A TWO-SECTOR MODEL OF ENDOGENOUS GROWTH WITH LEISURE

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Abstract

In this paper we analyze a class of endogenous growth models with physical and human capital and with three alternatives uses of time: unqualified leisure, work and education. In contrast to some other related models, we find that, even in the absence of technological externalities, there could be multiple balanced paths. We provide a characterization of the qualitative behavior of consumption, leisure, work and education over those balanced paths, and study their transitional dynamics.

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1. Introduction

This paper focuses on the equilibrium dynamics of an endogenous growth model with physical and human capital in which leisure considerations have a direct effect on the utility function. Subject to minor considerations, our model is taken from Lucas (1990) who carries out a quantitative analysis of the effects of several taxes on agents' welfare. The model is in turn a simple extension of the original setting of Uzawa (1965) and Lucas (1988). In the Uzawa-Lucas framework, time is devoted either to production in the goods sector or to improve the level of education. In the present framework, time may in addition be spent in leisure activities. In consequence, the amount of time engaged in productive occupations (goods production and education) is now an endogenous variable.

Growth models have become common devices for the study of macroeconomic problems. As a result, there are several important considerations that warrant the analysis of leisure in a choice theoretical framework of growth. First, leisure is a key variable in modern theories of business fluctuations since around two-thirds of the output variation over the business cycle can be accounted for by fluctuations in worked hours [cf., Kydland (1995)]. Also, leisure considerations are relevant in a theory of taxation since generally a tax on labor affects the time allocated to productive activities only if leisure considerations are present in the analysis. Finally, it has been of some concern to us how the intertemporal allocation of goods consumption, leisure, worked hours and education determine together the long-term growth of an economy and the transitional dynamics to a given steady state. With the exception of the simple model considered in Chase (1967), however, it appears that there are no systematic studies on the effects of leisure in the process of growth.

In our endogenous growth framework, there are several ways to model leisure depending on how the level of education affects its productivity. We consider here the extreme case in which the stock of human capital does not affect the marginal utility of leisure. As already remarked, this is the model studied in Lucas (1990), and it is consistent with some casual observations that technological progress has been slower in certain leisure activities —such as sleeping time or spending time with the family— than in productive occupations. Of course, alternative formulations with qualified leisure may be worth investigating. In this respect, Ortigueira (1994) analyzes a variant of the present model in which total effective leisure units are defined as the amount of time spent in leisure activities augmented by the level of education. It should be pointed out, however, that the presence of

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unqualified leisure in our endogenous growth framework leads to a non-necessarily concave optimization problem as the stock of human capital affects asymmetrically the time spent in the various activities.

In the original models of Uzawa (1965) and Lucas (1988), in the absence of externalities the long-term growth rate of the economy is solely determined by the discount rate, the elasticity of intertemporal substitution for consumption and the productivity of the human capital technology. Moreover, concavity of the primitive functions suffices to guarantee the uniqueness of the ray of balanced paths (or steady-state equilibria), and such ray is globally stable [see Caballé and Santos (1993), Chamley (1993), and Faig (1993)]. In contrast, in our simple model with leisure we find that even for the most common utility and production functional forms there could be multiple balanced paths with different growth rates. Hence, global stability is lost, and different economies may reach asymptotically different steady-state growth dynamics depending on their initial holdings of physical and human capital.

In addition to the aforementioned dynamical properties of the Uzawa-Lucas framework, the multiplicity of steady-state equilibria should likewise be confronted with the dynamical behavior of the standard neoclassical growth model with leisure and of the model with qualified leisure considered in Ortigueira (1994). In all these settings, under the general assumptions considered here there is always a unique globally stable steady-state equilibrium. Our model represents thus a minimal extension to obtain a multiplicity of steady states, and such property is unrelated to the fact that leisure may be an inferior commodity. Also, we would like to emphasize that the multiplicity of steady-state rays holds in the absence of technological externalities, and hence our findings are of a different nature from those reported in related models by Benhabib and Perli (1994), Chamley (1993) and Rustichini and Schmitz (1991). In all of our examples, all competitive equilibria are obtained as optimal solutions to a planning problem.

The possibility of multiple steady states in our setting is linked to the fact that the ratio of physical to human capital accumulated affects the opportunity cost of leisure. Thus, countries with a higher proportion of human capital may desire to reach a steady state with a higher rate of growth and lower proportions of consumption and leisure. Indeed, we shall present some simple examples of economies with several steady states such that if the relative endowment of physical capital is initially high then it becomes optimal not to invest in education. Hence, without resorting to externality-type arguments our model can account for countries with different rates of long-term growth. The disparity of permanent

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rates of growth is explained by the relative endowments of physical and human capital. Therefore, a different composition of wealth across countries not only has temporary effects on growth (as in the Uzawa-Lucas model) but may lead to permanent, increasing differences in income per capita.

It is also found that leisure has a noticeable effect on the transitional dynamics to a given steady-state or balanced growth path. If leisure activities are present, an increment in physical capital in the economy from a certain steady-state configuration induces an increase in both consumption and leisure. Agents find now more costly to spend time in the educational sector. As a result, it is now more plausible to obtain the so called paradoxical case discussed in Caballé and Santos (1993) in which a higher proportion of physical capital discourages human capital accumulation and leads the economy to a lower steady state. Some numerical computations will illustrate the range of parameters for which this transitional behavior is possible.

Although empirical work on growth has not addressed directly the possibility of multiple steady-state equilibria depending on relative endowments of physical and human capital, we should nevertheless point out that multiple patterns of behavior are observed in labor markets. Ríos-Rull (1993) reviews some stylized facts on labor supply for various skill levels, and documents that qualified people devote more time to work and education and less time to leisure activities. Moreover, earning profiles of qualified workers increase over time at a higher growth rate. Our model offers several insights on these issues, and links such patterns of growth to certain parameters and elasticities of the production and utility functions.

The outline of the paper is as follows. In Section 2 we introduce the model along with a basic discussion of the existence of a balanced path. In Section 3 we analyze the multiplicity of balanced paths in the context of some elementary production functions. For these simple functional forms, we also provide a characterization of the qualitative behavior of consumption, leisure, work, and education over the multiple balanced paths. Section 4 is devoted to the transitional dynamics of these variables toward a stable stationary solution. We conclude in Section 5 with a review of our main findings. Our analysis of the dynamic properties of our endogenous growth framework is supplemented with a technical appendix. In the first part of this appendix we reexamine the issue of the multiplicity of steady-state solutions in the standard exogenous growth model with leisure, and show that such multiplicity of steady-state equilibria is of a different nature from that observed in the present model. In the second part, we focus on the characterization of optimal solutions in our model. Even though the inclusion of unqualified

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leisure in our endogenous growth framework may lead to a non-concave optimization problem, we shall develop a method of analysis that allows to characterize optimal solutions from the standard first-order conditions of the Maximum Principle. This is an important technical result which has heretofore been neglected, and in some cases insures the optimality of the multiple steady states.

2. The Model

In this section we introduce a general model economy of endogenous growth with leisure. In the present setting, every optimal solution may be decentralized as a competitive equilibrium. Thus, without loss of generality we shall confine our analysis to the planner's problem.

The economy is populated by a continuum of identical infinitely lived households or dynasties that grow at an exogenously given rate, $n \geq 0$. Each household derives utility from the consumption of an aggregate good and leisure. The instantaneous utility function, $U[c(t), l(t)]$, is a C^2 mapping, strongly concave and increasing in both consumption, $c(t)$, and leisure, $l(t)$. Each agent discounts future utility at a constant positive rate, ρ.

Agents can allocate their available unit of time over three different margins: to produce the aggregate good, to accumulate human capital, or to engage in leisure activities. In the output sector, the technology is represented by a C^2 concave production function, $F(K, L)$, increasing and linearly homogeneous in physical capital, K, and labor, L. This function exhibits unbounded partial derivatives at the boundary, and capital as well as labor are essential factors in the production process. More precisely,

$$\lim_{L \to 0} F_L(\bar{K}, L) = \infty, \lim_{K \to 0} F_K(K, \bar{L}) = \infty, \text{ and } F(0, \bar{L}) = F(\bar{K}, 0) = 0 \qquad (2.1)$$

where the subindex denotes the variable with respect to which the partial derivative is taken, and $\bar{K} > 0$ and $\bar{L} > 0$ remain fixed. Also, $F_{KK}(K, L) < 0$ and $F_{LL}(K, L) < 0$ for all positive vectors (K, L).

If an agent devotes the fraction $u(t)$ of his available time to produce the physical good and the efficiency per unit of labor supplied is $h(t)$, then $L(t) = N(t)u(t)h(t)$, where $N(t)$ is the population size. Production of the aggregate good may be accumulated as physical capital or sold for consumption. Physical capital depreciates at a constant rate, $\pi \geq 0$. The resource constraint for the physical good may then be expressed as:

$$c(t) + \dot{k}(t) + (\pi + n)k(t) \leq F[k(t), u(t)h(t)] \qquad (2.2)$$

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where $k(t)$ is the average amount of physical capital, and $\dot{k}(t)$ is the time derivative.

For simplicity of the analysis, the production process in the educational sector will be restricted to a linear technology with constant marginal productivity, $\delta > 0$. As in Lucas (1988), we assume that educational capital accrues at no cost to newly-born individuals. The resource constraint for the educational sector is then written as:

$$\dot{h}(t) + \theta h(t) \leq \delta[1 - l(t) - u(t)]h(t) \qquad (2.3)$$

where $\theta \geq 0$ represents the depreciation rate of the average stock of human capital, $h(t)$, and $\dot{h}(t)$ is the time derivative.

In this economy, the optimization problem is to choose at each moment in time the amounts of consumption and investment, and fractions of time assigned to production, education and leisure activities, so as to maximize the infinite stream of discounted instantaneous utilities, given the resource constraints (2.2) and (2.3), and initial capital stocks, k_0 and h_0. For every such optimal solution, constraints (2.2) and (2.3) must always be binding.

DEFINITION 2.1: *An optimal solution for this economy is a set of paths $\{c(t), l(t), u(t), k(t), h(t)\}_{t=0}^{\infty}$ which solve the following maximization problem*

$$W(k_0, h_0) \;=\; \max_{c(t), l(t), u(t)} \int_0^{\infty} e^{-\rho t} U[c(t), l(t)] N(t) dt \qquad (P)$$

subject to

$$\dot{k}(t) = F[k(t), u(t)h(t)] - (\pi + n)k(t) - c(t)$$
$$\dot{h}(t) = \delta[1 - l(t) - u(t)]h(t) - \theta h(t)$$
$$c(t) \geq 0, k(t) \geq 0, \; h(t) \geq 0$$
$$l(t) \geq 0, \; u(t) \geq 0, \; u(t) + l(t) \leq 1$$
$$k(0), \; h(0) \; given, \; N(t) = N_0 e^{nt}, \; \rho - n > 0$$

As already pointed out, this is not generally a standard concave problem as the stock of human capital affects asymmetrically the time spent in the various activities. Indeed, let us temporarily define $h_l = lh, h_u = uh$ and $h_e = (1 - l - u)h$. Then for a concave utility function of the form $U(c, h_l)$ the entire optimization process would constitute a standard concave problem over the set of paths

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$\{c(t), k(t), h(t), \dot{k}(t), \dot{h}(t), h_l(t), h_u(t), h_e(t)\}$, since the utility function is concave, the above constraints are convex and $h_l(t) + h_u(t) + h_e(t) = h(t)$. However, the concavity of the optimization problem is not guaranteed in our case with unqualified leisure, since $U(c, l)$ may be written as $U(c, \frac{h_l}{h})$ and $\frac{1}{h}$ is a convex function.

DEFINITION 2.2: *A balanced path (or steady-state equilibrium) for this econ-omy is an optimal solution $\{c(t), l(t), u(t), k(t), h(t)\}$ to problem (P) for some initial conditions $k(0) = k_0$ and $h(0) = h_0$, such that $c(t)$, $k(t)$ and $h(t)$ grow at constant rates, $l(t)$ and $u(t)$ remain constant, and the output-capital ratio $F[k(t), u(t)h(t)]/k(t)$ is constant.*

It is readily shown that at a steady-state the equilibrium levels $c(t)$, $k(t)$ and $h(t)$ must all grow at the same rate, say ν. Furthermore, the existence of a balanced path imposes certain restrictions on the functional forms of the utility function and technological constraints [cf. King, Plosser and Rebelo (1988)]. In addition to joint concavity, the utility function must exhibit a constant elasticity of intertemporal substitution in consumption. Also, substitution effects associated with sustained growth in consumption and labor productivity must not alter the labor supply. Under the foregoing hypotheses, only the following functional forms for the utility function are possible along a balanced path:

$$U[c(t), l(t)] = \frac{1}{1 - \sigma} \left[c(t)^\alpha \psi(l(t)) \right]^{1 - \sigma} \text{ and } U[c(t), l(t)] = \alpha \log c(t) + \phi(l(t))$$

Here, $\sigma \neq 1$ and α are positive numbers and $\psi(\cdot)$ and $\phi(\cdot)$ are C^2 functions such that $U[c(t), l(t)]$ is jointly concave and increasing in both arguments.

3. Multiplicity of Balanced Paths

We shall proceed in our analysis with the above two families of utility functions compatible with the existence of a balanced path. For both types of utilities we show that there can be a multiplicity of steady-state rays. This result holds in the absence of external effects. Moreover, under the imposed functional restrictions such multiplicity of steady states does not arise in either the standard neoclas-sical model with leisure (see the Appendix) or in the endogenous growth model with time allocated between production and educational activities [Uzawa (1965), Lucas (1988)]. Therefore, this is a minimal extension to generate the non unique-ness result. If there are multiple balanced paths, then global stability is lost and the asymptotic behavior of an optimal orbit is determined by the initial ratio of physical to human capital.

These various balanced paths may feature different rates of growth, as well as different relative allocations of time devoted to leisure, work and education. Certain testable propositions will emerge from this analysis. As shown below (Prop. 4.4), an economy with a higher proportion of human capital will grow faster and devote less time to leisure activities and more time to schooling. The time share devoted to work will be higher only if the intertemporal elasticity of substitution for the composite commodity, σ^{-1}, is less than unity.

