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A TWO WELL LIOUVILLE THEOREM*

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Abstract. In this paper we analyse the structure of approximate solutions to the compatible two well problem with the constraint that the surface energy of the solution is less than some fixed constant. We prove a quantitative estimate that can be seen as a two well analogue of the Liouville theorem of Friesecke James Müller.

Let $H = \left(\begin{smallmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{smallmatrix}\right)$ for $\sigma > 0$. Let $0 < \zeta_1 < 1 < \zeta_2 < \infty$. Let $K := SO(2) \cup SO(2)H$. Let $u \in W^{2,1}(Q_1(0))$ be a C^1 invertible bilipschitz function with $\mathrm{Lip}(u) < \zeta_2$, $\mathrm{Lip}\left(u^{-1}\right) < \zeta_1^{-1}$.

There exists positive constants $\mathfrak{c}_1 < 1$ and $\mathfrak{c}_2 > 1$ depending only on σ, ζ_1, ζ_2 such that if $\epsilon \in (0, \mathfrak{c}_1)$ and u satisfies the following inequalities

$$\int_{Q_1(0)} \mathrm{d}(Du(z), K)\, \mathrm{d}L^2 z \leq \epsilon$$

$$\int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \leq \mathfrak{c}_1,$$

then there exists $J \in \{Id, H\}$ and $R \in SO(2)$ such that

$$\int_{Q_{\mathfrak{c}_1}(0)} |Du(z) - RJ|\, \mathrm{d}L^2 z \leq \mathfrak{c}_2 \epsilon^{\frac{1}{800}}.$$

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1. Introduction

We consider the following simple problem.

Problem A. Let E be a set of matrices and $F \notin E$. Let $q \geq 1$, and Ω be a Lipschitz domain in $\mathbb{R}^n$. Let $\mathrm{d}(\cdot, E)$ denote Euclidean distance from set E. Prove there exists constants $\epsilon_0 > 0$, $\beta_0 > 0$ such that any $u \in W^{2,q}(\Omega : \mathbb{R}^m)$ satisfying $u(x) = F(x)$ on $\partial\Omega$ and

$$\int_\Omega \mathrm{d}(Du(z), E)\, \mathrm{d}L^2 z \leq \epsilon \tag{1}$$

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for $\epsilon \in (0, \epsilon_0)$ has the property that

$$\int_{\Omega} \left|D^2 u(z)\right|^q \mathrm{d}L^2 z \geq \epsilon^{1-q-\beta_0}. \tag{2}$$

Problem A is solved only for sets of 2 or 3 matrices satisfying the following strong condition.

Definition 1.1. A set of matrices E is called restricted if and only if given any Lipschitz domain Ω there exists constant $c_1 > 0, \delta_0 > 0, \gamma_0 > 0$ such that if function $u \in \mathrm{Lip}$ satisfies $u = F$ on $\partial\Omega$ for $F \notin E$ and

$$\int_{\Omega} \mathrm{d}(Du(z), E)\, \mathrm{d}L^2 z < \delta_0$$

then u has the property

$$\sup\{|u(z) - F(z)| : x \in \Omega\} < c_1 \left(\int_{\Omega} \mathrm{d}(Du(z), E)\, \mathrm{d}L^2 z\right)^{\gamma_0}. \tag{3}$$

We briefly comment on how Problem A is solved for restricted sets of 2×2 matrices in order to motivate Definition 1.1. For restricted sets condition (1) forces the function to be pressed down uniformly close to the affine boundary condition F in the sense of (3). Let $v \in S^1$ be such that $(X - F)v \neq 0$ for any $X \in E$. Suppose we can find two points $a, b \in \Omega$ in direction v such that $Du_{\lfloor [a,b]} \approx X \in E$ then as $(X - F)(a - b) \approx (u - F)(a - b) \leq \|u - F\|_{L^\infty(\Omega)}$. So we have $|a - b| < c_2 \|u - F\|_{L^\infty(\Omega)} < c_2 c_1 \left(\int_{\Omega} \mathrm{d}(Du(z), K)\, \mathrm{d}L^2 x\right)^{\gamma_0}$. Thus for any line going through Ω there must be approximately $\left(c_2 c_1 \left(\int_{\Omega} \mathrm{d}(Du(x), K)\, \mathrm{d}L^2 x\right)^{\gamma_0}\right)^{-1}$ points at which Du jumps from one matrix inside E to another. Hence by Fubini (2) follows.

Solutions to problem A for restricted sets of 2 or 3 matrices appear in [6, 12]. For example the set $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ is restricted.

From the results of Müller, Šverák [15, 16] and Dacorogna, Marcellini [10] for the set of matrices $E = SO(2) \cup SO(2)H \subset M^{2\times 2}$, H diagonal there exists a large class of matrices $F \notin E$ for which we can solve the differential inclusion.

$$Du \in E \text{ for } a.e. \text{ and } u = F \text{ on } \partial\Omega.$$

Our goal is to solve Problem A with respect to this set of matrices. Our main theorem is following.

Theorem 1.1. *Let $0 < \zeta_1 < 1 < \zeta_2 < \infty$. Let $K := SO(2) \cup SO(2)H$ where $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$.*

Let $u \in W^{2,1}(Q_1(0))$ be a C^1 invertible bilipschitz function with $\mathrm{Lip}(u) < \zeta_2$, $\mathrm{Lip}(u^{-1}) < \zeta_1^{-1}$. There exists positive constants $c_1, c_3, c_4 < 1$ and $c_2, c_5 > 1$ depending only on σ, ζ_1, ζ_2 such that if $\kappa \in (0, c_1]$, $m_0 \geq c_2$ and u satisfies the following inequalities

$$\int_{Q_1(0)} \mathrm{d}(Du(z), K)\, \mathrm{d}L^2 z \leq \kappa^{m_0} \tag{4}$$

$$\int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \leq c_3 \kappa, \tag{5}$$

then there exists $J \in \{Id, H\}$ and $R \in SO(2)$ such that

$$\int_{Q_{c_4}(0)} |Du(z) - RJ|\, \mathrm{d}L^2 z \leq c_5 \kappa^{\frac{m_0}{800}}. \tag{6}$$

The integral $\int \mathrm{d}\left(Du\left(z\right),K\right)\mathrm{d}L^2z$ is known as the *bulk energy* and $\int \left|D^2u\left(z\right)\right|\mathrm{d}L^2z$ is known as the *surface energy*. To illustrate our theorem it is helpful to consider $\kappa = c_1$ and to take $m_0 \to \infty$ (this way we also obtain the theorem stated in the abstract). So for small but fixed surface energy, as the bulk energy decreases, the control of the derivative of the function in the central subsquare improves to some root power of the bulk energy. To state things more roughly, even though the surface energy is a small but *fixed* quantity, as the bulk energy decreases, the function in the central subsquare becomes increasingly flat.

The upper bound $c_5\kappa^{\frac{m_0}{800}}$ in (6) is far from optimal. The naive guess that the optimal bound is given by $c\kappa$ is false[1], this follows from the construction of [7], see [9] for more details.

The assumption that u is bilipschitz is a technical one, however it is used in an essential way many times in the proof. On the other hand the assumption u is C^1 is not necessary, it saves us some details to do with fine properties of Sobolev functions.

In another paper [13] we will use Theorem 1.1 to reduce Problem A to a kind of discrete ϵ free version of the problem[2].

As shown in the remark following Definition 1.1, for restricted sets E we can control the function *just* using bulk energy, then simply count up the surface energy. For our case with matrices $K = SO\left(2\right) \cup SO\left(2\right)H$ from the work of Dacorogna and Marcellini [10], Müller and Šverák [15], we have the existence of Lipschitz functions satisfying the affine boundary condition but for which $Du \in K$ *a.e.* in Ω. So there is *no* relation between small bulk energy (in this case zero bulk energy) and being pressed down close to the affine boundary. It is not possible to just use bulk energy, we have to control the function using bulk and surface energies in combination. Hence the need for Theorem 1.1.

Functionals of the form (4) for $K = SO\left(2\right) \cup SO\left(2\right)H$ have received much attention in non convex calculus of variations. From work of Ball, James [2, 3] and Chipot, Kinderlehrer [5] functionals of this form have been the basis of a well known model for solid-solid phase transformations. The basic idea was that deformations of the material will attempt to minimise an energy functional of the form

$$I\left(u\right) = \int_{\Omega} \phi\left(Du\left(x\right)\right)\mathrm{d}L^2x \tag{7}$$

where ϕ is the free energy per unit volume in Ω. Many features of minimising sequences can be understood from the set $\{F : \phi\left(F\right) = 0\}$. This set is known as the energy *wells* of the functional I. Certain natural assumptions on the behavior of ϕ, in particular frame indifference, imply that K has to be of the form

$$K = \{SO\left(3\right)A_i : i = 1, 2, \ldots m\} \tag{8}$$

where the A_i are symmetry related and depend on the action of the phase transition.

Functional I is not quasiconvex and so minimisers can not be found by lower semicontinuity, however as stated, from the work of Dacorogna and Marcellini, Müller and Šverák there exists absolute minimisers to I. It is the existence of these functions that make Problem A interesting.

A some what different but nevertheless relevant theorem is [11], Theorem 3.1.

Theorem 1.2 (Friesecke, James, Müller)**.** *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. The exists a constant $C\left(U\right)$ with the following property. For each $v \in W^{1,2}\left(U, \mathbb{R}^n\right)$ there exists an associated rotation $R \in SO\left(n\right)$ such that*

$$\left\|Dv - R\right\|_{L^2(U)} \leq C\left(U\right)\left\|\operatorname{dist}\left(Dv, SO\left(n\right)\right)\right\|_{L^2(U)}. \tag{9}$$

In [4] Theorem 1.2 was proved for the set $\widetilde{K} = SO\left(2\right) \cup SO\left(2\right)H$ where $H = \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots \lambda_n\right)$, $\lambda_i > 0$ is such that

$$\sum_{i=1}^{n}\left(1 - \lambda_i\right)\left(1 - \frac{\det\left(H\right)}{\lambda_i}\right) > 0. \tag{10}$$

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[1] Thanks to Sergio Conti for pointing this out.

[2] Though this discrete problem remains very much open.

Specifically it was shown that for each $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ there exists $R \in \widetilde{K}$ such that

$$\|Du - R\|_{L^2(\Omega)} \leq C(\Omega, H) \|\mathrm{dist}\left(Du, \widetilde{K}\right)\|_{L^2(\Omega)}.$$

Condition (10) forces the wells $SO(n)$ and $SO(n)H$ to be strongly incompatible, in particular H is not rank-1 connected to $SO(n)$.

In our case (where H is rank-1 connected to $SO(2)$) Theorem 1.2 is trivially false without additional conditions (a simple laminate being the counter example).

Our additional conditions are to bound $\|D^2u\|_{L^1(\Omega)}$ by a small but *fixed* constant and to constrain u to be bilipschitz[3], and we obtain the weaker bound.

$$\|Du - RJ\|_{L^1\left(Q_{c_4}(0)\right)} \leq c_4 \left(\|\mathrm{dist}(Du, K)\|_{L^1(Q_1(0))}\right)^{\frac{1}{800}}.$$

After this paper was submitted, we learned of the relevance of the work of Conti, Schweizer [8] on the Gamma limit of functional I with surface energy term, where I has linearised wells. Using methods of [9] (for the non-linear functional) Conti, Schweizer proved a strong generalisation of Theorem 1.1, their strategy was to use hypotheses (4) and (5) to deduce $\int_{Q_1(0)} \mathrm{d}(Du(z), SO(2)J) \leq \kappa^{m_0}$ for some $J \in \{Id, H\}$, the theorem then follows from Theorem 1.2. For a simple proof of Theorem 1.1 in the plane *via* application of Theorem 1.2, see [14].

2. Plan of proof

Strategy:

We will gain control of function u in a central subsquare by surrounding the central subsquare with a "diamond". Along the sides of the diamond we will show Du is L^1 close to a fixed rotation, the control in the central subsquare follows from this. Showing Du on a line l is L^1 close to a fixed rotation is "more or less" equivalent to showing $u(l)$ is "roughly" mapped to a straight (unstretched) line. We will develop methods that show that for many lines in $Q_1(0)$ (in the directions of the sides of the diamond), function u maps the lines to "roughly" straight (unstretched) lines.

2.1. The push over lemma

$H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$. To begin with note that there are two linearly independent vectors ϕ_1 and ϕ_2 such that $|H\phi_i| = 1$ for $i = 1, 2$. A short calculation gives that we can take $\phi_1 = \begin{pmatrix} \frac{1}{\sqrt{1+\sigma^2}} \\ \frac{\sigma}{\sqrt{1+\sigma^2}} \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} \frac{1}{\sqrt{1+\sigma^2}} \\ \frac{-\sigma}{\sqrt{1+\sigma^2}} \end{pmatrix}$. Let n_i denote the anticlockwise normal to ϕ_i for $i = 1, 2$.

Now the most basic example of a function satisfying the affine boundary condition that minimises bulk energy is a *laminate*. In the reference configuration this can be seen as a function defined on a collection of strips running parallel to either ϕ_1 or ϕ_2 for which the derivative of the *laminate* alternates from one strip to the next from being in $SO(2)$ to being in $SO(2)H$. For simplicity, let us suppose the strips are parallel to ϕ_1 and let us denote the *laminate* by u. Now if all our strips are of width w, by Fubini and the fact that $\det(H) = 1$ and $|H\phi_1| = 1$ we know that the images of our strips under the action of u will be strips of width w, as shown Figure 1.

For a general function v with small bulk energy (*i.e.* $\int_\Omega \mathrm{d}(Dv(x), SO(2) \cup SO(2)H)\, \mathrm{d}L^2x < \epsilon$) we will examine the behaviour of v on lines parallel to ϕ_i. Roughly speaking it will turn out that if l_1 is parallel to l_2

[3] Note that the fact we only have an L^1 bound on D^2u is important, for L^q bounds on D^2u a much stronger result is possible, see [14]. Also note that for a finite L^2 bound on D^2u the result can easily be deduced from Lemma 4 of [4].

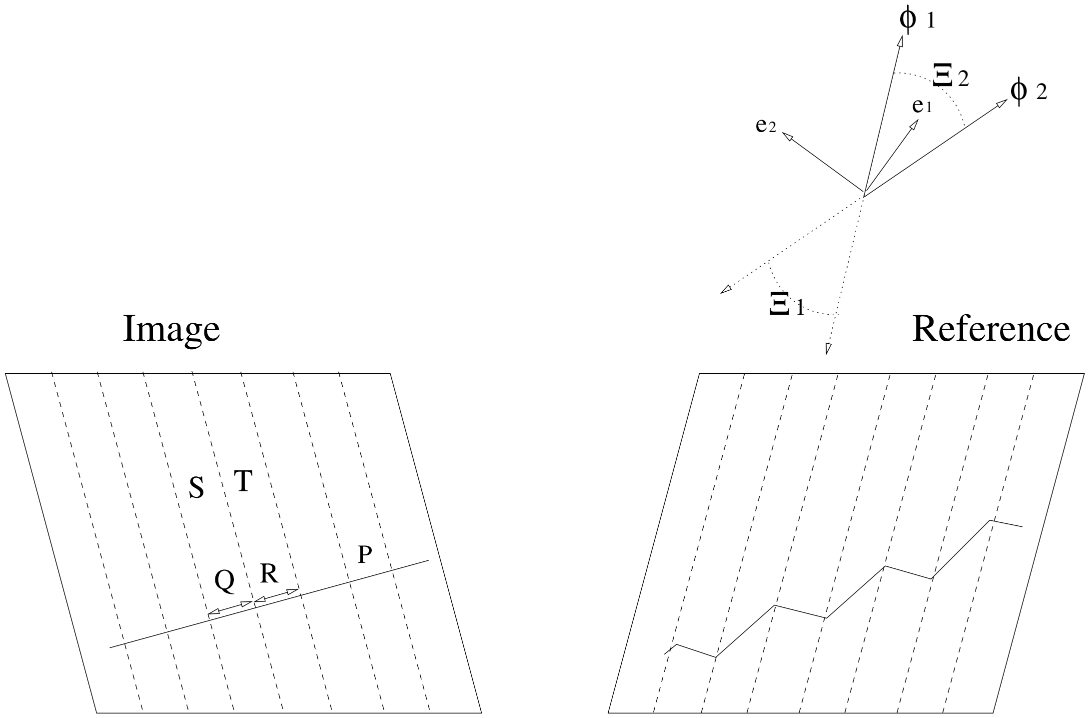


FIGURE 1. The pullback of a straight line by a laminate function.

and the two lines are distance w apart, then $v(l_1)$ will have to stay distance w away from $v(l_2)$. This is a consequence of the following inequality

$$|H\psi| \geq \psi \cdot n_i \text{ for all } \psi \in S^1. \tag{11}$$

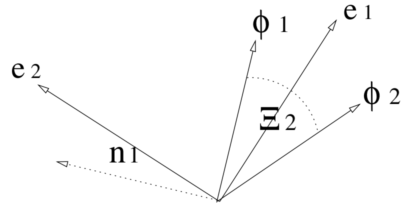
For the proof of which, see the argument following (27).

Firstly, suppose for two parallel lines l_1, l_2 in direction ϕ_1 that are distance w apart we have that $v(l_1)$, $v(l_2)$ are distance (much) less than w apart at some point, as shown in Figure 2 .

Let α be the line of length less than w joining $v(l_1)$ to $v(l_2)$. Using bilipschitzness of v and a Fubini argument, we can assume α is such that $\int_\alpha \mathrm{d}\left(Dv\left(v^{-1}(x)\right), SO(2) \cup SO(2)H\right) < \sqrt{\epsilon}$. We consider the preimage $v^{-1}(\alpha)$. We want to use the formula $H^1(\alpha) = \int_{v^{-1}(\alpha)} |Dv(x)\, t(x)| \,\mathrm{d}H^1 x$ and the fact that $H^1\left(v^{-1}(\alpha)\right) \geq w$ to get a contradiction from the assumption $H^1(\alpha) \ll w$.

Assume for simplicity $Dv\left(v^{-1}(x)\right) \in N_{\sqrt{\epsilon}}(SO(2) \cup SO(2)H)$ for all $x \in \alpha$. For each $x \in \alpha$ let $G(x) \in SO(2) \cup SO(2)H$ be the matrix such that $\left|Dv\left(v^{-1}(x)\right) - G(x)\right| = \mathrm{d}\left(Dv\left(v^{-1}(x)\right), SO(2) \cup SO(2)H\right)$, and let t_x denote the tangent to $v^{-1}(\alpha)$ at point x. We have

$$\begin{aligned}
H^1(\alpha) &= \int_{v^{-1}(\alpha)} \left|Dv\left(v^{-1}(x)\right) t_x\right| \mathrm{d}H^1 x \\
&\geq \int_{v^{-1}(\alpha)} |G(x)\, t_x| \,\mathrm{d}H^1 x - \sqrt{\epsilon} H^1\left(v^{-1}(\alpha)\right) \\
&\overset{(11)}{\geq} L^1\left(P_{\phi_i^\perp}\left(v^{-1}(\alpha)\right)\right) - \sqrt{\epsilon} H^1\left(v^{-1}(\alpha)\right) \\
&= w - \sqrt{\epsilon} H^1\left(v^{-1}(\alpha)\right).
\end{aligned}$$



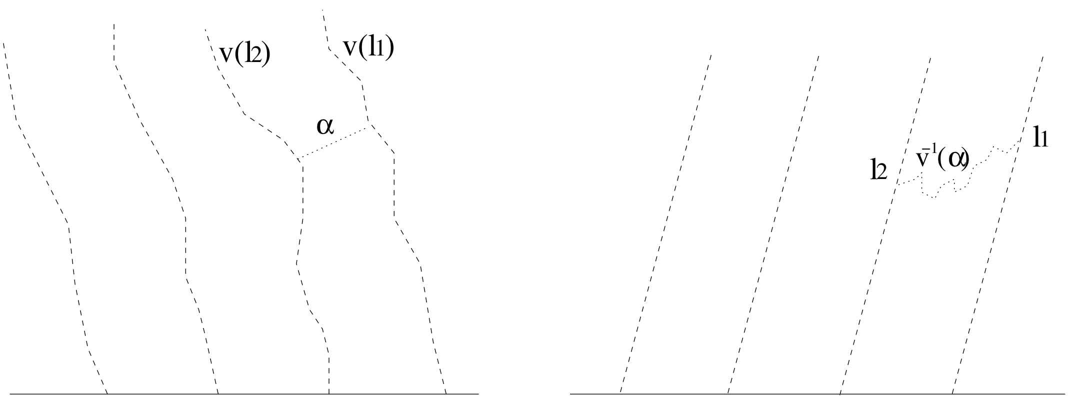


FIGURE 2. The pullback of the shortest line joining $v(l_2)$ to $v(l_1)$.

