf vector fields on S^3 have minimum volume among all unit vector fields. Thinking of S^3 as a Lie group, Hopf vector fields are exactly those with unit length which are left or right invariant, and TS^3 is a trivial vector bundle with a connection induced by the adjoint representation. This can be considered in the following more general setting: Let G be a Lie group and (V, ρ) a finite dimensional orthogonal real representation of G. Let $\pi : E = G \times V \to G$ be the trivial vector bundle. For $v \in V$, let $L_v : G \to E$ be the "constant" section $L_v(p) = (p, v)$.

There exists a unique connection ∇ on $E \to G$ such that

$$\nabla_Z L_v = L_{\frac{1}{2} d\rho(Z) v} \tag{1}$$

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for all $v \in V$ and all left invariant vector fields Z of G. The connection is easily seen to be compatible with the metric and is called the connection induced by (V, ρ).

On E one can define the canonical Sasaki metric induced by ∇ in such a way that the map

$$(d\pi, \mathcal{K})_\xi : T_\xi E \to T_q M \times E_q$$

is a linear isometry for each $\xi \in E$ (here $q = \pi(\xi)$ and $\mathcal{K}$ is the connection operator associated with ∇).

For example, the Levi Civita connection of a compact connected simple Lie group G with a bi-invariant metric may be obtained in this way: Let $\mathfrak{g}$ be the Lie algebra of G and consider on $E = G \times \mathfrak{g} \to G$ the connection induced by the adjoint representation. If ℓ_p denotes left multiplication by p, then the map

$$F : G \times \mathfrak{g} \to TG, \quad F(p, v) = d\ell_p v \tag{2}$$

is an affine vector bundle isomorphism. Moreover, it is an isometry if E and TG carry the corresponding Sasaki metrics.

For $v \in V$, let R_v the section of E defined by

$$R_v(g) = \left(g, \rho\left(g^{-1}\right) v\right).$$

The sections L_v and R_v are called left and right invariant, respectively, since in the particular case above they correspond to left and right invariant vector fields, respectively, via the isomorphism (2).

Let $\mathbf{H}$ denote the quaternions and consider on the sphere $S^3 = \{q \in \mathbf{H} \mid \|q\| = 1\}$ the canonical round metric induced from $\mathbf{R}^4$, or equivalently, thinking of S^3 as a Lie group, the bi-invariant metric with constant sectional curvature one. We prove the following analogue of the cited result of Gluck and Ziller.

Theorem. *Let $E = S^3 \times \mathbf{H} \to S^3$ be the vector bundle with the connection induced by the representation of S^3 on $\mathbf{H}$ given by $\rho(q) u = u\bar{q}$ and consider on E the associated Sasaki metric. Then the left and right invariant unit sections have minimum volume among all unit sections of $E \to S^3$ in their homology classes, and they are unique with this property.*

Let $E^1 = \{(q, v) \in E \mid \|v\| = 1\}$ be the associated sphere bundle, which as a differentiable manifold is $S^3 \times S^3$.

Remarks.

(a) The bundle $E^1 \to S^3$ has no parallel sections. In particular, the Sasaki metric is not the product metric.

(b) We prove the Theorem using a single calibration which calibrates both the left and right invariant sections. In particular, nonnegative integer combinations of them are homology mass minimizing.

(c) The submanifolds of E^1 determined by left and right invariant sections are totally geodesic round spheres.

(d) The fibers $\{q\} \times S^3$ have smaller volume than the left and right invariant sections, but perhaps the latter have minimum volume among all sections of $E^1 \to S^3$ (some of them determine submanifolds lying in other homology classes).

(e) Calibrations are rare jewels: Let G be a compact connected simple Lie group with a bi-invariant metric and $E \to G$ the affine bundle associated to a representation, as in the introduction. If E is endowed with the Sasaki metric, the submanifolds of E^1 determined by left invariant sections, in general are not critical points of the volume functional, let alone minima (see for example [6], where we deal with the case $E = TG$, that is $\rho = \mathrm{Ad}$).

