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*A Two Point Connection Problem for General Linear Ordinary Differential Equations**

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§1. Introduction.

In the theory of ordinary differential equations in the complex domain a study of analyzing global behaviours of solutions is of great importance, but extremely difficult. Specifically, such a global problem for linear ordinary differential equations consists in finding explicit connection formulas between local solutions. In fact, as is well known, a fundamental set of solutions of linear differential equations can be expressed in terms of linear combinations of another fundamental set of solutions with constant coefficients. But there exists no general way to evaluate the constant coefficients explicitly.

In 1858, B. Riemann [21] first investigated the connection problem for the so-called hypergeometric differential equation with three regular singularities and derived the complete results by the method of double-circuit contour integration. After that, many authors tackled and to some extent contributed to the global analysis of Fuchsian differential equations, though satisfactory results even for Heun's equation, a second order linear differential equation with four regular singularities have not yet been obtained. For topics on Heun's equation and Fuchsian differential equations, see [7], [5] and [17].

At the quite same time when B. Riemann wrote the above paper, G. G. Stokes [22] had been studying Airy's equation which has only a regular and an irregular singular point in the entire complex plane, and discovered a striking fact that constant coefficients appearing in asymptotic representations of solutions change discontinuously by a change of sectorial neighborhoods of an irregular singular point. This fact is now called the Stokes phenomenon. The Stokes phenomenon, as we explain below, can be completely worked out by the solution of a connection

* A part of the results in this paper was announced in [12] without proofs.

between two fundamental sets of solutions in respective neighborhoods of a regular and an irregular singular point. The detailed study of such a two point connection problem was initiated by G. D. Birkhoff [1] and followed by several authors [23], [13], [6], [8].

Our purpose of this paper also is to solve a two point connection problem for linear ordinary differential equations with an irregular singularity of an arbitrary rank, providing a method of evaluating constant coefficients in the linear relations between two fundamental sets of solutions.

Now we are concerned with a single n-th order linear ordinary differential equation with polynomial coefficients of the form

$$t^n \frac{d^n X}{dt^n} = \sum_{l=1}^{n} \left(\sum_{r=0}^{ql} a_{l,r} t^r \right) t^{n-l} \frac{d^{n-l} X}{dt^{n-l}} \tag{1.1}$$

which has obviously a regular singularity at the origin and an irregular singularity of rank q at infinity in the entire complex plane. As is easily verified, there exists a fundamental set of solutions expressed in terms of convergent power series

$$X_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \qquad (j=1, 2,\dots, n) \tag{1.2}$$

in the neighborhood of the regular singular point $t=0$. The constants ρ_j are the roots of the characteristic equation

$$I(\rho) \equiv [\rho]_n - \sum_{l=1}^{n} a_{l,0} [\rho]_{n-l} = 0 \tag{1.3}$$

where the notation $[\quad]_p$ means that

$$[\rho]_p = \rho(\rho-1)\cdots(\rho-p+1), \quad [\rho]_0 = 1. \tag{1.4}$$

According to the local theory of irregular singular points, we can calculate formal solutions of the form

$$X^k(t) = \exp\left(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t \right) t^{\mu_k} \sum_{s=0}^{\infty} h^k(s) t^{-s} \tag{1.5}$$
$$(k=1, 2,\dots, n)$$

and then prove the existence of an actual fundamental set of solutions $X_S^k(t)$ $(k=1, 2,\dots, n)$ with reference to every sector S with vertex at the origin and central angle not exceeding π/q such that

$$X_S^k(t) \sim X^k(t) \quad \text{as} \quad t \to \infty \quad \text{in} \quad S. \tag{1.6}$$

The last statement means the asymptotic relation in the sense of Poincaré's definition.

Here we remark that λ_k, $\alpha_{q-1}^k, \dots, \alpha_1^k$ and μ_k $(k=1, 2, \dots, n)$ in the formal power series (1.5) are the characteristic constants determined by algebraic processes, and, in particular, λ_k $(k=1, 2, \dots, n)$ are the roots of the simple equation

$$(1.7) \qquad J(\lambda) \equiv \lambda^n - \sum_{l=1}^{n} a_{l,ql}\lambda^{n-l} = 0.$$

We assume throughout this paper that $\rho_j - \rho_k \neq$ integer $(j \neq k)$, $\lambda_j \neq 0$ and $\lambda_j \neq \lambda_k$ $(j \neq k)$. The first assumption, which is not essential, excludes the case when the solutions in the neighborhood of the regular singular point $t=0$ involve logarithmic terms.

From now on we shall aim at deriving the asymptotic representations of the convergent power series solutions $X_j(t)$ in terms of the formal solutions $X^k(t)$ in the whole complex plane. Once we have the asymptotic relations

$$(1.8) \qquad X_j(t) \sim \sum_{k=1}^{n} T^{(j,k)}(S) X^k(t) \quad \text{as} \quad t \to \infty \qquad (j=1, 2, \dots, n)$$

in every sector S a finite number of which covers the whole complex plane, the constant coefficients $C^{(k,j)}(\hat{S})$ appearing in the connection formulas

$$(1.9) \qquad X_{\hat{S}}^k(t) = \sum_{j=1}^{n} C^{(k,j)}(\hat{S}) X_j(t) \qquad (k=1, 2, \dots, n)$$

in a sector $\hat{S} \subset S$ can readily be given by evaluating the inverse matrix of the matrix $\{T^{(j,k)}(S): j, k=1, 2, \dots, n\}$, since

$$X_{\hat{S}}^k(t) \sim \sum_{j=1}^{n} C^{(k,j)}(\hat{S}) \sum_{l=1}^{n} T^{(j,l)}(S) X^l(t) = X^k(t) \quad \text{as} \quad t \to \infty \quad \text{in} \quad \hat{S}.$$

And then, the Stokes phenomenon near the irregular singular point $t=\infty$ will be cleared up by the relations

$$(1.10) \qquad X_{\hat{S}}^k(t) \sim \sum_{j=1}^{n} C^{(k,j)}(S) \sum_{l=1}^{n} T^{(j,l)}(S') X^l(t) \quad \text{as} \quad t \to \infty \quad \text{in} \quad S' \qquad (k=1, 2, \dots, n).$$

The constant coefficients $T^{(j,k)}(S)$ varying with a sector S are called the Stokes multipliers.

We here give a short sketch of a method for the establishment of the asymptotic relations (1.8) together with the determination of the Stokes multipliers $T^{(j,k)}(S)$ for every sector S. Assume that the connection problem were solved. Then the convergent power series solution $X_j(t)$ may be written in the form

$$(1.11) \quad X_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m = \sum_{k=1}^{n} T^{(j,k)}(S) X_{\hat{S}}^k(t) = \sum_{k=1}^{n} \sum_{s=0}^{\infty} h^k(s) x^{(j,k)}(t, s)$$

where functions $x^{(j,k)}(t, s)$ may be considered to have the following behaviours:

$$\text{(1.12)}\qquad \begin{cases} x^{(j,k)}(t, s) \sim t^{\rho_j} & \text{near } t=0 \\ x^{(j,k)}(t, s) \sim T^{(j,k)} \exp\left(\dfrac{\lambda_k}{q}t^q + \dfrac{\alpha_{q-1}^k}{q-1}t^{q-1} + \cdots + \alpha_1^k t\right) t^{\mu_k - s} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{near } t=\infty. \end{cases}$$

Conversely, we take as the functions with just the same properties as (1.12) particular solutions

$$\text{(1.13)}\qquad x^{(j,k)}(t, s) = t^{\rho_j} \sum_{m=0}^{\infty} g^{(j,k)}(m, s) t^m$$

of first order nonhomogeneous differential equations

$$\text{(1.14)}\qquad t\frac{dx}{dt} = \{\lambda_k t^q + \alpha_{q-1}^k t^{q-1} + \cdots + \alpha_1^k t + (\mu_k - s)\} x + t^{\rho_j} P_{q-1}^{(j,k)}(t)$$

where $P_{q-1}^{(j,k)}(t)$ are appropriate polynomials of degree $(q-1)$. Then, we try to split the convergent power series solutions $X_j(t)$ into series of the form (1.11) and next make clear their global behaviours by means of the properties of the functions $x^{(j,k)}(t, s)$.

We call the functions $x^{(j,k)}(t, s)$ the fundamental functions associated with this two point connection problem, considering the fact that the first order nonhomogeneous differential equations (1.14) are determined solely by the characteristic constants. The global analysis of the fundamental functions is heavily based on the study of the coefficients $g^{(j,k)}(m, s)$ of the power series (1.13) which also are solutions of q-th order difference equations

$$\text{(1.15)}\qquad (m+s+\rho_j-\mu_k) g^{(j,k)}(m, s) = \alpha_1^k g^{(j,k)}(m-1, s) + \cdots$$
$$+ \alpha_{q-1}^k g^{(j,k)}(m-q+1, s) + \lambda_k g^{(j,k)}(m-q, s).$$

The method stated above was first applied to the two point connection problem for a differential system with an irregular singularity of rank 1 by K. Okubo [14] and then effectively used in a series of papers [15], [9], [11], [12]. In the case when the rank $q=1$, the corresponding coefficients $g^{(j,k)}(m, s)$ are the reciprocals of the Gamma functions and hence the analysis of $x^{(j,k)}(t, s)$ is due to the detailed investigation on the so-called generalized hypergeometric series by E. M. Wright [26] and others. In the paper [11] treating the case when the rank $q=2$, the global behaviours of the fundamental functions are cleared up with the help of the study of the modified Gamma function by N. G. de Bruijn [4].

Among the important papers cited before, H. L. Turrittin [23] treats a single n-th order differential equation with the extended form of Bessel's equation.

Although the rank is not necessarily one in that paper, the coefficients $G_j(m)$ of the corresponding convergent power series solutions are represented in terms of products of Gamma functions and their reciprocals, and therefore, as in the case of the rank 1, the analysis again reduces to that of the generalized hypergeometric series. We refer the reader to [2], [3] too.

Finally, we shall outline the steps to be taken in reaching our main theorem in this paper. We begin with the analysis of the associated fundamental functions and obtain beautiful results as to their global behaviours after a heavy computation. In order to proceed to the partition of the convergent power series solutions, the detailed investigation on the behaviours of the coefficients $G_j(m)$ and $h^k(s)$ for sufficiently large values of m and s are needed.

In § 3, by an ingenious method, we make clear the behaviours of $h^k(s)$ for sufficiently large values of s. In the last part of this section, we show how the characteristic constants λ_k, $\alpha_{q-1}^k, \dots, \alpha_1^k$ and μ_k are determined by the coefficients $a_{l,r}$ and find an invariant identity which plays an important role later.

§ 4 deals with the determination of the Stokes multipliers. There the coefficients $G_j(m)$ are considered as solutions of qn-th order difference equations and are expressed in terms of series in $h^k(s)$ and $g^{(j,k)}(m, s)$. Then such expressions will determine the Stokes multipliers.

In the last section we shall show the derivation of the asymptotic representation (1.8) by the method of decomposition.

§ 2. The global behaviour of the associated fundamental function.

We anew define the fundamental function $x(t, s)$ associated with the two point connection problem by the series

$$x(t, s) = \sum_{m=0}^{\infty} g(m+s)t^{m+\rho} \tag{2.1}$$

where the coefficient $g(m+s)$ satisfies the q-th order difference equation

$$\begin{aligned}(m+s+\rho-\mu)g(m+s) = {}& \alpha_1 g(m+s-1) + \cdots + \alpha_{q-1} g(m+s-q+1) \\ & + \lambda g(m+s-q)\end{aligned} \tag{2.2}$$

of just the same form as described in the statement (1.15), omitting all indices attached to the functions and the characteristic constants. Although in § 1 it was introduced as a particular solution of a differential equation, the fundamental function $x(t, s)$ defined as above satisfies a first order nonhomogeneous differential equation. In fact, we easily obtain a recurrence relation

$$\begin{aligned} x(t, s) &= g(s)t^{\rho} + \sum_{m=1}^{\infty} g(m+s)t^{m+\rho} \\ &= g(s)t^{\rho} + tx(t, s+1) \end{aligned}$$

and hence

$$x(t, s-r)=g(s-r)t^{\rho}+g(s-r+1)t^{\rho+1}+\cdots+g(s-1)t^{\rho+r-1}+t^{r}x(t, s). \tag{2.3}$$

We also have, using the difference equation (2.2),

$$\begin{aligned} t\frac{dx(t, s)}{dt} &= \sum_{m=0}^{\infty}\{(m+s+\rho-\mu)+(\mu-s)\}g\ (m+s)t^{m+\rho} \\ &= \sum_{m=0}^{\infty}\{\alpha_1 g(m+s-1)+\cdots+\alpha_{q-1}g(m+s-q+1)+\lambda g(m+s-q) \\ &\quad +(\mu-s)g(m+s)\}t^{m+\rho} \\ &=\alpha_1 x(t, s-1)+\cdots+\alpha_{q-1}x(t, s-q+1)+\lambda x(t, s-q) \\ &\quad +(\mu-s)x(t, s). \end{aligned} \tag{2.4}$$

Substituting the formula (2.3) into the right hand side of the relation (2.4), we then obtain the first order nonhomogeneous differential equation

$$\begin{aligned} t\frac{dx(t, s)}{dt} &= \{\lambda t^{q}+\alpha_{q-1}t^{q-1}+\cdots+\alpha_1 t+(\mu-s)\}x(t, s) \\ &\quad +[\lambda g(s-1)]t^{\rho+q-1} \\ &\quad +[\lambda g(s-2)+\alpha_{q-1}g(s-1)]t^{\rho+q-2} \\ &\quad + \\ &\quad \vdots \\ &\quad +[\lambda g(s-q+1)+\alpha_{q-1}g(s-q+2)+\cdots+\alpha_2 g(s-1)]t^{\rho+1} \\ &\quad +[\lambda g(s-q)+\alpha_{q-1}g(s-q+1)+\cdots+\alpha_2 g(s-2)+\alpha_1 g(s-1)]t^{\rho}. \end{aligned} \tag{2.5}$$

Therefore, the associated fundamental function may again be regarded as a particular solution of the differential equation (2.5) and its global behaviour will be analyzed through the integral representation of the solution.