Assuming v is bilipschitz (and so $H^1(v^{-1}(\alpha))$ is not too big) this implies the images of lines l_1 and l_2 must be (by at least $(1-c\sqrt{\epsilon})\,w$) "pushed over" from one another, *i.e.* we can not find a line α of length less than $(1-c\sqrt{\epsilon})\,w$ joining $v(l_1)$ to $v(l_2)$. This is our first restriction on the geometry of the function we want to study, just coming from smallness of bulk energy.

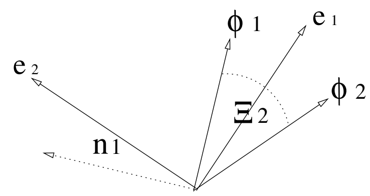
2.2. ODE method

We consider the same picture as before but from a different perspective. So $l_1, l_2, \ldots$ are lines in direction ϕ_1 going through Ω and we consider the images $v(l_1), v(l_2), \ldots$ Now supposing we were on a point $x \in v(l_1)$ and we wanted to get to $v(l_2)$ *via* a path in $v(\Omega)$ of the shortest length. If we start at point s the most obvious thing to do is to "draw a straight line" to the nearest point of $v(l_2)$. But supposing we are "blind" and we can not see which straight line to draw, suppose we have to find the path just using analytic information we have about v.

The most natural way to do it would be to consider the vector field given by the gradient of the function $\Psi_1 : v(\Omega) \to \mathbb{R}^2$ defined by $\Psi_1(x) := v^{-1}(x)\cdot n_1$ note $v(l_1)$, $v(l_2)$ are the level sets of Ψ_1. If we "follow" the vector field from point x it will indeed take us along the optimal path to $v(l_2)$. But "following" a vector field is exactly finding an integral curve for a vector field, which means solving the following ODE

$$X(0) = x \quad \frac{\mathrm{d}X}{\mathrm{d}t}(t_1) = D\Psi_1(X(t_1)). \tag{12}$$

Now if point $y \in \{X(t) : t > 0\}$ is such that $Dv(v^{-1}(y)) \in N_{\sqrt{\epsilon}}(SO(2) \cup SO(2)\,H)$ we calculate that $D\Psi_1(y) = Dv^{-T}(y)\cdot n_1$. Letting $R(v^{-1}(y))\,S(v^{-1}(y)) := Dv(v^{-1}(y))$ be the polar decomposition of $Dv(v^{-1}(y))$ (*i.e.* $R(v^{-1}(y)) \in SO(2)$ and $S(v^{-1}(y)) \in M^{2\times 2}_{\mathrm{sym}}$) we have $Dv^{-T}(y)\,n_1 = R(v^{-1}(y))\,S^{-1}(v^{-1}(y))\,n_1$ and as $S(v^{-1}(y)) \in N_{\sqrt{\epsilon}}(\{Id, H\})$ so either $S(v^{-1}(y)) \in N_{\sqrt{\epsilon}}(Id)$ and so



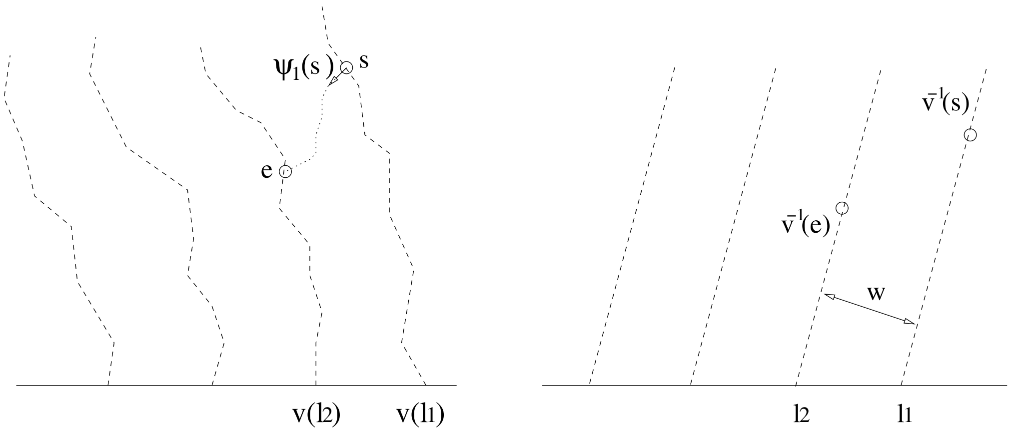


FIGURE 3. The integral path of the vector field Ψ_1.

$\left|S\left(v^{-1}(y)\right)n_1\right| \approx 1$ or $S\left(v^{-1}(y)\right) \in N_{\sqrt{\epsilon}}(H)$ and so $\left|S\left(v^{-1}(y)\right)n_1\right| \approx \left|H^{-1}n_1\right| = 1$. So assuming the path of the vector field is such that Dv stays close to the wells $SO(2) \cup SO(2)H$, if Λ is a connected subset of the set $\{X(t) : t > 0\}$ with end points $e \in v(l_2)$, $s \in v(l_1)$ then

$$|\Psi_1(e) - \Psi_1(s)| = \left|\left(v^{-1}(e) - v^{-1}(s)\right) \cdot n_1\right| \approx H^1(\Lambda). \tag{13}$$

So in Figure 3, as $v^{-1}(s) \in l_1$ and $v^{-1}(e) \in l_2$ then $H^1(\Lambda) \approx w$, but on the other hand, by the push over lemma we know that $|s - e|$ can not be much less than w, this implies Λ must be close to a straight line.

2.3. Finding lines in a grid of good subsquares

Suppose u is an invertible function with $\int_\Omega \mathrm{d}(Du(z), SO(2) \cup SO(2)H)\, \mathrm{d}L^2 z \leq \delta^2$ and $\int_\Omega \left|D^2 u(z)\right| \mathrm{d}L^2 z \leq \frac{1}{1000}$. It follows from the "push over lemma" and the "ODE method" that if we can find many paths $X(0) = x_0$, $\frac{\mathrm{d}X}{\mathrm{d}t}(t_0) = D\Psi_1(X(t_0))$ in $u(\Omega)$ where the path $\{X(t) : t > 0\}$ is mostly contained in the set

$$\left\{z \in u(\Omega) : \mathrm{d}\left(Du\left(u^{-1}(z)\right), K\right) < \delta\right\}$$

then we have that these paths are mostly straight and so we can control function u on the path, specifically u is L^∞ close to a rotation.

So the problem becomes how to find these paths. The key observation that allows us to find them is the following:

Suppose we have a point $x_0 \in u(\Omega)$ where the path

$$X(0) = x_0 \text{ and } \frac{\mathrm{d}X}{\mathrm{d}t}(t_0) = D\Psi_1(X(t_0)) \tag{14}$$

stays mostly in the set $\left\{z \in u(\Omega) : \mathrm{d}\left(Du\left(u^{-1}(z)\right), SO(2)\right) < \delta\right\}$ then from the study we made in Section 2.1 we know $u^{-1}(\{X(t) : t > 0\})$ will be "roughly" a line in direction n_1.

Conversely if we manage to find a line L in direction n_1 where Du on $\Omega \cap L$ stays mostly within $N_\delta(SO(2))$ then $u(\Omega \cap L)$ will "roughly" form an integral curve to $D\Psi_1$ and the path $u(\Omega \cap L)$ will stay mostly in the set $\left\{z \in u(\Omega) : \mathrm{d}\left(Du\left(u^{-1}(z)\right), SO(2)\right) < \delta\right\}$. So instead of trying to find paths $X : [a, b] \to u(\Omega)$ that satisfy (14) for which Du on $X([a, b])$ stays L^1 close to the wells $SO(2) \cup SO(2)H$, we can look for a straight lines in direction n_1 in Ω for which Du stays L^1 close to $SO(2)$. By Fubini there will be many lines L^1 close to $SO(2) \cup SO(2)H$ and by the bound on surface energy, many of these lines will either be L^1 close to $SO(2)$ or $SO(2)H$.

To summarise, what we have gained is that in the reference configuration (*i.e.* in Ω) we need only look for straight lines with low bulk energy, and by Fubini there will be plenty of these. The cost is that Du must stay close to the well $SO(2)$.

2.3.1. *The grid*

First we will repeat the idea given in Section 2.3 with a bit more detail. Let $\delta > 0$ be some small number and m be a large integer. Suppose we had an invertible function $u : Q_1(0) \to \mathbb{R}^2$ with

$$\int_{Q_1(0)} \mathrm{d}(Du(z), SO(2) \cup SO(2)H)\, \mathrm{d}L^2 z \leq \delta^2 \tag{15}$$

and

$$\int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \leq \frac{1}{1000}. \tag{16}$$

Suppose also we have an $m \times m$ grid of subsquares $T := \{Q_1, Q_2, \ldots Q_{m^2}\}$ that cover $Q_1(0)$ for which we have a subcollection G such that $\mathrm{Card}(T \backslash G) \leq (1 - \delta) m^2$ and G has the following property; for any $Q_k \in G$ there exists $R_k \in SO(2)$, $J_k \in \{Id, H\}$ such that $\int_{Q_k} |Du(z) - R_k J_k|\, \mathrm{d}L^2 z \leq \delta m^{-2}$. Then by the bound on surface energy (16) we must be able to find many lines L in direction n_1 such that $\{Q_k \in G : Q_k \cap L \neq \emptyset\}$ are all subsquares with Du close to either $SO(2)$ or all of them are such that Du is close to $SO(2)H$. If we know additionally that $\int_{Q_1(0)} \mathrm{d}(Du(z), SO(2))\, \mathrm{d}L^2 z \leq \int_{Q_1(0)} \mathrm{d}(Du(z), SO(2)H)\, \mathrm{d}L^2 z$ then we could in fact find many lines in direction n_1 (or direction n_2) on which Du stays close to $SO(2)$.

As we have argued, the u image of these lines will form paths which (roughly speaking) solve the ODE (14) and stay mostly inside the set $\left\{z \in u(Q_1(0)) : \mathrm{d}\left(Du\left(u^{-1}(z)\right), SO(2)\right) < \delta\right\}$ and hence by the push over lemma (*i.e.* using (13)) and the ODE method, these paths will form mostly straight lines.

Now given that there are many lines L in directions n_1 and n_2 on which Du stays close to a fixed (depending only on the line) rotation, it is easy to show that some central subsquare $\widetilde{S}$ (whose size is determined by the eigenvalues of matrix H) must be "surrounded" by the boundary of a "diamond" whose sides are parallel to n_1, n_2 and form subsets of these "controlled" lines (see Fig. 7). So on each of these four lines, (call them L_1, L_2, L_3, L_4) Du must be L^1 close to a fixed rotation R_k. One of the main reasons for working on the grid is that when two lines (say L_1, L_2) intersect on a "good" subsquare $Q_k \in T$ on which $Du \approx R_1$, $Du \approx R_2$ we have $R_1 \approx R_2$. So if we manage to find our four lines L_1, L_2, L_3, L_4 such that they only intersect on "good subsquares" function u on the boundary of the diamond will be L^1 close (with error $\delta^{\frac{1}{8}}$ say) to a fixed rotation. Since there are so many good subsquares finding these four lines is just a matter of careful counting.

Once this is established, by integrating the function in direction ϕ_1 (note $|Du(x)\phi_1| \approx 1$ for any $x \in Q_1(0)$ such that $Du(x)$ is close to the wells $SO(2) \cup SO(2)H$) from one side of the boundary of the diamond to the other we can show that inside the diamond, Du will be mostly close to a rotation R with error say $\delta^{\frac{1}{16}}$.

So if for some δ which is approximately a root power of κ^{m_0} if we can find such a grid we will be in a position to argue the statement of Theorem 1.1.

Ideas similar to this have been used in plate theories, specifically decomposing a region into squares on which a rigidity theorem is applied. See [11] Section 4, and [17].

2.4. The "weak" two well Liouville theorem

Recall our main theorem is a kind of Liouville theorem for functions with small (fixed) surface energy but *much much* smaller bulk energy, where the control of the derivative of the function inside a central subsquare is of some root power of the bulk energy.

We can have a "weaker" theorem of this type (weaker because the control of the derivative in the central subsquare will be bounded by the *surface energy*) as a simple corollary of the BV Poincaré inequality; by the inequality if we let $A = \frac{\int_{Q_1(0)} Du(z)\mathrm{d}L^2 z}{4}$ then we have

$$\int_{Q_1(0)} |Du(z) - A| \,\mathrm{d}L^2 z \leq c \int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \leq c\kappa.$$

And its easy to see $A \in N_\kappa (SO(2) \cup SO(2) H)$.

2.5. Carefully scaling of the "weak" two well Liouville Theorem

Suppose function u is such that $\int_{Q_1(0)} \mathrm{d}(Du(z), SO(2))\,\mathrm{d}L^2 z \leq \int_{Q_1(0)} \mathrm{d}(Du(z), SO(2)H)\,\mathrm{d}L^2 z$ and satisfies (4), (5).

Recall we want a grid of subsquares $T := \{Q_k : k = 1, 2, \dots\}$ that cover $Q_1(0)$ for which there is a subset $G \subset T$ such that for some (possible large) $q \in N$ we have

- $$\mathrm{Card}(T \backslash G) \leq \kappa^{\frac{m_0}{q}} \mathrm{Card}(T). \tag{17}$$
- For each $Q_k \in G$ there exists $R_k \in SO(2)$, $J_k \in \{Id, H\}$ such that
$$\int_{Q_k} |Du(z) - R_k J_k| \,\mathrm{d}L^2 z \leq \kappa^{\frac{m_0}{q}} L^2(Q_k). \tag{18}$$

Since m_0 can be arbitrarily big we can in effect have as much control of bulk energy as we like and so we need only concentrate on the surface energy. However surface energy being the gradient of Du means that it is "morally speaking" one dimension lower than the estimate on bulk energy. If we take a grid with elements of diameter h, we can think of the measure $A \to \int_A \left|D^2 u(z)\right| \mathrm{d}L^2 z$ as being a "one dimensional set" of length $\leq \kappa$ spread out across the elements of the grid.

So if we take the set of "bad" grid elements Q_k for which $\int_{Q_k} \left|D^2 u(z)\right| \mathrm{d}L^2 z \geq \kappa^{\frac{m_0}{q}} h$, the total sum of the lengths of the bad grid elements will be less than $\kappa^{1-\frac{m_0}{q}}$ which is $\kappa^{-\frac{m_0}{q}}$ times longer than the original "one dimensional" set of surface energy. However we are interested in establishing estimate (17) which is a "two dimensional" estimate because $\mathrm{Card}(T) \approx \frac{1}{h^2}$ so the set of bad grid elements is negligible.

Since by the bulk energy estimate we easily have that most of the elements Q_k are such that

$$\int_{Q_k} \mathrm{d}(Du(z), SO(2) \cup SO(2) H)\,\mathrm{d}L^2 z \leq \kappa^{\frac{m_0}{q}} h^2$$

we have the conditions to apply the "weak two well Liouville theorem" on "most" of the elements Q_k of the grid and this give us (17), (18). Hence we have the grid we need. Technicalities aside these are all the elements need for the proof.

We will prove Theorem 2.1, Theorem 1.1 follows by symmetry. Note that throughout the proof c will denote all unimportant constants depending only on σ, ζ_1, ζ_2.

Theorem 2.1. *Let* $0 < \zeta_1 < 1 < \zeta_2 < \infty$. *Let* $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ *for* $\sigma \in (0,1)$. *Let* $K := SO(2) \cup SO(2) H$. *Let* $u \in W^{1,2}(Q_1(0))$ *be a* C^1 *bilipschitz function with* $\mathrm{Lip}(u) < \zeta_2$, $\mathrm{Lip}(u^{-1}) < \zeta_1^{-1}$. *There exists positive* *constants* $\mathbf{c}_1, \mathbf{c}_3, \mathbf{c}_4 < 1$ *and* $\mathbf{c}_2, \mathbf{c}_5 > 1$ *depending on* σ, ζ_1, ζ_2 *such that if* $k \in (0, \mathbf{c}_1]$, $m_0 \geq \mathbf{c}_2$ *and function* u *satisfies*

$$\int_{Q_1(0)} \mathrm{d}\left(Du\left(z\right), K\right) \mathrm{d}L^2 z \leq \kappa^{m_0} \tag{19}$$

$$\int_{Q_1(0)} \left|D^2 u\left(z\right)\right| \mathrm{d}L^2 z \leq \mathbf{c}_3 \kappa \tag{20}$$

$$\int_{Q_{\mathbf{c}_4}(0)} \mathrm{d}\left(Du\left(z\right), SO\left(2\right) H\right) \mathrm{d}L^2 z \leq \int_{Q_{\mathbf{c}_4}(0)} \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) \mathrm{d}L^2 z \tag{21}$$

then there exists $R_1 \in SO(2)$ *such that*

$$\int_{Q_{\mathbf{c}_4}(0)} \left|Du\left(z\right) - R_1 H\right| \leq \mathbf{c}_5 \kappa^{\frac{m_0}{800}}.$$

3. Preliminary notation

Let $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ for $\sigma \in (0,1)$. Throughout all the lemmas we take

$$K := SO\left(2\right) \cup SO\left(2\right) H. \tag{22}$$

Let $0 < \zeta_1 < 1 < \zeta_2 < \infty$. Define

$$\mathcal{D}\left(\zeta_1, \zeta_2\right) := \left\{ M \in M^{2\times 2} : \inf_{v \in S^1} |Mv| \geq \zeta_1 \text{ and } \sup_{v \in S^1} |Mv| \leq \zeta_2 \right\}. \tag{23}$$

Given a C^1 invertible function $u : \Omega \to \mathbb{R}^2$, u being bilipschitz with $\mathrm{Lip}\left(u\right) \leq \zeta_2$, $\mathrm{Lip}\left(u^{-1}\right) \leq \zeta_1^{-1}$ is equivalent to

$$Du\left(z\right) \in \mathcal{D}\left(\zeta_1, \zeta_2\right) \text{ for all } z \in \Omega.$$

The latter formulation will be more convenient for us. Let

$$R\left(z, \alpha, \beta\right) := \left\{ x \in \mathbb{R}^2 : \left|\left(z - x\right) \cdot e_1\right| \leq \beta, \left|\left(z - x\right) \cdot e_2\right| \leq \alpha \right\}.$$

4. Push over lemma

This is the push over Lemma described in Section 2.1 of the introduction. The proof is essentially a calculation, see Section 2.1 for a explanation of why it works.

Lemma 4.1. *Let* $0 < \zeta_1 < 1 < \zeta_2 < \infty$. *Let* K *be as in (22). Let* $u \in W^{2,1}\left(Q_1\left(0\right)\right)$ *be a* C^1 *invertible function with the property that* $Du\left(x\right) \in \mathcal{D}\left(\zeta_1, \zeta_2\right)$ *for all* $x \in Q_1\left(0\right)$. *Let*

$$\phi_1 = \begin{pmatrix} \frac{1}{\sqrt{1+\sigma^2}} \\ \frac{\sigma}{\sqrt{1+\sigma^2}} \end{pmatrix}, \ \phi_2 = \begin{pmatrix} \frac{-1}{\sqrt{1+\sigma^2}} \\ \frac{\sigma}{\sqrt{1+\sigma^2}} \end{pmatrix} \text{ note that } \left|H\phi_i\right| = 1 \text{ for } i = 1, 2. \tag{24}$$

Let n_i *denote the anti-clockwise normal to* ϕ_i *for* $i = 1, 2$.