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Calibrations

A calibration on a Riemannian manifold M is a closed k-form ω on M such that for all $p \in M$ one has

(a) $\omega_p(\eta) \leq 1$ for all $\eta \in G_p(k)$ and

(b) $\omega_p(\xi) = 1$ for some $\xi \in G_p(k)$,

where $G_p(k)$ denotes the Grassmannian of oriented k-planes in T_pM. If ξ is as in (b), one says that ω_p calibrates ξ. The following well-known proposition is a direct consequence of Stokes' Theorem.

Proposition 1. *If M and ω are as above, an oriented submanifold N of M such that $\omega(T_qN) = 1$ for all $q \in N$, has minimum volume among all k-dimensional submanifolds in its homology class.*

The method of calibrations was developed in the fundamental paper [4], see further important applications for instance in [3].

Next we recall from [1] (see also [5]) a statement concerning 3-calibrations with constant coefficients in $\mathbf{R}^6$. It is a special case of double point calibrations of $\Lambda^3\left(\mathbf{R}^6\right)^*$ when the characterizing angles of the calibrated 3-planes are all equal.

Proposition 2 [1, Theorem 8]**.** *Let $\{e_i \mid i=1,\dots,6\}$ be an orthonormal basis of $\mathbf{R}^6$ and $\left\{e^i \mid i=1,\dots,6\right\}$ the dual basis. Let $\theta \in (0, 2\pi/3)$, $c=\cos\theta$, $s=\sin\theta$ and consider the oriented 3-plane*

$$\eta(\theta) = (ce_1+se_4)\wedge(ce_2+se_5)\wedge(ce_3+se_6).$$

There exists a unique calibration ω on $\mathbf{R}^6$ calibrating exactly the 3-planes $e_1\wedge e_2\wedge e_3$ and $\eta(\theta)$. The 3-form ω is given by

$$\omega = e^{123} + \lambda\left(e^{156}+e^{426}+e^{453}\right) + \mu e^{456},$$

where $e^{ijk}=e^i\wedge e^j\wedge e^k$ and

$$\lambda = \frac{c}{1+c} \quad \textit{and} \quad \mu = -\frac{1+2c}{s(1+c)}. \tag{3}$$

Left and right invariant sections and the Sasaki metric

From now on, $E = S^3\times\mathbf{H}$ with the connection ∇ induced by the representation ρ of S^3 on $\mathbf{H}$ given by $\rho(q)u = u\bar{q}$ (quaternionic multiplication). We have

$$d\rho(Z)u = \left.\frac{d}{dt}\right|_0 \rho(\exp tZ)u = \left.\frac{d}{dt}\right|_0 u\exp(-tZ) = -uZ.$$

Hence, in this particular case, (1) becomes

$$\nabla_Z L_v = L_{-vZ/2} \tag{4}$$

Let $\mathfrak{g} = \operatorname{Im}\mathbf{H} \cong T_1S^3$ be the Lie algebra of S^3.

Proposition 3. *The connection operator $\mathcal{K}_{(q,v)} : T_{(q,v)}E \to E_q = \{q\}\times\mathbf{H}$ is given by*

$$\mathcal{K}_{(q,v)}(d\ell_q Z, \xi) = (q, -vZ/2+\xi).$$

Proof. By definition of L_u and ∇ we have for $Z \in \mathfrak{g}$, $\xi \in \mathbf{H}$ that

$$\begin{aligned}
\mathcal{K}_{(q,v)}\left(d\ell_q Z, \xi\right) &= \left.\tfrac{D}{dt}\right|_0 \left(q \exp tZ, v + t\xi\right) \\
&= \left.\tfrac{D}{dt}\right|_0 L_v\left(q \exp tZ\right) + tL_\xi\left(q \exp tZ\right) \\
&= \left(\nabla_{d\ell_q Z} L_v\right)(q) + L_\xi(q) \\
&= L_{-\frac{1}{2}vZ}(q) + L_\xi(q),
\end{aligned}$$

from which the proposition follows. q.e.d.