3.1. Multiplicatively Separable Utility Functions

Let us first postulate a CES utility function of the form

$$U[c(t), l(t)] = \frac{1}{1 - \sigma} \left[c(t)^{\alpha} \psi(l(t)) \right]^{1-\sigma}$$

The monotonicity and joint concavity of U imposes further restrictions on the function $\psi(l)$ and parameter values σ and α. In particular, if $\psi(l) = l(t)^{1-\alpha}$, then these assumptions require that $\sigma > 0, \sigma \neq 1$, and $0 < \alpha < 1$.

Under this latter functional form, an interior optimal solution to problem (P) must satisfy in addition to the feasibility constraints (2.2) and (2.3) the following set of first order equations:

$$\alpha(c(t)^{\alpha} l(t)^{1-\alpha})^{-\sigma} c(t)^{\alpha-1} l(t)^{1-\alpha} = \gamma_1(t) \tag{3.1}$$

$$(1-\alpha)(c(t)^{\alpha} l(t)^{1-\alpha})^{-\sigma} c(t)^{\alpha} l(t)^{-\alpha} = \gamma_2(t) h(t) \delta \tag{3.2}$$

$$\gamma_1(t) F_L[k(t), u(t)h(t)] = \gamma_2(t) \delta \tag{3.3}$$

$$\frac{\dot{\gamma_1}(t)}{\gamma_1(t)} = \rho + \pi - F_K[k(t), u(t)h(t)] \tag{3.4}$$

$$\frac{\dot{\gamma_2}(t)}{\gamma_2(t)} = \rho - n - \delta u(t) - \delta[1 - l(t) - u(t)] + \theta \tag{3.5}$$

Then imposing steady-state conditions we can derive the long-run values for an interior solution from the following equations system:

$$\frac{1-\alpha}{\alpha} = l F_L\left(1, u\frac{h}{k}\right)\frac{h}{k}\frac{k}{c} \tag{3.6}$$

$$\rho + \pi + [1 - \alpha(1 - \sigma)]\nu = F_K\left(1, u\frac{h}{k}\right) \tag{3.7}$$

$$\rho - n = \alpha(1 - \sigma)\nu + \delta u \tag{3.8}$$

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$$\frac{c}{k} = F\left(1, u\frac{h}{k}\right) - (\pi + n + \nu) \qquad (3.9)$$

$$\nu = \delta(1 - l - u) - \theta \qquad (3.10)$$

where ν is the growth rate, and the ratios $\dfrac{c}{k}$ and $\dfrac{h}{k}$ remain constant along a given balanced path.

In order to illustrate that the system of equations (3.6)–(3.10) may contain multiple interior solutions, we first show that for a given equilibrium value for l^*, the equation subsystem (3.7)–(3.10) determines the remaining equilibrium values, u^*, $\left(\dfrac{h}{k}\right)^*$, $\left(\dfrac{c}{k}\right)^*$ and ν. That is, all these variables may be written as a function of l^*. Hence, the whole problem is reduced to the study of the existence and uniqueness of l^* in (3.6), provided that all other steady-state values fall within the feasible range.

For l^* given, we derive from equations (3.8) and (3.10) an equation on u,

$$\alpha(1 - \sigma)[\delta(1 - l^* - u)] + \delta u = \rho - n + \alpha(1 - \sigma)\theta$$

Assuming that $\alpha(1 - \sigma)[\delta - \theta] < \rho - n$ for $\rho - n > 0$,[1] then we obtain a unique value $0 < u^* \leq 1$, where $u^* = 1 - l^*$ only if $\delta(1 - l^*) \leq \rho - n + \alpha(1 - \sigma)\theta$. Moreover, from the above properties of F there exists a unique value $\left(\dfrac{h}{k}\right)^*$ that satisfies (3.7). The existence of $\left(\dfrac{c}{k}\right)^*$ and ν follows directly from (3.9) and (3.10), respectively. Finally, in order to prove the existence of l^*, we can express the right-hand side of (3.6) as a function of l. Let us represent such an expression by $\Upsilon(l)$. The existence of a steady-state then boils down to the existence of a solution to equation (3.6), for $\dfrac{1 - \alpha}{\alpha} = \Upsilon(l)$. These facts are formally summarized in the following proposition.

PROPOSITION 3.1: *Consider the optimization problem (P), where*

- *The production function $F(\cdot, \cdot)$ is a C^2 mapping, increasing, concave, linearly homogenous, and satisfies (2.1).*

- *The utility function $U(\cdot, \cdot)$ is CES, increasing, multiplicatively separable and jointly concave, $U(c, l) = \dfrac{1}{1 - \sigma}[c^\alpha l^{1-\alpha}]^{1-\sigma}$, with $\sigma > 0$, $\sigma \neq 1$, and $0 < \alpha < 1$.*

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[1]This is the transversality condition imposed in Uzawa (1965).

Assume that $\alpha(1 - \sigma)[\delta - \theta] < \rho - n$ *for* $\rho - n > 0$. *Then the following conditions are necessary and sufficient for the existence of an interior steady state* $\{\left(\frac{c}{k}\right)^{*}, l^{*}, u^{*}, \left(\frac{h}{k}\right)^{*}, \nu\}$,

(a) $\dfrac{1 - \alpha}{\alpha} \in (\min_{0<l<1} \Upsilon(\cdot), \max_{0<l<1} \Upsilon(\cdot))$,

(b) *For some* l^{*} *satisfying* $\dfrac{1 - \alpha}{\alpha} = \Upsilon(l^{*})$, *it must hold that* $\delta(1 - l^{*}) > \rho - n + \alpha(1 - \sigma)\theta$

Moreover, the number of interior steady-state rays is equal to the number of solutions l^{*}, *for* $\dfrac{1 - \alpha}{\alpha} = \Upsilon(l^{*})$, *satisfying condition* (b).

We observe that not all solutions satisfying condition (b) will conform a balanced path, since such solutions may not be optimal. Nothing guarantees, however, that there is only a unique solution that fulfils condition (b), and of those multiple solutions that only a unique one is optimal. This is presently illustrated for the simple Cobb-Douglas production function, $F(k, uh) = Bk^{\beta}(uh)^{1-\beta}, B > 0, 0 < \beta < 1$.

In a (β, σ)-plane Figure 1 displays different regions of existence of steady-state rays for parameter values $\alpha = 0.3, \rho = 0.05, n = 0, B = 1, \pi = 0, \delta = 0.23, \theta = 0$. The diagram is divided into four different regions. Region A comprises those economies with a unique interior steady state, whereas Region B contains those economies with a unique non-interior steady state (with no time allocated to education and growth). Region C may be discarded from the analysis on the grounds that at least one steady state violates the transversality condition, $\rho - n - \alpha(1 - \sigma)\nu > 0$. Finally, area D is made up of all economies with two interior steady states. One can observe from the figure that for the given parameterization the multiplicity of steady states appears for relatively high values for β and relatively low values for σ, although some of these parameters values do not seem relatively unrealistic (e.g., β around 0.35 and σ around 1).

Under the same parameterization, Figure 2 depicts in an analogous way various regions of existence of two steady states corresponding to different values of α. That is, for fixed α the dotted areas in the (β, σ)-plane refer to those economies with two interior steady states. Again, one can see that for neighboring values of $\alpha = 0.3$ there are economies containing two steady states with β close to 0.35 and

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σ close to 1.2. We now single out a representative economy of this given class and study its dynamic behavior. (This is point a in Figure 1.)

EXAMPLE 1: *Multiplicatively separable utility function,* $U(c, l) = \dfrac{(c^{\alpha} l^{1-\alpha})^{1-\sigma}}{1-\sigma}$.
In this example we consider the following parameter values

$$\sigma = 0.906, \alpha = 0.3, \rho = 0.05, n = 0, B = 1, \beta = 0.355, \pi = 0, \delta = 0.23, \theta = 0$$

For this particular case, the above equation $\dfrac{1-\alpha}{\alpha} = \Upsilon(l)$, where as before $\Upsilon(l)$ is the resulting function obtained by substituting out the remaining variables in (3.6), has two solutions, $l_1^* = 0.698$ and $l_2^* = 0.772$. (Observe that these values are relatively close to empirical estimates used in the literature.) Both solutions satisfy condition (b) of Prop. 3.1, and hence the economy contains two interior steady states. There is in addition a third steady-state, which is non-interior (i.e., not satisfying condition (b) of Prop. 3.1), with time devoted only to leisure and working activities and no time to education and growth. These stationary solutions are characterized by the following values:
Steady-State Ray 1:

$$\left(\frac{c}{h}\right)_1^* = 0.473, \ l_1^* = 0.698, \ u_1^* = 0.215, \ \left(\frac{k}{h}\right)_1^* = 2.694, \ \text{and} \ \nu_1 = 0.020$$

Steady-State Ray 2:

$$\left(\frac{c}{h}\right)_2^* = 0.612, \ l_2^* = 0.772, \ u_2^* = 0.217, \ \left(\frac{k}{h}\right)_2^* = 4.214, \ \text{and} \ \nu_2 = 0.003$$

Steady-State Ray 3:

$$\left(\frac{c}{h}\right)_3^* = 0.637, \ l_3^* = 0.783, \ u_3^* = 0.217, \ \left(\frac{k}{h}\right)_3^* = 4.522, \ \text{and} \ \nu_3 = 0$$

Observe that the different steady states generate reasonable values regarding per capita growth rates and time allocated among the various activities. In the Appendix we show that steady-state rays 1 and 3 are optimal solutions to planning problem (P) for the given initial conditions, and that steady state 2 is not optimal. This is not, however, the only possible configuration of multiple steady states for

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this class of economies. A further example is also given in the Appendix in which the three steady-state rays are all optimal solutions.[2]

Regarding the stability properties of these stationary solutions, we likewise show in the Appendix that steady states 1 and 3 are both saddle-path stable. The policy function features a simple discontinuity at a given "threshold point". Before such critical point all optimal paths converge to steady state 1, and for initial conditions beyond such point all optimal paths converge to steady state 3. Hence, without resorting to externality-type arguments the model features a "poverty trap" in the sense that an economy with a high ratio of physical to human capital may converge to a low growth steady state.

3.2. Additively Separable Utility Functions

We now study two families of additively separable utility functions which are compatible with the existence of a balanced path. The absence of cross effects renders the optimization problem easier to analyze. As a result, we shall provide an analytical characterization of those economies with multiple steady state rays.

We consider the following functional forms for the utility function:

$$
\begin{aligned}
U(c,l) &= \alpha \log c + (1-\alpha)\log l,\ 0 < \alpha < 1 \\
U(c,l) &= A \log c + l^{\mu},\ A > 0,\ 0 < \mu < 1
\end{aligned}
$$

For the most part, our analysis will focus on the first functional form with a logarithmic utility for leisure. As is well known, this utility function is the limiting case of the multiplicatively separable functional form, $U(c,l) = \dfrac{(c^{\alpha}l^{1-\alpha})^{1-\sigma} - 1}{1-\sigma}$, for $\sigma = 1$. Under this simple analytical expression, the marginal conditions for consumption and leisure become

$$
\begin{aligned}
\frac{\alpha}{c(t)} &= \gamma_1(t) \\
\frac{1-\alpha}{l(t)} &= \gamma_2 h(t)\delta
\end{aligned}
$$

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[2]Rustichini and Schmitz (1991) present a somewhat related model with also three possible uses of time and with two steady states. In their model, however, competitive allocations are not Pareto optimal. Also, for the optimal planning problem the authors simply conjecture that one steady state is non-optimal.

Furthermore, from the above system of first-order conditions we obtain that in this case all interior steady-state values $\{\left(\frac{c}{k}\right)^*, l^*, u^*, \left(\frac{h}{k}\right)^*, \nu\}$ must satisfy the following equations system:

$$\frac{1-\alpha}{\alpha} = lF_L(1, u\frac{h}{k})\frac{h}{k}\frac{k}{c} \tag{3.11}$$

$$\rho + \pi + \nu = F_k(1, u\frac{h}{k}) \tag{3.12}$$

$$\rho - n = \delta u \tag{3.13}$$

$$\frac{c}{k} = F(1, u\frac{h}{k}) - (\pi + n + \nu) \tag{3.14}$$

$$\nu = \delta(1 - l - u) - \theta \tag{3.15}$$

From these equations we can analogously establish the following results on existence of multiple stationary equilibria.

PROPOSITION 3.2: *Consider the optimization problem* (P), *where*

- *The production function* $F(\cdot, \cdot)$ *is a* C^2 *mapping, increasing, concave, linearly homogenous, and satisfies (2.1).*

- *The utility function* $U(\cdot, \cdot)$ *is an additively separable, increasing, concave function, logarithmic in consumption and leisure,* $U(c, l) = \alpha \log c + (1 - \alpha) \log l$, *with* $0 < \alpha < 1$.

Assume that $\rho - n > 0$. *Then the following conditions are necessary and sufficient for the existence of an interior steady state equilibrium* $\{\left(\frac{c}{k}\right)^*, l^*, u^*, \left(\frac{h}{k}\right)^*, \nu\}$,

(a) $\dfrac{1-\alpha}{\alpha} \in (\min_{0<l<1} \Psi(l), \max_{0<l<1} \Psi(l))$,

(b) *For some* l^* *satisfying* $\dfrac{1-\alpha}{\alpha} = \Psi(l^*)$, *it must hold that* $\delta(1 - l^*) > \rho - n$,

where

$$\Psi(l) = \frac{lF_L\left[F_K^{-1}[\rho+\pi-\theta+\delta(1-\frac{\rho-n}{\delta}-l)]\right]\frac{\delta}{\rho-n}F_K^{-1}[\rho+\pi-\theta+\delta(1-\frac{\rho-n}{\delta}-l)]}{F\left[F_K^{-1}[\rho+\pi-\theta+\delta(1-\frac{\rho-n}{\delta}-l)]\right]-\delta(1-\frac{\rho-n}{\delta}-l)-n-\pi+\theta} \tag{3.16}$$

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Moreover, the number of interior steady state rays is equal to the number of solutions l^*, *for* $\dfrac{1-\alpha}{\alpha} = \Psi(l^*)$, *satisfying condition* (b).