As is easily seen, a general solution of the differential equation (2.5) is written in the form

$$\begin{aligned} x(t, s) &= C\exp\left(\frac{\lambda}{q}t^{q}+\frac{\alpha_{q-1}}{q-1}t^{q-1}+\cdots+\alpha_1 t\right)t^{\mu-s} \\ &\quad +\int_0^t \exp\left(\frac{\lambda}{q}(t^{q}-\eta^{q})+\frac{\alpha_{q-1}}{q-1}(t^{q-1}-\eta^{q-1})+\cdots+\alpha_1(t-\eta)\right) \\ &\quad \times\{[\lambda g(s-1)]\eta^{\rho+q-1}+[\lambda g(s-2)+\alpha_{q-1}g(s-1)]\eta^{\rho+q-1} \\ &\quad +\cdots+[\lambda g(s-q)+\alpha_{q-1}g(s-q+1)+\cdots+\alpha_1 g(s-1)]\eta^{\rho}\}\left(\frac{\eta}{t}\right)^{s-\mu}\frac{d\eta}{\eta}. \end{aligned}$$

And the desirable solution is obtained by putting the integral constant $C=0$ since otherwise it does not behave like t^{ρ} near the regular singular point at the origin. Consequently, putting $\eta=t\tau$, we have the integral representation of the associated fundamental function $x(t, s)$ as follows:

$$
\begin{aligned}
(2.6)\quad x(t, s)=&[\lambda g(s-1)]t^{\rho+q-1}z(\lambda^{1/q}t: s-\mu+\rho+q-1: \alpha_k\lambda^{-(k/q)})\\
&+[\lambda g(s-2)+\alpha_{q-1}g(s-1)]t^{\rho+q-2}z(\lambda^{1/q}t: s-\mu+\rho+q-2: \alpha_k\lambda^{-(k/q)})\\
&+\\
&\vdots\\
&+[\lambda g(s-q)+\alpha_{q-1}g(s-q+1)+\cdots+\alpha_2 g(s-2)+\alpha_1 g(s-1)]t^{\rho}\\
&\times z(\lambda^{1/q}t: s-\mu+\rho: \alpha_k\lambda^{-(k/q)})
\end{aligned}
$$

where $\lambda^{1/q}=|\lambda|^{1/q}\exp\left(\frac{i}{q}\arg\lambda\right)$, $\lambda^{-(1/q)}=|\lambda|^{-(1/q)}\exp\left(-\frac{i}{q}\arg\lambda\right)$ and we put

$$
\begin{aligned}
(2.7)\qquad z(t: \nu: \gamma_k)&\equiv z(t: \nu: \gamma_{q-1},\dots, \gamma_k,\dots, \gamma_1)\\
&\equiv\int_0^1\exp\left(\frac{1}{q}t^q(1-\tau^q)+\frac{\gamma_{q-1}}{q-1}t^{q-1}(1-\tau^{q-1})+\cdots+\gamma_1 t(1-\tau)\right)\tau^{\nu-1}d\tau\,.
\end{aligned}
$$

Hereafter the above abbreviation will be used throughout, i.e., γ_k is representative of a set of constants $\gamma_{q-1}, \gamma_{q-2}, \dots, \gamma_1$ appearing in the integrand. And, for instance,

$$z(t: \nu: \alpha_k\lambda^{-(k/q)})=z(t: \nu: \alpha_{q-1}\lambda^{-((q-1)/q)},\dots, \alpha_k\lambda^{-(k/q)},\dots, \alpha_1\lambda^{-(1/q)}).$$

Owing to the integral representation (2.6), the study of the function $x(t, s)$ is reduced to that of the integral $z(t: \nu: \gamma_k)$.

Now we shall investigate the global behaviour of $z(t: \nu: \gamma_k)$ in the entire complex plane. In (2.7) the path of integration is the positive real axis and $\mathrm{Re}\,\nu>0$ is, for the moment, assumed, though this will be relaxed later.

At first, we consider the case when t lies in the sector

$$(2.8)\qquad S_1;\ -\frac{\pi}{q}\leqq\arg t\leqq\frac{\pi}{q}\,.$$

Setting the integral variable $\eta=t\tau$ and for brevity

$$p(t)=\frac{t^q}{q}+\frac{\gamma_{q-1}}{q-1}t^{q-1}+\cdots+\gamma_1 t,$$

we have

$$z(t: \nu: \gamma_k)=\int_0^t\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}$$

$$=\int_0^{|t|}\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}$$

$$+\int_{|t|}^{t}\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}$$

where the original path of integration, the straight line from the origin to t, is changed into the path (P_1) which consists of

(i)$_1$ the positive real axis from the origin to $|t|$

and

(ii)$_1$ the circular arc $|\eta|=|t|$ from $|t|$ to t.

Moreover, we have

$$z(t:\nu:\gamma_k)=\exp(p(t))t^{-\nu}\int_0^{\infty}\exp(-p(\eta))\eta^{\nu-1}d\eta \tag{2.9}$$

$$-\int_{|t|}^{\infty}\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}$$

$$+\int_{|t|}^{t}\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}.$$

The second and the third integrals in the right hand side of (2.9) are bounded for sufficiently large values of t. In fact, putting $t=|t|e^{i\theta}(|\theta|\leqq\pi/q,\ |t|\geqq 1)$,

$$\left|\int_{|t|}^{\infty}\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}\right|$$

$$=\left|\int_0^{\infty}\exp(p(t)-p(|t|+\xi))\left(\frac{|t|+\xi}{t}\right)^{\nu}\frac{d\xi}{|t|+\xi}\right|$$

$$\leqq\frac{e^{|\theta\operatorname{Im}\nu|}}{|t|}|\exp(p(t)-p(|t|))|\int_0^{\infty}|\exp(p(|t|)-p(|t|+\xi))|\left(\frac{|t|+\xi}{|t|}\right)^{\operatorname{Re}\nu-1}d\xi$$

$$\leqq\frac{e^{|\theta\operatorname{Im}\nu|}}{|t|}|\exp(p(t)-p(|t|))|\int_0^{\infty}\exp\left(-\frac{\xi^q}{q}+R_{q-1}(\xi,|t|)\right)(1+\xi)^{\operatorname{Re}\nu-1}d\xi$$

$$\leqq M \qquad (M\text{: a constant})$$

where $R_{q-1}(\xi,|t|)$ is a polynomial of degree $q-1$ in ξ and $|t|$ respectively with the form of

$$R_{q-1}(\xi,|t|)=-|t|^{q-1}\xi+\cdots+(-1)|t|\xi^{q-1}.$$

We proceed to the estimate of the third integral. We put $\eta=|t|e^{i\phi}$ along the circular arc (ii)$_1$ and then have

$$
\begin{aligned}
&|\exp(p(t)-p(\eta))| \\
&\leqq \left|\exp\left\{\frac{|t|^q}{q}(\cos q\theta-\cos q\phi)\right.\right. \\
&\quad \left.\left.+\sum_{k=1}^{q-1}\frac{|t|^k}{k}(\operatorname{Re}\gamma_k(\cos k\theta-\cos k\phi)+\operatorname{Im}\gamma_k(\sin k\phi-\sin k\theta))\right\}\right| \\
&\leqq 1
\end{aligned}
$$

for sufficiently large values of t, since $\cos q\theta-\cos q\phi\leqq 0$ for $0\leqq|\phi|\leqq|\theta|\leqq\pi/q$. Thus, we obtain

$$
\begin{aligned}
&\left|\int_{|t|}^{t}\exp(p(t)-p(\eta))\left(\frac{\eta}{t}\right)^{\nu}\frac{d\eta}{\eta}\right| \leqq \int_0^{|\theta|}|e^{(\theta-\phi)\operatorname{Im}\nu}|\,d\phi \\
&\leqq M\int_0^{\pi/q}d\phi = M\frac{\pi}{q} \qquad (M\text{: a constant}).
\end{aligned}
$$

Hence, if we set

$$
\begin{aligned}
(2.10) \qquad \phi(\nu:\gamma_k) &\equiv \phi(\nu:\gamma_{q-1},\dots,\gamma_k,\dots,\gamma_1) \\
&= \int_0^{\infty}\exp\left(-\frac{\eta^q}{q}-\sum_{k=1}^{q-1}\frac{\gamma_k}{k}\eta^k\right)\eta^{\nu-1}d\eta
\end{aligned}
$$

where we make use of the same notation as in (2.7), then we obtain the behaviour

$$
(2.11) \qquad z(t:\nu:\gamma_k)=\phi(\nu:\gamma_k)\exp(p(t))t^{-\nu}+O(1)
$$

for sufficiently large values of t in the sector S_1.

Next we consider the case when t lies in the sector

$$
(2.12) \qquad S_j;\ -\frac{3\pi}{q}+\frac{2\pi}{q}j \leqq \arg t \leqq -\frac{\pi}{q}+\frac{2\pi}{q}j \qquad (j=2, 3,\dots, q).
$$

In this case, we change the original path of integration, the straight line from the origin to t, into the path (P_j) which consists of

(i)$_j$ the straight line from the origin to $|t|e^{(2\pi(j-1)/q)i}$

and

(ii)$_j$ the circular arc $|\eta|=|t|$ from $|t|e^{(2\pi(j-1)/q)i}$ to t.

Let $\omega=e^{(2\pi/q)i}$ and hence $\omega^q=1$. By the same procedure as stated above, we obtain

$$(2.13)\quad \begin{aligned} &z(t: \nu: \gamma_k)\\ &=\int_0^{|t|}\exp\,(p(t)-p(\eta\omega^{j-1}))\Big(\frac{\eta}{t}\Big)^{\nu}\omega^{\nu(j-1)}\frac{d\eta}{\eta}\\ &\quad+\int_{|t|\exp(i\omega^{j-1})}^{|t|\exp(i\theta)}\exp\,(p(t)-p(\eta))\Big(\frac{\eta}{t}\Big)^{\nu}\frac{d\eta}{\eta}\\ &=p(t)t^{-\nu}\omega^{\nu(j-1)}\int_0^{\infty}\exp\Big(-\frac{\eta^q}{q}-\sum_{k=1}^{q-1}\frac{\gamma_k}{k}\omega^{k(j-1)}\eta^k\Big)\eta^{\nu-1}d\eta+O(1)\\ &=p(t)t^{-\nu}\omega^{\nu(j-1)}\phi(\nu: \gamma_k\omega^{k(j-1)})+O(1)\end{aligned}$$

for sufficiently large values of t in the sector S_j. We shall summarize these results in the form of

THEOREM 2.1. *For sufficiently large values of t in the sector*

$$S_j;\ -\frac{3\pi}{q}+\frac{2\pi}{q}j\leqq \arg t\leqq -\frac{\pi}{q}+\frac{2\pi}{q}j \qquad (j=1, 2,\dots, q),$$

the function $z(t: \nu: \gamma_k)$ has the behaviour

$$(2.14)\quad z(t: \nu: \gamma_k)=\phi(\nu: \gamma_k\omega^{k(j-1)})\,\omega^{\nu(j-1)}\exp\Big(\frac{t^q}{q}+\frac{\gamma_{q-1}}{q-1}t^{q-1}+\cdots+\gamma_1 t\Big)t^{-\nu}+O(1).$$

Here we make some remarks on the function $\phi(\nu: \gamma_k)$ of ν which was defined by the definite integral (2.10). By means of partial integration, we have

$$(2.15)\quad \nu\phi(\nu: \gamma_k)=\phi(\nu+q: \gamma_k)+\gamma_{q-1}\phi(\nu+q-1: \gamma_k)+\cdots+\gamma_2\phi(\nu+2: \gamma_k)+\gamma_1\phi(\nu+1: \gamma_k)$$

which is a q-th order difference equation in ν. And then, from the above difference equation, we easily see that the function $\phi(\nu: \gamma_k)$ defined in the half-plane $\mathrm{Re}\,\nu>0$ can be analytically continued over the whole complex ν-plane except for simple poles at $\nu=0, -1, -2,\dots$. Therefore the condition $\mathrm{Re}\,\nu>0$ assumed so far is replaced by the condition $\nu\neq$ *non-positive integer* and Theorem 2.1 also holds under the assumption that $\nu\neq$ *non-positive integer*. Such a function $\phi(\nu: \gamma_k)$ defined by the integral of the form (2.10) or the difference equation of the type (2.15) is called the modified Gamma function the asymptotic behaviour of which has been investigated in detail by means of the saddle point method by N. G. de Bruijn (4). We shall show and use N. G. de Bruijn's result later on.

We moreover consider the integral

$$(2.16)\quad \hat{z}(t: \nu: \gamma_k)=\int_0^1\exp\Big(\frac{1}{q}t^q(1-\tau^q)+\sum_{k=1}^{q-1}\frac{\gamma_k}{k}t^k(1-\tau^k)\Big)F(\tau)\tau^{\nu-1}d\tau$$

where the function $F(\tau)$ is holomorphic and bounded at least in the closed unit disk $|\tau|\leqq 1$. Since the asymptotic behaviour of the integral $\hat{z}(t: \nu: \gamma_k)$ depends mainly on the first constant term in the Taylor expansion of $F(\tau)$ near $\tau=0$, we readily obtain the following results.

THEOREM 2.2. *For sufficiently large values of t in the sector S_j, we have*

$$(2.17)\qquad \hat{z}(t: \nu: \gamma_k)=O\Big(\exp\Big(\frac{t^q}{q}+\frac{\gamma_{q-1}}{q-1}t^{q-1}+\cdots+\gamma_1 t\Big)t^{-\nu}\Big)+O(1)$$

and, more precisely,

$$(2.18)\qquad \hat{z}(t: \nu: \gamma_k)=\exp\Big(\frac{t^q}{q}+\frac{\gamma_{q-1}}{q-1}t^{q-1}+\cdots+\gamma_1 t\Big)t^{-\nu}$$
$$\times\Big(\sum_{l=0}^{m-1}\frac{F^{(l)}(0)}{l!}\phi(\nu+l: \gamma_k\omega^{k(j-1)})\,\omega^{(\nu+l)(j-1)}\Big)$$
$$+O\Big(\exp\Big(\frac{t^q}{q}+\frac{\gamma_{q-1}}{q-1}t^{q-1}+\cdots+\gamma_1 t\Big)t^{-\nu-m}\Big)+O(1)\,.$$

Now, in order to return to the analysis of the function $x(t, s)$, we must still make some preparations. If we apply Theorem 2.1 to the function $z(\lambda^{1/q}t: s-\mu+\rho: \alpha_k\lambda^{-(k/q)})$ in the integral representation (2.6), we have

$$(2.19)\qquad z(\lambda^{1/q}t: s-\mu+\rho: \alpha_k\lambda^{-(k/q)})$$
$$=\phi(s-\mu+\rho: \alpha_k\lambda^{-(k/q)}\omega^{k(j-1)})(\lambda^{-(1/q)}\omega^{j-1})^{s-\mu+\rho}$$
$$\times\exp\Big(\frac{\lambda}{q}t^q+\frac{\alpha_{q-1}}{q-1}t^{q-1}+\cdots+\alpha_1 t\Big)t^{-s+\mu-\rho}+O(1)$$

for sufficiently large values of t in the sector

$$(2.20)\qquad S_j(\lambda);\ -\frac{3\pi}{q}+\frac{2\pi}{q}j\leqq\arg\lambda^{1/q}t\leqq-\frac{\pi}{q}+\frac{2\pi}{q}j\qquad (j=1, 2,\dots, q)\,.$$

If we put

$$(2.21)\qquad \Phi_j(s)=\phi(s-\mu+\rho: \alpha_k\lambda^{-(k/q)}\omega^{k(j-1)})(\lambda^{-(1/q)}\omega^{j-1})^{s-\mu+\rho}\qquad (j=1, 2,\dots, q),$$

then we know that $\Phi_j(s)$ $(j=1, 2,\dots, q)$ are particular solutions of the difference equation

$$(2.22)\quad (s-\mu+\rho)\Phi(s)=\lambda\Phi(s+q)+\alpha_{q-1}\Phi(s+q-1)+\cdots+\alpha_2\Phi(s+2)+\alpha_1\Phi(s+1)$$

from the fact that $\phi(\nu: \gamma_k)$ satisfies the difference equation (2.15). Furthermore, we have the important result.

THEOREM 2.3. *Under the assumption that* $\rho-\mu\neq$ *integer*, $\Phi_j(s)$ $(j=1, 2,\dots, q)$ *make a fundamental set of solutions of the q-th order difference equation* (2.22).

This theorem teaches us the fact that the linearly independent solutions $\Phi_j(s)$ of the difference equation (2.22) appear one after another by the change of the sectors $S_j(\lambda)$ as the coefficients of the asymptotic representation of the function $z(\lambda^{1/q}t\colon s-\mu+\rho\colon \alpha_k\lambda^{-(k/q)})$.

In order to prove the above theorem, we need the following two lemmata.