Let $i \in \{1, 2\}$. *For any* $s, e \in u\left(Q_1\left(0\right)\right)$, *such that* $\eta := [s, e] \subset u\left(Q_1\left(0\right)\right)$ *and*

$$\int_{\eta} \mathrm{d}\left(Du\left(u^{-1}\left(z\right)\right), K\right) \mathrm{d}H^1 z < \alpha \left|s - e\right| \tag{25}$$

then

$$|s-e| > \left|\left(u^{-1}(s)-u^{-1}(e)\right)\cdot n_i\right| - \zeta_1^{-1}\alpha\,|s-e|\,. \tag{26}$$

Proof. We begin with the main inequality.

Step 1. Let $i\in\{1,2\}$, for any $\psi\in S^1$

$$|H\psi| \geq \psi\cdot n_i. \tag{27}$$

Proof of Step 1[4]. This follows by self adjointness of H and Cauchy Schwartz inequality, let $\psi^\flat$ denote the clockwise normal to ψ

$$\begin{aligned}\psi\cdot n_i &= \psi^\flat\cdot\phi_i\\ &= H^{-1}\psi^\flat\cdot H\phi_i\\ &\leq \left|H^{-1}\psi^\flat\right|\\ &= |H\psi|\,.\end{aligned}$$

Proof of Lemma. Let $J: u(Q_1(0))\to\mathbb{R}$ be defined by $J(x)=\mathrm{d}\left(Du\left(u^{-1}(x)\right),K\right)$. We let $t_x\in S^1$ denote the tangent to the curve $u^{-1}(\eta)$ at point x

$$\begin{aligned}\int_\eta J(z)\,\mathrm{d}H^1 z &= \int_{u^{-1}(\eta)} |Du(x)\,t_x|\,J(u(x))\,\mathrm{d}H^1 x\\ &\geq \zeta_1\int_{u^{-1}(\eta)} J(u(x))\,\mathrm{d}H^1 x.\end{aligned}$$

So using (25) we have

$$\zeta_1^{-1}\alpha\,|s-e| \geq \int_{u^{-1}(\eta)} \mathrm{d}(Du(x),K)\,\mathrm{d}H^1 x. \tag{28}$$

Now for each $x\in u^{-1}(\eta)$, let $G(x)\in K$ be the matrix such that $\mathrm{d}(Du(x),K)=|Du(x)-G(x)|$. Let $E(x)=Du(x)-G(x)$, note that $|E(x)|=\mathrm{d}(Du(x),K)$. So

$$\begin{aligned}|s-e| &= L^1(\eta)\\ &= \int_{u^{-1}(\eta)} |Du(x)\,t_x|\,\mathrm{d}H^1 x\\ &\geq \int_{u^{-1}(\eta)} |G(x)\,t_x| - |E(x)\,t_x|\,\mathrm{d}H^1 x\\ &\overset{(27)}{\geq} \int_{u^{-1}(\eta)} t_x\cdot n_i - \int_{u^{-1}(\eta)} |E(x)\,t_x|\,\mathrm{d}H^1 x\\ &\overset{(28)}{\geq} L^1\left(P_{\phi_i^\perp}\left(u^{-1}(\eta)\right)\right) - \zeta_1^{-1}\alpha\,|s-e|\\ &= \left|\left(u^{-1}(s)-u^{-1}(e)\right)\cdot n_i\right| - \zeta_1^{-1}\alpha\,|s-e|\,.\end{aligned}$$

□

5. Weak two well Liouville theorem

Lemma 5.1 is the "weak two well Liouville Theorem" described in Section 2.4 of the introduction. The proof is simply a matter of applying the BV Poincaré inequality.

[4] I would like to thank Laszlo Szekelyhidi for the following argument.

Lemma 5.1. *Suppose $u \in W^{2,1}(Q_1(0)) \cap C^1$ with the property that for constant $\zeta_2 > 1$ we have $Du(z) \in \mathcal{D}(0,\zeta_2)$ (see definition (23)) for all $z \in Q_1(0)$. Let K be as in (22). Suppose $\kappa > 0$ is a small number and that u satisfies the following inequalities*

$$\int_{Q_1(0)} \mathrm{d}(Du(z), K)\,\mathrm{d}L^2 z \le \kappa \tag{29}$$

$$\int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \le \kappa \tag{30}$$

then for some $R \in SO(2)$, $J \in \{H, Id\}$ we have

$$\int_{Q_1(0)} |Du(z) - RJ|\,\mathrm{d}L^2 z < c\kappa. \tag{31}$$

Proof. Let $A = \frac{1}{4}\int_{Q_1(0)} Du(z)\,\mathrm{d}L^2 z$. By the BV Poincaré inequality (see Th. 3.43 [1]) we have

$$\begin{aligned}\int_{Q_1(0)} |Du(z) - A|\,\mathrm{d}L^2 z &\le c\int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \\ &\le c\kappa.\end{aligned} \tag{32}$$

And

$$\begin{aligned}4\mathrm{d}(A, K) &\le \int_{Q_1(0)} |A - Du(z)|\,\mathrm{d}L^2 z + \int_{Q_1(0)} \mathrm{d}(Du(z), K)\,\mathrm{d}L^2 z \\ &\overset{(29),(32)}{\le} 2c\kappa.\end{aligned} \tag{33}$$

So there exists $R \in SO(2)$, $J \in \{H, Id\}$ such that $|A - RJ| = \mathrm{d}(A, K)$ and by (32) and (33) satisfies (31). ☐

Definition 5.2. Given vectors $v_1, v_2 \in S^1$ and $\delta > 0$ we define a grid $G(v_1, v_2, \delta)$ as follows.

$$G(v_1, v_2, \delta) := \{P(k_1\delta v_1 + k_2\delta v_2, v_1, v_2, \delta) : P(k_1\delta v_1 + k_2\delta v_2, v_1, v_2, \delta) \subset Q_1(0), k_1, k_2 \in \mathbb{Z}\}$$

where $P(x_1, v_1, v_2, \delta)$ is a parallelogram centered on x_1 whose sides are parrel to v_1, v_2 and of length δ. Note that the grid is the set of parallelograms inside $Q_1(0)$.

6. Scaling lemma

In this lemma we set up the grid described in Sections 2.3.1 and 2.5 of the introduction. The proof is a matter of simple scaling and counting.

Lemma 6.1. *Let $Q_1(0)$ be the unit square in $\mathbb{R}^2$. Let K be as in (22). Let integer m_0 be large. Given $u \in W^{1,2}(Q_1(0)) \cap C^1$ that for small $\kappa > 0$ satisfies the following properties,*

- $$Du(x) \in \mathcal{D}(0,\zeta_2) \text{ for all } x \in Q_1(0).$$
- $$\int_{Q_1(0)} \mathrm{d}(Du(z), K)\,\mathrm{d}L^2 z \le \kappa^{m_0} \tag{34}$$
- $$\int_{Q_1(0)} \left|D^2 u(z)\right| \mathrm{d}L^2 z \le 1. \tag{35}$$

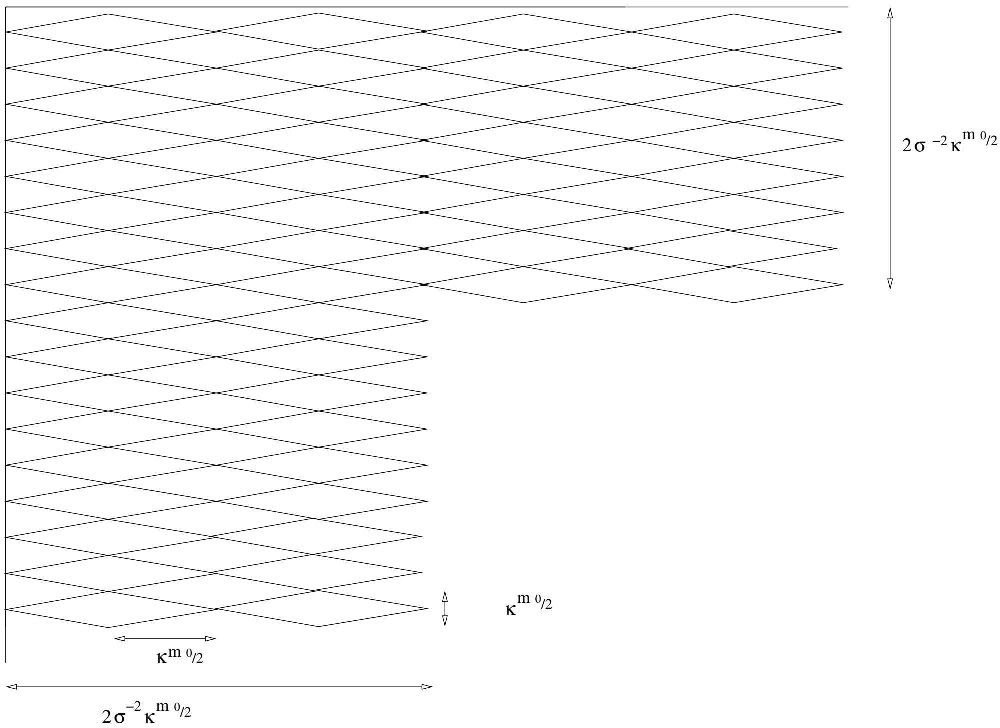


FIGURE 4. Elements of the grid $G\left(w_1, w_2, \kappa^{\frac{m_0}{2}}\right)$.

Let $w_1, w_2 \in S^1$ be vectors such that $w_1 \cdot w_2 \in \left(-1+2\sigma^6, 1-2\sigma^6\right)$. Then we can find a subcollection $\mathcal{G} \subset G\left(w_1, w_2, \kappa^{\frac{m_0}{2}}\right)$ with the following properties

- $\mathrm{Card}\left(G\left(w_1, w_2, \kappa^{\frac{m_0}{2}}\right) \backslash \mathcal{G}\right) \leq c\kappa^{-\frac{3m0}{4}}$.
- *For any $P \in \mathcal{G}$ there exists $R \in SO(2)$, $J \in \{H, Id\}$ such that*

$$\int_P |Du(z) - RJ| \, \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{4}} \kappa^{m_0}. \tag{36}$$

Proof. First note that $P(0, w_1, w_2, 1) \subset Q_1(0)$. We define $z_{k_1,k_2} := k_1 w_1 + k_2 w_2$.

Let $W := \left\{(k_1, k_2) : Q_{\kappa^{\frac{m_0}{2}}}(z_{k_1,k_2}) \subset Q_1(0)\right\}$. Let $\theta > 0$ be the angle between w_1 and w_2. Now $\left(\sin\frac{\theta}{2}\right)^2 = \frac{1-\cos\theta}{2} \geq \sigma^6$, so $\sin\frac{\theta}{2} \geq \sigma^3$. From this it follows that the width or height (which ever is smaller) of any parallelogram $P \in G(w_1, w_2, 1)$ is greater than σ^3. Now $G\left(w_1, w_2, \kappa^{\frac{m_0}{2}}\right) \backslash W$ are the set of parallelograms close to the boundary, as can easily be seen from Figure 4

$$\mathrm{Card}\left(G\left(w_1, w_2, \kappa^{\frac{m_0}{2}}\right)\right) - \mathrm{Card}(W) < c\kappa^{-\frac{m_0}{2}}. \tag{37}$$

Let

$$B_1 := \left\{(k_1, k_2) \in W : \int_{Q_{\kappa^{\frac{m_0}{2}}}(z_{k_1,k_2})} \mathrm{d}(Du(z), K) \, \mathrm{d}L^2 z \geq \kappa^{\frac{5m_0}{4}}\right\}. \tag{38}$$

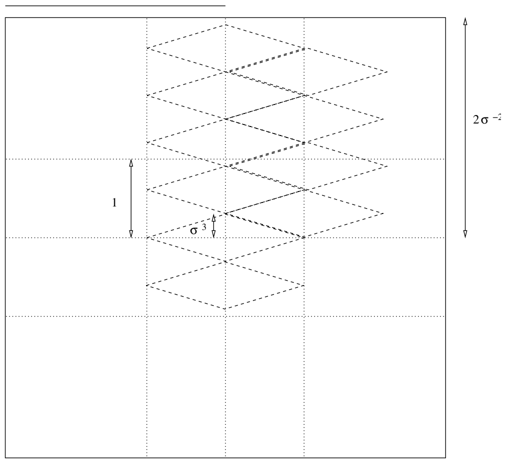


FIGURE 5. The squares $\left\{Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)\right\}$ centered on the elements of the grid.

Let

$$B_2 := \left\{(k_1,k_2) \in W : \int_{Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)} \left|D^2 u\left(z\right)\right| \mathrm{d}L^2 z \geq \kappa^{\frac{3m_0}{4}}\right\}. \tag{39}$$

Now as can seen from Figure 5, $\left\{Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right) : (k_1,k_2) \in W\right\}$ can not overlap by more than c times. Formally

$$\sum_{(k_1,k_2)\in W} \chi_{Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)}\left(z\right) \leq c. \tag{40}$$

So

$$\begin{aligned}
\mathrm{Card}\left(B_1\right)\kappa^{\frac{5m_0}{4}} &\leq \sum_{(k_1,k_2)\in B_1} \int_{Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)} \mathrm{d}\left(Du\left(z\right),K\right)\mathrm{d}L^2 z \\
&\overset{(40)}{\leq} c\int_{\bigcup_{(k_1,k_2)\in B_1} Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)} \mathrm{d}\left(Du\left(z\right),K\right)\mathrm{d}L^2 z \\
&\overset{(34)}{\leq} c\kappa^{m_0}.
\end{aligned}$$

Thus

$$\mathrm{Card}\left(B_1\right) \leq c\kappa^{-\frac{m_0}{4}}. \tag{41}$$

In the same way

$$\begin{aligned}
\mathrm{Card}\left(B_2\right)\kappa^{\frac{3m_0}{4}} &\leq \sum_{(k_1,k_2)\in B_2} \int_{Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)} \left|D^2 u\left(z\right)\right| \mathrm{d}L^2 z \\
&\leq c\int_{\bigcup_{(k_1,k_2)\in B_2} Q_{\kappa^{\frac{m_0}{2}}}\left(z_{k_1,k_2}\right)} \left|D^2 u\left(z\right)\right| \mathrm{d}L^2 z \\
&\overset{(35)}{\leq} c.
\end{aligned}$$

Thus

$$\text{Card}\,(B_2) \leq c\kappa^{-\frac{3m_0}{4}}. \tag{42}$$

Now for any $(k_1,k_2) \in W \setminus (B_1 \cup B_2)$ we can define function v on $Q_1(0)$ by

$$v(z) = u\left(\kappa^{\frac{m_0}{2}} z + z_{k_1,k_2}\right)\kappa^{-\frac{m_0}{2}}.$$

Since $(k_1,k_2) \notin B_1$ (see definition (38))

$$\begin{aligned}\int_{Q_1(0)} \mathrm{d}\,(Dv(z), K)\, \mathrm{d}L^2 z &= \int_{Q_{\kappa^{\frac{m_0}{2}}}(z_{k_1,k_2})} \mathrm{d}\,(Du(y), K)\,\kappa^{-m_0}\mathrm{d}L^2 y \\ &\leq \kappa^{\frac{m_0}{4}}.\end{aligned}$$

Now since $(k_1,k_2) \notin B_2$, (see (39))

$$\begin{aligned}\int_{Q_1(0)} \left|D^2 v(z)\right| \mathrm{d}L^2 z &= \int_{Q_{\kappa^{\frac{m_0}{2}}}(z_{k_1,k_2})} \left|D^2 u(y)\right| \kappa^{-\frac{m_0}{2}}\mathrm{d}L^2 y \\ &\leq \kappa^{\frac{m_0}{4}}.\end{aligned}$$

Now we can apply Lemma 5.1 to v on $Q_1(0)$ we can obtain that for some $R \in SO(2)$, $J \in \{Id, H\}$ we have

$$\int_{Q_1(0)} |Dv(z) - RJ|\, \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{4}}.$$

Since $P(0,w_1,w_2,1) \subset Q_1(0)$ this of course implies

$$\int_{P(0,w_1,w_2,1)} |Dv(z) - RJ|\, \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{4}}. \tag{43}$$

Now we scale this information back to learn about the derivative of u on $P\left(z_{k_1,k_2}, w_1, w_2, \kappa^{\frac{m_0}{2}}\right)$. Recall

$$u(z) = \kappa^{\frac{m_0}{2}} v\left(\frac{z - z_{k_1,k_2}}{\kappa^{\frac{m_0}{2}}}\right) \text{ for } z \in Q_{\kappa^{\frac{m_0}{2}}}(z_{k_1,k_2}).$$

So let $y = (z - z_{k_1,k_2})\,\kappa^{-\frac{m_0}{2}}$,

$$\begin{aligned}\int_{P\left(z_{k_1,k_2},w_1,w_2,\kappa^{\frac{m_0}{2}}\right)} |Du(z) - RJ|\, \mathrm{d}L^2 z \;&=\; \int_{P(0,w_1,w_2,1)} |Dv(y) - RJ|\,\kappa^{m_0}\mathrm{d}L^2 y \\ &\overset{(43)}{\leq}\; c\kappa^{\frac{m_0}{4}}\kappa^{m_0}.\end{aligned} \tag{44}$$

Let

$$G := \left\{P\left(z_{k_1,k_2}, w_1, w_2, \kappa^{\frac{m_0}{2}}\right) : (k_1,k_2) \in W \setminus (B_1 \cup B_2)\right\}$$

from (37), (41), (42) we have that

$$\text{Card}\left(G\left(w_1, w_2, \kappa^{\frac{m_0}{2}}\right) \setminus G\right) \leq c\kappa^{-\frac{3m_0}{4}}.$$

And by (44) any $P \in G$ satisfies (36) and this completes the proof. □

7. Following integral curves I

If we have a curve γ with endpoints a, b and $|a-b| > H^1(\gamma) - \delta$ we can show that the tangents (denote the tangent at point z by t_z) of the curve mostly point in direction $\frac{a-b}{|a-b|}$ by the following trick

$$\int_\gamma \left(t_z - \frac{a-b}{|a-b|} \right)^2 \mathrm{d}H^1 z = 2H^1(\gamma) - 2\left(\int_\gamma t_z \mathrm{d}H^1 z, \frac{a-b}{|a-b|} \right) = 2\left(H^1(\gamma) - |a-b| \right) < 2\delta. \tag{45}$$

Letting c_1, c_2 be the centres of P_1, P_{m_1} respectively, the curve we will be considering is given by $u([c_1, c_2])$. Analogously to what we discussed in Section 2.4 of the introduction, if we have a line L parallel to $H^{-1}n_1$ such that $\int_{L \cap Q_1(0)} \mathrm{d}(Du(z), SO(2)H) \mathrm{d}H^1 z \leq \delta$ then the curve $u([c_1, c_2])$ will form a small perturbation of an integral curve to the vector field $\Psi_1 : u(Q_1(0)) \to \mathbb{R}$ (recall $\Psi_1(x) := u^{-1}(x) \cdot n_1$). Since $||D\Psi_1(z)| - 1| < \delta$ for all z such that $\mathrm{d}\left(Du\left(u^{-1}(z)\right), K\right) < \delta$ we have $|(c_1 - c_2) \cdot n_1| = |\Psi_1(u(c_1)) - \Psi_1(u(c_2))| \approx H^1(u([c_1, c_2]))$. However by (50) this is also the distance between the end points of the path $u([c_1, c_2])$ and by a trick very similar to (45) this gives (51), (52). We will have to use Lemma 7.4 a couple of times, for this reason we formulate it in a more general way than would at first seem necessary.

Notation. Given a set of vectors $\{v_1, v_2, \ldots v_m\}$ let $\langle v_1, v_2, \ldots v_m \rangle$ denote the span of these vectors, *i.e.* $\langle v_1, v_2, \ldots v_m \rangle = \{\sum_{i=1}^m \lambda_i v_i : \lambda_i \in \mathbb{R}\}$.