For the basic Cobb-Douglas technology, $F(k, uh) = Bk^\beta(uh)^{1-\beta}$, $B > 0, 0 < \beta < 1$, equation $\dfrac{1-\alpha}{\alpha} = \Psi(l)$ leads to the following quadratic expression

$$\frac{(1-\alpha)}{\alpha}\left[\frac{\delta(1-l)(1-\beta)}{\beta} + \frac{(1-\beta)}{\beta}(\pi + n - \theta) + \rho - n\right]\left[\frac{\rho-n}{\delta}\right] - \frac{l(1-\beta)}{\beta}[(1-l)\delta + n + \pi - \theta] = 0 \tag{3.17}$$

This quadratic equation may contain two positive roots, $1 > l_2^* > l_1^* > 0$. Such roots are determined by the corresponding values,

$$\frac{\left[\frac{1-\alpha}{\alpha}(\rho-n)+\delta+n+\pi-\theta\right]\pm\left(\left[\frac{(1-\alpha)}{\alpha}(\rho-n)+\delta+n+\pi-\theta\right]^2 - 4\frac{(1-\alpha)}{\alpha}(\rho-n)\left[\delta+\pi+n-\theta+\frac{\beta}{1-\beta}(\rho-n)\right]\right)^{1/2}}{2\delta} \tag{3.18}$$

Of course, in order to guarantee that each of these roots generates an interior steady state, condition (b) in Prop. 3.2 must be satisfied. For this particular case, there is however available an alternative characterization since by equation (3.13) the time devoted to goods production is constant over all steady states, i.e., $u^* = \dfrac{\rho - n}{\delta}$. Thus, we have

(a) If $l_2 > \dfrac{\delta - \rho + n}{\delta}$ and $l_1 \in \left[0, \dfrac{\delta - \rho + n}{\delta}\right]$, then the economy has a unique interior steady-state ray.

(b) If $l_2 \in \left[0, \dfrac{(\delta - \rho + n)}{\delta}\right]$ and $l_1 \in \left[0, \dfrac{(\delta - \rho + n)}{\delta}\right]$, then the economy has two interior steady-state rays.

(c) In all other cases the economy has no interior steady states.

All these possibilities are summarized in the following proposition whose proof follows from direct inspection of (3.18).

PROPOSITION 3.3: *Consider the optimization problem* (P) *with* $U(c,l) = \alpha \log c + (1-\alpha)\log l$, *and* $F(k, uh) = Bk^\beta(uh)^{1-\beta}$, *where* $0 < \alpha < 1, 0 < \beta < 1$ *and* $B > 0$. *Assume that* $\delta > \rho - n > 0$. *Then*

(a) *If* $\dfrac{\beta}{1-\beta} < \dfrac{(\rho + \pi - \theta)(\alpha\delta - \rho + n)}{(1-\alpha)(\rho-n)^2}$, *then the economy has a unique interior steady-state ray.*

(b) *If* $\alpha \in \left(\dfrac{\rho - n}{\delta - \rho + \theta - \pi}, 1\right)$ *and* $0 < \beta < 1$ *is such that*

$\dfrac{\beta}{1-\beta} \in \left(\dfrac{[(1-\alpha)(\rho-n)-\alpha(\delta+n+\pi-\theta)]^2}{4\alpha(1-\alpha)(\rho-n)^2}, \dfrac{(\rho+\pi-\theta)(\alpha\delta-\rho+n)}{(1-\alpha)(\rho-n)^2}\right)$, *then the economy has two interior steady-state rays.*

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(c) In all other cases the economy has a unique non-interior steady-state ray with no time allocated to educational activities.

To see more transparently the nature of these results, Figure 3 portrays in a (α, β)-plane these three regions of existence of balanced paths for fixed parameter values $\rho = 0.05, n = 0, B = 1, \pi = 0, \delta = 0.25$ and $\theta = 0$. Observe from the figure that in the region of multiple steady states the values for α and β are monotonically related. Moreover, such monotonic relation is also associated with higher growth rates. Considering that α may take a value close to 0.3, then corresponding values for β seem to be relatively high although not extremely unrealistic. We now examine an example within this family. (This is point *b* in the diagram.)

EXAMPLE 2: *Additively separable utility function, logarithmic in consumption and leisure:* $U(c, l) = \alpha \log c + (1 - \alpha) \log l$. Consider the following parameter values

$$\alpha = 0.291, \rho = 0.05, n = 0, B = 1, \beta = 0.4, \dot{\pi} = 0, \delta = 0.25, \theta = 0$$

In this case, equation $\dfrac{1 - \alpha}{\alpha} = \Psi(l)$, given by (3.16), has two solutions $l_1 = 0.7162$ and $l_2 = 0.771$. Both solutions satisfy condition (b) of Prop. 3.3 , and hence the economy has two interior steady states. Moreover, there is also a non-interior steady state with time devoted only to leisure and working activities, and no time devoted to education. These stationary solutions are characterized by the following values.

Steady-State Ray 1:

$$\left(\frac{c}{h}\right)^{\bullet}_{1} = 0.5588, l^{\bullet}_1 = 0.7162, u^{\bullet}_1 = 0.2000, \left(\frac{k}{h}\right)^{\bullet}_{1} = 3.5738, \text{ and } \nu_1 = 0.0209$$

Steady-State Ray 2:

$$\left(\frac{c}{h}\right)^{\bullet}_{2} = 0.6939, l^{\bullet}_2 = 0.7710, u^{\bullet}_2 = 0.2000, \left(\frac{k}{h}\right)^{\bullet}_{2} = 5.1073, \text{ and } \nu_2 = 0.0072$$

Steady-State Ray 3:

$$\left(\frac{c}{h}\right)^{\bullet}_{3} = 0.7904, l^{\bullet}_3 = 0.8024, u^{\bullet}_3 = 0.1976, \left(\frac{k}{h}\right)^{\bullet}_{3} = 6.3232, \text{ and } \nu_3 = 0$$

Observe that these different steady-states generate reasonable values regarding per capita growth rates and time allocated among the various activities. As in

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Example 1, steady states 1 and 3 are both optimal solutions to the planning problem for given initial conditions, and steady state 2 is not optimal. Also, the dynamic behavior of optimal orbits in this economy is qualitatively the same as that of Example 1. There is a "threshold point" such that before such point all optimal paths converge to steady state 1, and beyond such a critical point all optimal paths converge to steady state 3.[3]

Our next example illustrates that the multiplicity of steady states also occurs for utility functions of the form, $U(c, l) = A \log c + l^\mu$. In contrast to the previous examples, we consider positive rates of depreciation for both stocks of capital, with the result that in a boundary balanced path the rate of growth must be negative. Again, in this example there are three steady states, and steady states 1 and 3 are optimal solutions to the planning problem. Also, the dynamic behavior of the model is qualitatively the same as that of the two previous examples.

EXAMPLE 3: *Additively separable utility function logarithmic in consumption,* $U(c, l) = A \log c + l^\mu$. Consider the following parameter values

$$A = 0.1786, \mu = 0.6, \rho = 0.05, n = 0, B = 1, \beta = 0.35, \pi = 0.04, \delta = 0.25, \theta = 0.02$$

This economy also contains three stationary solutions, which are characterized by the following values.

Steady-State Ray 1:

$$\left(\frac{c}{h}\right)^*_1 = 0.302, l^*_1 = 0.639, u^*_1 = 0.2, \left(\frac{k}{h}\right)^*_1 = 1.186, \text{ and } \nu_1 = 0.02$$

Steady-State Ray 2:

$$\left(\frac{c}{h}\right)^*_2 = 0.3754, l^*_2 = 0.750, u^*_2 = 0.2, \left(\frac{k}{h}\right)^*_2 = 1.8475, \text{ and } \nu_2 = -0.008$$

Steady-State Ray 3:

$$\left(\frac{c}{h}\right)^*_3 = 0.4216, l^*_3 = 0.8031, u^*_3 = 0.1969, \left(\frac{k}{h}\right)^*_3 = 2.3420, \text{ and } \nu_3 = -0.02$$

---

[3]The proof of these assertions is omitted as it follows from the same methods outlined in the Appendix.

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3.3. Comparative analysis

In the context of our simple model, we now provide further results on the behavior of our economic variables across interior steady states. We show that for those steady states with a higher physical capital ratio, optimizing agents would consume a higher proportion of output and devote more time to leisure activities and less time to education and growth. The time devoted to work is undetermined, and depends on whether the elasticity of intertemporal substitution is above or below unity.

PROPOSITION 3.4: *Let $U(c,l) = \dfrac{(c^\alpha l^{1-\alpha})^{1-\sigma}}{1-\sigma}$, for $\sigma \neq 1$, and $U(c,l) = \alpha \log c + (1-\alpha)\log l$ for $\sigma = 1$. Let $F(k,uh) = Bk^\beta(uh)^{1-\beta}$ for $B > 0$, and $0 < \beta < 1$. Let $\dot{h} = \delta(1 - l - u)h - \theta h$ for $\delta > 0$ and $\theta \geq 0$. Let*

$$\frac{\beta}{1-\beta}(\rho - n) > \min\{\alpha(\sigma - 1)(\rho + \pi), (\rho + \pi)\} \tag{3.19}$$

Assume that there are two interior balanced paths $\left\{\left(\dfrac{c}{k}\right)_1^*, l_1^*, u_1^*, \left(\dfrac{k_1}{h_1}\right)^*, \nu_1\right\}$ and $\left\{\left(\dfrac{c}{k}\right)_2^*, l_2^*, u_2^*, \left(\dfrac{k}{h}\right)_2^*, \nu_2\right\}$ such that $\left(\dfrac{k}{h}\right)_2^* > \left(\dfrac{k}{h}\right)_1^*$. Assume that $\nu_2 \geq 0$ and $\nu_1 \geq 0$. Then

(a) Leisure is higher in the physical-capital intensive balanced path: $l_2^* > l_1^*$.

(b) The time devoted to work depends on the elasticity of intertemporal substitution: $u_2^* < u_1^*$ for $\sigma > 1, u_2^* = u_1^*$ for $\sigma = 1, u_2^* > u_1^*$ for $\sigma < 1$.

(c) The rate of growth is lower in the physical-capital intensive balanced path: $\nu_2 < \nu_1$.

(d) For $h_1^* = h_2^* = 1$, consumption is higher in the physical-capital intensive balanced path: $c_2^* > c_1^*$. Moreover, the ratio of consumption to physical capital depends on the elasticity of intertemporal substitution for consumption and the elasticity of physical capital with respect to the marginal productivity of labor: $\left(\dfrac{c}{k}\right)_2^* < \left(\dfrac{c}{k}\right)_1^*$ for $1 - \alpha(1 - \sigma) > \beta$, $\left(\dfrac{c}{k}\right)_2^* = \left(\dfrac{c}{k}\right)_1^*$ for $1 - \alpha(1 - \sigma) = \beta$ and $\left(\dfrac{c}{k}\right)_2^* > \left(\dfrac{c}{k}\right)_1^*$ for $1 - \alpha(1 - \sigma) < \beta$.

We remark that (3.19) is merely a sufficient condition for these results to hold true, and such condition is automatically satisfied for $\sigma \leq 1$. Moreover, for $\sigma > 1$ the condition holds for standard calibrations of parameter values. Also, observe that for $\sigma > 1$ parts (a)-(c) are consistent with the evidence reported in Ríos-Rull

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(1993) in the sense that more qualified agents devote a higher fraction of their time to worked hours and education, and a smaller fraction to leisure activities. We are not aware of empirical evidence related to part (d).

PROOF OF PROPOSITION 3.4: (a) Without loss of generality, we assume that $h^* = 1$. Then we shall show from equations (3.6)-(3.10) that for the asserted functional forms the derivative $\dfrac{dl^*}{dk^*} > 0$. This later result will be established from a simple application of the inverse function theorem once we show that $\dfrac{dk^*}{dl^*} > 0$.

A straightforward manipulation of (3.7) yields that

$$k^* = u^* \left[\frac{\delta(1 - l^*) + n + \pi - \theta}{\beta} \right]^{\frac{1}{\beta - 1}} \tag{3.20}$$

Totally differentiating (3.20) with respect to l^*, and taking account of the fact from (3.8) and (3.10) that $\dfrac{du^*}{dl^*} = \dfrac{\alpha(1 - \sigma)}{[1 - \alpha(1 - \sigma)]}$, we obtain

$$\frac{dk^*}{dl^*} = \frac{\alpha(1 - \sigma)}{[1 - \alpha(1 - \sigma)]} \left(\frac{k^*}{u^*} \right) + \frac{\delta u^*}{\beta(1 - \beta)} \left(\frac{k^*}{u^*} \right)^{2 - \beta} \tag{3.21}$$

As proved below, (3.19) is a sufficient condition for expression (3.21) to be positive. Hence, this establishes part (a).

(b) As already pointed out, equations (3.8) and (3.10) imply that

$$\frac{du^*}{dl^*} = \frac{\alpha(1 - \sigma)}{[1 - \alpha(1 - \sigma)]}$$

Hence, the result now follows as a direct consequence of the chain rule and the fact that $\dfrac{dl^*}{dk^*} > 0$. This completes the proof of part (b).

(c) From equations (3.8) and (3.10) we also have that

$$\frac{d\nu}{dl^*} = \frac{-\delta}{[1 - \alpha(1 - \sigma)]}$$

Using again the chain rule, we obtain that $\nu_2 \lessgtr \nu_1$. This proves part (c).

(d) From equation (3.6) we obtain

$$c^* = \frac{(1 - \beta)\alpha}{(1 - \alpha)} l^* \left(\frac{k^*}{u^*} \right)^{\beta} \tag{3.22}$$

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Substituting out for $\frac{k^*}{u^*}$ from (3.20), equation (3.22) then implies that $\frac{dc^*}{dl^*} > 0$. Hence, $c_2^* > c_1^*$. Moreover, regarding the ratio $\left(\frac{c}{k}\right)^*$, a straightforward substitution of (3.7) into (3.9) implies

$$\left(\frac{c}{k}\right)^* = \frac{\rho + (1 - \beta)\pi}{\beta} + \nu \frac{1 - \alpha(1 - \alpha) - \beta}{\beta} - n$$

This expression yields directly the remaining results asserted in part (d).