LEMMA 2.1. (*N. G. de Bruijn*) *For sufficiently large values of s in the sector* $|\arg(s-\mu+\rho)|<\pi-\delta$ *where* δ *is a small positive number, we have*

$$\Phi_j(s)\sim(\lambda^{-(1/q)}\omega^{j-1})^{s-\mu+\rho}\left(\frac{2\pi}{q(s-\mu+\rho)}\right)^{1/2} \times\exp\left(-p(e^{\zeta_j})+(s-\mu+\rho)\left(\zeta_j-\frac{2\pi(j-1)}{q}i\right)\right)\sum_{k=0}^{\infty}d_{jk}s^{-(k/q)} \quad (j=1, 2,\dots, q) \tag{2.23}$$

where $d_{jo}=1$ *and*

$$p(u)=\frac{u^q}{q}+\frac{\alpha_{q-1}\lambda^{-((q-1)/q)}}{q-1}u^{q-1}+\cdots+\alpha_1\lambda^{-(1/q)}u\,. \tag{2.24}$$

Here the saddle point ζ_j *is a root of the equation*

$$e^{\zeta}p'(e^{\zeta})=s-\mu+\rho \tag{2.25}$$

close to $\frac{1}{q}\log(s-\mu+\rho)+\frac{2\pi(j-1)}{q}i$ *where the* log *has its principal value, and can be expressed in terms of the power series*

$$\zeta_j=\frac{1}{q}\log(s-\mu+\rho)+\frac{2\pi(j-1)}{q}i+c_{j1}s^{-(1/q)}+c_{j2}s^{-(2/q)}+\cdots \quad (j=1, 2,\dots, q) \tag{2.26}$$

for sufficiently large s.

LEMMA 2.2. *Let* $\zeta_1, \zeta_2,\dots, \zeta_q$ be the roots of an algebraic equation

$$u^q+\gamma_{q-1}u^{q-1}+\cdots+\gamma_1u+\gamma_0=0\,.$$

Let us define the sum σ_m *by*

$$\sigma_m=\zeta_1^m+\zeta_2^m+\cdots+\zeta_q^m \qquad (\sigma_0=q)\,. \tag{2.27}$$

Then, we have

$$\sigma_m = H_m(\gamma_{q-1}, \gamma_{q-2},\dots, \gamma_1, \gamma_0) \tag{2.28}$$

where $H_m(\gamma_{q-1},\dots, \gamma_0)$ *is a homogeneous function of* $\gamma_{q-1},\dots, \gamma_1$ *and* γ_0 *of degree* m *with integral coefficients. And that, if* $0<m<q$, $\gamma_{q-m-1},\dots, \gamma_1$ *and* γ_0 *are not included in* $H_m(\gamma_{q-1},\dots, \gamma_0)$, *i.e.*,

$$H_m(\gamma_{q-1}, \gamma_{q-2},\dots, \gamma_1, \gamma_0) = H_m(\gamma_{q-1}, \gamma_{q-2},\dots, \gamma_{q-m}). \tag{2.29}$$

The relation (2.28) *is called Newton's formula.*

PROOF OF THEOREM 2.3. A necessary and sufficient condition for the linear independence of the solutions $\Phi_1(s)$, $\Phi_2(s),\dots,$ $\Phi_q(s)$ of the difference equation (2.22) is the non-vanishing of the Casorati determinant

$$\mathscr{C}_\Phi(s) = \begin{vmatrix} \Phi_1(s) & \Phi_2(s) & \dots\dots\Phi_q(s) \\ \Phi_1(s+1) & \Phi_2(s+1) & \dots\dots\Phi_q(s+1) \\ \vdots & & \\ \Phi_1(s+q-1) & \Phi_2(s+q-1) & \dots\dots\Phi_q(s+q-1) \end{vmatrix}.$$

As is easily verified, the Casorati determinant $\mathscr{C}_\Phi(s)$ satisfies the first order difference equation

$$\lambda\mathscr{C}_\Phi(s+1) = (-1)^{q-1}(s-\mu+\rho)\mathscr{C}_\Phi(s)$$

and hence it has the explicit form

$$\mathscr{C}_\Phi(s) = \left(\frac{e^{(q-1)\pi i}}{\lambda}\right)^s \frac{\Gamma(s-\mu+\rho)}{\Gamma(-\mu+\rho)}\mathscr{C}_\Phi(0) \tag{2.30}$$

in the sector $|\arg(s-\mu+\rho)|<\pi-\delta$. Now, by virtue of (2.30), if we could show that $\mathscr{C}_\Phi(0)\neq 0$, then we could conclude that $\mathscr{C}_\Phi(s)\neq 0$ for all s except for $s-\mu+\rho$ $=$*non-positive integer*, in particular, every integer s under the assumption that $\rho-\mu\neq$ *integer*. The value of $\mathscr{C}_\Phi(0)$ will be evaluated by examining the behaviour of $\mathscr{C}_\Phi(s)$ for sufficiently large s. Since, according to Lemma 2.1,

$$\begin{aligned} \Phi_j(s) &\sim (\lambda^{-(1/q)}\omega^{j-1})^{s-\mu+\rho}\left(\frac{2\pi}{qs}\right)^{1/2} \\ &\times \exp\left(\frac{s}{q}\log s - \frac{s}{q} + s\sum_{k=1}^{\infty} \hat{c}_{jk}s^{-(k/q)}\right)\left(\sum_{k=0}^{\infty} \hat{d}_{jk}s^{-(k/q)}\right) \\ &\qquad (j=1, 2,\dots, q) \end{aligned} \tag{2.31}$$

holds for sufficiently large values of s in the sector $|\arg(s-\mu+\rho)|<\pi-\delta$, we have

$$(2.32)\qquad \frac{\Phi_j(s+r)}{\Phi_j(s)} \sim (\lambda^{-(1/q)}\omega^{j-1})^r \exp\Big(\frac{r}{q}\log s\Big)\{1+O(s^{-(1/q)})\}$$
$$(j=1,\, 2,\dots,\, q)$$

in the same sector.

Therefore, we obtain

$$(2.33)\qquad \mathscr{C}_\Phi(s) \sim [\prod_{j=1}^{q}\Phi_j(s)]\, V_q\Big(\Big(\frac{s}{\lambda}\Big)^{1/q}, \Big(\frac{s}{\lambda}\Big)^{1/q}\omega,\dots, \Big(\frac{s}{\lambda}\Big)^{1/q}\omega^{q-1}\Big)$$
$$\times\{1+O(s^{-(1/q)})\}$$
$$\sim [\prod_{j=1}^{q}\Phi_j(s)]\Big(\frac{s}{\lambda}\Big)^{(q-1)/2} V_q(1,\, \omega,\dots,\, \omega^{q-1})\{1+O(s^{-(1/q)})\}$$

where $V_q(x_1,\, x_2,\dots,\, x_q)$ is Vandermonde's determinant, i.e.,

$$V_q(x_1,\, x_2,\, \dots,\, x_q) = \begin{vmatrix} 1 & 1 & \dots\dots 1 \\ x_1 & x_2 & \dots\dots x_q \\ x_1^2 & x_2^2 & \dots\dots x_q^2 \\ \vdots & \vdots & \vdots \\ x_1^{q-1} & x_2^{q-1} & \dots\dots x_q^{q-1} \end{vmatrix}$$

which does not vanish if $x_i \neq x_j$ $(i\neq j)$. Using the asymptotic behaviour (2.23) again, we calculate the asymptotic behaviour of the product

$$(2.34)\qquad [\prod_{j=1}^{q}\Phi_j(s)] \sim (\lambda^{-1}\omega^{q(q-1)/2})^{s-\mu+\rho}\Big(\frac{2\pi}{q(s-\mu+\rho)}\Big)^{q/2}$$
$$\times \exp\Big\{-\sum_{j=1}^{q} p(e^{\zeta_j}) + (s-\mu+\rho)\sum_{j=1}^{q}\Big(\zeta_j - \frac{2\pi(j-1)}{q}i\Big)\Big\}$$
$$\times\{1+O(s^{-(1/q)})\}\,.$$

Since $e^{\zeta_j}(j=1,\, 2,\dots,\, q)$ are roots of the algebraic equation

$$u^q + \alpha_{q-1}(\lambda^{-(1/q)})^{q-1}u^{q-1} + \cdots + \alpha_1\lambda^{-(1/q)}u - (s-\mu+\rho) = 0\,,$$

we have, from (2.28) and (2.29) in Lemma 2.2,

$$\sum_{j=1}^{q} e^{m\zeta_j} = H_m(\alpha_{q-1}\lambda^{-((q-1)/q)},\dots,\, \alpha_{q-m}\lambda^{-((q-m)/q)})\qquad (1\leqq m<q)$$

and

$$\sum_{j=1}^{q} e^{q\zeta_j} = q(s-\mu+\rho) - \sum_{m=1}^{q-1}\alpha_m\lambda^{-(m/q)}H_m(\alpha_{q-1}\lambda^{-((q-1)/q)},\dots,\, \alpha_{q-m}\lambda^{-((q-m)/q)}).$$

Hence, it follows that

$$(2.35)\quad \sum_{j=1}^{q} p(e^{\zeta_j}) = (s-\mu+\rho)$$

$$+\sum_{m=1}^{q-1}\left(\frac{q-m}{mq}\right)\alpha_m\lambda^{-(m/q)}H_m(\alpha_{q-1}\lambda^{-((q-1)/q)},\dots,\alpha_{q-m}\lambda^{-((q-m)/q)})$$

$$\equiv (s-\mu+\rho)+R(\lambda,\alpha_{q-1},\dots,\alpha_1)$$

where $R(\lambda, \alpha_{q-1},\dots,\alpha_1)$ is a constant independent of s. Moreover we remark that

$$\prod_{j=1}^{q} e^{\zeta_j} = (-1)^{q-1}(s-\mu+\rho),\quad \text{i.e.,}$$

$$\sum_{j=1}^{q}\zeta_j = \log(s-\mu+\rho)+(q-1)\pi i.$$

We thus obtain the explicit asymptotic behaviour of (2.34)

$$(2.36)\quad [\prod_{j=1}^{q}\Phi_j(s)] \sim (\lambda^{-1}\omega^{q(q-1)/2})^{s-\mu+\rho}\left(\frac{2\pi}{q(s-\mu+\rho)}\right)^{q/2}$$

$$\times\exp\{-(s-\mu+\rho)+(s-\mu+\rho)\log(s-\mu+\rho)-R(\lambda,\alpha_{q-1},\dots,\alpha_1)\}$$

$$\times\{1+O(s^{-(1/q)})\}.$$

As a consequence of (2.30), (2.33), (2.36) and the asymptotic behaviour of the Gamma function

$$\frac{1}{\Gamma(s)}\sim \exp\left\{s-\left(s-\frac{1}{2}\right)\log s\right\}\left\{1+O\left(\frac{1}{s}\right)\right\},$$

we have

$$\mathscr{C}_\Phi(0)\sim\left(\frac{\lambda}{e^{(q-1)\pi i}}\right)^s\frac{\Gamma(-\mu+\rho)}{\Gamma(s-\mu+\rho)}\left(\frac{e^{(q-1)\pi i}}{\lambda}\right)^{s-\mu+\rho}\left(\frac{2\pi}{q(s-\mu+\rho)}\right)^{q/2}$$

$$\times\left(\frac{s}{\lambda}\right)^{(q-1)/2}\exp\{-(s-\mu+\rho)+(s-\mu+\rho)\log(s-\mu+\rho)\}$$

$$\times\exp\{-R(\lambda,\alpha_{q-1},\dots,\alpha_1)\}V_q(1,\omega,\dots,\omega^{q-1})\{1+O(s^{-(1/q)})\}$$

$$\sim\lambda^{\mu-\rho-(q-1)/2}e^{(q-1)(\rho-\mu)\pi i}\left(\frac{2\pi}{q}\right)^{q/2}\Gamma(-\mu+\rho)$$

$$\times\exp\{-R(\lambda,\alpha_{q-1},\dots,\alpha_1)\}V_q(1,\omega,\dots,\omega^{q-1})\{1+O(s^{-(1/q)})\}$$

for sufficiently large values of s in the sector $|\arg(s-\mu+\rho)|<\pi-\rho$. Letting s tend to infinity and noting that $\mathscr{C}_\Phi(0)$ is independent of s,

$$(2.37)\qquad \mathscr{C}_\Phi(0)=\lambda^{\mu-\rho-(q-1)/2}e^{(q-1)(\rho-\mu)\pi i}\left(\frac{2\pi}{q}\right)^{q/2}\Gamma(-\mu+\rho)$$
$$\times\exp\{-R(\lambda,\alpha_{q-1},\dots,\alpha_1)\}V_q(1,\omega,\dots,\omega^{q-1})$$

which does not vanish under the assumption that $\rho-\mu\neq$ *integer*. This completes the proof of Theorem 2.3.

A refined observation shows that the difference equation (2.22) has the converse form of the fundamental difference equation (2.2) described at the outset of this section. From now on we shall prove the existence of certain remarkable relations between two such difference equations.

Consider two difference equations

$$(2.38)\qquad a_0(x)\Phi(x+q)+a_1(x)\Phi(x+q-1)+\cdots+a_q(x)\Phi(x)=0,$$

$$(2.39)\qquad a_q(x+q)\Psi(x+q)+a_{q-1}(x+q-1)\Psi(x+q-1)+\cdots+a_0(x)\Psi(x)=0$$

where $a_0(x)\neq 0$ and $a_q(x)\neq 0$ for all x. One of such difference equations may be called the converse type of the other. Let us denote a fundamental set of solutions of (2.38) and that of (2.39) by

$$(2.40)\qquad \Phi_1(x),\ \Phi_2(x),\dots,\ \Phi_q(x)$$

and

$$(2.41)\qquad \Psi_1(x),\ \Psi_2(x),\dots,\ \Psi_q(x)$$

respectively. The respective Casorati determinants are denoted by $\mathscr{C}_\Phi(x)$ and $\mathscr{C}_\Psi(x)$ and, for brevity, the following notation of the determinant is put to use:

$$\mathscr{C}_\Phi(x)\equiv\operatorname*{Det}_i\begin{pmatrix}\Phi_i(x)\\ \Phi_i(x+1)\\ \vdots\\ \Phi_i(x+q-1)\end{pmatrix}$$
$$=\begin{vmatrix}\Phi_1(x) & \Phi_2(x) & \dots\Phi_q(x)\\ \Phi_1(x+1) & \Phi_2(x+1) & \dots\Phi_q(x+1)\\ \vdots & & \\ \Phi_1(x+q-1) & \Phi_2(x+q-1) & \dots\Phi_q(x+q-1)\end{vmatrix}.$$

The cofactor of the j-th column and k-th row element $\Phi_j(x+k-1)$ in the Casorati determinant $\mathscr{C}_\Phi(x)$ will be represented by $\Delta_\Phi^{(j,k)}(x)$ and, in particular, $\Delta_\Phi^{(j,1)}(x)$ by $D_j(x)$. Similarly, we define $\Delta_\Psi^{(j,k)}(x)$ and $\mathscr{D}_j(x)\equiv\Delta_\Psi^{(j,1)}(x)$. We often use the following notation

$$\Delta_{\Phi}^{(j,k)}(x)=(-1)^{j+k}\underset{i\neq j}{\mathrm{Det}}\begin{pmatrix}\Phi_i(x)\\ \vdots\\ \Phi_i(x+k-2)\\ \Phi_i(x+k)\\ \vdots\\ \Phi_i(x+q-1)\end{pmatrix}$$

which, as a matter of course, means that the j-th column and the k-th row are excluded from the Casorati determinant $\mathscr{C}_{\Phi}(x)$.

We are now in a position to state a theorem as to the important and very effective relations between (2.38) and (2.39). As is verified later, this theorem contributes to the derivation of the beautiful results for the asymptotic behaviour of the associated fundamental function.