Definition 7.1. A G-line inside grid $G(w_1, w_2, \alpha)$ is subset $\{P_1, P_2, \ldots P_{k_1}\} \subset G(w_1, w_2, \alpha)$ which form a connected line of parallelograms in direction w_1 or w_2. Formally, $\{P_1, P_2, \ldots P_{k_1}\}$ satisfies the following properties

- $\overline{P_k} \cap \overline{P_{k+1}} \neq \emptyset$ for $k \in \{1, 2, \ldots k_1 - 1\}$.
- If $C(P_k)$ denotes the center of the parallelogram P_k, then either

$$P_{w_1^\perp}(C(P_i)) = P_{w_1^\perp}(C(P_j)) \text{ for } i, j \in \{1, 2, \ldots k_1\}$$

or

$$P_{w_2^\perp}(C(P_i)) = P_{w_2^\perp}(C(P_j)) \text{ for } i, j \in \{1, 2, \ldots k_1\}$$

Definition 7.2. A complete G-line $\{P_1, P_2, \ldots P_{k_1}\}$ inside grid $G(w_1, w_2, \alpha)$ is a G-line with the property that $\mathrm{d}(P_1, \partial Q_1(0)) \leq 2\kappa^{\frac{m_0}{2}}$ and $\mathrm{d}(P_{k_1}, \partial Q_1(0)) \leq 2\kappa^{\frac{m_0}{2}}$. Informally, the G-line cuts right across the grid.

Definition 7.3. Given grid $G(w_1, w_2, \alpha)$, and G-line L we let

$$\widetilde{L} := \bigcup_{P \in L} P.$$

Lemma 7.4. *Let $u \in W^{1,2}\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right)$ be an invertible C^1 function with assumption that $Du(z) \in \mathcal{D}(\zeta_1, \zeta_2)$ for all $z \in Q_{16\zeta_1^{-1}\zeta_2}(0)$. Let K be defined by (22). Let $m_0 \geq 16$. Let $\kappa > 0$ be a small number (depending on σ, ζ_1, ζ_2), suppose function u satisfies the following properties:*

(1)
$$\int_{Q_{16\zeta_1^{-1}\zeta_2}(0)} \mathrm{d}(Du(z), K) \, \mathrm{d}L^2 z \leq \kappa^{m_0}. \tag{46}$$

(2) *There exist G-line $\{P_1, P_2, \ldots P_{m_1}\}$ parallel to $\frac{H^{-2}n_i}{|H^{-2}n_i|}$ inside grid $G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)$ and a subset $M_0 \subset \{P_1, P_2, \ldots P_{m_1}\}$ such that*
   - $$\mathrm{Card}(\{P_1, P_2, \ldots P_{m_1} \backslash M_0\}) \leq 2\kappa^{p_0}\kappa^{-\frac{m_0}{2}} \text{ for some } p_0 \geq \frac{m_0}{16}. \tag{47}$$

- $$\operatorname{dist}(P_1, P_{m_1}) > \frac{\sigma^2}{16}\cdot \tag{48}$$
- *For each $P \in M_0$ there exists $R \in SO(2)$ such that*

$$\int_P |Du(z) - RH| \, dL^2 z \leq c\kappa^{\frac{m_0}{4}} \kappa^{m_0}. \tag{49}$$

And with the property that for some points $x_1 \in P_1$ and $x_2 \in P_{m_1}$ where $\frac{x_2 - x_1}{|x_2 - x_1|} = \frac{H^{-2} n_i}{|H^{-2} n_i|}$ we have

$$||u(x_1) - u(x_2)| - |(x_1 - x_2) \cdot n_i|| < c\kappa^{q_0} \text{ for some } q_0 \geq \frac{m_0}{8}\cdot \tag{50}$$

Then let $R_0 \in SO(2)$ be such that $R_0 H^{-1} n_1 = \frac{u(x_2) - u(x_1)}{|u(x_2) - u(x_1)|}$, there exists a subset $M_1 \subset M_0$ with

$$\operatorname{Card}(M_0 \backslash M_1) \leq c\left(\kappa^{\frac{p_0}{2}} + \kappa^{\frac{q_0}{2}}\right) \kappa^{-\frac{m_0}{2}} \tag{51}$$

such that for any $P \in M_1$ we have

$$\int_P |Du(z) - R_0 H| \, dL^2 z \leq c\left(\kappa^{\frac{p_0}{4}} + \kappa^{\frac{q_0}{4}}\right) \kappa^{m_0}. \tag{52}$$

Proof.

Step 1. There exists $w_1 \in P_1$, $w_2 \in P_{m_1}$ such that if $v_1 := \frac{u(w_2) - u(w_1)}{|u(w_2) - u(w_1)|}$ then

$$\int_{w_1}^{w_2} \left| Du(x) H^{-2} n_i - v_1 \right|^2 dH^1 x < c(\kappa^{p_0} + \kappa^{q_0}). \tag{53}$$

Proof of Step 1. Define $O : M_0 \to SO(2)$ as follows. For each $P \in M_0$ let $O(P) \in SO(2)$ be a rotation (which by definition of M_0 we know exists) such that

$$\int_P |Du(z) - O(P) H| \, dL^2 z \leq c\kappa^{m_0} \kappa^{\frac{m_0}{4}}. \tag{54}$$

We define function

$$\widetilde{E}(z) := \begin{cases} |Du(z) - O(P) H| & z \in \widetilde{M_0} \\ 2\zeta_2 & z \notin \widetilde{M_0}. \end{cases} \tag{55}$$

So using (47)

$$\begin{aligned} \int_{\bigcup_{k=1}^{m_1} P_k} \widetilde{E}(z) \, dL^2 z &\leq c\kappa^{m_0} \kappa^{\frac{m_0}{4}} \operatorname{Card}(\{P_1, P_2, \dots P_{m_1}\} \cap M_0) \\ &\quad + 2\zeta_2 \kappa^{m_0} \operatorname{Card}(\{P_1, P_2, \dots P_{m_1}\} \backslash M_0) \\ &\overset{(47)}{\leq} c\kappa^{\frac{m_0}{4}} \kappa^{\frac{m_0}{2}} + 4\zeta_2 \kappa^{p_0} \kappa^{\frac{m_0}{2}} \\ &\leq c\kappa^{p_0} \kappa^{\frac{m_0}{2}}. \end{aligned} \tag{56}$$

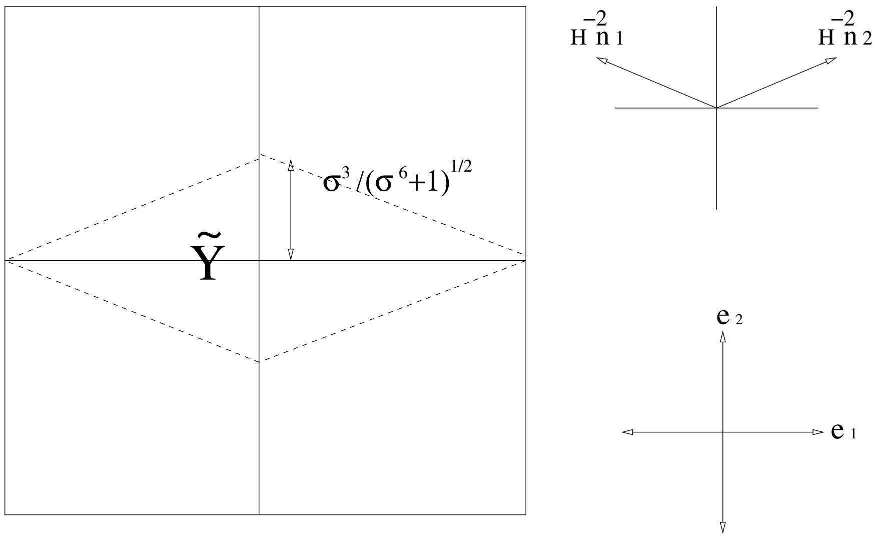


FIGURE 6. The central region $\widetilde{Y}$ in which we gain control of our function.

Now its a calculation to see $\frac{H^{-2}n_1}{|H^{-2}n_1|} = \begin{pmatrix} \frac{1}{\sqrt{1+\sigma^6}} \\ \frac{\sigma^3}{\sqrt{1+\sigma^6}} \end{pmatrix}$ so $\left|\frac{H^{-2}n_1}{|H^{-2}n_1|} \cdot e_2\right| \geq \frac{\sigma^3}{2}$. Now we will take a diamond of side length 1 with sides parallel to $\frac{H^{-2}n_i}{|H^{-2}n_i|}$ for $i = 1,2$. The length of the smallest projection will be greater than $\frac{\sigma^3}{2}$, see Figure 6. Now $\{P_k : k = 1, \dots m_1\}$ are diamonds of side length $\kappa^{\frac{m_0}{2}}$ with sides parallel to $\frac{H^{-2}n_i}{|H^{-2}n_i|}$ for $i = 1,2$. So

$$P_{(H^{-2}n_1)^{\perp}}(P_k) \geq \sigma^3 \frac{\kappa^{\frac{m_0}{2}}}{4}. \tag{57}$$

So by Fubini from (56) we must be able to find a point $w_1 \in P_1$ such that

$$\int_{(w_1 + \langle H^{-2}n_i \rangle) \cap \left(\bigcup_{k=1}^{m_1} P_k\right)} \widetilde{E}(z)\, \mathrm{d}L^1 z \leq c\kappa^{p_0}. \tag{58}$$

Take point $w_2 \in P_{m_1} \cap \left(w_1 + \langle H^{-2}n_i \rangle\right)$. Note that by Lipschitzness from (50) we have

$$||u(w_1) - u(w_2)| - |(w_1 - w_2) \cdot n_i|| \leq c\left(\kappa^{q_0} + \kappa^{\frac{m_0}{2}}\right). \tag{59}$$

For each $x \in [w_1, w_2]$ let $\Gamma(x) \in SO(2)$ be such that

$$\mathrm{d}(Du(x), SO(2)H) = |Du(x) - \Gamma(x)H|. \tag{60}$$

From (24) we know

$$\left|H^{-1}n_i\right| = \left|\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} \frac{\pm\sigma}{\sqrt{(1+\sigma^2)}} \\ \frac{1}{\sqrt{1+\sigma^2}} \end{pmatrix}\right| = \left|\begin{pmatrix} \frac{\pm 1}{\sqrt{(1+\sigma^2)}} \\ \frac{\sigma}{\sqrt{1+\sigma^2}} \end{pmatrix}\right| = 1. \tag{61}$$

Thus

$$\begin{aligned} \left|\left|Du(z)H^{-2}n_i\right| - \left|\Gamma(z)H^{-1}n_i\right|\right| &\leq 2\left|Du(z)H^{-1} - \Gamma(z)\right| \\ &\overset{(60)}{\leq} 4\sigma^{-1}\mathrm{d}(Du(x), SO(2)H). \end{aligned} \tag{62}$$

Hence from (55), (58), (62)

$$\begin{aligned}
&\left|\int_{[w_1,w_2]}\left|Du(z)\,H^{-2}n_i\right|^2 \mathrm{d}L^1 z-\int_{[w_1,w_2]}\left|\Gamma(z)\,H^{-1}n_i\right|^2 \mathrm{d}L^1 z\right|\\
&\quad\leq\left|\int_{[w_1,w_2]}\left(\left|Du(z)\,H^{-2}n_i\right|-\left|\Gamma(z)\,H^{-1}n_i\right|\right)\left(\left|Du(z)\,H^{-2}n_i\right|+\left|\Gamma(z)\,H^{-1}n_i\right|\right)\mathrm{d}L^1 x\right|\\
&\quad\leq\int_{[w_1,w_2]}2\left|\left|Du(z)\,H^{-2}n_i\right|-\left|\Gamma(z)\,H^{-1}n_i\right|\right|\zeta_2\sigma^{-1}\mathrm{d}L^1 z\\
&\quad\overset{(62)}{\leq} 8\zeta_2\sigma^{-2}\int_{[w_1,w_2]}\mathrm{d}\left(Du(z),SO(2)\,H\right)\mathrm{d}L^1 z\\
&\quad\overset{(55)}{\leq} 8\zeta_2\sigma^{-2}\int_{[w_1,w_2]}\widetilde{E}(z)\,\mathrm{d}L^1 z\\
&\quad\overset{(58)}{\leq} c\kappa^{p_0}.
\end{aligned}$$

Since from (61) we know $\left|\Gamma(x)\,H^{-1}n_i\right|=\left|H^{-1}n_i\right|=1$ we have

$$\left|\int_{[w_1,w_2]}\left|Du(x)\,H^{-2}n_i\right|^2 \mathrm{d}L^1 x-|w_1-w_2|\right|\leq c\kappa^{p_0}. \tag{63}$$

Let

$$v_1:=\frac{u(w_2)-u(w_1)}{|u(w_2)-u(w_1)|}. \tag{64}$$

So

$$\begin{aligned}
\int_{[w_1,w_2]}\left|Du(z)\,H^{-2}n_i-v_1\right|^2 \mathrm{d}L^1 z &= \int_{[w_1,w_2]}\left|Du(z)\,H^{-2}n_i\right|^2+|v_1|^2-2\left(Du(z)\,H^{-2}n_i,v_1\right)\mathrm{d}L^1 z\\
&\overset{(63)}{\leq} 2|w_1-w_2|+c\kappa^{p_0}-2\left|H^{-2}n_i\right|\left(\int_{[w_1,w_2]}Du(z)\,\frac{H^{-2}n_i}{|H^{-2}n_i|}\mathrm{d}L^1 z,v_1\right)\\
&= 2|w_1-w_2|+c\kappa^{p_0}-2\left|H^{-2}n_i\right|\left(u(w_1)-u(w_2),v_1\right).
\end{aligned} \tag{65}$$

Note from the definition of v_1, (64)

$$\left|H^{-2}n_i\right|\left(u(w_1)-u(w_2),v_1\right)=\left|H^{-2}n_i\right|\left|u(w_1)-u(w_2)\right|. \tag{66}$$

As $\frac{w_2-w_1}{|w_2-w_1|}=\frac{H^{-2}n_i}{|H^{-2}n_i|}$ so

$$|(w_1-w_2)\cdot n_i|=|w_1-w_2|\left|\frac{H^{-2}n_i}{|H^{-2}n_i|}\cdot n_i\right|. \tag{67}$$

Putting (67) together with (59) we get

$$\left||u(w_1)-u(w_2)|-|w_1-w_2|\left|\frac{H^{-2}n_i}{|H^{-2}n_i|}\cdot n_i\right|\right|\leq c\left(\kappa^{q_0}+\kappa^{\frac{m_0}{2}}\right). \tag{68}$$

Note by self adjointness

$$\begin{aligned} H^{-2}n_1 \cdot n_1 &= H^{-1}n_1 \cdot H^{-1}n_1 \\ &= \left|H^{-1}n_1\right|^2 \\ &\overset{(61)}{=} 1. \end{aligned} \tag{69}$$

In the same way we can see that $H^{-2}n_2 \cdot n_2 = 1$. So applying (69) to (68) we have

$$\left|\left|H^{-2}n_i\right| \left|u\left(w_1\right) - u\left(w_2\right)\right| - \left|w_1 - w_2\right|\right| \leq c\left(\kappa^{q_0} + \kappa^{\frac{m_0}{2}}\right). \tag{70}$$

So from (66) this implies

$$\left|\left|w_1 - w_2\right| - \left|H^{-2}n_i\right| \left(u\left(w_1\right) - u\left(w_2\right), v_1\right)\right| \leq c\left(\kappa^{q_0} + \kappa^{\frac{m_0}{2}}\right).$$

Applying this to (65) we get

$$\int_{[w_1,w_2]} \left|Du\left(z\right) H^{-2}n_i - v_1\right|^2 \mathrm{d}L^1 z \leq c\left(\kappa^{p_0} + \kappa^{q_0}\right). \tag{71}$$

This completes the proof of Step 1.

Proof of lemma continued.

Now recall from (58) we know that

$$\int_{[w_1,w_2]} \widetilde{E}\left(z\right) \mathrm{d}L^1 z \leq c\kappa^{p_0}. \tag{72}$$

So we can find a set of intervals $I_1, I_2, \dots I_{m_1-2} \subset [w_1, w_2]$ with $I_k := [w_1, w_2] \cap P_k$ for some $k \in \{1, 2, \dots m_1\}$ and $L^1\left([w_1, w_2] \setminus \left(\bigcup_{k=1}^{m_1-2} I_k\right)\right) \leq 3\kappa^{\frac{m_0}{2}}$. Let

$$A_1 := \left\{k \in \{1, 2, \dots m_1 - 2\} : \int_{I_k} \widetilde{E}\left(z\right) \mathrm{d}L^1 z \leq c\kappa^{\frac{p_0}{2}} L^1\left(I_k\right)\right\}. \tag{73}$$

Thus from (72)

$$\begin{aligned} c\kappa^{\frac{p_0}{2}}\left(\sum_{k\in\{1,2,\dots m_1-2\}\setminus A_1} L^1\left(I_k\right)\right) &\leq \sum_{k\in\{1,2,\dots m_1-2\}\setminus A_1} \int_{I_k} \widetilde{E}\left(z\right) \mathrm{d}L^1 z \\ &\overset{(72)}{\leq} c\kappa^{p_0}. \end{aligned}$$

So

$$c\kappa^{\frac{p_0}{2}} \geq \sum_{k\in\{1,2,\dots m_1-2\}\setminus A_1} L^1\left(I_k\right) = \mathrm{Card}\left(\{1, 2, \dots m_1 - 2\} \setminus A_1\right) \kappa^{\frac{m_0}{2}}.$$

Hence

$$\mathrm{Card}\left(\{1, 2, \dots m_1 - 2\} \setminus A_1\right) \leq c\kappa^{\frac{p_0}{2}} \kappa^{-\frac{m_0}{2}}. \tag{74}$$

Let

$$A_2 := \left\{k \in \{1, 2, \dots m_1 - 2\} : \int_{I_k} \left|Du\left(z\right) H^{-2}n_i - v_1\right|^2 \mathrm{d}L^1 z \leq c\left(\kappa^{q_0} + \kappa^{p_0}\right)^{\frac{1}{2}} \kappa^{\frac{m_0}{2}}\right\}.$$

So from (53)

$$\begin{aligned}\operatorname{Card}\left(\{1,2,\dots m_1-2\}\backslash A_2\right) c\left(\kappa^{q_0}+\kappa^{p_0}\right)^{\frac{1}{2}} \kappa^{\frac{m_0}{2}} \\ \leq \sum_{k\in\{1,2,\dots m_1-2\}\backslash A_2} \int_{I_k} \left|Du\left(z\right) H^{-2} n_i - v_1\right|^2 \mathrm{d}L^1 z \\ \leq c\left(\kappa^{p_0}+\kappa^{q_0}\right).\end{aligned}$$

Which implies

$$\operatorname{Card}\left(\{1,2,\dots m_1-2\}\backslash A_2\right) \leq c\left(\kappa^{\frac{q_0}{2}}+\kappa^{\frac{p_0}{2}}\right)\kappa^{-\frac{m_0}{2}}. \tag{75}$$

So for and $k\in\{1,2,\dots m_1-2\}\setminus(A_1\cup A_2)$, recalling definition (73) we have

$$\int_{I_k} \widetilde{E}\left(z\right)+\left|Du\left(z\right)H^{-2}n_i-v_1\right|^2 \mathrm{d}L^1 z \leq c\left(\kappa^{\frac{p_0}{2}}+\kappa^{\frac{q_0}{2}}\right)\kappa^{\frac{m_0}{2}}.$$

Hence there must exist a point $z_k\in I_k$ such that

$$\widetilde{E}\left(z_k\right)+\left|Du\left(z_k\right)H^{-2}n_i-v_1\right|^2 \leq c\left(\kappa^{\frac{p_0}{2}}+\kappa^{\frac{q_0}{2}}\right).$$