Let $\pi : E \to G$ be as before and Consider on E the Sasaki metric. Since on each fiber $\{q\}\times\mathbf{H}$ of E the Sasaki metric is given by the obvious identification of it with $\mathbf{H}$, one has clearly that $E^1 = S^3 \times S^3$, which is a Lie group. We denote the elements of the first and second factors by $p, q, \dots$ and $u, v, \dots$, respectively. Correspondingly, the tangent vectors to both factors will be denoted by Z or ξ, respectively, in order to emphasize their horizontal and vertical natures.

Let us denote

$$\phi_{(q,v)} = \left(d\pi_{(q,v)}, \mathcal{K}_{(q,v)}\right) : T_{(q,v)}E^1 \to T_q S^3 \times v^\perp$$

(we identify $\{q\} \times \mathbf{H} = \mathbf{H}$) and $\phi = \phi_{(1,1)} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. By Proposition 3 these maps are given by

$$\phi_{(q,v)}\left(d\ell_q Z, \xi\right) = \left(d\ell_q Z, -vZ/2 + \xi\right), \tag{5}$$

$$\phi\left(Z, \xi\right) = \left(Z, -Z/2 + \xi\right). \tag{6}$$

Proposition 4. *The Sasaki metric on $E^1 = S^3 \times S^3$ is the unique left invariant metric such that at the identity $(1,1)$ is the pull-back through ϕ of the product metric on $\mathfrak{g} \times \mathfrak{g}$.*

Proof. By definition of the Sasaki metric, we must show that in the commutative diagram

$$\begin{array}{ccc}
T_{(1,1)}E^1 & \xrightarrow{d\ell_{(q,v)}} & T_{(q,v)}E^1 \\
\downarrow \phi & & \downarrow \phi_{(q,v)} \\
\mathfrak{g} \times \mathfrak{g} & \xrightarrow{f} & T_q S^3 \times v^\perp,
\end{array}$$

the isomorphism f is a linear isometry. By (6), for $(Z, \xi) \in T_{(1,1)}E^1$ we compute $f\left(\phi\left(Z, \xi\right)\right) = f\left(Z, -Z/2 + \xi\right)$. On the other hand, by (5),

$$\phi_{(q,v)} d\ell_{(q,v)}\left(Z, \xi\right) = \phi_{(q,v)}\left(d\ell_q Z, v\xi\right) = \left(d\ell_q Z, -vZ/2 + v\xi\right).$$

Hence $f\left(Z, \eta\right) = \left(d\ell_q Z, v\eta\right)$ for all $(Z, \eta) \in \mathfrak{g}\times\mathfrak{g}$, which is clearly a linear isometry. q.e.d.

Lemma 5. $(\nabla_Z R_v)(q) = (q, vqZ/2)$.

Proof. We may assume that $Z \neq 0$, otherwise the assertion is trivial. For $t \sim 0$, let us define

$$\begin{aligned} V(q\exp tZ) &= (\cos t) L_{vq}(q\exp tZ) + (\sin t) L_{vqZ}(q\exp tZ) \\ &= (q\exp tZ, (\cos t) vq + (\sin t) vqZ). \end{aligned}$$

A straightforward computation shows that $V(q) = (q, vq) = R_v(q)$ and

$$\left.\frac{d}{dt}\right|_0 V(q\exp tZ) = (d\ell_q Z, vqZ) = \left.\frac{d}{dt}\right|_0 R_v(q\exp tZ).$$

Hence by (4) we have

$$\begin{aligned} (\nabla_Z R_v)(q) &= (\nabla_Z V)(q) \\ &= (\nabla_Z L_{vq})(q) + (L_{vqZ})(q) = L_{-\frac{1}{2}vqZ+vqZ}(q), \end{aligned}$$

from which the statement follows. q.e.d.