THEOREM 2.4. *We put*

$$E_j(x)=\frac{D_j(x)}{a_q(x)\mathscr{C}_{\Phi}(x)} \qquad (j=1, 2,\dots, q) \tag{2.42}$$

and

$$\mathscr{E}_j(x)=\frac{\mathscr{D}_j(x-q)}{a_0(x-q)\mathscr{C}_{\Psi}(x-q)} \qquad (j=1, 2,\dots, q)\ . \tag{2.43}$$

If the functions $\Phi_j(x)$ $(j=1, 2,\dots, q)$ form a fundamental set of solutions of (2.38), *then the functions $E_j(x)$ $(j=1, 2,\dots, q)$ form a fundamental set of solutions of* (2.39). *Conversely, the functions $\mathscr{E}_j(x)$ $(j=1, 2,\dots, q)$ make a fundamental set of solutions of* (2.38) *under the assumption that $\mathscr{C}_{\Psi}(x)\neq 0$. Moreover, for each j,*

$$\begin{aligned} &a_q(x+k)E_j(x+k)+a_{q-1}(x+k-1)E_j(x+k-1)+\cdots+a_{q-k}(x)E_j(x)\\ &=\frac{\Delta_{\Phi}^{(j,k+1)}(x)}{\mathscr{C}_{\Phi}(x)} \qquad (k=0, 1,\dots, q-1)\end{aligned} \tag{2.44}$$

and

$$\begin{aligned} &a_0(x-q+k)\mathscr{E}_j(x+k)+a_1(x-q+k)\mathscr{E}_j(x+k-1)+\cdots+a_k(x-q+k)\mathscr{E}_j(x)\\ &=\frac{\Delta_{\Psi}^{(j,k+1)}(x-q)}{\mathscr{C}_{\Psi}(x-q)} \qquad (k=0, 1,\dots, q-1)\end{aligned} \tag{2.45}$$

hold.

PROOF. We easily see that

$$\mathscr{C}_{\Phi}(x+1)=(-1)^{q}\frac{a_q(x)}{a_0(x)}\mathscr{C}_{\Phi}(x) \tag{2.46}$$

and

$$\mathscr{C}_{\Psi}(x+1)=(-1)^{q}\frac{a_0(x)}{a_q(x+q)}\mathscr{C}_{\Psi}(x)\,. \tag{2.47}$$

Now we shall prove the relations (2.44) by induction. The relation is valid for $k=0$ just by the definition of $E_j(x)$. Assume that the relation (2.44) is valid for k. Since

$$a_0(x)\Delta_{\Phi}^{(j,k+1)}(x+1)=(-1)^{j+k+1}\underset{i\neq j}{\mathrm{Det}}\begin{pmatrix}\Phi_i(x+1)\\ \vdots\\ \Phi_i(x+k)\\ \Phi_i(x+k+2)\\ \vdots\\ \Phi_i(x+q-1)\\ a_0(x)\Phi_i(x+q)\end{pmatrix}$$

$$=(-1)^{j+k+1}\underset{i\neq j}{\mathrm{Det}}\begin{pmatrix}\Phi_i(x+1)\\ \vdots\\ \Phi_i(x+k)\\ \Phi_i(x+k+2)\\ \vdots\\ \Phi_i(x+q-1)\\ -a_{q-k-1}(x)\Phi_i(x+k+1)-a_q(x)\Phi_i(x)\end{pmatrix}$$

$$=a_{q-k-1}(x)(-1)^{q-1}D_j(x)+a_q(x)(-1)^q\Delta_{\Phi}^{(j,k+2)}(x),$$

we have, multiplying the both sides of the above relation by the reciprocal of (2.46),

$$\frac{\Delta_{\Phi}^{(j,k+1)}(x+1)}{\mathscr{C}_{\Phi}(x+1)}=-a_{q-k-1}(x)E_j(x)+\frac{\Delta_{\Phi}^{(j,k+2)}(x)}{\mathscr{C}_{\Phi}(x)}$$

which means that

$$\begin{aligned}&a_q(x+k+1)E_j(x+k+1)+a_{q-1}(x+k)E_j(x+k)\\&\quad+\cdots+a_{q-k}(x+1)E_j(x+1)+a_{q-k-1}(x)E_j(x)\\&=\frac{\Delta_{\Phi}^{(j,k+1)}(x+1)}{\mathscr{C}_{\Phi}(x+1)}+a_{q-k-1}(x)E_j(x)=\frac{\Delta_{\Phi}^{(j,k+2)}(x)}{\mathscr{C}_{\Phi}(x)}\,.\end{aligned}$$

Hence the relation (2.44) is proved. In particular, from the relation (2.44) for $k=q-1$ and $\Delta_\Phi^{(j,q)}(x+1)=(-1)^{q-1}D_j(x)$, we obtain

$$a_q(x+q)E_j(x+q)+a_{q-1}(x+q-1)E_j(x+q-1)+\cdots+a_1(x+1)E_j(x+1)$$
$$=\frac{\Delta_\Phi^{(j,q)}(x+1)}{\mathscr{C}_\Phi(x+1)}=\frac{a_0(x)(-1)^{q-1}D_j(x)}{(-1)^q a_q(x)\mathscr{C}_\Phi(x)}$$
$$=-a_0(x)E_j(x).$$

We have thus proved that $E_j(x)$ $(j=1, 2,\dots, q)$ are particular solutions of the difference equation (2.39). In order to prove that $E_j(x)$ $(j=1, 2,\dots, q)$ form a fundamental set of solutions, we must consider the Casorati determinant $\mathscr{C}_E(x)$ constructed by them. Since

$$\sum_{j=1}^{q} E_j(x)\Phi_j(x)=\frac{1}{a_q(x)} \tag{2.48}$$

and

$$\sum_{j=1}^{q} E_j(x)\Phi_j(x+k)=0 \qquad (1\leqq k\leqq q-1), \tag{2.49}$$

we have

$$\begin{pmatrix} \Phi_1(x) & \Phi_2(x) & \dots\Phi_q(x) \\ \Phi_1(x+1) & \Phi_2(x+1) & \dots\Phi_q(x+1) \\ \vdots & & \\ \Phi_1(x+q-1) & \Phi_2(x+q-1) & \dots\Phi_q(x+q-1) \end{pmatrix} \begin{pmatrix} E_1(x) & E_1(x+1)\dots E_1(x+q-1) \\ E_2(x) & E_2(x+1)\dots E_2(x+q-1) \\ \vdots & \\ E_q(x) & E_q(x+1)\dots E_q(x+q-1) \end{pmatrix}$$
$$=\begin{pmatrix} 1/a_q(x) & & 0 \\ & 1/a_q(x+1) & \\ & & \ddots \\ 0 & & 1/a_q(x+q-1) \end{pmatrix}$$

whence we obtain

$$\mathscr{C}_E(x)=\left[\prod_{k=1}^{q}\frac{1}{a_q(x+k-1)}\right]\frac{1}{\mathscr{C}_\Phi(x)}\neq 0.$$

Similarly, we can prove the relation (2.45) and the fact that $\mathscr{E}_j(x)$ $(j=1, 2,\dots, q)$ form a fundamental set of solutions of (2.38). Thus, we reach the required results.

Here we shall apply Theorem 2.4 to the difference equations

(2.22) $\lambda\Phi(s+q)+\alpha_{q-1}\Phi(s+q-1)+\cdots+\alpha_1\Phi(s+1)-(s-\mu+\rho)\Phi(s)=0$

and

(2.2) $-(s+q-\mu+\rho)g(s+q)+\alpha_1 g(s+q-1)+\cdots+\alpha_{q-1}g(s+1)+\lambda g(s)=0.$

Since, according to Theorem 2.3, the functions $\Phi_j(s)$ $(j=1, 2,\dots, q)$ defined as in (2.21) form a fundamental set of solutions of (2.22), we immediately obtain a fundamental set of solutions of (2.2)

$$E_j(s)=\frac{D_j(s)}{-(s-\mu+\rho)\mathscr{C}_\Phi(s)} \tag{2.50}$$

$$=\left(\frac{\lambda}{e^{(q-1)\pi i}}\right)^{s-\mu+\rho}\frac{D_j(s)}{\Gamma(s-\mu+\rho+1)}\Omega_j(\lambda, \alpha_{q-1},\dots, \alpha_1)$$

$$(j=1, 2,\dots, q)$$

where

$$\Omega_j(\lambda, \alpha_{q-1},\dots, \alpha_1)=\frac{-\lambda^{(q-1)/2}\exp(R(\lambda, \alpha_{q-1},\dots, \alpha_1))}{V_q(1, \omega,\dots, \omega^{q-1})}\left(\frac{q}{2\pi}\right)^{q/2}. \tag{2.51}$$

The last statement was derived from (2.30) and (2.37)

In the discussion thus far we have made use of only the difference equation (2.2) and no properties of its solutions. And so the definition of the functions $\Phi_j(s)$ $(j=1, 2,\dots, q)$ was independent of the choice of a fundamental set of solutions of (2.2).

We therefore put

$$g_l(s)=-E_l(s) \qquad (l=1, 2,\dots, q) \tag{2.52}$$

and denote the associated fundamental function defined in (2.1) by attaching the corresponding suffix, i.e.,

$$x_l(t, s)=\sum_{m=0}^{\infty} g_l(m+s)t^{m+\rho}. \tag{2.53}$$

Then we have the main theorem of this section concerning the asymptotic behaviour of $x_l(t, s)$ in the entire complex plane.

THEOREM 2.5. *Each function* $x_l(t, s)$ $(l=1, 2,\dots, q)$ *has the asymptotic behaviour of the exponential type only in one sector* $S_l(\lambda)$;

$$x_l(t, s)=\exp\left(\frac{\lambda}{q}t^q+\frac{\alpha_{q-1}}{q-1}t^{q-1}+\cdots+\alpha_1 t\right)t^{-s+\mu}+O(t^{\rho+q-1}) \tag{2.54}$$

as $t\to\infty$ *in* $S_l(\lambda)$

and for $j \neq l$

$$x_l(t, s) = O(t^{\rho+q-1}) \quad as \quad t \to \infty \quad in \quad S_j(\lambda). \tag{2.55}$$

PROOF. From the integral representation (2.6) of $x_l(t, s)$ and the asymptotic behaviour (2.19), we easily obtain

$$\begin{aligned} x_l(t, s) = \{&[\lambda g_l(s-1)]\Phi_j(s+q-1) \\ &+[\lambda g_l(s-2)+\alpha_{q-1}g_l(s-1)]\Phi_j(s+q-2) \\ &+ \\ &\vdots \\ &+[\lambda g_l(s-q+1)+\alpha_{q-1}g_l(s-q+2)+\cdots+\alpha_2 g_l(s-1)]\Phi_j(s+1) \\ &+[\lambda g_l(s-q)+\alpha_{q-1}g_l(s-q+1)+\cdots+\alpha_2 g_l(s-2)+\alpha_1 g_l(s-1)]\Phi_j(s)\} \\ &\times \exp\left(\frac{\lambda}{q}t^q+\frac{\alpha_{q-1}}{q-1}t^{q-1}+\cdots+\alpha_1 t\right)t^{-s+\mu}+O(t^{\rho+q-1}) \end{aligned} \tag{2.56}$$

for sufficiently large values of t in the sector $S_j(\lambda)$. Considering the relation (2.44) and the definition (2.52), we have

$$\begin{aligned} &\lambda g_l(s-q+k-1)+\alpha_{q-1}g_l(s-q+k)+\cdots+\alpha_k g_l(s-1) \\ &=-\lambda E_l(s-q+k-1)-\alpha_{q-1}E_l(s-q+k)-\cdots-\alpha_k E_l(s-1) \\ &=\alpha_{k-1}E_l(s)+\cdots+\alpha_1 E_l(s+k-2)-(s+k-1-\mu+\rho)E_l(s+k-1) \\ &=\frac{\Delta_\Phi^{(l,k)}(s)}{\mathscr{C}_\Phi(s)} \end{aligned}$$

and consequently

$$\begin{aligned} &[\lambda g_l(s-1)]\Phi_j(s+q-1) \\ &+[\lambda g_l(s-2)+\alpha_{q-1}g_l(s-1)]\Phi_j(s+q-2) \\ &+ \\ &\vdots \\ &+[\lambda g_l(s-q+1)+\alpha_{q-1}g_l(s-q+2)+\cdots+\alpha_2 g_l(s-1)]\Phi_j(s+1) \\ &+[\lambda g_l(s-q)+\alpha_{q-1}g_l(s-q+1)+\cdots+\alpha_2 g_l(s-2)+\alpha_1 g_l(s-1)]\Phi_j(s) \\ &=\frac{\Delta_\Phi^{(l,q)}(s)}{\mathscr{C}_\Phi(s)}\Phi_j(s+q-1)+\frac{\Delta_\Phi^{(l,q-1)}(s)}{\mathscr{C}_\Phi(s)}\Phi_j(s+q-2) \\ &+\cdots+\frac{\Delta_\Phi^{(l,2)}(s)}{\mathscr{C}_\Phi(s)}\Phi_j(s+1)+\frac{\Delta_\Phi^{(l,1)}(s)}{\mathscr{C}_\Phi(s)}\Phi_j(s) \end{aligned}$$

$$=\begin{cases} 1 & \text{for } j=l \\ 0 & \text{for } j\neq l. \end{cases}$$

We have thus completed the proof of Theorem 2.5.

§ 3. The anaylsis of the coefficients of the formal solutions.

This section deals with the behaviour of the coefficients $h^k(s)$ of the formal power series solutions (1.5) when s is sufficiently large. The coefficient $h(s)$, the index k being omitted again, satisfies a certain difference equation which will, in general, be derived by insertion of the power series of the form (1.5) into the differential equation (1.1). Since such direct substitution is very complicated, we overcome the difficulty by a method of detour used in the previous paper [11].

Let us define $X_p(t)$ by

$$(3.1)\qquad X_p(t)=t^{-(q-1)p}\frac{d^pX(t)}{dt^p}$$

$$=\exp\left(\frac{\lambda}{q}t^q+\frac{\alpha_{q-1}}{q-1}t^{q-1}+\cdots+\alpha_1t\right)t^\mu\sum_{s=0}^{\infty}h_p(s)t^{-s}.$$

Substituting the expression in the right hand side of (3.1) into an obvious relation

$$(3.2)\qquad X_p(t)=t^{-(q-1)}\frac{d}{dt}X_{p-1}+(q-1)(p-1)t^{-q}X_{p-1}(t)$$

and then identifying the coefficients of like powers of t in both sides, we have the *first* difference equation

$$(3.3)\qquad h_p(s)=\lambda h_{p-1}(s)+\alpha_{q-1}h_{p-1}(s-1)+\cdots+\alpha_1h_{p-1}(s-q+1)$$
$$+(\mu-s+(q-1)p+1)h_{p-1}(s-q)\qquad (p=1, 2,\ldots, n).$$

Similarly, substituting $X_p(t)$ into the differential equation (1.1), we obtain the *second* difference equation

$$(3.4)\qquad h_n(s)=\sum_{l=1}^{n}\sum_{r=0}^{ql}a_{l,r}h_{n-l}(s+r-ql).$$

There and hereafter the assumption $h(s)\equiv h_0(s)=0$, a posteriori, $h_p(s)=0$ for $s<0$ is made. Then, the difference equation satisfied by the coefficient $h(s)\equiv h_0(s)$ will be obtained by substituting the *first* difference equation into the *second* one. In fact, on account of the form of the *first* difference equation (3.3), we easily see that $h_p(s)$ can be written in the form

$$h_p(s)=\sum_{\nu=0}^{qp} M(p:\nu:s)h(s-\nu) \qquad (p=1, 2,\dots, n). \tag{3.5}$$

If we substitute the expressions of $h_p(s)$ and $h_{p-1}(s)$ into (3.3) and identify the coefficients of $h(s-\nu)$ in both sides, we obtain the following three types of difference equations for the coefficient $M(p: \nu: s)$:

Case (I) when $0\leqq \nu\leqq q-1$

$$\begin{aligned} M(p:\nu:s)=&\lambda M(p-1:\nu:s)+\alpha_{q-1}M(p-1:\nu-1:s-1)\\ &+\cdots+\alpha_{q-\nu}M(p-1:0:s-\nu), \end{aligned} \tag{3.6}$$

Case (II) when $q\leqq \nu\leqq q(p-1)$

$$\begin{aligned} M(p:\nu:s)=&\lambda M(p-1:\nu:s)+\alpha_{q-1}M(p-1:\nu-1:s-1)\\ &+\cdots+\alpha_1 M(p-1:\nu-q+1:s-q+1)\\ &+(\mu-s+(q-1)p+1)M(p-1:\nu-q:s-q), \end{aligned} \tag{3.7}$$

Case (III) when $q(p-1)+1\leqq \nu\leqq qp$

$$\begin{aligned} M(p:\nu:s)=&\alpha_{qp-\nu}M(p-1:q(p-1):s-\nu+q(p-1))\\ &+\cdots+\alpha_1 M(p-1:\nu-q+1:s-q+1)\\ &+(\mu-s+(q-1)p+1)M(p-1:\nu-q:s-q). \end{aligned} \tag{3.8}$$

Then, solving the above difference equations for $M(p: \nu: s)$, we obtain the difference equation to determine the coefficient $h(s)$, though the derivation of its explicit form is not an easy work. For the purpose of our study on the behaviour of $h(s)$, we need not seek the explicit form of $M(p: \nu: s)$ and only attempt to obtain the asymptotic behaviour of $M(p: \nu: s)$ for sufficiently large values of s.