So if $P\in\{P_1,P_2,\dots P_{m_1}\}\setminus(A_1\cup A_2)$ by definition of $\widetilde{E}$ (see (55)) we have

$$\left|Du\left(z_k\right)-O\left(P\right)H\right| \leq c\left(\kappa^{\frac{p_0}{2}}+\kappa^{\frac{q_0}{2}}\right) \tag{76}$$

and

$$\left|Du\left(z_k\right)H^{-2}n_i-v_1\right|^2 \leq c\left(\kappa^{\frac{p_0}{2}}+\kappa^{\frac{q_0}{2}}\right). \tag{77}$$

Now (76) implies

$$\left|Du\left(z_k\right)H^{-2}n_i-O\left(P\right)H^{-1}n_i\right| < c\left(\kappa^{\frac{p_0}{2}}+\kappa^{\frac{q_0}{2}}\right). \tag{78}$$

And (77) implies

$$\left|Du\left(z_k\right)H^{-2}n_i-v_1\right| < c\left(\kappa^{\frac{p_0}{4}}+\kappa^{\frac{q_0}{4}}\right) \tag{79}$$

so adding (78) and (79) together gives

$$\left|O\left(P\right)H^{-1}n_i-v_1\right| \leq c\left(\kappa^{\frac{p_0}{4}}+\kappa^{\frac{q_0}{4}}\right). \tag{80}$$

Let $M_1 := \{P_k : k\in\{1,2,\dots m_1-2\}\setminus(A_1\cup A_2)\}\cap M_0$. Note by (75) and (74) we have

$$\operatorname{Card}\left(M_0\backslash M_1\right) \leq c\left(\kappa^{\frac{p_0}{2}}+\kappa^{\frac{q_0}{2}}\right)\kappa^{-\frac{m_0}{2}}. \tag{81}$$

Let $R_1\in SO\left(2\right)$ be the rotation such that

$$R_1 H^{-1} n_i = v_1, \tag{82}$$

recall (64) for a reminder of the definition of v_1. Since $|w_1 - x_1| < \kappa^{\frac{m_0}{2}}$ and $|w_2 - x_2| < \kappa^{\frac{m_0}{2}}$ and from (48) $|x_1 - x_2| > \frac{\sigma^2}{32}$ (recall $x_1 \in P_1$, $x_2 \in P_{m_1}$). By bilipschitzness, from (64) making obvious estimates we obtain

$$\left| \frac{u(x_2) - u(x_1)}{|u(x_2) - u(x_1)|} - v_1 \right| \leq c\kappa^{\frac{m_0}{2}}. \tag{83}$$

Now recall the definition of R_0 in the statement of the lemma, $R_0 H^{-1} n_i := \frac{u(x_2)-u(x_1)}{|u(x_2)-u(x_1)|}$. Hence from (83) $\left|R_0 H^{-1} n_i - R_1 H^{-1} n_i\right| < c\kappa^{\frac{m_0}{2}}$ which implies

$$|R_0 - R_1| \leq c\kappa^{\frac{m_0}{2}}. \tag{84}$$

So (80) implies that for any $P \in \{P_1, P_2, \ldots P_{m_1}\} \setminus (A_1 \cup A_2)$,

$$\begin{aligned} |O(P) - R_0| &\leq \left|O(P) H^{-1} n_i - R_0 H^{-1} n_i\right| \\ &\overset{(82),(84)}{\leq} \left|O(P) H^{-1} n_i - v_1\right| + c\kappa^{\frac{m_0}{2}} \\ &\overset{(80)}{\leq} c\left(\kappa^{\frac{p_0}{4}} + \kappa^{\frac{q_0}{4}}\right). \end{aligned} \tag{85}$$

To summarise, by (81) we can find a set $M_1 \subset M_0$ such that

- $\mathrm{Card}(M_0 \backslash M_1) \leq c\left(\kappa^{\frac{p_0}{2}} + \kappa^{\frac{q_0}{2}}\right)\kappa^{-\frac{m_0}{2}}$
- From (85), for each $P \in M_1$ we have $|O(P) - R_0| \leq c\left(\kappa^{\frac{p_0}{4}} + \kappa^{\frac{q_0}{4}}\right)$ and so putting this together with (54)

$$\int_P |Du(z) - R_0 H| \, \mathrm{d}L^2 z \leq c\left(\kappa^{\frac{p_0}{4}} + \kappa^{\frac{q_0}{4}}\right)\kappa^{m_0}.$$

Thus M_1 satisfies all the properties we want and hence we have established the lemma. □

8. Following integral curves II

As explained in the introduction to Lemma 7.4, Hypotheses (88) and (89) imply $|(c_1 - c_2) \cdot n_1| \approx H^1(u([c_1, c_2]))$ where c_1, c_2 denote the centres of P_1, P_{n_1} respectively. To recall, this is essentially because (88), (89) imply $u([c_1, c_2])$ is close to an integral curve of the vector field $\Psi_1(x)$ where $\Psi_1 : u(Q_1(0)) \to \mathbb{R}$ is defined by $\Psi_1(x) := u^{-1}(x) \cdot n_1$.

Now by the "push over" lemma, *i.e.* Lemma 4.1 (see Sect. 2.1 of the introduction) if we know

$$\int_{u^{-1}([u(c_1), u(c_2)])} \mathrm{d}(Du(z), K) \, \mathrm{d}H^1 z \text{ is small} \tag{86}$$

then $|u(c_1) - u(c_2)|$ is (with some small error) greater than $|(c_1 - c_2) \cdot n_1|$ and so the endpoints of $u([c_1, c_2])$ are pushed far enough apart to make $u([c_1, c_2])$ an "almost" straight line, then we can simply apply Lemma 7.4 to arrive at conclusions (90) and (91). The only issue is establishing (86) *via* the area formula, a Fubini argument and Lipschitzness.

Lemma 8.1. *Let $u \in W^{2,1}\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right) \cap C^1$ be invertible with the assumption that $Du(z) \in \mathcal{D}(\zeta_1, \zeta_2)$ for all $z \in Q_{16\zeta_1^{-1}\zeta_2}(0)$. Let K be defined by (22). Let m_0 be a big integer. Let $\kappa > 0$ be a small number (depending on σ, ζ_1, ζ_2), suppose function u satisfies the following properties:*

(1)

$$\int_{Q_{16\zeta_1^{-1}\zeta_2}(0)} \mathrm{d}(Du(z), K)\, \mathrm{d}L^2 z \le \kappa^{m_0}. \tag{87}$$

(2) *There exists G-line $\{P_1, P_2, \dots P_{m_1}\}$ parallel to $\frac{H^{-2}n_i}{|H^{-2}n_i|}$ inside grid $G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)$ and a subset $M \subset \{P_1, P_2, \dots P_{m_1}\}$ such that*

- $$\mathrm{Card}(\{P_1, P_2, \dots P_{m_1}\} \backslash M) \le 2\kappa^{\frac{m_0}{16}} \kappa^{-\frac{m_0}{2}} \tag{88}$$
- $$\mathrm{dist}(P_1, P_{m_1}) > \frac{\sigma^3}{8}.$$
- *For each $P \in M$ there exists $R \in SO(2)$ such that*

$$\int_P |Du(z) - RH|\, \mathrm{d}L^2 z \le c\kappa^{\frac{m_0}{4}} \kappa^{m_0}. \tag{89}$$

Then there exists a set $M_0 \subset M$ and fixed $R_0 \in SO(2)$ such that

- $$\mathrm{Card}(M \backslash M_0) \le c\kappa^{\frac{m_0}{32}} \kappa^{-\frac{m_0}{2}}. \tag{90}$$
- *Every $P \in M_0$ satisfies the inequality*

$$\int_P |Du(z) - R_0 H|\, \mathrm{d}L^2 z \le c\kappa^{\frac{m_0}{64}} \kappa^{m_0}. \tag{91}$$

Proof.

Step 1. Let $i \in \{1,2\}$ be such that the G-line $\{P_1, P_2, \dots P_{m_1}\}$ is parallel to $\frac{H^{-2}n_i}{|H^{-2}n_i|}$. We will we show that for any point $x_1 \in P_1$ and any point $x_2 \in P_{m_1}$ such that $\frac{x_2 - x_1}{|x_2 - x_1|} = \frac{H^{-2}n_i}{|H^{-2}n_i|}$ we have the following inequality

$$|u(x_1) - u(x_2)| \le |(x_1 - x_2) \cdot n_i| + c\kappa^{\frac{m_0}{16}}. \tag{92}$$

Proof of Step 1. We define the function $E : \bigcup_{k=1}^{m_1} P_k \to \mathbb{R}$ by

$$E(x) = \begin{cases} |Du(x) - R_k H| & \text{for } x \in P_k \in M \text{ where } R_k \in SO(2) \text{ satisfies (89)} \\ 2\zeta_2 & \text{for } x \in \left(\bigcup_{k=1}^{m_1} P_k\right) \backslash M. \end{cases} \tag{93}$$

From (88), (89) we know

$$\begin{aligned} \int_{\bigcup_{k=1}^{m_1} P_k} E(x)\, \mathrm{d}L^2 x &\le \sum_{P_k \in M} \int_{P_k} |Du(z) - R_k H|\, \mathrm{d}L^2 z + c\kappa^{m_0} \mathrm{Card}(\{P_1, P_2, \dots P_{m_1}\} \backslash M) \\ &\le c\kappa^{\frac{m_0}{4}} \kappa^{\frac{m_0}{2}} + c\kappa^{\frac{m_0}{2}} \kappa^{\frac{m_0}{16}} \\ &\le c\kappa^{\frac{m_0}{16}} \kappa^{\frac{m_0}{2}}. \end{aligned}$$

Now in the same way as we deduced inequality (58) from inequality (56) in Lemma 7.4. Here we again use the fact that $P_{H^{-2}n_i^\perp}(\bigcup_{k=1}^{m_1} P_k) \geq \sigma^3 \frac{\kappa^{\frac{m_0}{2}}}{4}$. So by Fubini, we must be able to find point $z_1 \in P_1$ and $z_2 \in P_{m_1}$ such that $\frac{z_2-z_1}{|z_2-z_1|} = \frac{H^{-2}n_i}{|H^{-2}n_i|}$ and

$$\int_{z_1}^{z_2} E(x)\, \mathrm{d}H^1 x \leq c\kappa^{\frac{m_0}{16}}. \tag{94}$$

We will first show the inequality for z_1, z_2. It will then follow by Lipschitzness. First note

$$\left|\int_{z_1}^{z_2} Du(x) \cdot H^{-2}n_i\right| = \left|H^{-2}n_i\right| |u(z_2) - u(z_1)|. \tag{95}$$

Now for each $x \in [z_1, z_2]$, let $\Gamma(x) \in SO(2)H$ be such that $|Du(x) - \Gamma(x)| = \mathrm{d}(Du(x), SO(2)H)$. Note that $|Du(x) - \Gamma(x)| \leq E(x)$.

From (61) we have $\left|\Gamma(x) \cdot H^{-2}n_i\right| = \left|H^{-1}n_i\right| = 1$ and so from (94)

$$\begin{aligned}
\left|\int_{z_1}^{z_2} Du(x) \cdot H^{-2}n_i \mathrm{d}L^1 x\right| &\leq \left|\int_{z_1}^{z_2} \Gamma(x) \cdot H^{-2}n_i \mathrm{d}L^1 x\right| + \sigma^{-1}\int_{z_1}^{z_2} E(x)\, \mathrm{d}L^1 x \\
&\leq \int_{z_1}^{z_2} \left|\Gamma(x) \cdot H^{-2}n_i\right| \mathrm{d}L^1 x + c\kappa^{\frac{m_0}{16}} \\
&\leq |z_1 - z_2| + c\kappa^{\frac{m_0}{16}}.
\end{aligned}$$

By (95) this implies

$$\left|H^{-2}n_i\right| |u(z_2) - u(z_1)| \leq |z_1 - z_2| + c\kappa^{\frac{m_0}{16}}. \tag{96}$$

Recall from (69) we have $H^{-2}n_i \cdot n_i = 1$. So since $\frac{z_2-z_1}{|z_2-z_1|} = \frac{H^{-2}n_i}{|H^{-2}n_i|}$ from (96) we have

$$\begin{aligned}
|u(z_1) - u(z_2)| &\leq \left|H^{-2}n_i\right|^{-1} \left(|z_1 - z_2| + c\kappa^{\frac{m_0}{16}}\right) \\
&= \left|\frac{H^{-2}n_i}{|H^{-2}n_i|} \cdot n_i\right| \left(|z_1 - z_2| + c\kappa^{\frac{m_0}{16}}\right) \\
&\leq |(z_1 - z_2) \cdot n_i| + c\kappa^{\frac{m_0}{16}}
\end{aligned}$$

and this completes the proof of this inequality for z_1, z_2. Since $x_1 \in B_{\kappa^{\frac{m_0}{2}}}(z_1)$ and $x_2 \in B_{\kappa^{\frac{m_0}{2}}}(z_2)$ inequality (92) in the statement of Step 1 follows by Lipschitzness.

Step 2. We will show that for any point $x_1 \in P_1$ and any point $x_2 \in P_{m_1}$ such that $\frac{x_2-x_1}{|x_2-x_1|} = \frac{H^{-2}n_i}{|H^{-2}n_i|}$ we have the following inequality

$$|u(x_1) - u(x_2)| \geq |(x_1 - x_2) \cdot n_i| - c\kappa^{\frac{m_0}{4}} |x_1 - x_2|. \tag{97}$$

Proof of Step 2. Let $J(z) = \mathrm{d}\left(Du\left(u^{-1}(z)\right), K\right)$. So by the area formula

$$\begin{aligned}
\int_{u\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right)} J(z) \left|\det\left(Du\left(u^{-1}(z)\right)\right)\right|^{-1} \mathrm{d}L^2 z &= \int_{Q_{16\zeta_1^{-1}\zeta_2}(0)} J(u(z))\, \mathrm{d}L^2 z \\
&= \int_{Q_{16\zeta_1^{-1}\zeta_2}(0)} \mathrm{d}(Du(x), K)\, \mathrm{d}L^2 x \\
&\leq \kappa^{m_0}.
\end{aligned}$$

Since $Du \in \mathcal{D}(\zeta_1, \zeta_2)$ we know $|\det(Du(z))| \leq \zeta_2^2$ for all $z \in Q_{16\zeta_1^{-1}\zeta_2}(0)$. So

$$\int_{u\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right)} J(z)\,\mathrm{d}L^2 z \leq \zeta_2^2 \kappa^{m_0}. \tag{98}$$

Now as we know u is invertible and by assumption since $\|Du^{-1}(x)\| < \zeta_1^{-1}$, so u^{-1} is ζ_1^{-1}-Lipschitz. So $Q_{4\zeta_2}(u(0)) \subset u\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right)$ since otherwise there would be a point $q \in \partial Q_{16\zeta_1^{-1}\zeta_2}(0)$ with $|u(0) - u(q)| < 4\zeta_2$. And hence $|0 - q| \geq 2\zeta_1^{-1}|u(0) - u(q)|$ which contradicts ζ_1^{-1}-Lipschitzness of u^{-1}.

Similarly, as u is ζ_2-Lipschitz, so $Q_1(0) \subset u^{-1}(Q_{4\zeta_2}(u(0)))$. So as for any two points, $x_1 \in P_1$, $x_2 \in P_{m_1}$ we know that $u(x_1), u(x_2) \in Q_{4\zeta_2}(u(0))$. Since $Q_{4\zeta_2}(u(0))$ is convex

$$[u(x_1), u(x_2)] \subset Q_{4\zeta_2}(u(0)) \subset u\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right).$$

By a Fubini argument using (98) we must be able to find points $z_1 \in B_{\kappa^{\frac{m_0}{2}}}(u(x_1))$ and $z_2 \in B_{\kappa^{\frac{m_0}{2}}}(u(x_2))$ such that

$$\int_{z_1}^{z_2} J(z)\,\mathrm{d}L^1 z \leq \zeta_1^{-2}\kappa^{\frac{m_0}{2}}.$$

Now since $x_1 \in P_1$ and $x_2 \in P_2$ we know $|x_1 - x_2| \geq \frac{\sigma^3}{16}$. By bilipschitzness this implies $|u(x_1) - u(x_2)| > \frac{\zeta_1\sigma^3}{16}$ so $|z_1 - z_2| > \frac{\zeta_1\sigma^3}{32}$. Hence

$$\int_{z_1}^{z_2} J(z)\,\mathrm{d}L^1 z \leq c\,|z_1 - z_2|\,\kappa^{\frac{m_0}{2}}.$$

We apply Lemma 4.1 to conclude that

$$|z_1 - z_2| \geq \left|\left(u^{-1}(z_1) - u^{-1}(z_2)\right) \cdot n_i\right| - c\kappa^{\frac{m_0}{2}}|z_1 - z_2|. \tag{99}$$

Now $|z_1 - u(x_1)| < \kappa^{\frac{m_0}{2}}$, $|z_2 - u(x_2)| < \kappa^{\frac{m_0}{2}}$ which implies $\left|u^{-1}(z_1) - x_1\right| < \zeta_1^{-1}\kappa^{\frac{m_0}{2}}$ and $\left|u^{-1}(z_2) - x_2\right| < \zeta_1^{-1}\kappa^{\frac{m_0}{2}}$, so applying this to (99) gives Step 2.

Note, by putting Step 1 (92) and Step 2 (97) together we have

$$\left||u(x_1) - u(x_2)| - |(x_1 - x_2) \cdot n_i|\right| \leq c\kappa^{\frac{m_0}{16}}. \tag{100}$$

Notice that for $p_0 = \frac{m_0}{16}$, $q_0 = \frac{m_0}{16}$, (88), (100) give us the hypotheses to apply Lemma 7.4. So by Lemma 7.4 there exists a set $M_0 \subset M$ and some fixed $R_0 \in SO(2)$ such that

$$\mathrm{Card}(M \backslash M_0) \leq c\kappa^{\frac{m_0}{32}}\kappa^{-\frac{m_0}{2}}$$

and every $P \in M_0$ satisfies the inequality

$$\int_P |Du(z) - R_0 H|\,\mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}. \qquad \square$$

9. Transferring orientation across lines

Now from hypotheses (101), (102), (103) and by Lemmas 6.1, 7.4, 8.1 we have the existence of a grid G and many lines L in directions $H^{-2}n_1$ and $H^{-2}n_2$ for which Du on $\{P : P \in G, P \cap L \neq \emptyset\}$ is "mostly" orientated by $R(L)H$, $R(L) \in SO(2)$. See Section 2.3.1 for a basic outline of the idea. What we would like to do

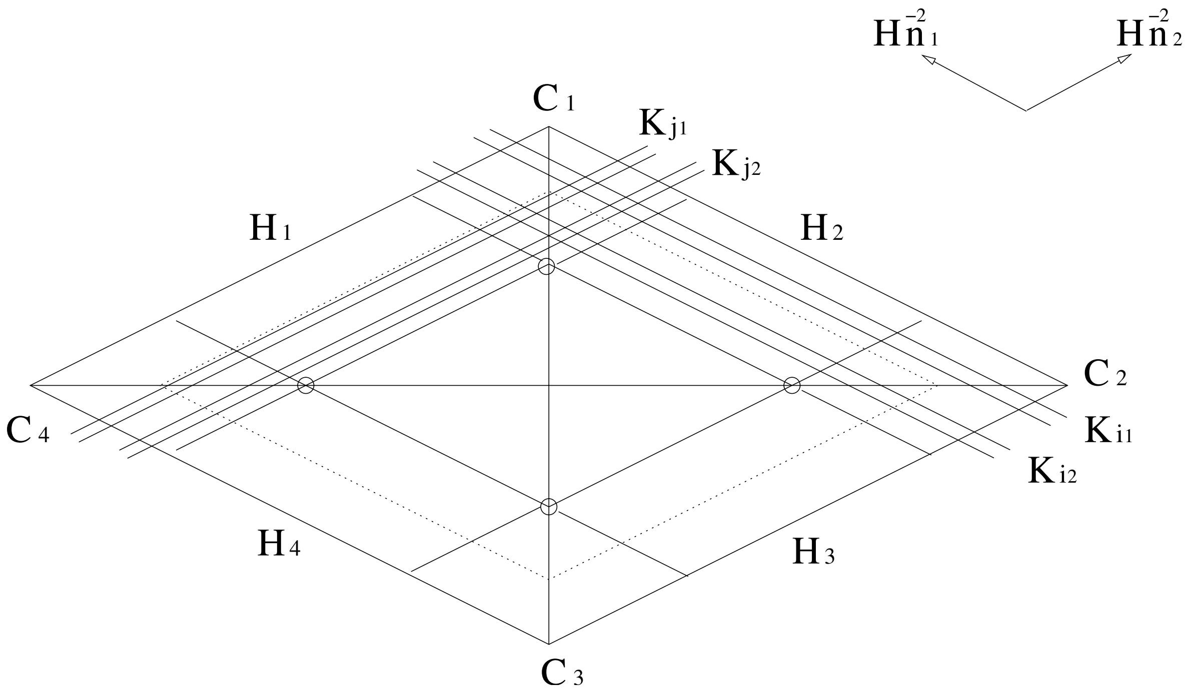


FIGURE 7. Transferring orientation across G-lines.

is to surround a central subsquare in $Q_1(0)$ with a "diamond" whose boundary is contained in the union of lines L_1, L_2, L_3, L_4 in directions $H^{-2}n_1$, $H^{-2}n_2$ (see Fig. 6) such that Du on the set

$$\{P : P \in G, P \cap L_i \neq \emptyset \text{ for some } i \in \{1,2,3,4\}\}$$

is "mostly" orientated by RH for some fixed R.