Let $\mathcal{L}_v = L_v(S^3)$ and $\mathcal{R}_v = R_v(S^3)$ be the submanifolds of E^1 determined by the sections L_v and R_v, respectively, with the corresponding orientations induced by that of S^3.

Proposition 6.

(a) *The submanifolds $\mathcal{L}_1$ and $\mathcal{R}_1$ of E^1 meet at $e = (1,1) \in E^1$ and one has*

$$\begin{aligned} (7)\quad \phi(T_e\mathcal{L}_1) &= \phi(\mathfrak{g}\times\{0\}) = \{(Z, -Z/2) \mid Z \in \mathfrak{g}\} \quad \textit{and} \\ \phi(T_e\mathcal{R}_1) &= \phi(\{(Z,Z) \mid Z \in \mathfrak{g}\}) = \{(Z, Z/2) \mid Z \in \mathfrak{g}\}. \end{aligned}$$

(b) *The submanifolds $\mathcal{L}_v$ and $\mathcal{R}_{v\bar{q}}$ of E^1 meet at $e = (q,v) \in E^1$ and one has*

$$T_{(q,v)}\mathcal{L}_v = \left(d\ell_{(q,v)}\right)_e (T_e\mathcal{L}_1) \quad \textit{and} \quad T_{(q,v)}\mathcal{R}_{v\bar{q}} = \left(d\ell_{(q,v)}\right)_e T_e\mathcal{R}_1.$$

Proof. A generic element of $T_{(q,v)}\mathcal{R}_{v\bar{q}}$ has the form

$$\left.\frac{d}{dt}\right|_0 (q\exp tZ, v\bar{q}q\exp tZ) = (d\ell_q Z, vZ),$$

for $Z \in \mathfrak{g}$. Hence, if $(q,v) = e$ one obtains the expression for $\phi(T_e\mathcal{R}_1)$ in (7), since by (6) one has $\phi(Z,Z) = (Z, Z/2)$. Similarly for left invariant sections. The second statement is clear from the preceding. q.e.d.

Proof of the Theorem

The Theorem is an immediate consequence of Proposition 1 and the following.

Proposition 7. *Let Ω be the left invariant 3-form on $S^3 \times S^3$ such that $\Omega_e = \phi^*\omega$, where ω is the 3-form of Proposition 2 with $c = 3/5$ and $s = 4/5$, for an appropriate identification of $\mathbf{R}^6$ with $\mathfrak{g} \times \mathfrak{g}$. Then Ω is a calibration which calibrates exactly the left and right invariant sections of $E^1 \to S^3$.*

Proof. Let $\{f_1, f_2, f_3\}$ be a positively oriented orthonormal basis of $\mathfrak{g}$ and $r = \sqrt{5}/2$. If $e_i = (f_i, -f_i/2)/r$ and $e_{i+3} = (f_i/2, f_i)/r$ for $i = 1, 2, 3$, then $\{e_1, \ldots, e_6\}$ is an orthonormal basis of $\mathfrak{g} \times \mathfrak{g}$. By (7), $\{e_1, e_2, e_3\}$ and

$$\{(f_i, f_i/2)/r \mid i = 1, 2, 3\} = \{(3e_i + 4e_{i+3})/5 \mid i = 1, 2, 3\}$$

are positively oriented orthonormal bases of $\phi(T_e\mathcal{L}_1)$ and $\phi(T_e\mathcal{R}_1)$, respectively. Hence the characteristic angles of this pair of 3-planes are all equal to $\arccos(3/5)$. Thus, by Proposition 2, Ω_e calibrates exactly $T_e\mathcal{L}_1$ and $T_e\mathcal{R}_1$. Now, by Proposition 6 (b), Ω calibrates exactly the left and right invariant unit sections. It remains only to show that Ω is closed.