In Case (I), $M(p: \nu: s)$ $(0\leqq \nu\leqq q-1)$ will be given successively by the formula of summation

$$M(p:\nu:s)=\lambda^p+\lambda^p \mathop{\mathfrak{S}}_{j=1}^{p} \frac{\sum_{i=1}^{\nu}\alpha_{q-i}M(j-1:\nu-1:s-i)}{\lambda^j} \tag{3.9}$$

where the symbol $\mathfrak{S}$ means the summation in j. It is easily seen that the first $M(p: \nu: s)$ $(0\leqq \nu\leqq q-1)$ are expressed in terms of the characteristic constants not depending on the variable s.

In Case (II), using the values of $M(p: \nu: s)$ derived in Case (I), we have

$$(3.10)\qquad M(p: v: s)=\lambda^{p}+\lambda^{p}\mathop{\mathfrak{S}}_{j=1}^{p}\frac{\sum_{i=1}^{q-1}\alpha_{q-i}M(j-1: v-i: s-i)}{\lambda^{j}}$$

$$+\lambda^{p}\mathop{\mathfrak{S}}_{j=1}^{p}\frac{(\mu-s+(q-1)j+1)M(j-1: v-q: s-q)}{\lambda^{j}}.$$

In the last Case (III), the formula (3.8) give the remaining value of $M(p: v: s)$ $(q(p-1)+1\leqq v\leqq qp)$. By means of the above formulas of summation, we first obtain the following

THEOREM 3.1. *Let s tend to the positive infinity. Then*

$$(3.11)\qquad \lim_{s\to\infty}\frac{1}{s^{k}}M(p: kq: s)=(-1)^{k}\lambda^{p-k}\binom{p}{k}\qquad(0\leqq k\leqq p)$$

and

$$(3.12)\qquad \lim_{s\to\infty}\frac{1}{s^{k}}M(p: kq+i: s)=c_{p,kq+i}\qquad(0\leqq k\leqq p-1,\ 1\leqq i\leqq q-1)$$

where $c_{p,kq+i}$ are the constant numbers. The round brackets $\binom{}{}$ mean that

$$\binom{p}{q}=\begin{cases}\dfrac{p!}{q!(p-q)!} & \textit{for } p\geqq q\\ 0 & \textit{for } p<q\,.\end{cases}$$

PROOF. The proof will be done by induction with respect to p and k.

Since

$$M(1: 0: s)=\lambda,\ M(1: 1: s-1)=\alpha_{q-1},\dots,\ M(1: q-1: s-q+1)=\alpha_{1},$$

$$M(1: q: s-q)=(\mu-s+q),$$

the relations (3.11) and (3.12) are valid for $p=1$ and $k=0$ and moreover, the relation (3.11) holds for $p=1$ and $k=1$. Now we shall prove that the relations (3.11) and (3.12) hold for p under the assumption of the validity of them for $1, 2,\dots, p-1$. From the fact that

$$M(p: 0: s)=\lambda^{p}\qquad(p\geqq 1)$$

and that $M(p: v: s)$ $(1\leqq v\leqq q-1)$ are constants independent of s, the theorem is obvious for $k=0$. Moreover we have

$$\lim_{s\to\infty}\frac{1}{s}M(p: q: s)=\lim_{s\to\infty}\frac{1}{s}\left\{\lambda^{p}+\lambda^{p}\mathop{\mathfrak{S}}_{j=1}^{p}\frac{\sum_{i=1}^{q-1}\alpha_{q-i}M(j-1: q-i: s-i)}{\lambda^{j}}\right\}$$

$$+\lim_{s\to\infty}\lambda^p \mathop{\mathfrak{S}}_{j=1}^{p} \frac{\mu-s+(q-1)j+1}{s}\frac{M(j-1:0:s-q)}{\lambda^j}$$

$$=(-1)\lambda^{p-1}\binom{p}{1}.$$

Next, under the assumption that the relations (3.11) and (3.12) are valid for k, we prove that they hold for $k+1$. For $1\leqq k+1\leqq p-2$ and $1\leqq i\leqq q$, we use the formula of summation (3.10) to obtain

$$\lim_{s\to\infty}\frac{1}{s^{k+1}}M(p:(k+1)q+l:s)$$

$$=\lambda^p \mathop{\mathfrak{S}}_{j=1}^{p} \frac{\sum_{i=1}^{q-1}\alpha_{q-i}}{\lambda^j}\left(\lim_{s\to\infty}\frac{M(j-1:(k+1)q+l-i:s-i)}{s^{k+1}}\right)$$

$$+\lambda^p \mathop{\mathfrak{S}}_{j=1}^{p} \frac{1}{\lambda^j}\frac{\mu-s+(q-1)j+1}{s}\lim_{s\to\infty}\left(\frac{M(j-1:kq+l:s-q)}{s^k}\right)$$

$$=\text{const.}$$

for $1\leqq l\leqq q-1$. In particular, for $l=q$, we have

$$\lim_{s\to\infty}\frac{1}{s^{k+2}}M(p:(k+1)q+q:s)$$

$$=\lambda^p \mathop{\mathfrak{S}}_{j=1}^{p} \frac{1}{\lambda^j}\frac{\mu-s+(q-1)j+1}{s}\lim_{s\to\infty}\left(\frac{M(j-1:(k+1)q:s-q)}{s^{k+1}}\right)$$

$$=\lambda^p \mathop{\mathfrak{S}}_{j=1}^{p} \frac{(-1)}{\lambda^j}(-1)^{k+1}\lambda^{j-1-k-1}\binom{j-1}{k+1}$$

$$=(-1)^{k+2}\lambda^{p-k-2}\mathop{\mathfrak{S}}_{j=1}^{p}\binom{j-1}{k+1}=(-1)^{k+2}\lambda^{p-k-2}\binom{p}{k+2}.$$

For $k+1=p-1$ and $1\leqq i\leqq q$, we must use the difference equation (3.8) in order to obtain

$$\lim_{s\to\infty}\frac{1}{s^{k+1}}M(p:(k+1)q+i:s)$$

$$
\begin{aligned}
&=\alpha_{q-i}\lim_{s\to\infty}\frac{1}{s^{k+1}}M(p-1: q(p-1): s-i)\\
&+\\
&\vdots\\
&+\alpha_1\lim_{s\to\infty}\frac{1}{s^{k+1}}M(p-1: kq+i+1: s-q+1)\\
&+\lim_{s\to\infty}\left\{\frac{\mu-s+(q-1)p+1}{s}\frac{M(p-1: kq+i: s-q)}{s^k}\right\}\\
&=\text{const.}
\end{aligned}
$$

and moreover

$$
\begin{aligned}
\lim_{s\to\infty}\frac{M(p: pq: s)}{s^p}&=\lim_{s\to\infty}\frac{\mu-s+(q-1)p+1}{s}\frac{M(p-1: q(p-1): s-q)}{s^{p-1}}\\
&=(-1)^p.
\end{aligned}
$$

Thus the proof of Theorem 3.1 is completed.

Now we shall consider the difference equation for $h(s)$. If we regard that

$$M(p: \nu: s)\equiv 0 \quad \text{for} \quad \nu>qp \quad \text{and} \quad \nu<0$$

and

$$a_{l,r}\equiv 0 \quad \text{for} \quad r>ql \quad \text{and} \quad r<0,$$

then the difference equation will be written in the form

$$\sum_{\nu=0}^{nq}\{M(n: \nu: s)-\sum_{r=0}^{nq}\sum_{l=1}^{n}a_{l,ql-r}M(n-l: \nu-r: s-r)\}h(s-\nu)=0. \tag{3.13}$$

But the coefficients corresponding to $h(s-\nu)$ ($\nu=0, 1,\dots, q-1$) are identically zero, because such q relations determine the values of the characteristic constants.

$$
\begin{aligned}
J(\lambda)&\equiv M(n: 0: s)-\sum_{l=1}^{n}a_{l,ql}M(n-l: 0: s)\\
&=\lambda^n-\sum_{l=1}^{n}a_{l,ql}\lambda^{n-l}=0
\end{aligned}
$$

is the characteristic equation determining the characteristic constants λ_k. By the relations

$$M(n: \nu: s)-\sum_{r=0}^{\nu}\sum_{l=1}^{n}a_{l,ql-r}M(n-l: \nu-r: s-r)=0 \qquad (1\leqq\nu\leqq q-1), \tag{3.14}$$

the characteristic constants $\alpha^k_{p-\nu}(1 \leqq \nu \leqq q-1)$ are determined. Moreover, the relation

$$(3.15) \qquad M(n\colon q\colon q) - \sum_{r=0}^{q} \sum_{l=1}^{n} a_{l,ql-r} M(n-l\colon q-r\colon q-r) = 0$$

gives the characteristic constant μ_k. Hence, the difference equation for $h(s)$ is of order $q(n-1)$ and its coefficient of the highest order can be evaluated explicitly. In fact, taking into consideration that $M(p\colon \nu\colon s)$ $(0 \leqq \nu \leqq q-1)$ are constants and subtracting (3.15) from the coefficient of $h(s-q)$, we have

$$(3.16) \qquad M(n\colon q\colon s) - \sum_{r=0}^{q} \sum_{l=1}^{n} a_{l,ql-r} M(n-l\colon q-r\colon s-r)$$

$$= M(n\colon q\colon s) - M(n\colon q\colon q) - \sum_{l=1}^{n} a_{l,ql}(M(n-l\colon q\colon s) - M(n-l\colon q\colon q))$$

$$= \lambda^n \mathop{\mathfrak{S}}_{j=1}^{n} \frac{(q-s)\lambda^{j-1}}{\lambda^j} - \sum_{l=1}^{n} a_{l,ql} \lambda^{n-l} \mathop{\mathfrak{S}}_{j=1}^{n-l} \frac{(q-s)\lambda^{j-1}}{\lambda^j}$$

$$= (q-s)\left\{\lambda^{n-1}\binom{n}{1} - \sum_{l=1}^{n} a_{l,ql} \lambda^{n-l-1}\binom{n-l}{1}\right\}$$

$$= (q-s) J'(\lambda)\,.$$

As a result of (3.16), putting $h^k(0)=1$, we can calculate the coefficients $h^k(s)$ of the formal solutions (1.5) successively under the assumption that $J'(\lambda_k) \neq 0$, i.e., $\lambda_j \neq \lambda_k$ $(j \neq k)$. As regards the other coefficients of the difference equation (3.13) we know the following important properties.

THEOREM 3.2. *For sufficiently large positive values of s,*

$$(3.17) \qquad M(n\colon \nu\colon s) - \sum_{r=0}^{\nu} \sum_{l=1}^{n} a_{l,ql-r} M(n-l\colon \nu-r\colon s-r)$$

$$= (-1)^{[\nu/q]} J^{([\nu/q])}(\lambda) s^{[\nu/q]}\left\{c_\nu + O\left(\frac{1}{s}\right)\right\} \qquad (q+1 \leqq \nu \leqq nq)$$

where we make use of the Gauss symbol [ ], *i.e.*, $[\nu/q]$ *means the integer such that*

$$\frac{\nu}{q} \leqq \left[\frac{\nu}{q}\right] < \frac{\nu}{q} + 1$$

Among the constants c_ν $(q+1 \leqq \nu \leqq nq)$ *in the statement* (3.17), $c_{kq}=1$ $(k=2, 3, .., n)$ *especially.*

PROOF. From Theorem 3.1, we easily see that

$$\lim_{s\to\infty}\frac{1}{s^{[\nu/q]}}\{M(n:\nu:s)-\sum_{l=1}^{n}a_{l,ql}M(n-l:\nu:s-r)$$

$$-\sum_{r=1}^{\nu}\sum_{l=1}^{n}a_{l,ql-r}M(n-l:\nu-r:s-r)\}$$

$$=\tilde{c}_\nu(=\text{const}).$$

In particular, for $\nu=kq$

$$\tilde{c}_\nu=(-1)^k\lambda^{n-k}\binom{n}{k}-\sum_{l=1}^{n}a_{l,ql}(-1)^k\lambda^{n-l-k}\binom{n-l}{k}$$

$$=(-1)^kJ^{(k)}(\lambda).$$

Using the results just derived above, we can rewrite the difference equation (3.13) in the form

$$(3.18)\quad J'(\lambda)sh(s)+\sum_{\nu=1}^{q-1}s\{\tilde{c}_\nu+\tilde{M}_\nu(s)\}h(s-\nu)$$

$$+(-1)J''(\lambda)s^2\{1+\tilde{M}_q(s)\}h(s-q)+\sum_{\nu=q+1}^{2q-1}s^2\{\tilde{c}_\nu+\tilde{M}_\nu(s)\}h(s-\nu)$$

$$+$$

$$\vdots$$

$$+(-1)^{n-2}J^{(n-1)}(\lambda)s^{n-1}\{1+\tilde{M}_{(n-2)q}(s)\}h(s-(n-2)q)$$

$$+\sum_{\nu=(n-2)q+1}^{(n-1)q-1}s^{n-1}\{\tilde{c}_\nu+\tilde{M}_\nu(s)\}h(s-\nu)$$

$$+(-1)^{n-1}J^{(n)}(\lambda)s^n\{1+\tilde{M}_{(n-1)q}(s)\}h(s-(n-1)q)=0$$

where $\tilde{c}_\nu$ $(\nu=1, 2,\dots, (n-1)q-1)$ are constant numbers and

$$(3.19)\qquad \tilde{M}_\nu(s)=O\left(\frac{1}{s}\right)\qquad (\nu=1, 2,\dots, (n-1)q)$$

for sufficiently large positive values of s.

For our object of deriving the behaviour of $h(s)$, there fortunately exists a result obtained by O. Perron (18) (19) (20). We here describe O. Perron's theorem for the asymptotic behaviour of the solution of the difference equation in such a fashion that we can easily apply it to the above difference equation (3.18).

THEOREM (Perron-Poincaré). *Consider a difference equation*

$$(3.20)\qquad \Phi(s+q)+a_1(s)\Phi(s+q-1)+\cdots+a_q(s)\Phi(s)=0$$

where the coefficients $a_j(s)$ *have the following properties:*

$$\lim_{s\to\infty}\frac{a_j(s)}{s^{k_j}}=\hat{a}_j\ (=\text{constant})\qquad (j=1,2,\dots,q)\,. \tag{3.21}$$

We construct an upward convex polygon, known as the Newton-Puiseux polygon, such that the $q+1$ *points with the coordinates*

$$(0,0),\ (1,k_1),\dots,(q,k_q) \tag{3.22}$$

either lie upon the line or below the line.