In order to complete the proof of the theorem, we need to establish that (3.21) is positive. After simple manipulations we have that (3.21) is positive if and only if

$$\delta u^*[1 - \alpha(1 - \sigma)] > (1 - \beta)(\sigma - 1)\alpha\beta \left(\frac{k^*}{u^*}\right)^{\beta-1} \tag{3.23}$$

Moreover, plugging in $\frac{k^*}{u^*}$ from (3.20) into (3.23) yields that

$$\delta u^*[1 - \alpha(1 - \sigma)] > (1 - \beta)(\sigma - 1)\alpha[\pi + \rho + [1 - \alpha(1 - \sigma)]\nu^*] \tag{3.24}$$

Also, from equation (3.8) the growth rate can be expressed in terms of worked hours as $\nu = \frac{\rho - n - \delta u^*}{\alpha(1 - \sigma)}$. Then, after some simple rearrangements, (3.24) is equivalent to

$$\frac{\beta}{(1 - \beta)}\delta u^*[1 - \alpha(1 - \sigma)] > \alpha(\sigma - 1)(\rho + \pi) - [1 - \alpha(1 - \sigma)](\rho - n)$$

Since the last term of the right-hand side of this expression is negative, we must have

$$\frac{\beta}{(1 - \beta)}\delta u^*[1 - \alpha(1 - \sigma)] > \alpha(\sigma - 1)(\rho + \pi)$$

Finally, we observe that for $\sigma \geq 1$ equation (3.8) implies that $\delta u^* \geq \rho - n$. Hence,

$$\frac{\beta}{(1 - \beta)}(\rho - n) + \frac{\beta}{(1 - \beta)}\alpha(\sigma - 1)(\rho - n) > \alpha(\sigma - 1)(\rho + \pi) \tag{3.25}$$

One readily checks that (3.25) is true under condition (3.19). Therefore, this rather long argumentation leads us to the conclusion that (3.21) is positive under (3.19). The proof is complete.

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4. Transitional Dynamics: The Case of Additively Separable Utility Functions

We now examine the behavior of our economic variables along the transition to an interior, locally stable balanced path. For expositional convenience, we shall focus on the simpler case of additively separable, logarithmic utility functions in consumption and leisure, $U(c, l) = \alpha \log c + (1 - \alpha) \log l$, with $0 < \alpha < 1$. As in Caballé and Santos (1993), our analysis will be restricted to the case of a sudden increase in the stock of physical capital near a given steady state solution. (Symmetric conclusions may be drawn for a sudden decrement in the level of physical capital, or equivalently for a sudden increase in the level of human capital.)

A sudden increase in the stock of physical capital sets up a transitional process for consumption and investment, leisure, worked hours, education, and the rate of growth. After an appropriate normalization of the stock variables, we find that an increment in physical capital leads to an immediate increase in consumption and leisure. Then along the transition the levels of consumption, leisure and physical capital go down. The transitional dynamics for worked hours and education are still undetermined. Indeed, without further restrictions on utilities and technologies it is possible to obtain the following three cases: (a) *The normal case*, after a sudden increase in k the time devoted to education goes up, and the economy converges to a higher steady state; (b) *The exogenous growth case*, after a sudden increase in k the time devoted to education remains unchanged, and the economy converges back to the same steady state; (c) *The paradoxical case*, after a sudden increase in k the time devoted to education goes down and the economy converges to a lower steady state.

In contrast to the analogous analysis of Caballé and Santos (1993) of the Uzawa-Lucas model, these three cases are not solely determined by the elasticity of intertemporal substitution, σ^{-1}, and the elasticity of the marginal productivity of labor with respect to capital, β. Thus, we shall illustrate from some numerical computations that other important parameters of the model such as the rate of discount, ρ, the rate of population growth, n, the relative weight of leisure in the instantaneous utility, α, and the productivity of the human capital technology, δ, also play a relevant role to single out these three growth cases. Indeed, the paradoxical case is even plausible for a logarithmic utility function.[4]

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[4] In Caballé and Santos (1993) the normal case is obtained for $\sigma > \beta$, the paradoxical case for $\sigma < \beta$, and the exogenous growth case implies that $\sigma = \beta$. Hence, for $0 < \beta < 1$ the paradoxical case cannot arise under an instantaneous logarithmic utility function; i.e., for $\sigma = 1$.

We first proceed with a re-scaling of our level variables in order to render our dynamic problem time invariant. Let

$$
\begin{aligned}
\tilde{c}(t) &= c(t)e^{-\nu t} \\
\tilde{k}(t) &= k(t)e^{-\nu t} \\
\tilde{h}(t) &= h(t)e^{-\nu t}
\end{aligned}
$$

where ν is the rate of growth at a given balanced path. Hence, the normalized values $\tilde{c}, \tilde{k}$ and $\tilde{h}$ remain constant over such a stationary solution.

Under this redefinition of our variables the first-order conditions and feasible constraints for an interior solution may be written as

$$
\frac{\alpha}{\tilde{c}(t)} = \gamma_1(t) \tag{4.1}
$$

$$
\frac{(1-\alpha)}{l(t)} = \gamma_2(t)\tilde{h}(t)\delta \tag{4.2}
$$

$$
\gamma_1(t)F_L[\tilde{k}(t), u(t)\tilde{h}(t)] = \gamma_2(t)\delta \tag{4.3}
$$

$$
\frac{\dot{\gamma_1}(t)}{\gamma_1(t)} = \rho + \pi + \nu - F_K[\tilde{k}(t), u(t)\tilde{h}(t)] \tag{4.4}
$$

$$
\frac{\dot{\gamma_2}(t)}{\gamma_2(t)} = \rho - n + \theta + \nu - \delta u(t) - \delta[1 - u(t) - l(t)] \tag{4.5}
$$

$$
\dot{\tilde{k}}(t) = F[\tilde{k}(t), u(t)\tilde{h}(t)] - (\pi + n + \nu)\tilde{k}(t) - \tilde{c}(t) \tag{4.6}
$$

$$
\dot{\tilde{h}}(t) = \delta[1 - l(t) - u(t)]\tilde{h}(t) - (\theta + \nu)\tilde{h}(t) \tag{4.7}
$$

We now assume that the economy is at a stable interior steady state $\{\frac{\tilde{c}^*}{\tilde{k}^*}, l^*, u^*, \frac{\tilde{h}^*}{\tilde{k}^*}, \nu\}$ and examine the behavior of our economic variables after a small positive shock in the level of physical capital. For convenience, we suppose that $F(k, uh) = k^\beta(uh)^{1-\beta}$ with $0 < \beta < 1$.

Leisure and worked hours: *After a sudden increment in physical capital, leisure cannot decrease, and in the normal growth case worked hours go down.* We first show by a contradictory argument that l cannot decrease. Assume that l decreases. It follows then from (4.2) that γ_2 must increase. As shown in the Appendix, the derivative of the value function $DW(k, h) = (\gamma_1, \gamma_2)$. Since such derivative is in the case of a logaritmic utility homogeneous of degree -1, we have that

$$
W_{kk}(k, h)k + W_{kh}(k, h)h = -\gamma_1 \tag{4.8}
$$

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where subscripts connote partial differentiation with respect to the corresponding variables. Therefore, $\dfrac{\partial \gamma_1}{\partial k}\dfrac{k}{\gamma_1} + \dfrac{\partial \gamma_2}{\partial k}\dfrac{h}{\gamma_1} = -1$. Thus, if $\dfrac{\partial \gamma_2}{\partial k} \geq 0$ then the elasticity of $\dfrac{\gamma_1}{\gamma_2}$ with respect to k is no less than unity in absolute value. Observe that the elasticity of F_L with respect to k is $0 < \beta < 1$. Hence, (4.3) implies that u must go down, and this is impossible in the paradoxical and exogenous growth cases. Moreover, regarding the normal case, if l goes down we have from (4.5) that $\dot{\gamma}_2(t) > 0$. Hence, $\dfrac{\partial \gamma_2(t)}{\partial k} \leq 0$ as in the normal case $\gamma_2(t)$ converges to a steady state with a lower value for γ_2 [cf. Caballé and Santos (1993)]. But $\dfrac{\partial \gamma_2}{\partial k} \leq 0$ implies from (4.2) that l cannot go down. We then conclude that l cannot decrease, and consequently in the normal case u must go down since in such case $(1 - l - u)$ goes up.

Consumption: *Consumption jumps up immediately and then goes down along the transition.* For the normal and exogenous growth cases, the immediate jump in consumption is readily shown from our preceding arguments and (4.3), since γ_1 goes down as k goes up. For the paradoxical case, just notice that if c goes down, as $\tilde{k} < 0$ in such case. equation (4.6) implies that u cannot go up in the same proportion as k. We then have that $F_L(\tilde{k}, u\tilde{h})$ increases. Hence, from (4.3) we obtain that $\dfrac{\partial \gamma_1}{\partial k} < 0$, since $\dfrac{\partial \gamma_2}{\partial k} \leq 0$ by the previous paragraph. This necessarily entails that c must increase. Moreover, from (4.8) we have that $-\dfrac{\partial \gamma_1}{\partial k}\dfrac{k}{\gamma_1} \leq 1$; and thus, the elasticity $\dfrac{\partial c}{\partial k}\dfrac{k}{c} \leq 1$. Along the transition, $\dot{\tilde{c}}(t) < 0$, as (4.4) implies that $\dot{\gamma}_1(t) > 0$.

Physical Capital : *Physical capital accumulation is negative along the transition.* This claim can also be proved by a contradiction argument. Assume that physical capital accumulation is non-negative along the transition; for example, that the economy goes from point a to b, in Figure 4. Observe that as shown previously the cross-partial derivative of the value function $W_{hk}(k, h) = \dfrac{\partial \gamma_2}{\partial k} \leq 0$, and hence $W_{kh}(k, h) = \dfrac{\partial \gamma_1}{\partial h} \leq 0$. However, along the transition $\dot{\gamma}_1(t) > 0$, and if the economy goes from point a to b, we obtain from the preceding paragraph that $\dfrac{\partial \gamma_1}{\partial h} > 0$. This contradiction shows that physical capital accumulation must be negative.

<u>Human Capital</u> : *Human capital may go up or down depending upon parameter values*. Given our simple linear technology for human capital accumulation, this means that the time devoted to education is undetermined along the transition. Thus, even in the case of a logarithmic utility function, we shall presently show that it is possible to obtain the exogenous and paradoxical growth cases. In the exogenous growth case $\frac{\partial u}{\partial k} = -\frac{\partial l}{\partial k}$. Hence, taking *logs* in equations (4.1)–(4.3), and after some simple arrangements, we obtain the following equation for the exogenous growth case

$$\left(1 + \frac{\beta l^*}{u^*}\right) \frac{\partial \gamma_2}{\partial k} \frac{k}{\gamma_2} - \frac{\partial \gamma_1}{\partial k} \frac{k}{\gamma_1} - \beta = 0 \tag{4.9}$$

As already shown, $0 \le -\frac{\partial \gamma_1}{\partial k} \frac{k}{\gamma_1} \le 1$, $\frac{\partial \gamma_2}{\partial k} \frac{k}{\gamma_2} \le 0$. Moreover, without further restrictions (4.9) may hold with equality. This is illustrated in the following numerical exercises that trace out a set of parameter values to single out the three growth cases.

In a (α, β)-plane, Figure 5 shows in the context of a reasonably calibrated economy that for α-values about 0.3 the paradoxical case may arise for β-values close to 0.3. As it is to be expected, the region of paradoxical growth gets smaller for decrements of the relative weight of leisure in the instantaneous utility (i.e., for higher values for α). Similarly, Figure 6 illustrates the trade-off between the marginal productivity of the human capital sector, δ, and parameter β. A more productive human capital technology makes more attractive the time spent in that sector, and so the normal growth case becomes more likely. Analogous results are available for the rate of discount, ρ, and the rate of population growth, n; along the transition a more patient economy attaches a higher value to education, and this renders more plausibility to the normal growth case.

5. Concluding Remarks

In this paper we have focused on the equilibrium dynamics of an endogenous growth model with physical and human capital accumulation and with three alternative uses of available time: unqualified leisure, work and education. The model provides a general equilibrium framework to address issues related to growth theory, taxation, business cycles and labor economies —on how various policies may affect the intertemporal allocation of consumption, leisure, worked hours, and education.

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From a technical point of view, the inclusion of unqualified leisure in our endogenous growth framework leads to a non-concave optimization problem. In the Appendix we outline a general method of proof in which for all cases studied optimal solutions can be characterized from the usual variational conditions. Moreover, such results insure in our setting the equivalence between competitive allocations and optimal solutions of the given planning problem.

Even for the most basic technologies and utilities, we find that our model may contain a multiplicity of optimal balanced paths. Unlike related literature in this area, such multiplicity holds in the absence of any type of externalities. A country with a higher ratio of human capital may choose to grow faster, consume initially less, and devote less time to leisure activities. A higher stock of human capital increases the productivity in the goods sector, and results thus in a higher opportunity cost for leisure. As a consequence, the economy may allocate a smaller amount of time to leisure activities and a greater proportion to work and education. Therefore, policies that bring about changes in the ratio of physical to human capital may vary the long-term rate of growth of an economy.

The multiplicity of steady states resembles certain patterns of behavior observed in labor markets. It has been documented [cf. Ríos-Rull (1993)] that skilled people devote more time to work and education, and less time to leisure activities. Although these issues are more rigorously approached in an economic model with heterogeneous agents, our simple framework may still offer some insights about such empirical regularities. First, our analysis has shown that it is optimal for skilled agents to choose a higher rate of growth, since such agents face an increased opportunity cost for leisure. Second, testable propositions have been derived about some properties of the different steady-state configurations, and such patterns of behavior are related to parameters and elasticities of the model.