For $i = 1, 2, 3$, let $X_i = (f_i, 0)$ and $X_{i+3} = (0, f_i)$. We compute

$$\left(\phi^* e^j\right)(X_i) = e^j \phi(f_i, 0) = e^j(f_i, -f_i/2) = e^j(re_i) = r\delta_{i,j}$$

and $(0, f_i) = 2r(-e_i + 2e_{i+3})/5$, from which

$$\left(\phi^* e^j\right)(X_{i+3}) = e^j \phi(0, f_i) = e^j(0, f_i) = 2r(-\delta_{i,j} + 2\delta_{i+3,j})/5.$$

Therefore, if $\left\{\xi^i \mid i = 1, \ldots, 6\right\}$ is the dual basis of $\{X_i \mid i = 1, \ldots, 6\}$, calling $a = 2/5$, we have

$$\phi^* e^j = \begin{cases} r\left(\xi^j - a\xi^{j+3}\right) & \text{if } 1 \leq j \leq 3, \\ rs\xi^j & \text{if } 4 \leq j \leq 6. \end{cases} \tag{8}$$

From now on we think of ξ^j as a left invariant form on E^1. Let us denote $\xi^{ijk} = \xi^i \wedge \xi^j \wedge \xi^k$, $\theta_1 = \xi^{156} + \xi^{426} + \xi^{453}$ and $\theta_2 = \xi^{423} + \xi^{153} + \xi^{126}$. By (8), a straightforward computation yields

$$\Omega = r^3\left(\xi^{123} + \overline{\mu}\xi^{456} + \left(a^2 + \lambda s^2\right)\theta_1 - a\theta_2\right)$$

for some $\overline{\mu} \in \mathbf{R}$. Now, $\lambda = 3/8$ by (3) and one easily computes $a^2 + \lambda s^2 = a$. Denoting $\omega_1 = \xi^{156} - \xi^{423}$, $\omega_2 = \xi^{426} - \xi^{153}$ and $\omega_3 = \xi^{453} - \xi^{126}$, one has

$$\begin{aligned}\Omega &= r^3 \left(\xi^{123} + \overline{\mu}\xi^{456} + a\left(\theta_1 - \theta_2\right)\right) \\ &= r^3 \left(\xi^{123} + \overline{\mu}\xi^{456} + a\left(\omega_1 + \omega_2 + \omega_3\right)\right).\end{aligned}$$

We compute $d\xi^1 = -2\xi^2 \wedge \xi^3$ (X_2, X_3 may be thought of as orthonormal left invariant vector fields on S^3, hence $[X_2, X_3] = 2X_1$). Similarly, $d\xi^4 = -2\xi^5 \wedge \xi^6$, hence $d\left(\xi^5 \wedge \xi^6\right) = 0$. Therefore

$$\begin{aligned}d\xi^{156} &= d\xi^1 \wedge \left(\xi^5 \wedge \xi^6\right) - \xi^1 \wedge d\left(\xi^5 \wedge \xi^6\right) \\ &= -2\xi^2 \wedge \xi^3 \wedge \xi^5 \wedge \xi^6.\end{aligned}$$

Analogously, $d\xi^{423} = -2\xi^5 \wedge \xi^6 \wedge \xi^2 \wedge \xi^3$ and consequently $d\omega_1 = 0$. In the same way one obtains $d\omega_2 = d\omega_3 = 0$. On the other hand, clearly ξ^{123} and ξ^{456} are closed, since they are pull-backs of 3-forms on the first and second factors of $S^3 \times S^3$, respectively. This implies that Ω is closed. q.e.d.

Remark. The submanifolds $\mathcal{L}_u$ and $\mathcal{R}_v$ lie in different homology classes. Although the second remark to the Theorem shows the convenience of calibrating them simultaneously, one can ask oneself whether they can be calibrated separately. We do not know if this is possible. For instance, the 3-form on E^1 obtained by projecting orthogonally onto $T\mathcal{L}_u$ or $T\mathcal{R}_v$ is not closed. In particular, the fibrations of E^1 with fibers $\mathcal{L}_u$ or $\mathcal{R}_v$ (which are totally geodesic by the second remark to the Theorem) is not the fibration of parallel submanifolds $N \times \{x\}$ induced by a Riemannian product decomposition of E^1. Next we check the validity of these assertions.