If the Newton-Puiseux polygon consists of one straight line, then we have

$$\overline{\lim_{s\to\infty}}\left(\frac{|\Phi(s)|}{|\Gamma(s+1)^{\tau}|}\right)^{1/s}=|\gamma_j| \tag{3.23}$$

where τ *is the directional coefficient of the straight line. The constant* γ_j *is one of the roots of the equation*

$$t^q+\hat{a}_{i_1}t^{q-i_1}+\hat{a}_{i_2}t^{q-i_2}+\cdots+\hat{a}_q=0 \tag{3.24}$$

where $i_1,\ i_2,\dots,q$ *are such numbers that the coordinates* $(i_1,k_{i_1}),\ (i_2,k_{i_2}),\dots,(q,k_q)$ *lie on the straight line.*

In this case when the Newton-Puiseux polygon is one straight line, we obtain, with the aid of the transformation

$$\Phi(s)=\Gamma(s+1)^{\tau}\hat{\Phi}(s),$$

the so-called Poincaré's difference equation

$$\hat{\Phi}(s+q)+b_1(s)\hat{\Phi}(s+q-1)+\cdots+b_q(s)\hat{\Phi}(s)=0$$

where the coefficients $b_j(s)$ tend to finite limits as $s\to\infty$, i.e.,

$$\lim_{s\to\infty}b_j(s)=\hat{b}_j.$$

And then, according to Perron's theorem (19), the value of

$$\overline{\lim_{s\to\infty}}\ |\hat{\Phi}(s)|^{1/s}$$

is equal to the absolute value of a root of the equation

$$t^q+\hat{b}_1t^{q-1}+\cdots+\hat{b}_q=0.$$

Now we shall apply Perron-Poincaré's theorem to the difference equation (3.18). The coordinates of the points corresponding to (3.22) are

$$(3.25)\quad \begin{cases} (0,0),\ (1,0), \dots, (q-1,0) \\ (q,1),\ (q+1,1), \dots, (2q-1,1), \\ \dots\dots \\ ((n-2)q,\ n-2),\ ((n-2)q+1,\ n-2), \dots, ((n-1)q-1,\ n-2) \\ ((n-1)q,\ n-1) \end{cases}$$

and therefore, the Newton-Puiseux polygon is the straight line with the directional coefficient $\tau=1/q$. The constant γ_j in (3.23) is given by the equation

$$(3.26)\quad J'(\lambda)t^{q(n-1)}+(-1)J''(\lambda)t^{q(n-2)} + \cdots + (-1)^{k-1}J^{(k)}(\lambda)t^{q(n-k)} + \cdots + (-1)^{n-1}J^{(n)}(\lambda)=0$$

which has, for a fixed λ_k, the roots

$$(3.27)\quad (\gamma_j)^q=\left(\frac{1}{\lambda_k-\lambda_j}\right) \qquad (j=1, 2,\dots, n\colon j\neq k).$$

Consequently, we obtain the main results of this section.

THEOREM 3.3. *For each k, the coefficients $h^k(s)$ of the formal solutions* (1.5) *have the behaviours*

$$(3.28)\quad \overline{\lim_{s\to\infty}}\left(\frac{|h^k(s)|}{|\Gamma(s+1)^{1/q}|}\right)^{1/s} \leqq \frac{1}{|\hat{\lambda}_k-\lambda_k|^{1/q}} \qquad (k=1,2,\dots,n)$$

where

$$(3.29)\quad |\hat{\lambda}_k-\lambda_k| = \min_{j\neq k}|\lambda_j-\lambda_k|.$$

Lastly, we shall make some remarks as to the characteristic constants λ_k, α_{q-1}^k, $\alpha_{q-2}^k,\dots, \alpha_1^k$ and μ_k $(k=1, 2,\dots, n)$ which are successively determined by the relations (1.7), (3.14) and (3.15). In fact, from the formula of summation (3.9), we know that

$$(3.30)\quad M_k(p\colon v\colon s) = \lambda_k^p + \binom{p}{1}\lambda_k^{p-1}(\alpha_{q-1}^k+\alpha_{q-2}^k+\cdots+\alpha_{q-v}^k) + \hat{M}_k(p\colon \lambda_k, \alpha_{q-1}^k, \alpha_{q-2}^k,\dots, \alpha_{q-v+1}^k)$$

where $\hat{M}_k(p\colon \lambda_k, \alpha_{q-1}^k,\dots, \alpha_{q-v+1}^k)$ include the powers of λ_k of degree not greater than $p-2$, and then obtain, substituting (3.30) into the relation (3.14),

$$(3.31)\qquad \begin{aligned}&(\alpha_{q-1}^k+\alpha_{q-2}^k+\cdots+\alpha_{q-\nu}^k)J'(\lambda_k)\\ &=-\hat{M}_k(n:\lambda_k,\alpha_{q-1}^k,\ldots,\alpha_{q-\nu+1}^k)\\ &\quad-\sum_{l=1}^n a_{l,ql}\hat{M}_k\,(n-l:\lambda_k,\alpha_{q-1}^k,\ldots,\alpha_{q-\nu+1}^k)\\ &\quad+\sum_{r=1}^{\nu}\sum_{l=1}^n a_{l,ql-r}M_k(n-l:\nu-r:s-r)\\ &\qquad\qquad\qquad(\nu=1,2,\ldots,q-1)\,.\end{aligned}$$

This relation gives the value of the characteristic constant $\alpha_{q-\nu}^k$ depending on $\lambda_k, \alpha_{q-1}^k,\ldots, \alpha_{q-\nu+1}^k$.

Moreover, substituting the formula

$$(3.32)\qquad \begin{aligned}M_k(p:q:q)=&\lambda_k^p+\binom{p}{1}\lambda_k^{p-1}(\alpha_{q-1}^k+\alpha_{q-2}^k+\cdots+\alpha_1^k)\\ &+\lambda_k^{p-1}\left\{\mu_k\binom{p}{1}+(q-1)\binom{p}{2}\right\}\\ &+\hat{M}_k(p:\lambda_k,\alpha_{q-1}^k,\ldots,\alpha_1^k)\end{aligned}$$

into the relation (3.15), we again obtain the equation determining the characteristic constant μ_k

$$(3.33)\qquad \begin{aligned}&(\alpha_{q-1}^k+\alpha_{q-2}^k+\cdots+\alpha_1^k+\mu_k)J'(\lambda_k)\\ &=-(q-1)\left\{\binom{n}{2}\lambda_k^{n-1}-\sum_{l=1}^n a_{l,ql}\binom{n-l}{2}\lambda_k^{n-l-1}\right\}\\ &\quad-\hat{M}_k(n:\lambda_k,\alpha_{q-1}^k,\ldots,\alpha_1^k)\\ &\quad-\sum_{l=1}^n a_{l,ql}M_k(n-l:\lambda_k,\alpha_{q-1}^k,\ldots,\alpha_1^k)\\ &\quad+\sum_{r=1}^{q}\sum_{l=1}^n a_{l,ql-r}M_k(n-l:q-r:q-r)\,.\end{aligned}$$

From the equation (3.31), we here obtain the important relations

$$(3.34)\qquad \sum_{k=1}^n(\alpha_{q-1}^k+\alpha_{q-2}^k+\cdots+\alpha_{q-\nu}^k)=\sum_{r=1}^{\nu}a_{1,q-r}\qquad(\nu=1,2,\ldots,q-1)$$

which are derived by the method of residue calculation: In order to obtain (3.34), we may only integrate the function of λ

$$\frac{1}{J(\lambda)}[-\hat{M}(n:\lambda,\dots,\alpha_{q-v+1})-\sum_{l=1}^{n}a_{l,ql}\hat{M}(n-l:\lambda,\dots,\alpha_{q-v+1})$$

$$+\sum_{r=1}^{v}\sum_{l=1}^{n}a_{l,ql-r}M(n-l:v-r:s-r)]$$

along a sufficiently large circle around the origin, including all poles $\lambda_1, \lambda_2,\dots, \lambda_n$ in its interior, in the complex λ-plane. Similarly, we obtain from the equation (3.33)

$$\sum_{k=1}^{n}(\alpha_{q-1}^{k}+\alpha_{q-2}^{k}+\cdots+\alpha_{1}^{k}+\mu_k)=\sum_{r=1}^{q}a_{1,q-r}-(q-1)\binom{n}{2}. \tag{3.35}$$

Hence, subtracting the relation (3.34) for $v=q-1$ from both sides of (3.35),

$$\sum_{k=1}^{n}\mu_k=a_{1,0}-(q-1)\binom{n}{2} \tag{3.36}$$

is obtained. On the other hand, it is easy to obtain the relation

$$\sum_{j=1}^{n}\rho_j=a_{1,0}+\binom{n}{2} \tag{3.37}$$

from the characteristic equation (1.3).

We at last obtain one relation which plays an important role in the next section and describe it in the form of

LEMMA 3.1. *With respect to the n-th order linear differential equation (1.1) with two singular points one of which is irregular, there exists one invariant identity*

$$\sum_{j=1}^{n}\rho_j-\sum_{k=1}^{n}\mu_k=q\binom{n}{2}. \tag{3.38}$$

§4. The determination of the Stokes multipliers.

In this section, we shall investigate the coefficient $G_j(m)$ of the convergent power series solution (1.2) with the purpose of the determination of the Stokes multipliers.

Taking account of the relation

$$t^p\frac{d^p}{dt^p}G_j(m)t^{m+\rho_j}=G_j(m)[m+\rho_j]_p\,t^{m+\rho_j}$$

where the notation (1.4) is used again, it is easy to see that the substitution of the convergent power series solution $X_j(t)$ into the differential equation (1.1) yields

$$\sum_{m=0}^{\infty} G_j(m)[m+\rho_j]_n t^{m+\rho_j} = \sum_{m=0}^{\infty} \Big(\sum_{l=1}^{n} \sum_{r=0}^{ql} a_{l,r} G_j(m-r)[m-r+\rho_j]_{n-l}\Big) t^{m+\rho_j}$$

under the assumption that $G_j(r)=0$ for $r<0$. Therefore, by comparison of the coefficients of like powers of t in both sides of the above formula, we have the qn-th order difference equation

$$[m+\rho_j]_n G_j(m) = \sum_{l=1}^{n} \sum_{r=0}^{ql} a_{l,r}[m-r+\rho_j]_{n-l} G_j(m-r) \tag{4.1}$$

or

$$I(m+\rho_j)G_j(m) = \sum_{l=1}^{n} \sum_{r=1}^{ql} a_{l,r}[m-r+\rho_j]_{n-l} G_j(m-r) \qquad (j=1, 2,\dots, n)\,. \tag{4.2}$$

The formulas for $m=0$

$$I(\rho_j)G_j(0)=0 \qquad (j=1, 2,\dots, n) \tag{4.3}$$

are the identities because of the characteristic equation (1.3). Hence, if we put

$$G_j(0)=1 \qquad (j=1, 2,\dots, n), \tag{4.4}$$

then $G_j(m)$ can be successively determined by (4.2) since $I(m+\rho_j)$ never vanish under the assumption that $\rho_i-\rho_j \neq integer$ $(i\neq j)$.

Here we shall define new functions $f_l^{(j,k)}(m)$ expressed in terms of the series as follows:

$$f_l^{(j,k)}(m) = \sum_{s=0}^{\infty} h^k(s) g_l^{(j,k)}(m+s) \qquad (j,\ k=1, 2,\dots, n:\ l=1, 2,\dots, q) \tag{4.5}$$

where the functions

$$g_1^{(j,k)}(m),\ g_2^{(j,k)}(m),\dots,\ g_q^{(j,k)}(m) \qquad (j,\ k=1, 2,\dots, n)$$

form the fundamental set of solutions of the difference equations (2.2) with the coefficients of the characteristic constants λ_k, $\alpha_{q-1}^k,\dots, \mu_k$, ρ_j and are written as in (2.52). The reason why such functions are introduced will be obvious according to the relations (1.11); for, by quite a formal calculation, we have

$$\begin{aligned}\sum_{s=0}^{\infty} h^k(s) x^{(j,k)}(t, s) &= \sum_{s=0}^{\infty} h^k(s) \sum_{m=0}^{\infty} g^{(j,k)}(m+s) t^{m+\rho_j} \\ &= t^{\rho_j} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} h^k(s) g^{(j,k)}(m+s) t^m \\ &= t^{\rho_j} \sum_{m=0}^{\infty} f^{(j,k)}(m) t^m\,.\end{aligned}$$

For the moment, we assume the well-definedness of the functions $f_l^{(j,k)}(m)$. Then, we can define the well-defined functions

$$(4.6)\qquad f_{l,p}^{(j,k)}(m)=\sum_{s=0}^{\infty}h_p^k(s)g_l^{(j,k)}(m+s),$$

$$f_{l,0}^{(j,k)}(m)=f_l^{(j,k)}(m)\qquad (p=0,1,\dots,n)$$

one after another by means of the *first* difference equations (3.3). In fact, if we multiply both sides of the *first* difference equation

$$\begin{aligned}h_p^k(s)=&\lambda_k h_{p-1}^k(s)+\alpha_{q-1}^k h_{p-1}^k(s-1)+\cdots+\alpha_1^k h_{p-1}^k(s-q+1)\\&+(m+\rho_j+(q-1)p+1)h_{p-1}^k(s-q)\\&-(m+s+\rho_j-\mu_k)h_{p-1}^k(s-q)\end{aligned}$$

by $g_l^{(j,k)}(m+s)$ and sum them up over s from zero to infinity, we obtain, again using (2.2),

$$\begin{aligned}f_{l,p}^{(j,k)}(m)=&\lambda_k f_{l,p-1}^{(j,k)}(m)+\alpha_{q-1}^k f_{l,p-1}^{(j,k)}(m+1)+\cdots+\alpha_1^k f_{l,p-1}^{(j,k)}(m+q-1)\\&+(m+\rho_j+(q-1)p+1)f_{l,p-1}^{(j,k)}(m+q)\\&-\sum_{s=0}^{\infty}(m+s+\rho_j-\mu_k)g_l^{(j,k)}(m+s)h_{p-1}^k(s-q)\\=&\lambda_k f_{l,p-1}^{(j,k)}(m)+\alpha_{q-1}^k f_{l,p-1}^{(j,k)}(m+1)+\cdots+\alpha_1^k f_{l,p-1}^{(j,k)}(m+q-1)\\&+(m+\rho_j+(q-1)p+1)f_{l,p-1}^{(j,k)}(m+q)\\&-\sum_{s=0}^{\infty}\{\alpha_1^k g_l^{(j,k)}(m+s-1)+\cdots+\alpha_{q-1}^k g_l^{(j,k)}(m+s-q+1)\\&+\lambda_k g_l^{(j,k)}(m+s-q)\}h_{p-1}^k(s-q)\\=&(m+q+\rho_j+(q-1)(p-1))f_{l,p-1}^{(j,k)}(m+q)\,.\end{aligned}$$

Hence, the functions $f_{l,p}^{(j,k)}(m)$ are well-defined and

$$(4.7)\qquad f_{l,p}^{(j,k)}(m)=[m+\rho_j+qp]_p f_l^{(j,k)}(m+qp)\qquad (p=1,2,\dots,n)$$

hold.

The same procedure as stated above can also be applied to the *second* difference equation (3.4), obtaining the relations

$$(4.8)\qquad f_{l,n}^{(j,k)}(m)=\sum_{u=1}^{n}\sum_{r=0}^{qu}a_{u,r}f_{l,n-u}^{(j,k)}(m+qu-r)\,.$$

Substituting (4.7) into (4.8), we at last obtain

$$(4.9)\qquad [m+qn+\rho_j]_n f_l^{(j,k)}(m+qn)$$
$$=\sum_{u=1}^{n}\sum_{r=0}^{qu} a_{u,r}[m+qn-r+\rho_j]_{n-u} f_l^{(j,k)}(m+qn-r)$$
$$(j,\ k=1,2,\dots,n:\ l=1,2,\dots,q)\,.$$

The above relations show that for a fixed j, the functions $f_l^{(j,k)}(m)$ $(k=1, 2,\dots, n: l=1, 2,\dots, q)$ are particular solutions of the qn-th order difference equations (4.1) satisfied by $G_j(m)$.