In the particular context of growth theory, it is yet to be explored that the various patterns of behavior are linked to relative endowments of physical wealth and education. As pointed out in Becker et. al. (1990), at least since S. Mill (1848) it has been observed that countries with a higher proportion of human capital display higher growth rates. It remains as an empirical investigation to determine the economic conditions under which such higher growth rates are transitory or permanent. Our model suggests that for small perturbations in the ratio of physical to human capital these higher rates of growth are transitory, but for large deviations from the given ratio an economy may move to a different balanced path with a higher long-run growth rate. Further research on these issues will no doubt improve our understanding of the process of convergence and

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the dynamics of growth.

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6. Appendix

The purpose of this appendix is to establish two different kind of results invoked in the text. In Part I we address the issue of the multiplicity of steady states in a simple version of the exogenous growth model with leisure. We show that under our previous assumptions on utility and production functions there is at most a unique steady-state solution. In Part II we reconsider our endogenous growth model. We outline a general method of proof that allows to characterize optimal solutions from the first-order conditions derived from the Maximum Principle. This analysis shows that in some cases all multiple steady states can be optimal solutions to the planning problem for the specified initial conditions. We shall also study the stability properties of the multiple balanced paths.

Part I

Consider the following simple version of the exogenous growth model with leisure. Find a continuous path $\{c(t), l(t), k(t)\}_{t=0}^{\infty}$ as a solution to

$$\max \int_0^{\infty} e^{-(\rho-n)t} U[c(t), l(t)] dt$$

subject to
$$\dot{k}(t) = F[k(t), (1 - l(t))h(t)] - (\pi + n)k(t) - c(t)$$
$$\overleftarrow{h}(t) = e^{\nu t}h(0), \rho - n > \nu \geq 0$$
$$c(t) \geq 0, \ 0 \leq l(t) \leq 1, k(t) \geq 0, \ h(t) \geq 0$$
$$k(0), \ h(0) \ given$$

As in Section 2, we assume that $U[c(t), l(t)] = \dfrac{[c(t)^{\alpha}\psi(l(t))]^{1-\sigma}}{1 - \sigma}$ is an increasing, strongly concave, C^2 mapping with $\alpha > 0$ and $\sigma > 0$. Also, $F[k(t), (1 - l(t))h(t)]$ is an increasing, concave, linearly homogeneous, C^2 mapping that satisfies (2.1).

This is a standard concave problem that features a unique optimal solution for every initial pair of non-negative capitals $(k(0), h(0))$. Moreover, the methods applied in Section 3 [cf. equations (3.6)-(3.10)] imply that an interior optimal sta-

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tionary solution $\left\{ \left(\frac{c}{k}\right)^*, l^*, \left(\frac{h}{k}\right)^* \right\}$ must satisfy the following equations system,

$$\frac{c}{\alpha h} \frac{\psi'(l)}{\psi(l)} = F_L\left(1, \frac{(1-l)h}{k}\right) \tag{6.1}$$

$$\rho + \pi + [1 - \alpha(1-\sigma)]\nu = F_K\left(1, \frac{(1-l)h}{k}\right) \tag{6.2}$$

$$\frac{c}{k} = F\left(1, \frac{(1-l)h}{k}\right) - (\pi + n + \nu) \tag{6.3}$$

Since the left-hand side in (6.2) is a given number, it follows that $F_K\left(1, \frac{(1-l)h}{k}\right)$ must be constant over all possible steady states. Furthermore, $\frac{(1-l)h}{k}$ and $F_L\left(1, \frac{(1-l)h}{k}\right)$ must also remain unchanged.

Let us normalize $h = 1$, and consider two different stationary solutions $\{c_1, l_1, k_1\}$ and $\{c_2, l_2, k_2\}$ with $k_1 > k_2$. By virtue of (6.2) we obtain that $l_1 < l_2$. Furthermore, from (6.3) it follows that $c_1 > c_2$. Also, the instantaneous utility must be higher in the steady state with a greater amount of physical capital. That is,

$$\frac{[c_1^\alpha \psi(l_1)]^{1-\sigma}}{1-\sigma} > \frac{[c_2^\alpha \psi(l_2)]^{1-\sigma}}{1-\sigma} \tag{6.4}$$

Observe that the right-hand side of (6.1) is constant, and so multiple steady states are not possible if both consumption and leisure are normal goods. Moreover, since $\frac{[c^\alpha \psi(l)]^{1-\sigma}}{1-\sigma}$ is a concave function, (6.4) implies that the consumer will lower the amount of leisure from steady state 1 to steady state 2 if

$$\frac{c}{\alpha h} \frac{\psi'(l_1)}{\psi(l_1)} < F_L\left(1, \frac{(1-l)h}{k}\right) \tag{6.5}$$

where $c < c_1$. That is, for every pair (c, l_1) with $c < c_1$ the marginal relation of substitution $\frac{U_l}{U_c}$ must be smaller than the marginal productivity F_L. But (6.5) follows immediately from (6.1) evaluated at (c_1, l_1, k_1), for $c < c_1$ and F_L constant across steady states. This implies that $l_1 > l_2$, which is a contradiction to our previous assertion that $l_2 > l_1$.

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This contradictory argument then establishes that under the present assumptions there is at most a unique steady state solution in the above version of the exogenous growth model with leisure. Moreover, the result also illustrates that the existence of multiple balanced paths in our endogenous growth framework is not directly related to the fact that leisure may be an inferior good.

Part II

This subsection is concerned with the existence and characterization of optimal solutions in our endogenous growth model. As shown in Lucas (1990), every optimal solution may be decentralized as a competitive equilibrium. Moreover, from the methods developed here it readily follows that every competitive equilibrium defines an optimal allocation. Hence, our framework preserves the traditional equivalence between competitive and optimal solutions, even though unqualified leisure is a potential source of non-convexities.

Our method of proof rests upon the underlying basic assumption that the instantaneous objective is concave in the control variables –although such functional is not necessarily jointly concave in the state and control variables.[5] The strategy of proof is first to construct a "candidate" mapping for the value function from the first-order variational conditions. Then we check that the resulting mapping satisfies the Bellman equation. Since Bellman's functional equation has a unique fixed point, we thus obtain that such mapping is the true value function that characterizes the corresponding optimal solution. This somewhat roundabout procedure is essentially what Fleming and Rishel (1975, Ch. IV) term the "verification theorem", and it may be of particular interest in related applications.

With the aid of these methods, we then examine the optimality of the various steady-state rays. In all of our examples the optimality of these stationary solutions is related to its stability properties: Only unstable steady states with complex roots may be non-optimal. As a consequence, there are economies where all the steady-state rays are optimal solutions to the planning problem for the given initial conditions.

For the sake of convenience, the proof of these facts has been structured in a series of claims.

(1) We first embed our model in a standard reduced form, and verify the concavity of the instantaneous objective in the controls. Let

$$v(k, h, \dot{k}, \dot{h}) \;=\; \max_{c,l,u} \; U(c,l)$$

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[5]Concavity in the optimal control variables plays a major role in standard proofs of existence of optimal solutions (cf. Fleming and Rishel, 1975, Ch. III).

$$s.\ t.$$
$$\dot{k} = F(k, uh) - (n + \pi)k - c$$
$$\dot{h} = \delta(1 - l - u)h - \theta h$$

In the case that such optimization problem has no solution, let $v(k, h, \dot{k}, \dot{h}) = -\infty$. It follows then from our asserted hypotheses that $v(k, h, \dot{k}, \dot{h})$ is upper semicontinuous, and concave in $(\dot{k}, \dot{h})$. Moreover, $\{k(t), h(t)\}$ is an optimal solution to the problem (P) if and only if it is an optimal solution to

$$W'(k(0), h(0)) = \max \int_0^\infty e^{-(\rho - n)t} v(k(t), h(t), \dot{k}(t), \dot{h}(t)) dt \qquad (P')$$

$k(0), h(0)$ given, and $\rho - n > 0$.

(2) The existence of an absolutely continuous, optimal path $\{k(t), h(t)\}_{t \geq 0}$ to problem (P') follows from the standard theory [cf., Fleming and Rishel (1975), Carlson and Haurie (1987) and Toman (1989)].

In order to apply directly these methods the space of feasible solutions must be bounded. However, this condition is easily obtained after a normalization of the variables $\{c, h, k\}$ in the way proposed in Section 4. Also, observe that a crucial condition for the existence of optimal solutions is the concavity of the mapping $v(k, h, \cdot, \cdot)$ in the controls $(\dot{k}, \dot{h})$ for every fixed pair of state variables (k, h).

(3) We now focus on the dynamics of solutions of the Euler equations converging to the steady-state ray. After substituting out in (2.2)-(2.3) and (3.1)-(3.5) for the control l and the co-state variables γ_1 and γ_2 we obtain the following system of differential equations in the variables c, u, k and h,

$$\sigma \frac{\dot{c}}{c} = (1 - \alpha)(1 - \sigma)[\delta u + \pi + n] + A\beta \left(\frac{k}{uh}\right)^{\beta - 1} [\sigma + \alpha(1 - \sigma)] \qquad (6.6)$$
$$-\rho - \pi + (1 - \sigma)\nu$$

$$\beta \frac{\dot{u}}{u} = (1 - \beta)(\pi + n - \theta) + \delta(1 - \beta)(1 - l) + \beta \delta u - \beta \left(\frac{c}{k}\right) \qquad (6.7)$$

$$\frac{\dot{k}}{k} = A \left(\frac{k}{uh}\right)^{\beta - 1} - (\pi + n) - \frac{c}{k} \qquad (6.8)$$

$$\frac{\dot{h}}{h} = \delta(1 - u - l) - \theta \qquad (6.9)$$

Moreover, for those situations of a non-interior solution, where the time devoted

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to education is equal to zero, the system becomes

$$[\alpha(1-\sigma)-1]\left(\frac{\dot{c}}{c}\right) = \pi + \rho + \theta[1-\alpha(1-\sigma)] - \beta A\left(\frac{k}{u}\right)^{\beta-1} + \frac{(1-\alpha)(1-\sigma)u}{[[\alpha(1-\sigma)-1]\beta(1-u)-\sigma u]}$$

$$\left\{\alpha(1-\sigma)\beta A\left(\frac{k}{u}\right)^{\beta-1} - [\alpha(1-\sigma)-1]\beta[n+\pi+\tfrac{c}{k}] - \pi - \rho - \theta[1-\alpha(1-\sigma)]\right\} \tag{6.10}$$

$$\left(\frac{\dot{u}}{u}\right)\left[\frac{[\alpha(1-\sigma)-1]\beta(1-u)-\sigma u}{(1-u)}\right] = \alpha(1-\sigma)\beta A\left(\frac{k}{u}\right)^{\beta-1} - \beta[\alpha(1-\sigma)-1] \tag{6.11}$$

$$[n+\pi+\frac{c}{k}] - \pi - \rho - \theta[1-\alpha(1-\sigma)]$$

$$\frac{\dot{k}}{k} = A\left(\frac{k}{u}\right)^{\beta-1} - (\pi+n) - \tfrac{c}{k} \tag{6.12}$$

From these equations, we define $z = \dfrac{c}{h}$ and $x = \dfrac{k}{h}$. One can easily show that all steady-state rays are those solutions to (6.6)-(6.12) such that $\dot{z}(t) = 0, \dot{u} = 0$ and $\dot{x}(t) = 0$.

Figure 7 portrays the dynamics for state variable x for Example 1 of Section 3, for those solutions of the Euler equations that converge to a given steady state ray. After computing the eigenvalues in that model, we find that steady-state rays 1 and 3 are saddle-path stable; thus, following a standard numerical technique we can trace out the stable manifolds of the system with an arbitrary degree of accuracy.[6] Since steady-state 2 has two complex roots, these paths cycle when approaching such steady state (see Figure 8). Following the same procedure, Figure 9 depicts the dynamics of the converging trajectories for an economy in which all steady states have only real roots.

(4) From these stable trajectories, we now construct a certain mapping that will correspond to the value function. We first outline the construction of such mapping and then study its differentiability properties.

For given x_0, define $\varphi(x_0)$ as the value of the objective in (P') for a trajectory satisfying (6.6)-(6.9) [when the solution reaches the boundary, (6.10)-(6.12)

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[6]In this computational procedure, system (6.10)-(6.12) becomes effective, once $l + u = 1$ in (6.6)-(6.9). Our computations are effected by a standard Euler method [see, e.g., Gerald and Wheatley (1990, Ch. 5)].

becomes effective] and with initial conditions $(k_0, h_0) = (x_0, 1)$. If as in Figure 7 several trajectories start from a given x_0 then $\varphi(x_0)$ is defined as the maximum value over all possible trajectories. After some straightforward calculations we find that such trajectories correspond roughly to the dotted line in Figures 7 and 8; hence, by construction function $\varphi(x_0)$ is continuous over the set of positive numbers, and in this case, the unstable steady state is non-optimal.

Let us define the function $\hat{W}(k, h) = \dfrac{h^{1-\sigma}}{1-\sigma}\varphi(x)$, where $x = \dfrac{k}{h}$. Then this function is well defined and continuous over $\mathcal{R}_+^2$. We now prove that such function is differentiable at almost every point in the domain.

LEMMA A.1: *Let x be such that $\dot{x}$ is uniquely defined. Let $(k, h) = (hx, h)$. Then the function $\hat{W}(k, h)$ is C^1 at (k, h), and the derivative $D\hat{W}(m) = -D_2v(m, \dot{m})$, where $m = (k, h)$ and $\dot{m}$ is the time derivative along the trajectory.*

Here $D_2v(m, \dot{m})$ refers to the derivative of v with respect to $\dot{m} = (\dot{k}, \dot{h})$. By $\dot{x}$ uniquely defined we mean that given x there is only one possible trajectory defining the function $\hat{W}(k, h)$. Thus, $\dot{x}$ is not uniquely defined in point $\underline{a}$ in Figure 7, since function $\hat{W}(k, h)$ has the same value along both trajectories. Observe that the derivative $D\hat{W}(k_0, h_0)$ takes on the same value as in the standard concave model (cf. Benveniste and Scheinkman, 1982). In this latter case, however, a simpler proof is available based upon the concavity of the value function.