It would then be a relatively elementary matter to show that most of the elements of the grid inside the central subsquare are such that Du is orientated by RH; we just need to notice that function u is fixed on the endpoints of the lines in direction $H^{-2}n_i$ intersected with the diamond, so we can apply Lemma 7.4 to them.

We only need to find the "diamond". Note that if line L_1 in direction $H^{-2}n_1$ and line L_2 in direction $H^{-2}n_2$ intersect (inside $Q_1(0)$) and at the intersection they have an element of the grid G for which Du is orientated both by $R(L_1)H$ and $R(L_2)H$, then $R(L_1) \approx R(L_2)$. Our strategy for the proof is to find lines L_1, L_2, L_3, L_4 where we have this intersection of grid elements on which Du is orientated by $R(L_i)$ and $R(L_{i+1})$ occurs between L_1 and L_2, between L_2 and L_3 and between L_3 and L_4. The reason we can find these lines is that there are so many lines in direction $H^{-2}n_1$ and $H^{-2}n_2$ which have most of the grid elements where Du along them is orientated by a fixed rotation, so to find four lines that intersect three times on (mutually) orientated grid elements is just a matter of careful counting. See Figures 7, 8, 9 for an impression of how we do this.

Recall definition (7.3), given a G-line L, we define $\widetilde{L}$ to be the set given by the union of all the parallelograms in L.

Lemma 9.1. *Let $u \in W^{2,1}\left(Q_{16\zeta_1^{-1}\zeta_2}(0)\right)$ be C^1 invertible with the assumption that $Du(z) \in \mathcal{D}(\zeta_1, \zeta_2)$ for all $z \in Q_{16\zeta_1^{-1}\zeta_2}(0)$. Let K be as defined in (22). There exists constant c_1 depending on σ, ζ_1, ζ_2 such that if*

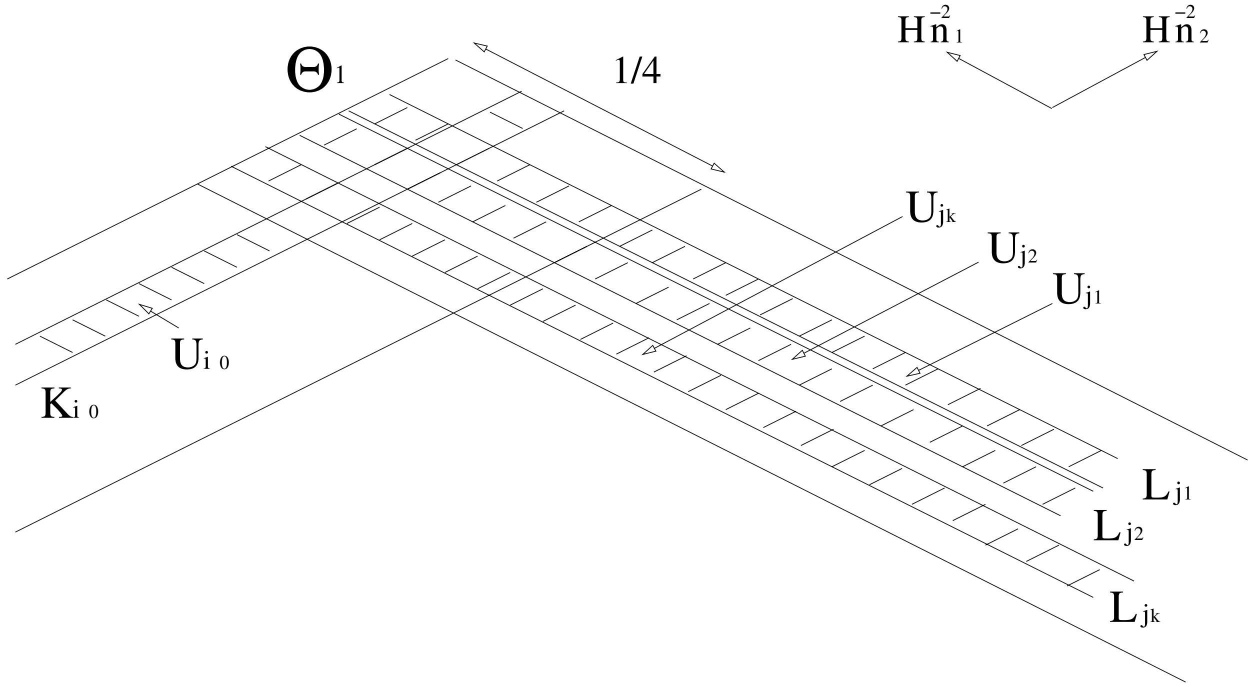


FIGURE 8. A closer view of how orientation is transfered.

function u satisfies the following inequalities

$$\int_{Q_{16\zeta_1^{-1}\zeta_2}(0)} \mathrm{d}\left(Du\left(z\right),K\right)\mathrm{d}L^2 z \leq \kappa^{m_0} \tag{101}$$

$$\int_{Q_{16\zeta_1^{-1}\zeta_2}(0)} \left|D^2 u\left(z\right)\right| \mathrm{d}L^2 z \leq c_1 \tag{102}$$

$$\int_{Q_{\frac{\sigma^3}{2\sqrt{\sigma^6+1}}}(0)} \mathrm{d}\left(Du\left(z\right),SO\left(2\right)H\right)\mathrm{d}L^2 z \leq \int_{Q_{\frac{\sigma^3}{2\sqrt{\sigma^6+1}}}(0)} \mathrm{d}\left(Du\left(z\right),SO\left(2\right)\right)\mathrm{d}L^2 z. \tag{103}$$

Given grid $G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|},\frac{H^{-2}n_2}{|H^{-2}n_2|},\kappa^{\frac{m_0}{2}}\right)$ *there exists a complete G-lines* K_{i_1},K_{i_3} *in direction* $H^{-2}n_2$ *and complete G-lines* K_{i_2},K_{i_4} *in direction* $H^{-2}n_1$ *which satisfy the following properties.*

- *The connected component of* $Q_1\left(0\right)\setminus\left(\widetilde{K_{i_0}}\cup\widetilde{K_{i_1}}\cup\widetilde{K_{i_2}}\cup\widetilde{K_{i_3}}\right)$ *containing zero also contains* $Q_{\frac{\sigma^3}{2\sqrt{\sigma^6+1}}}(0)$.
- *There exists a subset* $M\subset K_{i_0}\cup K_{i_1}\cup K_{i_2}\cup K_{i_3}$ *with the property that*

$$\mathrm{Card}\left(\left(K_{i_0}\cup K_{i_1}\cup K_{i_2}\cup K_{i_3}\right)\backslash M\right) < c\kappa^{\frac{m_0}{32}}\kappa^{-\frac{m_0}{2}} \tag{104}$$

and for some fixed $R\in SO\left(2\right)$, *for any* $P\in M$ *we have*

$$\int_P \left|Du\left(z\right)-RH\right|\mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}. \tag{105}$$

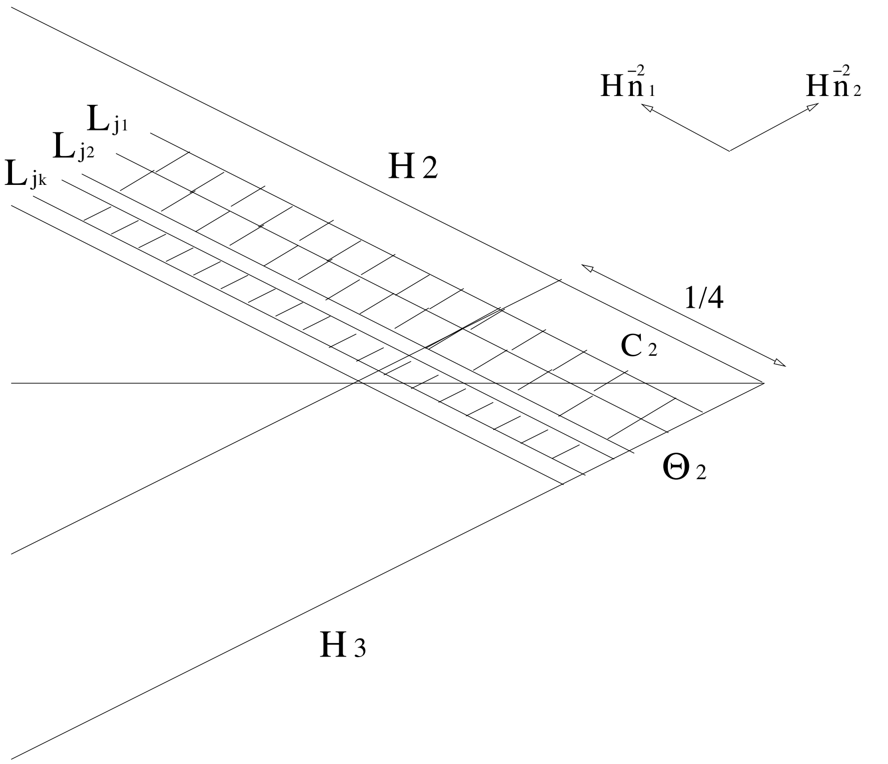


FIGURE 9. The oriented G-lines reach the C_2 corner.

Proof. To start with we know by Lemma 6.1 there is a subset G of the grid $G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)$ with the following properties

- $$\mathrm{Card}\left(G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)\backslash G\right) \leq \kappa^{\frac{m_0}{4}}\kappa^{-m_0}. \tag{106}$$
- For any $P \in G$ there exists $R \in SO(2)$, $J \in \{Id, H\}$ such that
$$\int_P |Du(z) - RJ| \, \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{4}}\kappa^{m_0}. \tag{107}$$

Let v_i denote the anticlockwise rotation of $\frac{H^{-2}n_i}{|H^{-2}n_i|}$ for $i = 1, 2$. Now $G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)$ is made up of a union of complete G-lines in direction $H^{-2}n_1$. We denote them $K_1, K_2, \ldots K_{n_2}$ where n_2 is of order $\kappa^{-\frac{m_0}{2}}$. And in the same way $G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)$ is made of the union of complete G-lines in direction $H^{-2}n_2$. We denote them $K_{n_2+1}, K_{n_2+2}, \ldots K_{2n_2}$.

Observe Figure 6. It should be clear that there exists some constant $\mathfrak{a}_\sigma > 0$ such that for any two G-lines K_i, K_j such that

$$\widetilde{K_i} \cap \langle e_2 \rangle \subset [-\mathfrak{a}_\sigma e_2, \mathfrak{a}_\sigma e_2], \; \widetilde{K_j} \cap \langle e_2 \rangle \subset [-\mathfrak{a}_\sigma e_2, \mathfrak{a}_\sigma e_2]$$

must be such that $\widetilde{K_i} \cap \widetilde{K_j} \neq \emptyset$. Its a calculation to see

$$\frac{H^{-2}n_1}{|H^{-2}n_1|} = \begin{pmatrix} \frac{1}{\sqrt{1+\sigma^6}} \\ \frac{\sigma^3}{\sqrt{1+\sigma^6}} \end{pmatrix}. \tag{108}$$

As can been seen from Figure 6 we can take

$$\mathfrak{a}_{\sigma}=\frac{H^{-2}n_1}{\left|H^{-2}n_1\right|}\cdot\begin{pmatrix}0\\1\end{pmatrix}=\frac{\sigma^3}{\sqrt{\sigma^6+1}}. \tag{109}$$

Let $\widetilde{Y}$ denote the region enclosed by the lines

$$\left\{\frac{\sigma^3 e_2}{\sqrt{\sigma^6+1}}+\langle H^{-2}n_1\rangle,\frac{\sigma^3 e_2}{\sqrt{\sigma^6+1}}+\langle H^{-2}n_2\rangle,\frac{-\sigma^3 e_2}{\sqrt{\sigma^6+1}}+\langle H^{-2}n_2\rangle,\frac{-\sigma^3 e_2}{\sqrt{\sigma^6+1}}+\langle H^{-2}n_1\rangle\right\} \tag{110}$$

as shown in Figure 6.

Its a routine calculation to see that

$$\mathrm{d}\left(SO\left(2\right),SO\left(2\right)H\right)\geq\sigma^{-1}+\sigma-2=:\varepsilon_{\sigma}.^{5} \tag{111}$$

Step 1. Let

$$E_1:=\left\{\begin{array}{l} k\in\{1,2,\dots n_2\}:\ \text{There exists } P_1,P_2\in K_k\cap G \text{ with} \\ \qquad\qquad\int_{P_1}\mathrm{d}\left(Du\left(z\right),SO\left(2\right)H\right)\mathrm{d}L^2z<c\kappa^{\frac{m_0}{4}}\kappa^{m_0} \\ \qquad\qquad\int_{P_2}\mathrm{d}\left(Du\left(z\right),SO\left(2\right)\right)\mathrm{d}L^2z<c\kappa^{\frac{m_0}{4}}\kappa^{m_0}\end{array}\right\},$$

$$F_1:=\left\{\begin{array}{l} i\in\{n_2+1,n_2+2,\dots n_2\}:\ \text{There exists } Q_1,Q_2\in K_i\cap M \text{ with} \\ \qquad\qquad\int_{Q_1}\mathrm{d}\left(Du\left(z\right),SO\left(2\right)H\right)\mathrm{d}L^2z<c\kappa^{\frac{m_0}{4}}\kappa^{m_0} \\ \qquad\qquad\int_{Q_2}\mathrm{d}\left(Du\left(z\right),SO\left(2\right)\right)\mathrm{d}L^2z<c\kappa^{\frac{m_0}{4}}\kappa^{m_0}\end{array}\right\},$$

$$E_2:=\left\{k\in\{1,2,\dots n_2\}:\mathrm{Card}\left(K_k\backslash G\right)\geq\kappa^{\frac{m_0}{8}}\kappa^{-\frac{m_0}{2}}\right\}, \tag{112}$$

and

$$F_2:=\left\{i\in\{n_2+1,n_2+2,\dots 2n_2\}:\mathrm{Card}\left(K_i\backslash G\right)\geq\kappa^{\frac{m_0}{8}}\kappa^{-\frac{m_0}{2}}\right\}. \tag{113}$$

We will show

$$\mathrm{Card}\left(E_1\right)\leq\frac{4c_1}{\varepsilon_{\sigma}\sigma^3}\kappa^{-\frac{m_0}{2}}. \tag{114}$$

$$\mathrm{Card}\left(F_1\right)\leq\frac{4c_1}{\varepsilon_{\sigma}\sigma^3}\kappa^{-\frac{m_0}{2}}. \tag{115}$$

$$\mathrm{Card}\left(E_2\right)\leq c\kappa^{\frac{m_0}{8}}\kappa^{-\frac{m_0}{2}}. \tag{116}$$

$$\mathrm{Card}\left(F_2\right)\leq c\kappa^{\frac{m_0}{8}}\kappa^{-\frac{m_0}{2}}. \tag{117}$$

Proof of Step 1. First we estimate the cardinality of E_1. Let $k_1\in E_1$ and let $P_1,P_2\in K_{k_1}\cap G$ such that

$$\int_{P_1}\mathrm{d}\left(Du\left(z\right),SO\left(2\right)H\right)\mathrm{d}L^2z\leq c\kappa^{\frac{m_0}{4}}\kappa^{m_0}, \tag{118}$$

$$\int_{P_2}\mathrm{d}\left(Du\left(z\right),SO\left(2\right)\right)\mathrm{d}L^2z\leq c\kappa^{\frac{m_0}{4}}\kappa^{m_0}.$$

[5] Identifying 2×2 matrices with 4 vectors in the obvious way, its enough to notice that the projection of

$$\left\{\begin{pmatrix}\sigma\sin\alpha\\ \sigma^{-1}\cos\alpha\\ \sigma^{-1}\sin\alpha\\ -\sigma\cos\alpha\end{pmatrix}:\alpha\in[0,2\pi)\right\}\text{ onto the subspace } \mathit{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}$$

forms as circle of radius $\sigma+\sigma^{-1}$.

Note that, as we have seen before (see (57), Lem. 7.4)

$$L^1\left(P_{v_1^\perp}(P_1)\right) \geq \frac{\sigma^3}{4}\kappa^{\frac{m_0}{2}}. \tag{119}$$

Let

$$B_1 := \left\{x \in P_{v_1^\perp}(P_1) : \inf\left\{\mathrm{d}\left(Du(z), SO(2)H\right) : z \in P^{-1}_{v_1^\perp}(x) \cap P_1\right\} \geq \kappa^{\frac{m_0}{8}}\right\}.$$

So by (118) $L^1(B_1)\kappa^{\frac{m_0}{8}}\kappa^{\frac{m_0}{2}} \leq c\kappa^{\frac{m_0}{4}}\kappa^{m_0}$. Which implies

$$L^1(B_1) \leq c\kappa^{\frac{m_0}{8}}\kappa^{\frac{m_0}{2}}. \tag{120}$$

Let

$$B_2 := \left\{x \in P_{v_1^\perp}(P_2) : \inf\left\{\mathrm{d}\left(Du(z), SO(2)\right) : z \in P^{-1}_{v_1^\perp}(x) \cap P_1\right\} \geq \kappa^{\frac{m_0}{8}}\right\}.$$

In the same way we have that

$$L^1(B_2) \leq c\kappa^{\frac{m_0}{8}}\kappa^{\frac{m_0}{2}}. \tag{121}$$

Now for any $x \in P_{v_1^\perp}(P_1)\backslash(B_1 \cup B_2)$ we have a point $p(x) \in P_1$ such that $\mathrm{d}\left(Du(p(x)), SO(2)H\right) < \kappa^{\frac{m_0}{8}}$ and $q(x) \in P_2$ such that $\mathrm{d}\left(Du(p(x)), SO(2)\right) < \kappa^{\frac{m_0}{8}}$ and thus by using (111) we have

$$\begin{aligned}\left|\int_{q(x)}^{p(x)} D^2u(z)\frac{H^{-2}n_1}{|H^{-2}n_1|}\right| &= |Du(p(x)) - Du(q(x))| \\ &\geq \frac{\varepsilon_\sigma}{2}.\end{aligned}$$

So by Fubini and (119), (121), (120)

$$\begin{aligned}\int_{\bigcup_{P\in K_k} P}\left|D^2u(z)\right|\mathrm{d}L^2z &\geq \frac{\varepsilon_\sigma}{2}L^1\left(P_{v^\perp}(P_1)\setminus(B_1\cup B_2)\right) \\ &\geq \frac{\varepsilon_\sigma}{4}\sigma^3\kappa^{\frac{m_0}{2}}.\end{aligned}$$

Thus from (102) we have

$$\begin{aligned}c_1 &\geq \int_{Q_1(0)}\left|D^2u(z)\right|\mathrm{d}L^2z \\ &\geq \frac{\varepsilon_\sigma}{4}\sigma^3\mathrm{Card}(E_1)\kappa^{\frac{m_0}{2}}.\end{aligned}$$

And thus we have (114). In exactly the same way we obtain the upper bound (115).