Now we shall proceed to the proofs of the well-definedness of the functions $f_l^{(j,k)}(m)$, i.e., the convergence of the series in the right hand side of (4.5) and the linear independence of qn particular solutions $f_l^{(j,k)}(m)$ $(k=1, 2,\dots, n: l=1, 2,\dots, p)$ of the difference equation (4.1) for a fixed j. For that purpose, we need some informations on the behaviours of the modified Gamma functions $g_l^{(j,k)}(m)$ for sufficiently large values of m and the series expanded in terms of them.

We immediately obtain

LEMMA 4.1. *If m is sufficiently large in the sector $|\arg(m-\mu_k+\rho_j)|<\pi-\delta$, δ being a small positive number, then*

$$(4.10)\qquad g_l^{(j,k)}(m)\sim(-1)^{q+l}\lambda_k^{-(1/q)}\frac{V_{q-1}(1,\omega,\dots,\omega^{l-2},\omega^l,\dots,\omega^{q-1})}{V_q(1,\omega,\dots,\omega^{q-1})}$$
$$\times\frac{m^{-((q-1)/q)}}{\Phi_l^{(j,k)}(m+1)}\{1+O(m^{-(1/q)})\}$$
$$(j,\ k=1,2,\dots,n:\ l=1,2,\dots,q)\,.$$

PROOF. Dropping the indices (j, k), the functions $g_l(m)$ are explicitly written as

$$g_l(m)=-E_l(m)=\frac{(-1)^{q-1}D_l(m)}{\lambda\mathscr{C}_\Phi(m+1)}\qquad(l=1,2,\dots,q)\,.$$

By the same method of calculation as done in § 2, it is easily obtained from the asymptotic formula (2.32) that

$$(4.11)\qquad \mathscr{C}_\Phi(m+1)\sim\Big[\prod_{j=1}^{q}\Phi_j(m+1)\Big]\Big(\frac{m}{\lambda}\Big)^{(q-1)/2}V_q(1,\omega,\dots,\omega^{q-1})$$
$$\times\{1+O(m^{-(1/q)})\}$$

and

$$(4.12)\qquad D_l(m)\sim(-1)^{l+1}\Big[\prod_{\substack{j=1\\ j\neq l}}^{q}\Phi_j(m+1)\Big]\Big(\frac{m}{\lambda}\Big)^{(q-1)(q-2)/(2q)}$$
$$\times V_{q-1}(1,\omega,\dots,\omega^{l-2},\omega^l,\dots,\omega^{q-1})\{1+O(m^{-(1/q)})\}\,.$$

Hence, Lemma 4.1 is established.

Next we show a result as to the series of the form (4.5) which is expanded by means of the modified Gamma functions.

LEMMA 4.2. *Suppose that the series defining the functions* $f_l^{(j,k)}(m_0)$ *are absolutely convergent for a certain number* m_0. *Then the series* $f_l^{(j,k)}(m)$ *are also absolutely convergent in the right half-plane* $\mathrm{Re}\, m \geqq \mathrm{Re}\, m_0 + \varepsilon$, ε *being any positive small number. Moreover, in that right half-plane, the asymptotic relations*

$$f_l^{(j,k)}(m) \sim g_l^{(j,k)}(m)\{1 + O(m^{-(1/q)})\} \tag{4.13}$$

$$(j, k = 1, 2,\dots, n:\ l = 1, 2,\dots, q)$$

hold.

PROOF. For a sufficiently large positive integer σ, we put

$$R(m: \sigma) = \frac{1}{g(m+\sigma)} \sum_{s=\sigma+1}^{\infty} h(s) g(m+s), \tag{4.14}$$

dropping the indices (j, k) and l again. Then we may only prove that the remaining series $R(m: \sigma)$ is absolutely convergent and uniformly bounded in the right half-plane $\mathrm{Re}\, m \geqq \mathrm{Re}\, m_0 + \varepsilon$. First, applying formally Abel's transformation, we have

$$\begin{aligned} R(m: \sigma) &= \frac{1}{g(m_0+\sigma)} \sum_{s=\sigma+1}^{\infty} h(s) g(m_0+s) \frac{g(m_0+\sigma) g(m+s)}{g(m+\sigma) g(m_0+s)} \\ &= \frac{1}{g(m_0+\sigma)} \sum_{s=\sigma+1}^{\infty} h(s) g(m_0+s) \{1 + \sum_{p=\sigma+1}^{s} q(m, m_0: \sigma: p)\} \\ &= \frac{1}{g(m_0+\sigma)} \sum_{s=\sigma+1}^{\infty} h(s) g(m_0+s) \\ &\quad + \frac{1}{g(m_0+\sigma)} \sum_{p=\sigma+1}^{\infty} q(m, m_0: \sigma: p) \{\sum_{s=p}^{\infty} h(s) g(m_0+s)\} \end{aligned} \tag{4.15}$$

where

$$q(m, m_0: \sigma: p) = \frac{g(m_0+\sigma)}{g(m+\sigma)} \left\{ \frac{g(m+p)}{g(m_0+p)} - \frac{g(m+p-1)}{g(m_0+p-1)} \right\}.$$

From Lemma 4.1 and the asymptotic relation (2.32), we can obtain

$$\begin{aligned} \frac{g(m_0+\sigma)}{g(m+\sigma)} \frac{g(m+p)}{g(m_0+p)} &\sim \frac{\Phi(m+\sigma+1)}{\Phi(m_0+\sigma+1)} \frac{\Phi(m_0+p+1)}{\Phi(m+p+1)} \\ &\times \{1 + O(\sigma^{-(1/q)})\} \sim \left(\frac{p}{\sigma}\right)^{(m_0-m)/q} \{1 + O(\sigma^{-(1/q)})\} \end{aligned}$$

and hence

$$q(m, m_0: \sigma: p) \sim \left(\frac{m_0-m}{q}\right)\left(\frac{p}{\sigma}\right)^{(m_0-m)/q}\left(\frac{1}{p}\right)\{1+O(\sigma^{-(1/q)})\}$$

for sufficiently large σ.

Then we have

$$(4.16) \qquad \sum_{p=\sigma+1}^{\infty} |q(m, m_0: \sigma: p)|$$

$$\leqq M\left|\frac{m_0-m}{q}\right| \sum_{p=\sigma+1}^{\infty} \left|\left(\frac{p}{\sigma}\right)^{(m_0-m)/q}\left(\frac{1}{p}\right)\right|$$

$$\leqq \frac{M}{\sigma+1}\left|\frac{m_0-m}{q}\right|\left(\frac{\sigma+1}{\sigma}\right)^{\mathrm{Re}(m_0-m)} \sum_{p=\sigma+1}^{\infty}\left(\frac{\sigma+1}{p}\right)^{1+\varepsilon}.$$

Since the series in the right hand side of (4.16) is convergent and, as is easily seen, approaches zero exponentially as m tends to infinity, the series in the left hand side of (4.16) is absolutely convergent and therefore Abel's transformation is valid. Concerning the asymptotic behaviour, we instantly obtain

$$f(m) = \sum_{s=0}^{\sigma} h(s)g(m+s) + g(m+\sigma)R(m: \sigma)$$

$$= g(m)\left\{1 + \sum_{s=1}^{\sigma} h(s)\frac{g(m+s)}{g(m)} + \frac{g(m+\sigma)}{g(m)} R(m: \sigma)\right\}$$

$$\sim g(m)\{1+O(m^{-(1/q)})\}$$

since

$$(4.17) \qquad \frac{g_l(m+s)}{g_l(m)} \sim (\lambda^{-(1/q)}\omega^{l-1})^{-s} m^{-(s/q)}\{1+O(m^{-(1/q)})\}$$

$$(l=1, 2,\dots, q).$$

Thus we obtain the required results in Lemma 4.2.

We have made preparations for stating the theorem as to the well-definedness and linear independence of the functions (4.5).

THEOREM 4.1. *Under the condition that*

$$(4.18) \qquad 0<|\lambda_j|<|\lambda_j-\lambda_k| \qquad (j\neq k;\ k=1, 2,\dots, n),$$

the functions $f_l^{(j,k)}(m)$ $(j, k=1, 2,\dots, n: l=1, 2,\dots, q)$ *are well-defined for every integer* $m \geqq -qn+1$. *And that, for a fixed* j, *the functions* $f_l^{(j,k)}(m)$ $(k=1, 2,\dots, n: l=1, 2,\dots, q)$ *make a fundamental set of solutions of the qn-th order difference equation* (4.1).

PROOF. To begin with, we prove the absolute convergence of the series defining $f_l^{(j,k)}(-qn)$, using Lemma 4.1 and Theorem 3.3. For sufficiently large s, we have

$$
\begin{aligned}
& g_l^{(j,k)}(s-qn)\Gamma(s+1)^{1/q} \\
& \sim (\lambda_k^{-(1/q)}\omega^{l-1})^{-s}\exp\left\{-\left(\frac{s-qn+1}{q}\right)\log s+\frac{s}{q}+\frac{1}{2}\log s\right. \\
& \quad \left.-\frac{q-1}{q}\log s - sO(s^{-(1/q)})\right\}\exp\left\{\frac{1}{q}\left(s+\frac{1}{2}\right)\log s-\frac{s}{q}\right\} \\
& \quad \times\{\tilde{d}_l^{(j,k)}+O(s^{-(1/q)})\} \\
& \sim (\lambda_k^{-(1/q)}\omega^{l-1})^{-s}\exp\left\{\left(n-\frac{1}{2}+\frac{1}{2q}\right)\log s - sO(s^{-(1/q)})\right\} \\
& \quad \times\{\tilde{d}_l^{(j,k)}+O(s^{-(1/q)})\}
\end{aligned}
$$

whence

$$
(4.19)\qquad \overline{\lim_{s\to\infty}}\,|g_l^{(j,k)}(s-qn)\Gamma(s+1)^{1/q}|^{1/s}\leqq|\lambda_k^{1/q}\omega^{-l+1}|=|\lambda_k|^{1/q}.
$$

Consequently, we have, from (3.28),

$$
\begin{aligned}
(4.20)\qquad & \overline{\lim_{s\to\infty}}\,|h^k(s)g_l^{(j,k)}(s-qn)|^{1/s} \\
& \leqq \overline{\lim_{s\to\infty}}\,(|h^k(s)/\Gamma(s+1)^{1/q}|)^{1/s}\,\overline{\lim_{s\to\infty}}\,(|g_l^{(j,k)}(s-qn)\Gamma(s+1)^{1/q}|)^{1/s} \\
& \leqq \frac{|\lambda_k|^{1/q}}{|\hat{\lambda}_k-\lambda_k|^{1/q}}.
\end{aligned}
$$

If the value in the last statement is less than 1, i.e.,

$$
(4.21)\qquad 0<|\lambda_k|^{1/q}<|\hat{\lambda}_k-\lambda_k|^{1/q}\leqq|\lambda_j-\lambda_k|^{1/q}\qquad (j\neq k;\, j=1, 2,\dots, n),
$$

then the series $f_l^{(j,k)}(-qn)$ are absolutely convergent. Hence, the functions $f_l^{(j,k)}(m)$ are all well-defined for every integer $m\geqq -qn+1$ because of Lemma 4.2.

Since we have already proved that the functions $f_l^{(j,k)}(m)$ ($k=1, 2,\dots, n$: $l=1, 2,\dots, q$) are particular solutions of the difference equation (4.1), we shall, from now on, prove their linear independence i.e., the nonvanishing of their Casorati determinant

$$\mathscr{C}_f(m)=\operatorname*{Det}_{\substack{1\leqq k\leqq n\\1\leqq l\leqq q}}\begin{pmatrix}f_l^{(j,k)}(m)\\f_l^{(j,k)}(m+1)\\\vdots\\f_l^{(j,k)}(m+qn-1)\end{pmatrix}. \tag{4.22}$$

In this case, the Casorati determinant satisfies the first order difference equation

$$\mathscr{C}_f(m+1)=\frac{(-1)^{qn-1}a_{n,qn}}{I(m+qn+\rho_j)}\mathscr{C}_f(m). \tag{4.23}$$

Considering the relations

$$I(m+qn+\rho_j)=\prod_{k=1}^{n}(m+qn+\rho_j-\rho_k), \tag{4.24}$$

$$\prod_{k=1}^{n}\lambda_k=(-1)^{n-1}a_{n,qn} \tag{4.25}$$

which are immediately derived from (1.3) and (1.7), we then have the solution of the Casorati difference equation (4.23) of the form

$$\mathscr{C}_f(m)=(-1)^{(q-1)n(m+qn)}\left[\prod_{k=1}^{n}\frac{\lambda_k^{m+qn}\Gamma(\rho_j-\rho_k+1)}{\Gamma(m+qn+\rho_j-\rho_k)}\right]\mathscr{C}_f(-qn+1). \tag{4.26}$$

Therefore, in order to prove the nonvanishing of $\mathscr{C}_f(m)$ for $m\geqq -qn+1$, we may show that $\mathscr{C}_f(-qn+1)\neq 0$, evaluating its explicit value by means of the asymptotic behaviour of $\mathscr{C}_f(m)$ for sufficiently large m.

From Lemmata 4.1 and 4.2, it is easy to see that

$$\begin{aligned}\frac{f_l^{(j,k)}(m+r)}{f_l^{(j,k)}(m)}&\sim\frac{g_l^{(j,k)}(m+r)}{g_l^{(j,k)}(m)}\{1+O(m^{-(1/q)})\}\\&\sim\frac{\Phi_l^{(j,k)}(m+1)}{\Phi_l^{(j,k)}(m+r+1)}\{1+O(m^{-(1/q)})\}\\&\sim(\lambda_k^{-(1/q)}\omega^{l-1})^{-r}m^{-(r/q)}\{1+O(m^{-(1/q)})\}.\end{aligned} \tag{4.27}$$

Using (4.27), we then obtain

$$\begin{aligned}\mathscr{C}_f(m)\sim{}&\Big[\prod_{k=1}^{n}\prod_{l=1}^{q}g_l^{(j,k)}(m)\Big]m^{-(1/q)(qn(qn-1)/2)}\\&\times V_{qn}(\lambda_1^{1/q},\lambda_1^{1/q}\omega^{-1},\dots,\lambda_1^{1/q}\omega^{-q+1},\dots,\lambda_n^{1/q},\lambda_n^{1/q}\omega^{-1},\dots,\lambda_n^{1/q}\omega^{-q+1})\\&\times\{1+O(m^{-(1/q)})\}\end{aligned} \tag{4.28}$$

where Vandermonde's determinant never vanishes since $\lambda_j \neq \lambda_k (j \neq k)$, and moreover, from (4.10) and (2.36),

$$(4.29) \quad [\prod_{k=1}^{n} \prod_{l=1}^{q} g_l^{(j,k)}(m)]$$

$$\sim \left[\prod_{k=1}^{n} \prod_{l=1}^{q} \frac{(-1)^{q+l} \lambda_k^{-(1/q)}}{\Phi_l^{(j,k)}(m+1)}\right] m^{-((q-1)qn/q)} \left(\frac{V_{q-1}}{V_q}\right)^{qn}$$

$$\times \{1+O(m^{-(1/q)})\}$$

$$\sim (-1)^{qn(3q+1)/2} \left[\prod_{k=1}^{n} \lambda_k^{-(1/q)} (\lambda_k \omega^{-(q(q-1)/2)})^{m+1-\mu_k+\rho_j}\right.$$

$$\times \exp(-R(\lambda_k, \alpha_{q-1}^k, \ldots, \alpha_1^k))$$

$$\left.\times \exp\left(-\left(m+1-\mu_k+\rho_j-\frac{q}{2}\right) \log m+m\right)\right]$$

$$\times m^{-n(q-1)} \left(\frac{q}{2\pi}\right)^{qn/2} \left(\frac{V_{q-1}}{V_q}\right)^{qn} \{1+O(m^{-(1/q)})\}.$$

Denoting, for simplicity, the nonzero constant by C_0, we at last obtain

$$(4.30) \quad \mathscr{C}_f(-qn+1)$$

$$\sim \prod_{k=1}^{n} \left[\frac{\Gamma(m+qn+\rho_j-\rho_k)}{\Gamma(\rho_j-\rho_k+1)} \exp\left\{-\left(m+1-\mu_k+\rho_j-\frac{q}{2}\right) \log m+m\right\}\right]$$

$$\times m^{-(n(qn+2q-3)/2)} \{C_0+O(m^{-(1/q)})\}$$

$$\sim \prod_{k=1}^{n} \left[\frac{1}{\Gamma(\rho_j-\rho_k+1)} \exp\left\{\left(\mu_k-\rho_k+\frac{q}{2}-\frac{3}{2}\right) \log m\right\}\right]$$

$$\times m^{-(n(qn+2q-3)/2)} \{C_0+O(m^{-(1/q)})\}$$

$$\sim \left[\prod_{k=1}^{n} \frac{1}{\Gamma(\rho_j-\rho_k+1)}\right] \exp\left\{\left[\sum_{k=1}^{n} (\mu_k-\rho_k)+q\binom{n}{2}\right] \log m\right\}$$

$$\times \{C_0+O(m^{-(1/q)})\}.$$

The last expression namely means that

$$(4.31) \qquad \mathscr{C}_f(-qn+1) = \left[\prod_{k=1}^{n} \frac{1}{\Gamma(\rho_j-\rho_k+1)}\right] C_0$$

because of the invariant identity (3.38) in Lemma 3.1. Thus, we obtain $\mathscr{C}_f(m) \neq 0$ for $m \geqq -qn+1$ under the assumption that $\rho_j-\rho_k \neq$ *integer* $(j \neq k)$. Obviously, from (4.23), $\mathscr{C}_f(-qn)=0$ and hence, $\mathscr{C}_f(m)=0$ for $m \leqq -qn$. The proof of Theorem 4.1 is thus completed.