PROOF OF THE LEMMA: Define the mapping $(k(t), h(t)) = \phi(k_0, h_0, t)$, given by the composition of the mappings

$$(k_0, h_0, t) \rightarrow (x_0, h_0, t) \rightarrow (\eta(x_0, t), h(t)) \rightarrow (h(t)\eta(x_0, t), h(t))$$

where $\eta(x_0, t) = \dfrac{k(t)}{h(t)}$ and $h(t)$ are obtained from (6.6)-(6.9) [or (6.10)-(6.12)] corresponding to those trajectories that define function $\hat{W}(k_0, h_0)$. Then ϕ is well defined and it is infinitely differentiable at every point (k_0, h_0) such that $\dot{x}_0$ is uniquely defined, for $x_0 = \dfrac{k_0}{h_0}$.[7] Let

$$\hat{W}^T(k_0, h_0) \equiv \int_0^T e^{-(\rho-n)t}v(k(t), h(t), \dot{k}(t), \dot{h}(t))dt$$

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[7]Even if a trajectory switches from system (6.6)-(6.9) to system (6.10)-(6.12) function η is still infinitely differentiable, since it can be expressed as the composition of two infinitely differentiable mappings.

where the values $(k(t), h(t), \dot{k}(t), \dot{h}(t))$ are defined by the mapping ϕ. Then differentiating under the integral sign we obtain

$$D\hat{W}^T(k_0, h_0) \equiv \int_0^T e^{-(\rho-n)t}[D_1 v(m(t), \dot{m}(t)) \cdot D_1\phi(m_0, t) + D_2 v(m(t), \dot{m}(t)) \cdot D_{21}\phi(m_0, t)]dt$$

Since the mapping ϕ has been defined from the Euler equations (6.6)-(6.12), and the matrix of cross-partial derivatives $D_{12}\phi(m_0, t) = D_{21}\phi(m_0, t)$, it follows from a well known argument based upon an integration by parts (cf. Luenberger 1968, Ch. 1) that

$$\begin{aligned} D\hat{W}^T(k_0, h_0) &\equiv e^{-(\rho-n)t} D_2 v(m(t), \dot{m}(t)) \cdot D_1\phi(m_0, t) \big|_{t=0}^T \qquad (6.13) \\ &\equiv e^{-(\rho-n)T} D_2 v(m(T), \dot{m}(T)) \cdot D_1\phi(m_0, T) - D_2 v(m(0), \dot{m}(0)) \end{aligned}$$

Now, observe that $\{\hat{W}^T(k_0, h_0)\}_{T \geq 1}$ is a sequence of continuous functions that converge uniformly to $\hat{W}(k_0, h_0)$ on every compact set. Also, one can easily establish that the first term in (6.13) converges uniformly to zero. Hence, under the above hypotheses the function $\hat{W}(k_0, h_0)$ is C^1 at (k_0, h_0) and $D\hat{W}(m_0) = -D_2 v(m_0, \dot{m}_0)$ for $m_0 = (k_0, h_0)$.

(5) We now show that the function $\hat{W}$ is equal to the value function W as defined in (P) and (P'), and hence those trajectories defining $\hat{W}$ are optimal solutions to the planning problem. Write

$$\hat{W}(k_0, h_0) \equiv \int_0^T e^{-(\rho-n)t} v(k(t), h(t), \dot{k}(t), \dot{h}(t))dt + e^{-(\rho-n)T}\hat{W}(k(T), h(T)) \quad (6.14)$$

Where the variables $(k(t), h(t))$ are evaluated along a given trajectory. Then totally differentiating (6.14) with respect to T, and evaluating such derivative at $T = 0$, we obtain

$$0 \equiv v(k(0), h(0), \dot{k}(0), \dot{h}(0)) + D\hat{W}(k(0), h(0)) \cdot (\dot{k}(0), \dot{h}(0)) - (\rho-n)\hat{W}(\dot{k}(0), \dot{h}(0)) \quad (6.15)$$

From the fact that $D\hat{W}(m_0) = -D_2 v(m_0, \dot{m}_0)$ we obtain that the first order conditions with respect to the variables $(\dot{k}, \dot{h})$ in (6.15) are necessarily satisfied at the point $(k(0), h(0), \dot{k}(0), \dot{h}(0))$. Since from part (1) the mapping $v(k(0), h(0), \cdot, \cdot)$ is concave in $(\dot{k}, \dot{h})$, it follows from (6.15) that

$$(\rho-n)\hat{W}(k(0), h(0)) = \max_{\dot{k}, \dot{h}} v(k(0), h(0), \dot{k}, \dot{h}) + D\hat{W}(k(0), h(0)) \cdot (\dot{k}, \dot{h}) \quad (6.16)$$

Observe that equation (6.16) is the Bellman equation. By virtue of the "verification theorem" (cf. Fleming and Rishel, 1975, Ch. IV), the mapping $\hat{W}$ must

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be identical to the value function W, and so the point $(\dot{k}(0), \dot{h}(0))$ is an optimal solution at every $(k(0), h(0))$.[8]

(6) For a given model economy, the above method determines the global behavior of an optimal trajectory from those trajectories converging to a given steady-state ray. We have applied this technique to various examples with either a unique or several steady-states, and in all cases our algorithm has characterized globally the optimal path. For those economies with a unique steady-state ray, or in which for all unstable steady state rays there are no complex eigenvalues (as in Figure 9), all stable trajectories from the first order variational conditions define the policy function, and such function is continuous.

It seems difficult to provide a more general method to single out an optimal trajectory regardless of the dynamic behavior of the selected, stable orbits. This is because optimal orbits may feature some discontinuities near an unstable steady state.