Now we estimate the cardinality of E_2. From (106)

$$\begin{aligned}\mathrm{Card}(E_2)\kappa^{\frac{m_0}{8}}\kappa^{-\frac{m_0}{2}} &\leq \mathrm{Card}\left(G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}, \frac{H^{-2}n_2}{|H^{-2}n_2|}, \kappa^{\frac{m_0}{2}}\right)\backslash G\right) \\ &\leq c\kappa^{\frac{m_0}{4}}\kappa^{-m_0}\end{aligned}$$

and thus we have (116). In exactly the same way we have (117).

Step 2. We will show that for any $i \in \{1, 2, \ldots n_2\}\setminus(E_1 \cup E_2)$ and for any $P \in K_i \cap G$ we have

$$\int_P \mathrm{d}\left(Du(z), SO(2)H\right)\mathrm{d}L^2z \leq c\kappa^{\frac{m_0}{4}}\kappa^{m_0}.$$

And for any $j\in\{n_2+1,n_2+2,\dots 2n_2\}\setminus(F_1\cup F_2)$ we have

$$\int_P \mathrm{d}\left(Du\left(z\right),SO\left(2\right)H\right)\mathrm{d}L^2z\le c\kappa^{\frac{m_0}{4}}\kappa^{m_0}\text{ for any }P\in K_j\cap G. \tag{122}$$

Proof of Step 2. Let

$$\Delta_1:=\left\{i\in\{1,2,\dots n_2\}:\widetilde{K_i}\cap Q_{\frac{\sigma^3}{2\sqrt{1+\sigma^6}}}\left(0\right)\neq\emptyset\right\}.$$

Let

$$\Delta_2:=\left\{j\in\{n_2+1,n_2+2,\dots 2n_2\}:\widetilde{K_j}\cap Q_{\frac{\sigma^3}{2\sqrt{1+\sigma^6}}}\left(0\right)\neq\emptyset\right\}.$$

Let

$$\Psi_H:=\left\{P:P\in G,\int_P \mathrm{d}\left(Du\left(z\right),SO\left(2\right)H\right)\mathrm{d}L^2z\le c\kappa^{m_0}\kappa^{\frac{m_0}{4}},\ P\subset Q_{\frac{\sigma^3}{2\sqrt{1+\sigma^6}}}\left(0\right)\right\}.$$

Let

$$\Psi_R:=\left\{P:P\in G,\int_P \mathrm{d}\left(Du\left(z\right),SO\left(2\right)\right)\mathrm{d}L^2z\le c\kappa^{m_0}\kappa^{\frac{m_0}{4}},\ P\subset Q_{\frac{\sigma^3}{2\sqrt{1+\sigma^6}}}\left(0\right)\right\}. \tag{123}$$

First note that if there exists $i_0\in\Delta_1\setminus(E_1\cup E_2)$ such that $K_{i_0}\cap G\cap\Psi_H\neq\emptyset$ then by definition of E_1, every $P_1\in K_{i_0}\cap G$ will be such that $P_1\in\Psi_H$. Now take $j\in\Delta_2\setminus(F_1\cup F_2)$ such that $K_j\cap K_{i_0}\cap G\neq\emptyset$ then by definition of F_1, for every $P_2\in K_j\cap G$ we must also have $P_2\in\Psi_H$. Note

$$\{j\in\Delta_2\setminus(F_1\cup F_2):K_j\cap K_{i_0}\cap G\neq\emptyset\}=\{j\in\Delta_2\setminus(F_1\cup F_2)\}\setminus\{j:K_j\cap K_{i_0}\cap G=\emptyset\}$$

and as $\mathrm{Card}\left(\{j:K_j\cap K_{i_0}\cap G=\emptyset\}\right)\le\mathrm{Card}\left(K_{i_0}\backslash G\right)$ so from (115), (117) and definition (113) we have

$$\begin{aligned}\mathrm{Card}\left(\{j\in\Delta_2\setminus(F_1\cup F_2):K_j\cap K_{i_0}\cap G\neq\emptyset\}\right)&\ge\mathrm{Card}\left(\{j\in\Delta_2\setminus(F_1\cup F_2)\}\right)\\&\quad-\mathrm{Card}\left(K_{i_0}\backslash G\right)\\&\overset{(115),(117),(113)}{\ge}\mathrm{Card}\left(\Delta_2\right)-\frac{8c_1}{\varepsilon_\sigma\sigma^3}\kappa^{-\frac{m_0}{2}}.\end{aligned}$$

We have a large number of G-lines in $\{K_j:j\in\Delta_2\setminus(F_1\cup F_2)\}$ with all the $P\in K_j\cap G$ being such that Du on P is close to $SO\left(2\right)H$. From this, using similar arguments its easy to show that *all* G-lines K_j with $j\in\Delta_1\setminus(E_1\cup E_2)$ satisfy (122). And consequently *all* G-lines K_i with $i\in\{1,2,\dots n_1\}\setminus(F_1\cup F_2)$ also satisfy (122).

Thus we only need to argue the case where

$$\bigcup_{i\in\Delta_1\setminus(E_1\cup E_2)}\{P:P\in K_i\cap G\}\subset\Psi_R. \tag{124}$$

Let

$$\Theta_0:=\left\{P\in G\left(\frac{H^{-2}n_1}{|H^{-2}n_1|},\frac{H^{-2}n_2}{|H^{-2}n_2|},\kappa^{\frac{m_0}{2}}\right):P\subset Q_{\frac{\sigma^3}{2\sqrt{1+\sigma^6}}}\left(0\right)\right\}. \tag{125}$$

Since from inequalities (114), (116) and definition (112)

$$\begin{aligned}&\mathrm{Card}\left(\Theta_0\backslash\left(\bigcup_{i\in\Delta_1\setminus(E_1\cup E_2)}\{P:P\in K_i\cap G\}\right)\right)\\&\quad\le\mathrm{Card}\left(E_1\cup E_2\right)\kappa^{-\frac{m_0}{2}}+c\kappa^{\frac{m_0}{8}}\kappa^{-\frac{m_0}{2}}\\&\quad\le\frac{16c_1\kappa^{-m_0}}{\varepsilon_\sigma\sigma^3}.\end{aligned}$$

So from (124) we have

$$\operatorname{Card}\left(\Theta_0 \backslash \Psi_R\right) \leq \frac{16 c_1 \kappa^{-m_0}}{\varepsilon_\sigma \sigma^3}\cdot$$

Since obviously $\Psi_H \cap \Psi_R = \emptyset$ so

$$\begin{aligned} \operatorname{Card}\left(\Psi_H\right) &\leq \operatorname{Card}\left(\Theta_0 \backslash \Psi_R\right) \\ &\leq \frac{16 c_1 \kappa^{-m_0}}{\varepsilon_\sigma \sigma^3}\cdot \end{aligned} \tag{126}$$

Note that for any $P \in \Psi_R$

$$\begin{aligned} \kappa^{\frac{m_0}{8}} L^2\left(\left\{x \in P : \mathrm{d}\left(Du\left(x\right), SO\left(2\right)\right) \geq \kappa^{\frac{m_0}{8}}\right\}\right) &\leq \int_P \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) \mathrm{d}L^2 z \\ &\leq c\kappa^{m_0} \kappa^{\frac{m_0}{4}}. \end{aligned}$$

So $E\left(P\right) := \left\{x \in P : \mathrm{d}\left(Du\left(x\right), SO\left(2\right)\right) < \kappa^{\frac{m_0}{8}}\right\}$ is such that

$$L^2\left(E\left(P\right)\right) \geq L^2\left(P\right) - c\kappa^{\frac{m_0}{8}} \kappa^{m_0}.$$

Note that for each $x \in E\left(P\right)$, $\mathrm{d}\left(Du\left(x\right), SO\left(2\right) H\right) > \frac{3\varepsilon_\sigma}{4}$ and hence

$$\int_P \mathrm{d}\left(Du\left(x\right), SO\left(2\right) H\right) \mathrm{d}L^2 x \geq \left(L^2\left(P\right) - c\kappa^{\frac{m_0}{8}} \kappa^{m_0}\right) \frac{3\varepsilon_\sigma}{4}\cdot$$

Thus since $P \in \Psi_R$ (recall definition (123)) we have

$$\begin{aligned} \int_P \mathrm{d}\left(Du\left(z\right), SO\left(2\right) H\right) - \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) \mathrm{d}L^2 z &\geq \left(L^2\left(P\right) - c\kappa^{\frac{m_0}{8}} \kappa^{m_0}\right) \frac{3\varepsilon_\sigma}{4} - c\kappa^{\frac{m_0}{8}} \kappa^{m_0} \\ &\geq \frac{\varepsilon_\sigma}{2} L^2\left(P\right). \end{aligned}$$

Multiplying by -1 gives

$$\int_P \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) - \mathrm{d}\left(Du\left(z\right), SO\left(2\right) H\right) \mathrm{d}L^2 z \leq -\frac{\varepsilon_\sigma}{2} L^2\left(P\right). \tag{127}$$

Let $A := L^2\left(P\right)$ for any $P \in G\left(\frac{H^{-2} n_1}{\left|H^{-2} n_1\right|}, \frac{H^{-2} n_2}{\left|H^{-2} n_2\right|}, \kappa^{\frac{m_0}{2}}\right)$. So using (103) and (106)

$$\begin{aligned} 0 &\overset{(103)}{\leq} \int_{Q_{\frac{\sigma^3}{2\sqrt{\sigma^6+1}}}(0)} \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) - \mathrm{d}\left(Du\left(z\right), SO\left(2\right) H\right) \mathrm{d}L^2 z \\ &\leq \sum_{P \in \Psi_R} \int_P \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) - \mathrm{d}\left(Du\left(z\right), SO\left(2\right) H\right) \mathrm{d}L^2 z \\ &\quad + \sum_{P \in \Psi_H} \int_P \mathrm{d}\left(Du\left(z\right), SO\left(2\right)\right) - \mathrm{d}\left(Du\left(z\right), SO\left(2\right) H\right) \mathrm{d}L^2 z \\ &\quad + 2\zeta_2 \operatorname{Card}\left(G\left(\frac{H^{-2} n_1}{\left|H^{-2} n_1\right|}, \frac{H^{-2} n_2}{\left|H^{-2} n_2\right|}, \kappa^{\frac{m_0}{2}}\right) \backslash G\right) \kappa^{m_0} \\ &\overset{(106),(127)}{\leq} -\operatorname{Card}\left(\Psi_R\right) \frac{\varepsilon_\sigma}{2} A + 2\zeta_2 \operatorname{Card}\left(\Psi_H\right) A + c\kappa^{\frac{m_0}{4}}. \end{aligned}$$

Thus

$$\mathrm{Card}\,(\Psi_R)\,\frac{\varepsilon_\sigma}{2}A \leq 2\zeta_2 \mathrm{Card}\,(\Psi_H)\,A + c\kappa^{\frac{m_0}{4}}.$$

Now as we have seen before (see (109)) $A \geq \frac{\sigma^3}{2}\kappa^{m_0}$ so using (126)

$$\begin{aligned}\mathrm{Card}\,(\Psi_R)\,\frac{\varepsilon_\sigma}{2} &\leq 2\zeta_2\mathrm{Card}\,(\Psi_H) + c\kappa^{\frac{m_0}{4}}\kappa^{-m_0}\\ &\overset{(126)}{\leq} \frac{32c_1\zeta_2\kappa^{-m_0}}{\varepsilon_\sigma\sigma^3} + c\kappa^{\frac{m_0}{4}}\kappa^{-m_0}\\ &\leq \frac{64c_1\zeta_2\kappa^{-m_0}}{\varepsilon_\sigma\sigma^3}.\end{aligned}$$

Thus

$$\mathrm{Card}\,(\Psi_R) \leq \frac{128c_1\zeta_2\kappa^{-m_0}}{\varepsilon_\sigma^2\sigma^3}. \tag{128}$$

Since $G = \Psi_H \cup \Psi_R$ we know from (106) $\mathrm{Card}\,(\Theta_0 \backslash (\Psi_H \cup \Psi_R)) \leq c\kappa^{\frac{m_0}{4}}\kappa^{-m_0}$ so using (126), (128)

$$\begin{aligned}\mathrm{Card}\,(\Theta_0) &\leq \mathrm{Card}\,(\Psi_H) + \mathrm{Card}\,(\Psi_R) + c\kappa^{\frac{m_0}{4}}\kappa^{-m_0}\\ &\leq \frac{256\zeta_2 c_1}{\varepsilon_\sigma^2\sigma^3}\kappa^{-m_0}.\end{aligned}$$

Since from definition (125) we know

$$\mathrm{Card}\,(\Theta_0) \geq \left(\frac{\sigma^3\kappa^{-\frac{m_0}{2}}}{2\sqrt{1+\sigma^6}}\right)^2 \geq \frac{\sigma^6}{8}\kappa^{-m_0}$$

so we know

$$\frac{\sigma^6\kappa^{-m_0}}{8} \leq \frac{256\zeta_2 c_1}{\varepsilon_\sigma^2\sigma^3}\kappa^{-m_0},$$

and assuming sufficient smallness of c_1 we have a contradiction. So we have established Step 2.

Notation for Step 3.

Firstly we note that for any $k \in \{1, 2, \dots n_1\} \backslash E_1 \cup E_2$ by definition of E_2 (see (112) and (107)) we have the hypotheses (88) and (89) of Lemma 8.1, by (101) we also have hypothesis (87) so by the lemma there there exists a subset $U(k) \subset K_k$ with the following properties

- For fixed $R_k \in SO(2)$ we have for any $P \in U(k)$

$$\int_P |Du(z) - R_k H|\,\mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}. \tag{129}$$

-

$$\mathrm{Card}\,(K_k \backslash U(k)) \leq c\kappa^{\frac{m_0}{32}}\kappa^{-\frac{m_0}{2}}. \tag{130}$$

Similarly for any $i \in \{n_1, n_1+1, \dots 2n_1\} \backslash (F_1 \cup F_2)$ there is a subset $U(i) \subset K_i$ with the following properties

- For fixed $R_i \in SO(2)$ we have for any $P \in U(i)$

$$\int_P |Du(z) - R_i H|\,\mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}. \tag{131}$$

-

$$\mathrm{Card}\,(K_i \backslash U(i)) \leq c\kappa^{\frac{m_0}{32}}\kappa^{-\frac{m_0}{2}}. \tag{132}$$

Observe the Figure 7.

Let $V_i := \frac{H^{-2}n_i}{|H^{-2}n_i|}$ for $i = 1, 2$. Define

$$\begin{aligned} H_1 &:= P_{V_2^\perp}^{-1}\left(P_{V_2^\perp}\left(\left[\frac{V_1}{4}, \frac{V_1}{2}\right]\right)\right), \\ H_2 &:= P_{V_1^\perp}^{-1}\left(P_{V_1^\perp}\left(\left[\frac{V_2}{4}, \frac{V_2}{2}\right]\right)\right), \\ H_3 &:= P_{V_2^\perp}^{-1}\left(P_{V_2^\perp}\left(\left[-\frac{V_1}{2}, -\frac{V_1}{4}\right]\right)\right), \\ H_4 &:= P_{V_1^\perp}^{-1}\left(P_{V_1^\perp}\left(\left[-\frac{V_2}{2}, -\frac{V_2}{4}\right]\right)\right). \end{aligned} \tag{133}$$

And define

$$\begin{aligned} J_1 &:= \left\{k \in \{1, 2, \dots n_2\} : \widetilde{K_k} \subset H_1, k \notin E_1 \cup E_2\right\}, \\ J_3 &:= \left\{k \in \{1, 2, \dots n_2\} : \widetilde{K_k} \subset H_3, k \notin E_1 \cup E_2\right\}, \\ J_2 &:= \left\{i \in \{n_2 + 1, n_2 + 2, \dots 2n_2\} : \widetilde{K_i} \subset H_2, i \notin F_1 \cup F_2\right\}, \\ J_4 &:= \left\{i \in \{n_2 + 1, n_2 + 2, \dots 2n_2\} : \widetilde{K_i} \subset H_4, i \notin F_1 \cup F_2\right\}. \end{aligned}$$

Step 3. We will show we can find $i_0 \in J_1$, $j_1, j_2, \dots j_{\xi_1} \in J_2$ where

$$\xi_1 \geq \frac{\kappa^{-\frac{m_0}{2}}}{128} \tag{134}$$

such that for some fixed $\widetilde{R} \in SO(2)$, for any $P \in U(i_0) \cup \bigcup_{k=1}^{\xi_1} U(j_k)$ we have

$$\int_P \left|Du(z) - \widetilde{R}H\right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}. \tag{135}$$

Proof of Step 3. Its helpful to observe Figure 8.

As shown in Figure 7. We let $C_1 := H_1 \cap H_2$, $C_2 := H_2 \cap H_3$, $C_3 := H_3 \cap H_4$, $C_4 := H_1 \cap H_4$.