Let θ be the volume form on S^3 and $\pi_2 : S^3 \times S^3 \to S^3$ the projection onto the second factor. One computes easily that $\pi_2 \circ L_u \equiv u$ and $\pi_2 \circ R_v = \ell_v$. Hence,

$$\int_{\mathcal{L}_u} \pi_2^*\theta = \int_{S^3} L_u^*\pi_2^*\theta = 0 \qquad \text{and}$$
$$\int_{\mathcal{R}_v} \pi_2^*\theta = \int_{S^3} R_v^*\pi_2^*\theta = \int_{S^3} \ell_v^*\theta = \int_{S^3} \theta = \text{ vol }\left(S^3\right).$$

Therefore $\mathcal{L}_u$ and $\mathcal{R}_v$ lie in different homology classes, since $\pi_2^*\theta$ is clearly closed. On the other hand, the 3-form on E^1 obtained by projecting orthogonally at (q,v) onto $T_{(q,v)}\mathcal{L}_v$ is the left invariant form on E^1 given at the identity (with the notation of the proof of the Theorem) by

$$\phi^*\left(e^1 \wedge e^2 \wedge e^3\right) = r^3\left(\xi^1 - a\xi^4\right) \wedge \left(\xi^2 - a\xi^5\right) \wedge \left(\xi^3 - a\xi^6\right),$$

which is not closed by the same arguments used in the proof of the Theorem. Similarly for $\mathcal{R}_v$ instead of $\mathcal{L}_u$, by Lemma 8 below.

Lemma 8. *The map $F : E^1 \to E^1$ defined by $F(q,v) = (\bar{q}, -v\bar{q})$ is an isometry satisfying $F(\mathcal{L}_u) = \mathcal{R}_{-u}$ for all $u \in T_1S^3$.*

Proof. Fix $(q,v) \in E^1$ and let $\widetilde{F} : T_qS^3 \times v^\perp \to T_{\bar{q}}S^3 \times (v\bar{q})^\perp$ be defined by

$$\widetilde{F}(d\ell_q Z, v\xi) = -\left(dr_{\bar{q}}Z, v\xi\bar{q}\right),$$

for $Z, \xi \in T_1S^3$, where r_p denotes right multiplication by p. The map $\widetilde{F}$ is clearly an isometry with respect to the product metric. By definition of the Sasaki metric, we must verify that the following diagram commutes:

$$\begin{array}{ccc} T_{(q,v)}E^1 & \xrightarrow{dF_{(q,v)}} & T_{(\bar{q},-v\bar{q})}E^1 \\ \downarrow \phi_{(q,v)} & & \downarrow \phi_{(\bar{q},-v\bar{q})} \\ T_qS^3 \times v^\perp & \xrightarrow{\widetilde{F}} & T_{\bar{q}}S^3 \times (v\bar{q})^\perp . \end{array}$$

In fact, by (5), the image of

$$\begin{aligned} dF_{(q,v)}(d\ell_q Z, v\xi) &= \tfrac{d}{dt}\Big|_0 F(q\exp tZ, v\exp t\xi) \\ &= \tfrac{d}{dt}\Big|_0 \left(\exp(-tZ)\,\bar{q}, -v\exp(t\xi)\exp(-tZ)\,\bar{q}\right) \\ &= \left(-d\ell_{\bar{q}}(qZ\bar{q}), -v\xi\bar{q} + vZ\bar{q}\right), \end{aligned}$$

under $\phi_{(\bar{q},-v\bar{q})}$ is $\widetilde{F} \circ \phi_{(q,v)}(d\ell_q Z, v\xi)$, as desired. q.e.d.

Comments on the remarks to the Theorem.