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[8]The "verification theorem" holds in the case studied since $\hat{W}$ is piece-wise C^1.
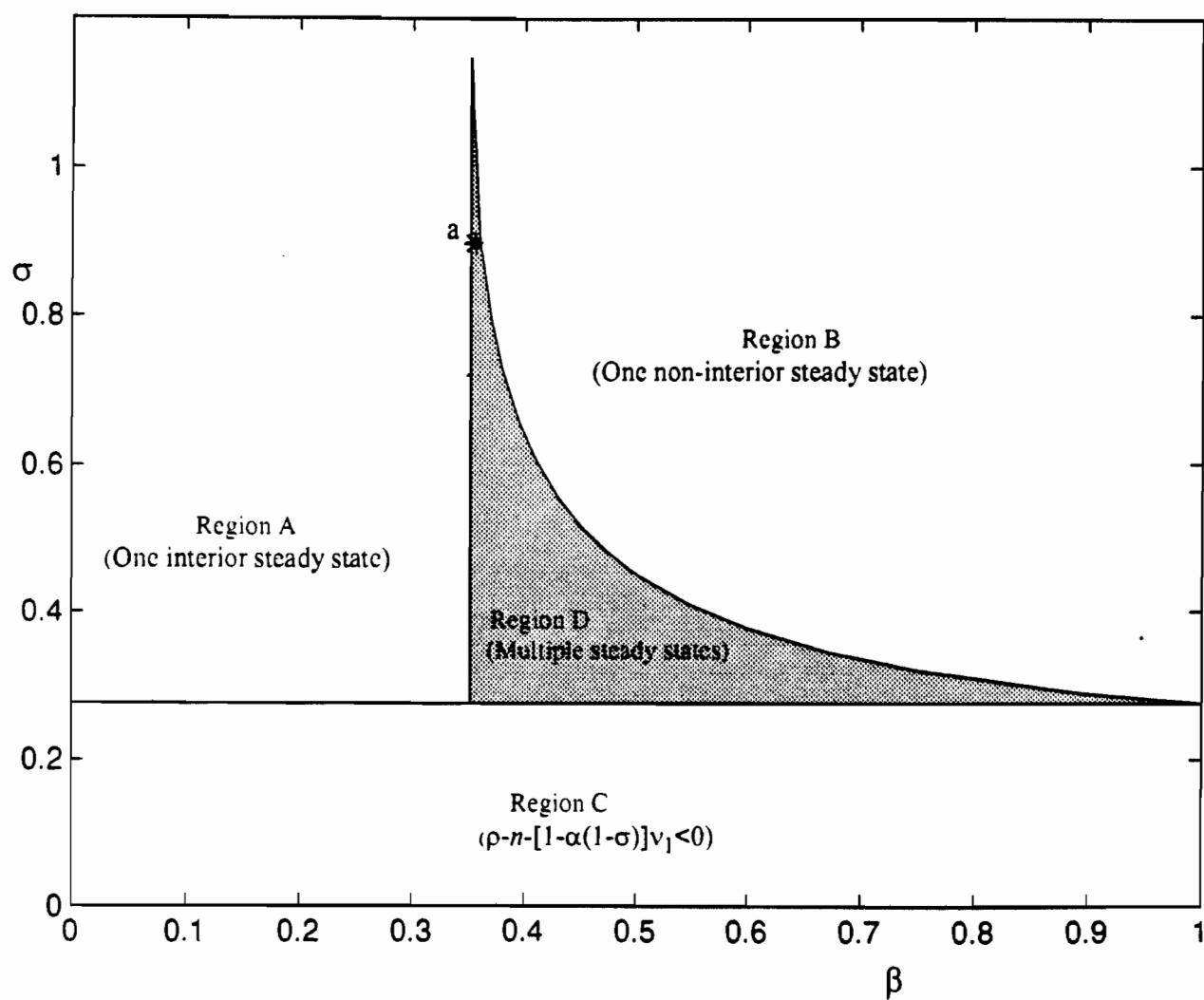
Figure 1.- Regions of existence of balanced paths in a (β,σ)-plane for fixed parameter values, $\alpha=0.3$, $\rho=0.05$, $n=0$, $B=1$, $\pi=0$, $\delta=0.23$ and $\theta=0$.
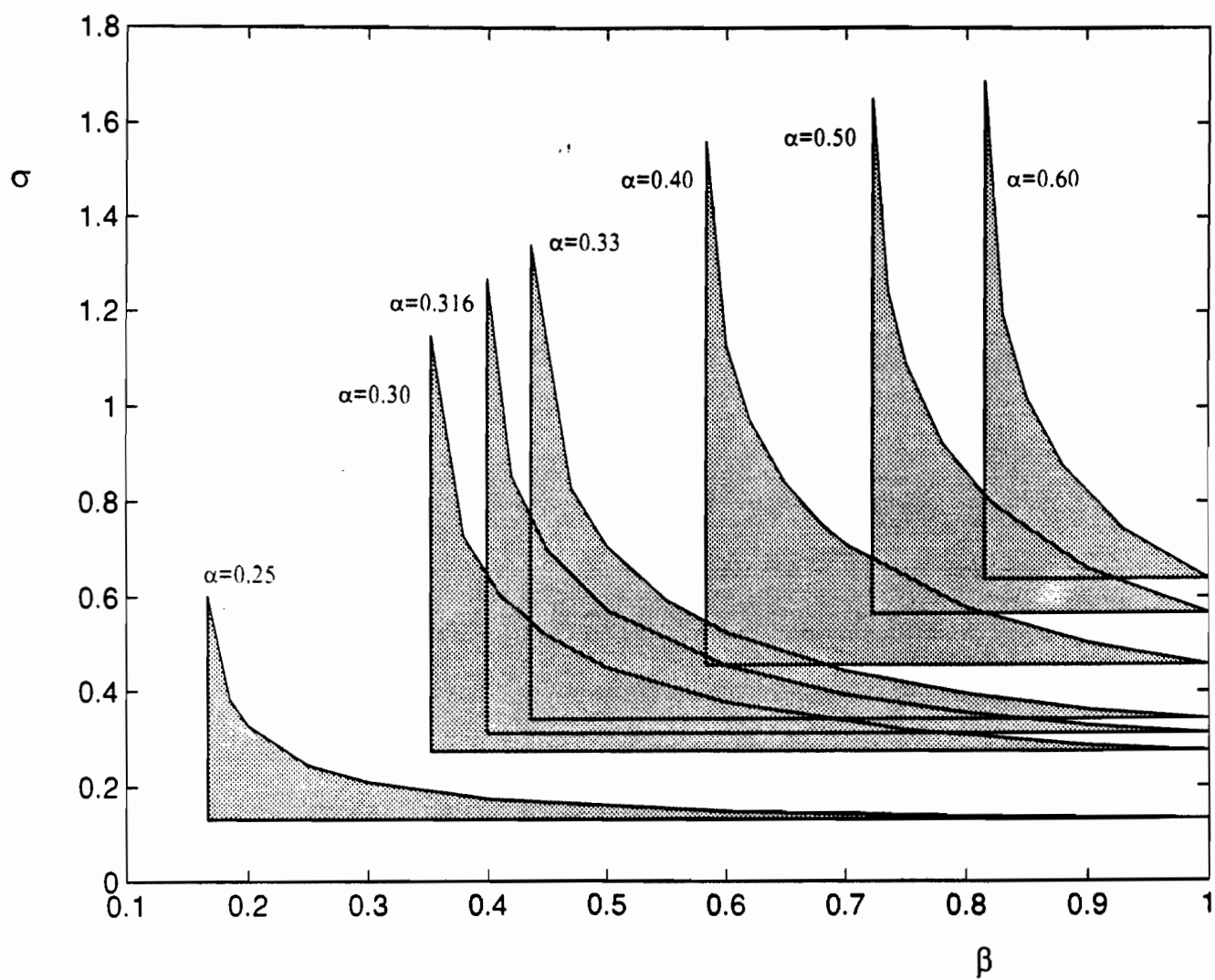
Figure 2.- Regions of existence of balanced paths in a (β,σ)-plane for several values of α, and fixed parameters values, $\rho=0.05$, $n=0$, $B=1$, $\pi=0$, $\delta=0.23$, $\theta=0$.
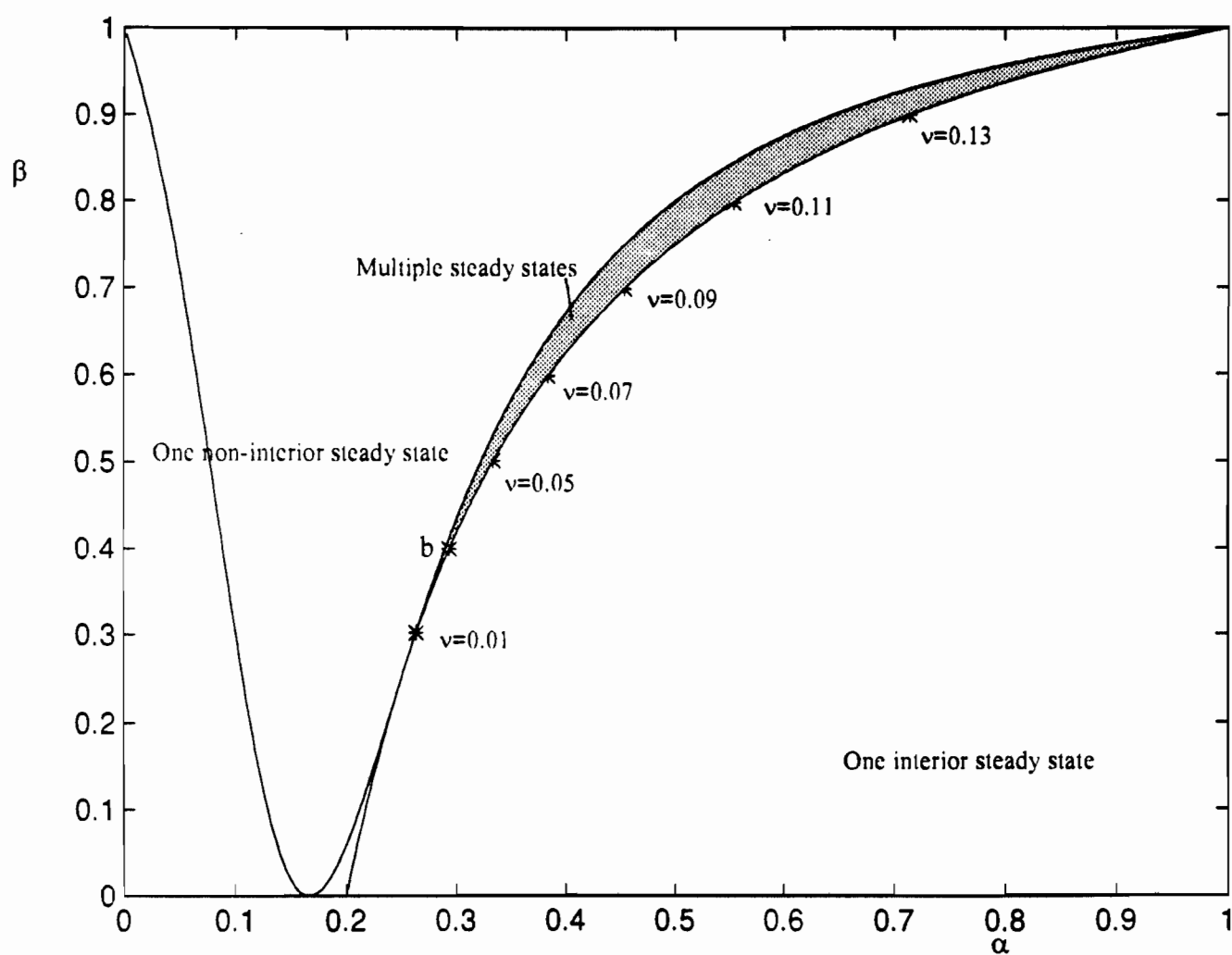
Figure 3.- Regions of existence of balanced paths in a (α,β)-plane for the case of the logarithmic utility function in consumption and leisure, and parameter values ρ=0.05, n=0, B=1. π=0. δ=0.25 and θ=0.
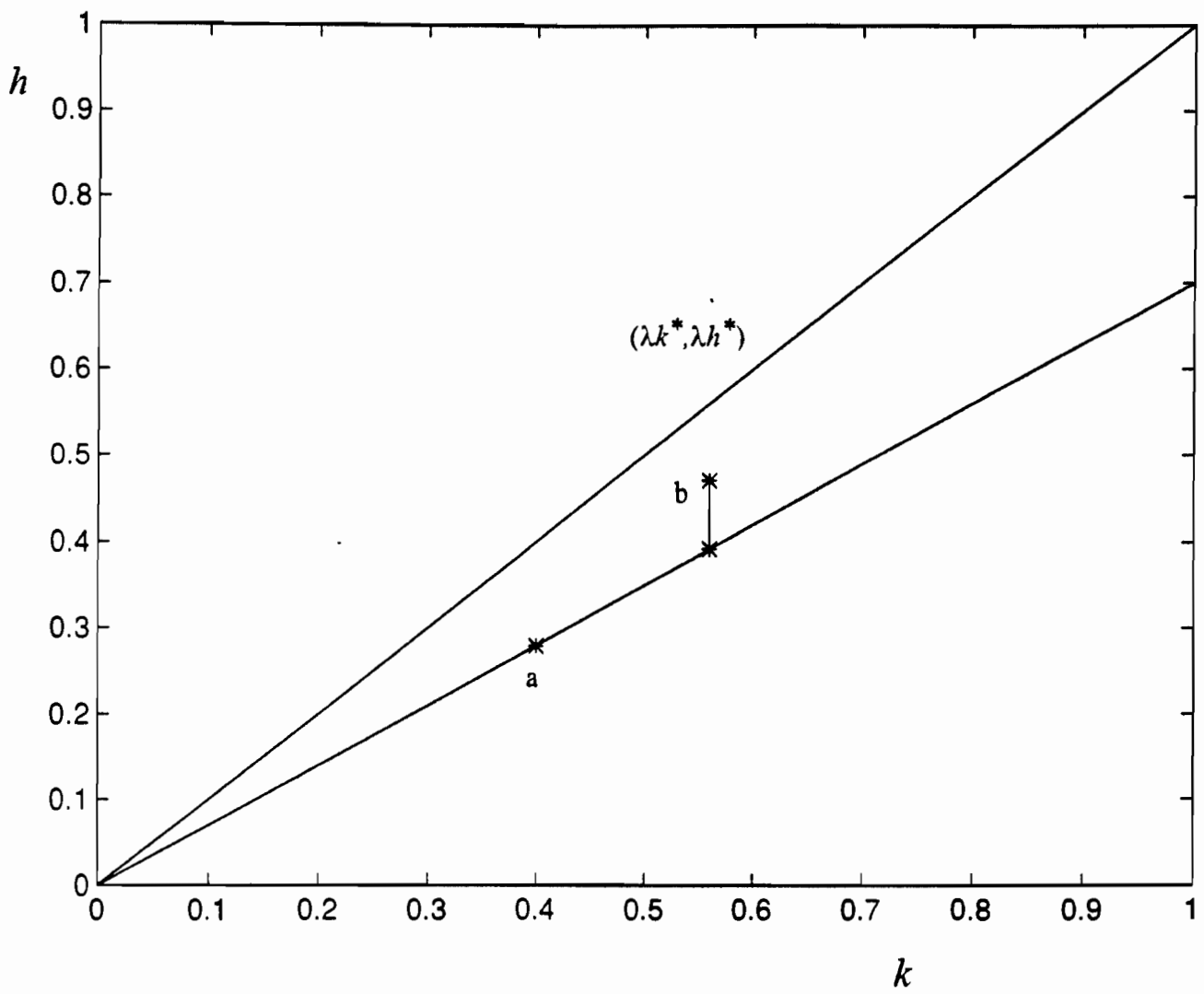
Figure 4.- Local convergence to the balanced path $(\lambda k^*, \lambda h^*)$, $\lambda > 0$.
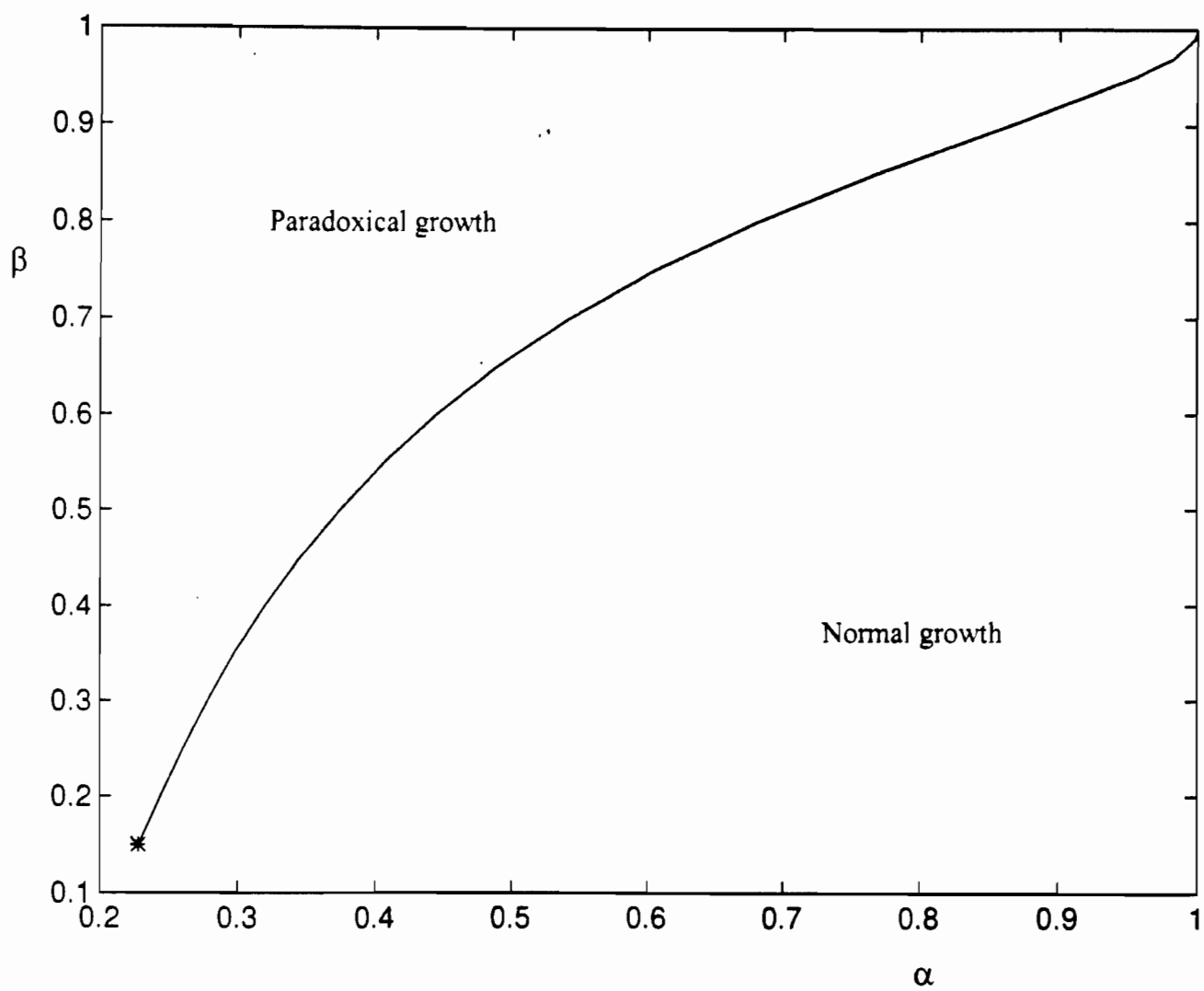
Figure 5.- Growth regions in a (α,β)-plane for $\sigma=1$, $\rho=0.05$, $n=0$, $B=1$, $\pi=0$, $\delta=0.25$ and $\theta=0$.