Now we shall state how to determine the Stokes multipliers in the following

THEOREM 4.2. *The Stokes multipliers* $T_l^{(j,k)}$ $(k=1, 2,\dots, n: l=1, 2,\dots, q)$ *are determined by the linear equations*

$$(4.32)\qquad G_j(r)=\sum_{l=1}^{q}\sum_{k=1}^{n} T_l^{(j,k)} f_l^{(j,k)}(r) \qquad (r=-qn+1, -qn+2,\dots, 0)$$

where

$$G_j(0)=1 \quad\text{and}\quad G_j(r)=0 \quad\text{for}\quad r<0.$$

Then, the coefficients $G_j(m)$ $(j=1, 2,\dots, n)$ *are expressed in terms of the linear combinations*

$$(4.33)\qquad G_j(m)=\sum_{l=1}^{q}\sum_{k=1}^{n} T_l^{(j,k)} f_l^{(j,k)}(m) \qquad (j=1, 2,\dots, n).$$

§5. Main results for the two point connection problem.

Now, making use of Theorem 4.2 in the previous section, we shall attempt to partition the convergent power series solutions $X_j(t)$ into qn fundamental functions $x_l^{(j,k)}(t, s)$ as follows:

$$\begin{aligned}(5.1)\qquad X_j(t) &= \sum_{m=0}^{\infty} G_j(m)t^{m+\rho_j}\\ &= \sum_{l=1}^{q}\sum_{k=1}^{n} T_l^{(j,k)}\Big(\sum_{m=0}^{\infty} f_l^{(j,k)}(m)t^{m+\rho_j}\Big)\\ &= \sum_{l=1}^{q}\sum_{k=1}^{n} T_l^{(j,k)}\sum_{s=0}^{\infty} h^k(s)\Big(\sum_{m=0}^{\infty} g_l^{(j,k)}(m+s)t^{m+\rho_j}\Big)\\ &= \sum_{l=1}^{q}\sum_{k=1}^{n} T_l^{(j,k)}\sum_{s=0}^{\infty} h^k(s)x_l^{(j,k)}(t, s) \qquad (j=1, 2,\dots, n)\end{aligned}$$

where we, of course, put

$$(5.2)\qquad x_l^{(j,k)}(t, s)=\sum_{m=0}^{\infty} g_l^{(j,k)}(m+s)t^{m+\rho_j}$$

$$(j, k=1, 2,\dots, n: l=1, 2,\dots, q).$$

In the above statement (5.1), the interchangeability of the order of the sums by s and m is guaranteed by the absolute convergence of the double series

$$(5.3) \qquad X_l^{(j,k)}(t) \equiv \sum_{m=0}^{\infty} f_l^{(j,k)}(m) t^{m+\rho_j}$$

$$= \sum_{s=0}^{\infty} h^k(s) x_l^{(j,k)}(t, s)$$

$$(j, k = 1, 2, \dots, n : l = 1, 2, \dots, q) .$$

In fact, we easily see

$$|\sum_{m=0}^{\infty} (\sum_{s=0}^{\infty} h^k(s) g_l^{(j,k)}(m+s)) t^{m+\rho_j}|$$

$$\leqq |\sum_{s=0}^{\sigma} h^k(s) \sum_{m=0}^{\infty} g_l^{(j,k)}(m+s) t^{m+\rho_j}| + |\sum_{m=0}^{\infty} g_l^{(j,k)}(m+\sigma) R_l^{(j,k)}(m:\sigma) t^{m+\rho_j}|$$

$$\leqq \sum_{s=0}^{\sigma} |h^k(s)| \, |\sum_{m=0}^{\infty} g_l^{(j,k)}(m+s) t^{m+\rho_j}| + M \sum_{m=0}^{\infty} |g_l^{(j,k)}(m+\sigma) t^{m+\rho_j}|$$

since the functions $R_l^{(j,k)}(m:\sigma)$ are uniformly bounded as stated in the proof of Lemma 4.2 and the functions $x_l^{(j,k)}(t, s)$ are entire.

Here we shall investigate the global behaviours of the functions $X_l^{(j,k)}(t)$ defined as in (5.3). From the global properties of the fundamental functions $x_l^{(j,k)}(t, s)$, we immediately obtain the following results.

THEOREM 5.1. *Let σ be an arbitrarily large positive integer. In the entire complex plane, the functions $X_l^{(j,k)}(t)$ $(j, k = 1, 2, \dots, n)$ admit the asymptotic behaviours as follows:*

$$(5.4) \qquad X_l^{(j,k)}(t) = \exp\left(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\right) t^{\mu_k}$$

$$\times \{ \sum_{s=0}^{\sigma} h^k(s) t^{-s} + O(t^{-\sigma-1}) \} + O(t^{\rho_j + q - 1})$$

$$\text{as } t \to \infty \text{ in } S_l(\lambda_k)$$

and

$$(5.5) \qquad X_l^{(j,k)}(t) = O\left(\exp\left(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\right) t^{\mu_k - \sigma - 1} \right)$$

$$+ O(t^{\rho_j + q - 1}) \qquad \text{as } t \to \infty \text{ in } S_i(\lambda_k) \quad (i \neq l).$$

PROOF. By means of the integral representation (2.6) for the fundamental function, we have

$$(5.6)\quad X_l^{(j,k)}(t)=\sum_{s=0}^{\sigma}h^k(s)x_l^{(j,k)}(t,s)+\sum_{s=\sigma+1}^{\infty}h^k(s)x_l^{(j,k)}(t,s)$$

$$=\sum_{s=0}^{\sigma}h^k(s)x_l^{(j,k)}(t,s)$$

$$+t^{\rho_j+q-1}\int_0^1\exp\Big(\frac{\lambda_k}{q}t^q(1-\tau^q)+\sum_{u=1}^{q-1}\frac{\alpha_u^k}{u}t^u(1-\tau^u)\Big)$$

$$\times[\lambda_k F_l^{(j,k)}(\tau:\,-1)]\tau^{-\mu_k+\rho_j+q-2}d\tau$$

$$+t^{\rho_j+q-2}\int_0^1\exp\Big(\frac{\lambda_k}{q}t^q(1-\tau^q)+\sum_{u=1}^{q-1}\frac{\alpha_u^k}{u}t^u(1-\tau^u)\Big)$$

$$\times[\lambda_k F_l^{(j,k)}(\tau:\,-2)+\alpha_{q-1}^k F_l^{(j,k)}(\tau:\,-1)]\tau^{-\mu_k+\rho_j+q-3}d\tau$$

$$+$$

$$\vdots$$

$$+t^{\rho_j}\int_0^1\exp\Big(\frac{\lambda_k}{q}t^q(1-\tau^q)+\sum_{u=1}^{q-1}\frac{\alpha_u^k}{u}t^u(1-\tau^u)\Big)[\lambda_k F_l^{(j,k)}(\tau:\,-q)$$

$$+\alpha_{q-1}^k F_l^{(j,k)}(\tau:\,-q+1)+\cdots+\alpha_2^k F_l^{(j,k)}(\tau:\,-2)+$$

$$+\alpha_1^k F_l^{(j,k)}(\tau:\,-1)]\tau^{-\mu_k+\rho_j-1}d\tau$$

where we put

$$(5.7)\qquad F_l^{(j,k)}(\tau:m)=\sum_{s=\sigma+1}^{\infty}h^k(s)g_l^{(j,k)}(s+m)\tau^s\,.$$

In the above calculation (5.6) we used the termwise integration which is again guaranteed by the fact that the power series in the right hand side of (5.7) is absolutely and uniformly convergent in any compact domain of

$$0\leqq|\tau|<\frac{|\hat{\lambda}_k-\lambda_k|^{1/q}}{|\lambda_k|^{1/q}},\quad |\hat{\lambda}_k-\lambda_k|=\min_{j\neq k}|\lambda_j-\lambda_k|$$

and hence, in the unit disk $|\tau|\leqq 1$ according to Theorem 4.1 and its proof.

Applying Theorem 2.5 to the first σ terms and Theorem 2.2 to the remaining integrals in the right hand side of (5.6), we have

$$(5.8)\qquad \sum_{s=0}^{\sigma}h^k(s)x_l^{(j,k)}(t,s)=\exp\Big(\frac{\lambda_k}{q}t^q+\frac{\alpha_{q-1}^k}{q-1}t^{q-1}+\cdots+\alpha_1^k t\Big)t^{\mu_k}\sum_{s=0}^{\sigma}h^k(s)t^{-s}$$

$$+O(t^{\rho_j+q-1})\,,$$

$$(5.9)\qquad \sum_{s=\sigma+1}^{\infty}h^k(s)x_l^{(j,k)}(t,s)=O\Big(\exp\Big(\frac{\lambda_k}{q}t^q+\frac{\alpha_{q-1}^k}{q-1}t^{q-1}+\cdots+\alpha_1^k t)t^{\mu_k-\sigma-1}\Big)$$

$$+O(t^{\rho_j+q-1})$$

for sufficiently large values of t in the sector $S_l(\lambda_k)$ and

$$(5.10) \qquad \sum_{s=0}^{\sigma} h^k(s) x_l^{(j,k)}(t, s) = O(t^{\rho_j+q-1}),$$

$$(5.11) \qquad \sum_{s=\sigma+1}^{\infty} h^k(s) x_l^{(j,k)}(t, s) = O\Big(\exp\Big(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\Big) t^{\mu_k-\sigma-1}\Big) + O(t^{\rho_j+q-1})$$

in the sector $S_i(\lambda_k)$ for $i \neq l$. Thus the statements in Theorem 5.1 are derived by combining (5.8) with (5.9) and (5.10) with (5.11).

Now we put

$$(5.12) \qquad X^{(j,k)}(t) \equiv \sum_{l=1}^{q} T_l^{(j,k)} X_l^{(j,k)}(t) \qquad (j, k = 1, 2, \dots, n)$$

and consider their asymptotic behaviours in the entire complex plane.

For a fixed index k, the sectors $S_1(\lambda_k)$, $S_2(\lambda_k)$, ..., $S_{q-1}(\lambda_k)$ and $S_q(\lambda_k)$ cover the whole complex plane. Hence, if t is sufficiently large, then t necessarily lies in some sector $S_l(\lambda_k)$ and, according to Theorem 5.1, we have

$$\begin{aligned}(5.13) \quad X^{(j,k)}(t) &\sim T_l^{(j,k)} \exp\Big(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\Big) t^{\mu_k} \\ &\quad \times \{\sum_{s=0}^{\sigma} h^k(s) t^{-s} + O(t^{-\sigma-1})\} \\ &\quad + O\Big(\sum_{\substack{i=1 \\ i\neq l}}^{q} T_i^{(j,k)} \exp\Big(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\Big) t^{\mu_k-\sigma-1}\Big) \\ &\quad + O(t^{\rho_j+q-1}) \\ &\sim T_l^{(j,k)} \exp\Big(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\Big) t^{\mu_k} \\ &\quad \times \{\sum_{s=0}^{\sigma} h^k(s) t^{-s} + O(t^{-\sigma-1})\} + O(t^{\rho_j+q-1}).\end{aligned}$$

In other words, we can say that among q functions expressing the function $X^{(j,k)}(t)$ one and only one function has the principal asymptotic behaviour and the remaining $(q-1)$ functions are absorbed in that principal one.

We can at last obtain the main results of this paper by means of the decomposition relation (5.1), i.e.,

$$X_j(t) = \sum_{k=1}^{n} X^{(j,k)}(t) \qquad (j = 1, 2, \dots, n)$$

and state them in the following final theorem.

THEOREM 5.2. *Suppose that*

(i) $\rho_j - \rho_k \neq$ *integer* $(j \neq k)$,

(ii) $\rho_j - \mu_k \neq$ *integer, and*

(iii) $0 < \dfrac{|\lambda_k|}{|\lambda_j - \lambda_k|} < 1 \qquad (j \neq k) \qquad (j, k = 1, 2, \dots, n).$

If t is sufficiently large, then t necessarily lies in some sector

$$S(l_1, l_2, \dots, l_n) = S_{l_1}(\lambda_1) \cap S_{l_2}(\lambda_2) \cap \cdots \cap S_{l_n}(\lambda_n) \tag{5.14}$$

where $1 \leqq l_1, l_2, \dots, l_n \leqq q$ *and* $\cap$ *means the usual notation of the intersection, and*

$$X_j(t) \sim \sum_{k=1}^{n} T_{l_k}^{(j,k)} \exp\left(\frac{\lambda_k}{q} t^q + \frac{\alpha_{q-1}^k}{q-1} t^{q-1} + \cdots + \alpha_1^k t\right) t^{\mu_k} \tag{5.15}$$

$$\times \{ \sum_{s=0}^{\sigma} h^k(s) t^{-s} + O(t^{-\sigma-1}) \} + O(t^{\rho_j + q - 1})$$

$$\text{as} \quad t \to \infty \quad \text{in} \quad S(l_1, l_2, \dots, l_n) \qquad (j = 1, 2, \dots, n).$$

In particular, if the exponential functions in the asymptotic expression (5.15) *for all k are dominant over the powers of t in the sector* $S(l_1, l_2, \dots, l_n)$, *we then have the exactly desirable asymptotic relations*

$$X_j(t) \sim \sum_{k=1}^{n} T_{l_k}^{(j,k)} X^k(t) \tag{5.16}$$

$$\text{as} \quad t \to \infty \quad \text{in} \quad S(l_1, l_2, \dots, l_n) \qquad (j = 1, 2, \dots, n).$$

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