(a) Let V any smooth unit section of $E^1 \to S^3$ with $V(1) = (1,v)$. Let $\{x,y\}$ be an orthonormal subset of $T^1S^3 = \operatorname{Im}\mathbf{H}$ and let X, Y be the corresponding left invariant vector fields. By definition of ∇ (see (4)), and using that $xy = -yx$ since $\langle x,y\rangle = 0$, the curvature

$$\begin{aligned} R(x,y)(1,v) &= \left(\nabla_X\nabla_Y L_v - \nabla_Y\nabla_X L_v - \nabla_{[X,Y]}L_v\right)(1) \\ &= \left(1, \tfrac{1}{4}vyx - \tfrac{1}{4}vxy + \tfrac{1}{2}v(xy - yx)\right) \\ &= \left(1, \tfrac{1}{2}vxy\right) \end{aligned}$$

does not vanish. Hence V is not parallel.

(c) Clearly $L_1 : S^3 \to \mathcal{L}_1$ is a diffeomorphism. By (7),

$$\|(dL_1)_1 Z\|_S = \|(Z,0)\|_S = \|\phi(Z,0)\| = \|(Z,-Z/2)\| = \sqrt{5}\,\|Z\|/2,$$

where $\| \|_S$ denotes the norm on TE^1 associated to the Sasaki metric. By left invariance, the metric on S^3 induced by L_1 is $5/4$ times the canonical one. Thus, $\mathcal{L}_1$ (and hence also $\mathcal{L}_u$ for any u) is a round sphere of Riemannian diameter $\sqrt{5}\pi/2$.

Next we compute the shape operator of $\mathcal{L}_1$ at e. We know that $T_e\mathcal{L}_1 = \{(Z,0) \mid Z \in \mathfrak{g}\}$. Hence,

$$(T_e\mathcal{L}_1)^\perp = \{(U, 5U/2) \mid U \in \mathfrak{g}\},$$

since it has dimension three and

$$\langle \phi(Z,0), \phi(U,5U/2)\rangle = \langle (Z,-Z/2), (U,2U)\rangle = 0.$$

By the well-known formula for the Levi Civita connection, which simplifies if the metric and the vector fields are left invariant, if $z = (Z,0)$, $u = (U, 5U/2)$ and $x = (X,0)$, we have

$$2\langle \nabla_z u, x\rangle = \langle \phi[z,u], \phi x\rangle - \langle \phi[u,x], \phi z\rangle + \langle \phi[x,z], \phi u\rangle.$$

A straightforward computation using (6) yields

$$2\left\langle \nabla_{(Z,0)}(U,5U/2), (X,0)\right\rangle = \tfrac{5}{4}\left(\langle [Z,U], X\rangle - \langle [U,X], Z\rangle\right),$$

which vanishes, since $\operatorname{ad} U$ is skew symmetric. Therefore $\mathcal{L}_1$ (and also $\mathcal{L}_u$ for all unit u) is totally geodesic in E^1, since it is a Lie subgroup and the metric is left invariant. By Lemma 8, all the facts above are valid for the submanifolds determined by right invariant sections.

References

[1] J. Dadok & R. Harvey, *Calibrations on* $\mathbf{R}^6$, Duke Math. J. **50(4)** (1983) 1231–1243.

[2] H. Gluck & W. Ziller, *On the volume of a unit vector field on the three sphere*, Comm. Math. Helv. **61** (1986) 177–192.

[3] R. Harvey, *Spinors and calibrations*, Perspectives in Mathematics, Academic Press, 1990.

[4] R. Harvey & H. Lawson, *Calibrated geometries*, Acta Math. **148** (1982) 47–157.

[5] Morgan, *On the singular structure of three-dimensional, area-minimizing surfaces in* $\mathbf{R}^n$, Trans. Am. Math. Soc. **276** (1983) 137–143.

[6] M. Salvai, *On the volume of a unit vector field on a compact Lie group*, preprint.

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