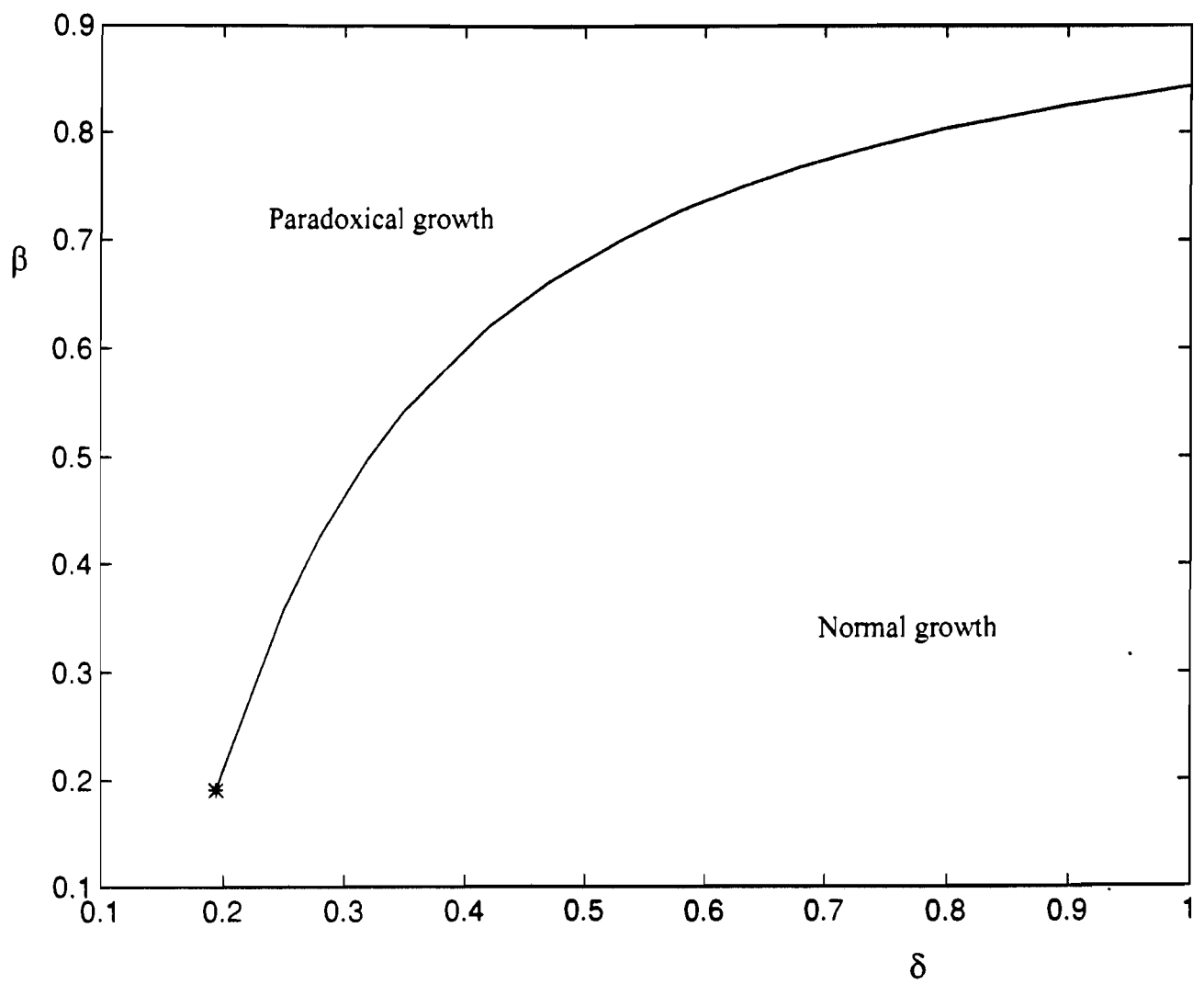
Figure 6.- Growth regions in a (δ,β)-plane for $\sigma=1$, $\alpha=0.3$, $\rho=0.05$, $n=0$, $B=1$, $\pi=0$ and $\theta=0$.
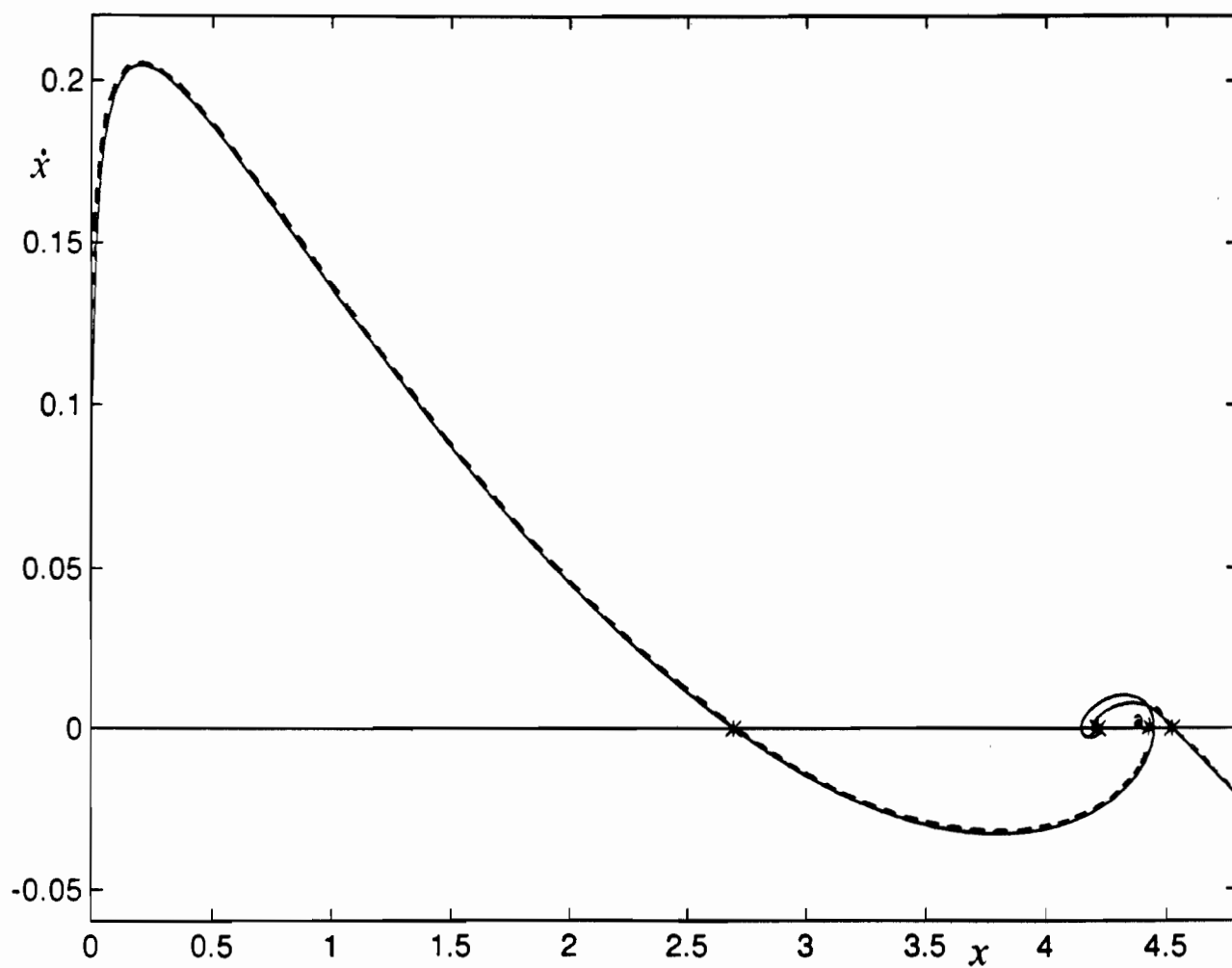
Figure 7.- Global dynamics for state variable $x=k/h$ for the model economy of Example 1..
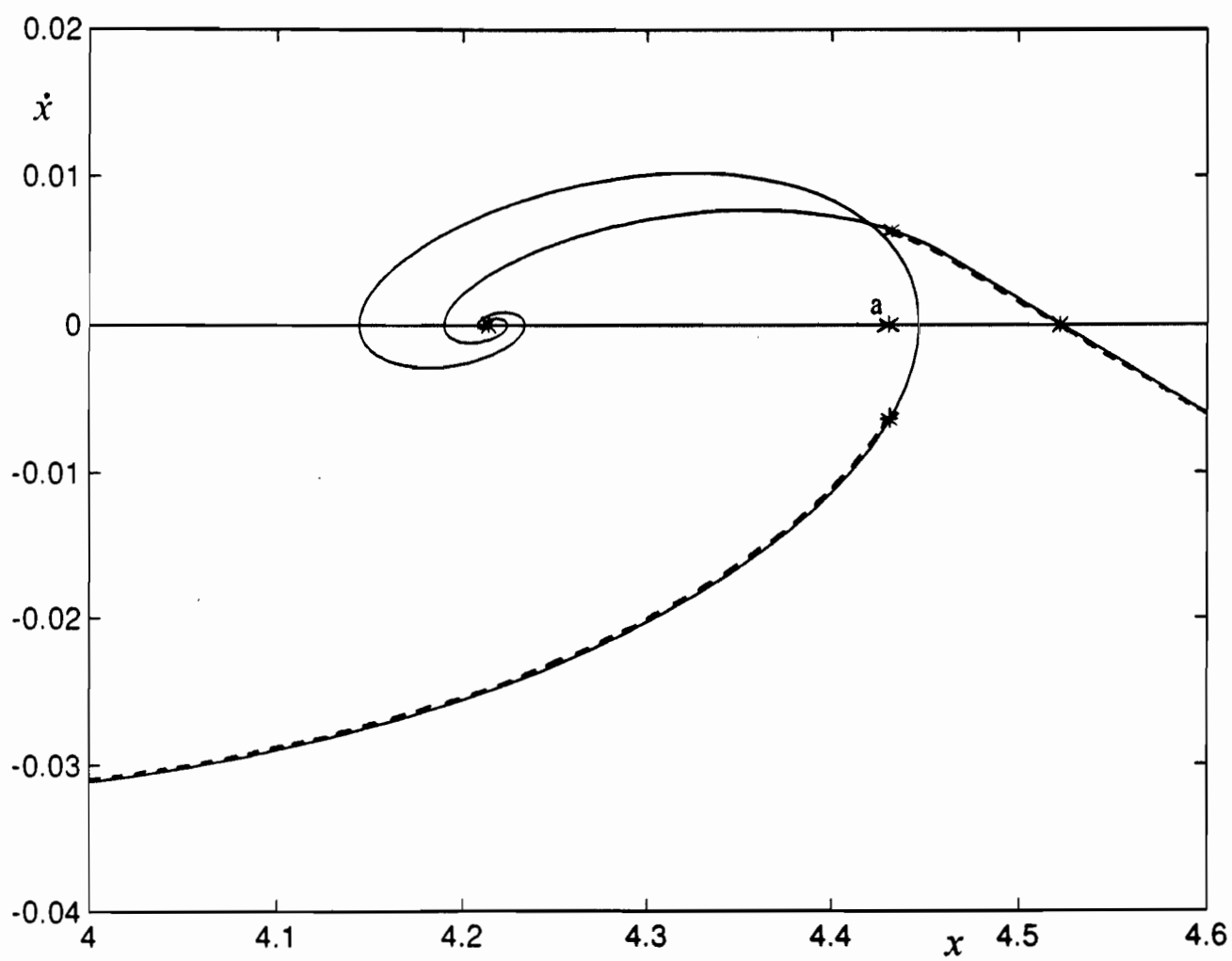
Figure 8.- Local dynamics of converging trajectories for state variable $x=k/h$ around steady state 2 for the model economy of Example 1.
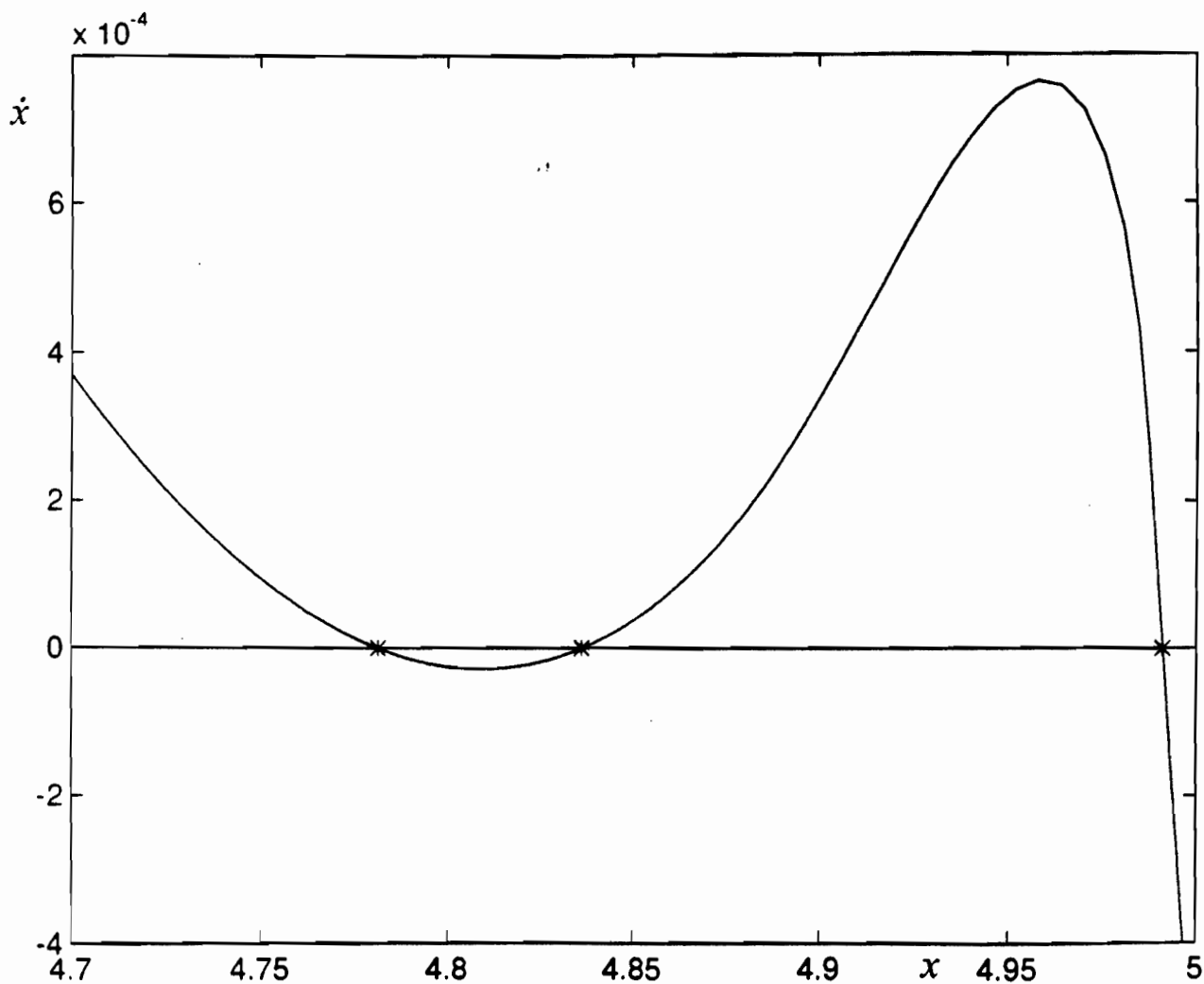
Figure 9.- Dynamics near the steady states for state variable $x=k/h$, for an economy with multiplicatively separable utility and parameter values

$\sigma=0.99547$, $\alpha=0.34$, $\rho=0.05$, $n=0$, $B=1$, $\beta=0.3496$, $\pi=0$, $\delta=0.1992$, $\theta=0$

The steady states are defined by the following values:

$(c/h)_1^*=0.697$, $l_1^*=0.742$, $u_1^*=0.251$, $(k/h)_1^*=4.782$ and $v_1=0.0014$

$(c/h)_2^*=0.701$, $l_2^*=0.744$, $u_2^*=0.251$, $(k/h)_2^*=4.836$ and $v_2=0.0010$

$(c/h)_3^*=0.714$, $l_3^*=0.749$, $u_3^*=0.251$, $(k/h)_3^*=4.991$ and $v_3=0$