Its easy to see the convex hull of the set $\{C_1, C_2, C_3, C_4\}$ will be contained the region $\widetilde{Y}$ shown of Figure 6, see (110) and (133) for definitions. As shown in Figure 8, let

$$\Theta_1 := \{P : P \in U(i) \text{ for some } i \in J_2, P \subset C_1\}.$$

We start by estimating the cardinality of Θ_1. Let

$$Z_1 := \left\{P \in G\left(V_1, V_2, \kappa^{\frac{m_0}{2}}\right) : P \subset C_1\right\}.$$

Note that

$$\mathrm{Card}(Z_1) \geq \frac{\kappa^{-m_0}}{32}. \tag{136}$$

If $P \in Z_1 \backslash \Theta_1$ then either $P \in K_i$ for some $i \in F_1 \cup F_2$ or $P \in K_i \backslash U(i)$ for some some $i \in J_2$. Formally;

$$Z_1 \backslash \Theta_1 \subset \left(\bigcup_{i \in F_1 \cup F_2} K_i\right) \cup \left(\bigcup_{i \in J_2} K_i \backslash U(i)\right).$$

So from (115), (117), (132)

$$\begin{aligned}\operatorname{Card}(Z_1\backslash\Theta_1) &\leq \operatorname{Card}(F_1\cup F_2)\,\kappa^{-\frac{m_0}{2}}+\sum_{i\in J_2}\operatorname{Card}(K_i\backslash U(i))\\ &\overset{(132),(115),(117)}{\leq} \frac{8c_1}{\varepsilon_\sigma\sigma^3}\kappa^{-m_0}+c\kappa^{\frac{m_0}{32}}\kappa^{-m_0}\\ &\leq \frac{16c_1}{\varepsilon_\sigma\sigma^3}\kappa^{-m_0}. \end{aligned} \tag{137}$$

Let $\Psi_1 := \{P : P \in K_i \text{ for some } i \in J_1\}$. So from (114), (116)

$$\begin{aligned}\operatorname{Card}(Z_1\backslash\Psi_1) &\leq \operatorname{Card}(E_1\cup E_2)\,\kappa^{-\frac{m_0}{2}}\\ &\leq \frac{16c_1}{\varepsilon_\sigma\sigma^3}\kappa^{-m_0}. \end{aligned} \tag{138}$$

Note from (137), (138), (136) (assuming c_1 is small enough)

$$\begin{aligned}\operatorname{Card}(\Psi_1\cap\Theta_1) &\overset{(137),(138)}{\geq} \operatorname{Card}(Z_1)-\frac{32c_1}{\varepsilon_\sigma\sigma^3}\kappa^{-m_0}\\ &\overset{(136)}{\geq} \frac{\kappa^{-m_0}}{32}-\frac{32c_1}{\varepsilon_\sigma\sigma^3}\kappa^{-m_0}\\ &\geq \frac{\kappa^{-m_0}}{64}. \end{aligned} \tag{139}$$

Now we have the obvious estimate $\operatorname{Card}(J_1)\leq\kappa^{-\frac{m_0}{2}}$. And as

$$\Psi_1\cap\Theta_1=\bigcup_{i\in J_1}K_i\cap\Theta_1$$

so (139) implies there must exist $i_0\in J_1$ such that

$$\operatorname{Card}(K_{i_0}\cap\Theta_1)\geq\frac{\kappa^{-\frac{m_0}{2}}}{64}.$$

So using (130) we have

$$\operatorname{Card}(K_{i_0}\cap\Theta_1\cap U(i_0))\geq\frac{\kappa^{-\frac{m_0}{2}}}{128}. \tag{140}$$

Now by definition of $U(i_0)$ (since $i_0\in J_1$) there exists $\widetilde{R}\in SO(2)$ such that for every $P\in U(i_0)$ we have

$$\int_P\left|Du(z)-\widetilde{R}H\right|\mathrm{d}L^2z\leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}. \tag{141}$$

Let $\{P_1,P_2,\dots P_{\xi_1}\}:=K_{i_0}\cap\Theta_1\cap U(i_0)$, so of course from (140) we know $\xi_1\geq\frac{\kappa^{-\frac{m_0}{2}}}{128}$. By definition of Θ_1 for every $k\in\{1,2,\dots\xi_1\}$ we have that $P_k\in U(j_k)$ for some $j_k\in J_2$. And by definition of $U(j_k)$ we have for some fixed $R(j_k)\in SO(2)$ such that for any $\widetilde{P}\in U(j_k)$

$$\int_{\widetilde{P}}|Du(z)-R(j_k)H|\,\mathrm{d}L^2z\leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}\text{ for some fixed }R(j_k)\in SO(2).$$

So putting this together with (141)

$$\int_{P_k} \left| Du(z) - \widetilde{R}H \right| + \left| Du(z) - R(j_k) H \right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}$$

and hence as $L^2(P_k) \geq \frac{\sigma^3}{2\sqrt{1+\sigma^6}} \kappa^{m_0}$ (see (109)) there must be a point $z_k \in P_k$ such that

$$\left| Du(z_k) - \widetilde{R}H \right| + \left| Du(z_k) - R(j_k) H \right| \leq c\kappa^{\frac{m_0}{64}}$$

which implies

$$\left| R(j_k) - \widetilde{R} \right| \leq c\kappa^{\frac{m_0}{64}}.$$

From this and (141) Step 3 follows.

Step 4. Let

$$T_3 := \left\{ i \in J_3 : \mathrm{Card}\left(K_i \cap \Theta_2\right) \geq c\kappa^{\frac{m_0}{32}} \kappa^{-\frac{m_0}{2}} \right\}. \tag{142}$$

We will show we can find $r_1, r_2, \dots r_{\xi_2} \in T_3$ with $\xi_2 > \frac{\kappa^{-\frac{m_0}{2}}}{2048}$ with the property that for any $P \in \bigcup_{i=1}^{\xi_2} U(r_i)$ satisfies inequality

$$\int_P \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}. \tag{143}$$

Proof of Step 4. Let

$$\Theta_2 := \left\{ P : P \in U(j_k) \text{ for } k \in \{1, 2, \dots \xi_1\}, P \in C_2 \right\}. \tag{144}$$

From (132), (134) and Figure 9 we see that

$$\begin{aligned} \mathrm{Card}(\Theta_2) &\overset{(132),\text{Figure 9}}{\geq} \xi_1 \left(\frac{1}{4} - c\kappa^{\frac{m_0}{32}} \right) \kappa^{-\frac{m_0}{2}} \\ &\overset{(134)}{\geq} \frac{\kappa^{-m_0}}{1024}. \end{aligned} \tag{145}$$

Since for any $i \in H_3$ we have trivially that $\mathrm{Card}(K_i \cap \Theta_2) \leq \kappa^{-\frac{m_0}{2}}$. So

$$\mathrm{Card}(\Theta_2) \leq \mathrm{Card}(T_3)\, \kappa^{-\frac{m_0}{2}} + c\kappa^{\frac{m_0}{32}} \kappa^{-\frac{m_0}{2}} \mathrm{Card}(J_3 \backslash T_3).$$

Hence from (145) and the trivial estimate $\mathrm{Card}(J_3 \backslash T_3) \leq \kappa^{-\frac{m_0}{2}}$.

$$\frac{\kappa^{-m_0}}{1024} \leq \mathrm{Card}(T_3)\, \kappa^{-\frac{m_0}{2}} + c\kappa^{\frac{m_0}{32}} \kappa^{-m_0}$$

we have

$$\mathrm{Card}(T_3) \geq \frac{\kappa^{-\frac{m_0}{2}}}{2048}. \tag{146}$$

Now from (130) since (definition (142)) $T_3 \subset J_3 \subset \{1, 2, \dots n_2\} \setminus (E_1 \cup E_2)$ so by (129), (130) for any $i \in T_3$, $\mathrm{Card}(K_i \backslash U(i)) \leq c\kappa^{\frac{m_0}{32}} \kappa^{-\frac{m_0}{2}}$. So by definition of T_3, $U(i) \cap \Theta_2 \neq \emptyset$ so we can pick $P_0 \in U(i) \cap \Theta_2$. Now by definition of Θ_2, (see (144)) and of the set $\{j_1, j_2, \dots j_{\xi_1}\}$ (see (134), (135)) we have

$$\int_{P_0} \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}.$$

Also by definition of $U(i)$, (see (129), (130)) we know there exists $R_i \in SO(2)$ such that

$$\int_{\widetilde{P}} |Du(z) - R_i H| \, \mathrm{d}L^2 \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0} \text{ for all } \widetilde{P} \in U(i).$$

Hence as we have argued before (since $P_0 \in U(i)$), there must be a point $z_0 \in P_0$ such that

$$\begin{aligned} \left|\widetilde{R} - R_i\right| &\leq \left|Du(z_0) - \widetilde{R}H\right| + |Du(z_0) - R_i H| \\ &\leq c\kappa^{\frac{m_0}{64}}. \end{aligned}$$

And so for all $P \in U(i)$

$$\int_P \left|Du(z) - \widetilde{R}H\right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}.$$

Let $\{r_1, r_2, \ldots r_{\xi_2}\}$ be an ordering of T_3. Note that from (146) we have

$$\xi_2 \geq \frac{\kappa^{-\frac{m_0}{2}}}{2048}. \tag{147}$$

So we have shown all the $P \in G$ inside the set of G-lines $\left\{K_{r_1}, K_{r_2}, \ldots K_{r_{\xi_2}}\right\}$ are such that Du on P is orientated by $\widetilde{R}$. This completes the proof of Step 4.

Step 5. We will show we can find $i_0 \in J_1$, $i_1 \in J_2$, $i_2 \in J_3$ and $i_3 \in J_4$ such that for some fixed $\widetilde{R} \in SO(2)$, for any $P \in U(i_1) \cup U(i_3) \cup U(i_2) \cup U(i_4)$ we have

$$\int_P \left|Du(z) - \widetilde{R}H\right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}.$$

Proof of Step 5. Let

$$\Theta_3 := \{P : P \in U(r_i), \text{ for } i = 1, 2, \ldots \xi_2, P \in C_3\}.$$

We make the same estimates as before, from (147)

$$\begin{aligned} \mathrm{Card}(\Theta_3) &\geq \xi_2 \frac{\kappa^{-\frac{m_0}{2}}}{8} \\ &\geq \frac{\kappa^{-m_0}}{16\,384}. \end{aligned}$$

Let $T_4 := \left\{i \in H_4 : \mathrm{Card}(K_i \cap \Theta_3) \geq c\kappa^{-\frac{m_0}{2}}\right\}$, as before

$$\mathrm{Card}(\Theta_3) \leq \mathrm{Card}(T_4)\, \kappa^{-\frac{m_0}{2}} + c\kappa^{\frac{m_0}{32}} \kappa^{-m_0}.$$

So $\frac{\kappa^{-m_0}}{22768} \leq \mathrm{Card}(T_4)\, \kappa^{-\frac{m_0}{2}}$ which implies $\frac{\kappa^{-\frac{m_0}{2}}}{22768} \leq \mathrm{Card}(T_4)$.

So as in Step 4 since $T_4 \subset J_4 \subset \{1, 2, \ldots n_1\} \setminus (F_1 \cup F_2)$ so by (131), (132) we must be able to find a G-line K_{l_0} where $l_0 \in T_4$ and $\mathrm{Card}(K_{l_0} \backslash U(l_0)) \leq c\kappa^{\frac{m_0}{32}} \kappa^{-\frac{m_0}{2}}$. Hence $U(l_0) \cap \Theta_3 \neq \emptyset$ so as before we have the property that there exists $R_{l_0} \in SO(2)$ such that for any $P \in U(l_0) \cap \Theta_3$

$$\int_P \left|Du(z) - \widetilde{R}H\right| + |Du(z) - R_{l_0} H| \, \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}.$$

So there must exists a point $z_0 \in P$ such that

$$\left|Du(z_0) - \widetilde{R}H\right| + |Du(z_0) - R_{l_0}H| \leq c\kappa^{\frac{m_0}{64}}.$$

Hence $\left|\widetilde{R} - R_{l_0}\right| \leq c\kappa^{\frac{m_0}{64}}$ and thus for every $\widetilde{P} \in U(l_0)$ we have

$$\int_{\widetilde{P}} \left|Du(z) - \widetilde{R}H\right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}}\kappa^{m_0}.$$

We have already chosen i_0 in Step 3, see (134). Let i_1 be any member of $\{j_1, j_2, \dots j_{\xi_1}\}$ (see again Step 3) and let i_2 be any member of $\{r_1, r_2, \dots r_{\xi_2}\}$ (see Step 4) and let $i_3 = l_0$. Now i_0, i_1, i_2, i_3 satisfy all the properties required.

Proof of Lemma continued.

Now since the G-line K_{l_0} must intersect the original G-line K_{i_0}. And since any G-line K_{r_k}, $k \in \{1, 2, \dots \xi_2\}$ must intersect any G-line K_{j_k} for $k \in \{1, 2, \dots \xi_1\}$. So the G-lines K_{i_1}, K_{i_2}, K_{i_3}, K_{i_4} from Step 5 (and inequalities (130), (132)) satisfy all the properties of the statement of the lemma. □

10. Proof of Theorem 2.1

The strategy of the proof of Theorem 2.1 is as has been outlined in the introduction to Lemma 9.1. Lemma 9.1 gives us four lines L_1, L_2, L_3, L_4 (parallel either to $H^{-2}n_1$ or $H^{-2}n_2$) that contain the boundary of a "diamond" surrounding a central subsquare. These lines have the property that "most" of the grid elements that intersect them are such that Du on these elements will be L^1 close to matrix RH for some fixed $R \in SO(2)$.

We will be considering lines in direction $H^{-2}n_1$ that start and end on the boundary of the diamond. However before applying Lemma 7.4 we need to know that "most"of the grid elements along the line are such that Du is close to a matrix in the well $SO(2)H$. Note that we know from Lemma 6.1 that most of the grid elements are such that Du is either close to a matrix in the well $SO(2)$ or close to a matrix in the well $SO(2)H$.

So we need to rule out the possibility that there are many grid elements inside the diamond for which Du is close to $SO(2)$. Now note that $|He_2| = \sigma^{-1} > 1$, so if for some line Q (inside the diamond) in direction e_2, many of the grid elements intersecting Q are such that Du is close to a matrix in $SO(2)$, letting a, b denote the endpoints of Q where (say) $a \in L_1$ and $b \in L_3$ we would have $H^1(u([a,b])) \ll |He_2||a-b|$ which is a contradiction because $u([a,b])$ has to connect $u(a)$ to $u(b)$ and integrating from a to $L_1 \cap L_3$, then from $L_1 \cap L_3$ to b we see that $|u(a) - u(b)| \approx |RH(a-b)| = |He_2||a-b|$. Thus there can not be many grid elements in the diamond for which Du is close to a matrix in $SO(2)$ and thus we can apply Lemma 7.4 to control "most" of the lines in direction $H^{-2}n_1$.

Proof of Theorem 2.1. First note by Lemma 6.1 there exists $G \subset G\left(\frac{H^{-2}\phi_1}{|H^{-2}\phi_1|}, \frac{H^{-2}\phi_2}{|H^{-2}\phi_2|}, \kappa^{\frac{m_0}{2}}\right)$ with the following properties

- $$\mathrm{Card}\left(G\left(\frac{H^{-2}\phi_1}{|H^{-2}\phi_1|}, \frac{H^{-2}\phi_2}{|H^{-2}\phi_2|}, \kappa^{\frac{m_0}{2}}\right) \backslash G\right) \leq c\kappa^{\frac{m_0}{4}}\kappa^{-m0}. \tag{148}$$
- For any $P \in G$ there exists $R \in SO(2)$, $J \in \{H, Id\}$ such that

$$\int_P |Du(z) - RJ| \,\mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{4}}\kappa^{m_0}. \tag{149}$$

By Lemma 9.1 there exists G-lines K_{i_1}, K_{i_3} in direction $H^{-2}n_1$ and G-lines K_{i_2}, K_{i_4} in direction $H^{-2}n_2$ which satisfy the following properties.

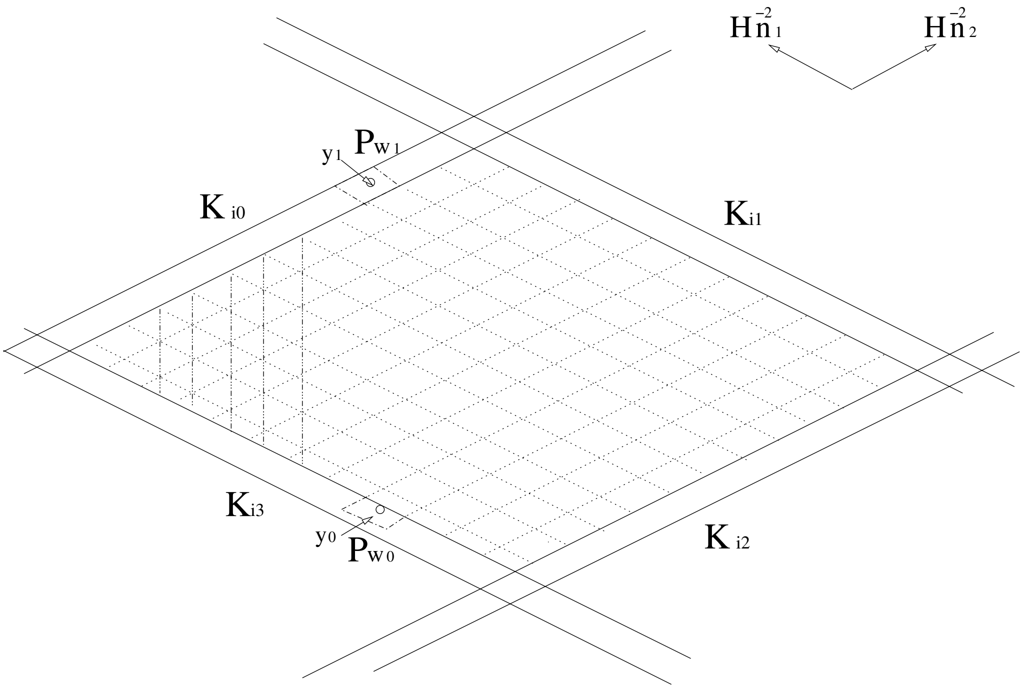


FIGURE 10. Vertical lines connecting the oriented G-lines.

- Let W be the connected component of $Q_1(0) \setminus \left(\widetilde{K_{i_0}} \cup \widetilde{K_{i_1}} \cup \widetilde{K_{i_2}} \cup \widetilde{K_{i_3}}\right)$ containing zero, then

$$Q_{\frac{\sigma^3}{2\sqrt{\sigma^6+1}}}(0) \subset W. \tag{150}$$

- There exists a subset $M \subset K_{i_0} \cup K_{i_1} \cup K_{i_2} \cup K_{i_3}$ with the property that

$$\mathrm{Card}\left(\left(K_{i_0} \cup K_{i_1} \cup K_{i_2} \cup K_{i_3}\right) \setminus M\right) < c\kappa^{\frac{m_0}{32}} \kappa^{-\frac{m_0}{2}} \tag{151}$$

and for some fixed $\widetilde{R} \in SO(2)$, for any $P \in M$ we have

$$\int_P \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0}. \tag{152}$$

Let

$$\mathbb{B} := \left\{ P \in G\left(\frac{H^{-2}\phi_1}{|H^{-2}\phi_1|}, \frac{H^{-2}\phi_2}{|H^{-2}\phi_2|}, \kappa^{\frac{m_0}{2}} \right) : P \subset W \right\}.$$

And let

$$\mathbb{D} := \left\{ P \in G \cap \mathbb{B} : \int_P |Du(z) - R| \, \mathrm{d}L^2 z \leq c\kappa^{m_0} \kappa^{\frac{m_0}{4}} \text{ for some } R \in SO(2) \right\}.$$

Part 1. We will show

$$\mathrm{Card}(\mathbb{D}) \leq 5\kappa^{\frac{m_0}{100}} \kappa^{-m_0}. \tag{153}$$

Proof of Part 1. Suppose not, so

$$\mathrm{Card}(\mathbb{D}) \geq 5\kappa^{\frac{m_0}{100}} \kappa^{-m_0}. \tag{154}$$

Let $C(P)$ denote the center of each P. We can partition $\mathbb{B}$ into columns parallel to e_2 in the following way. Let

$$\mathbb{R}(\alpha) := \{P \in W : C(P) \cdot e_1 = \alpha\}.$$

As we can see from Figure 10, for some constant $\varrho_\sigma > 0$ we have

$$\mathbb{B} \subset \bigcup_{k=-2\left[\kappa^{-\frac{m_0}{2}}\right]}^{2\left[\kappa^{-\frac{m_0}{2}}\right]} \mathbb{R}\left(k\varrho_\sigma \kappa^{\frac{m_0}{2}}\right).$$

Let

$$\Phi := \left\{k \in \left\{-2\left[\kappa^{-\frac{m_0}{2}}\right], \ldots 2\left[\kappa^{\frac{m_0}{2}}\right]\right\} : \mathrm{Card}\left(\mathbb{R}\left(k\varrho_\sigma \kappa^{\frac{m_0}{2}}\right) \cap \mathbb{D}\right) \geq \kappa^{\frac{m_0}{100}} \kappa^{-\frac{m_0}{2}}\right\}. \tag{155}$$

By (154) $5\kappa^{\frac{m_0}{100}}\kappa^{-m_0} \leq 2\kappa^{-\frac{m_0}{2}}\mathrm{Card}(\Phi) + \kappa^{\frac{m_0}{100}}\kappa^{-m_0}$ so we have

$$\mathrm{Card}(\Phi) \geq 2\kappa^{\frac{m_0}{100}}\kappa^{-\frac{m_0}{2}}. \tag{156}$$

Step 1.1. We claim we must be able to find $k_1 \in \Phi$ such that

$$\mathrm{Card}\left(\mathbb{R}\left(j\varrho_\sigma \kappa^{\frac{m_0}{2}}\right) \backslash G\right) \leq \kappa^{\frac{m_0}{20}}\kappa^{-\frac{m_0}{2}} \text{ for } j \in \{k_1 - 1, k_1, k_1 + 1\}. \tag{157}$$

Proof of Step 1.1. Suppose not. So we have a subset $\widetilde{\Phi} \subset \Phi$ with

$$\begin{aligned} \mathrm{Card}\left(\widetilde{\Phi}\right) &\geq \frac{\mathrm{Card}(\Phi)}{3} - 2 \\ &\overset{(156)}{\geq} \frac{\kappa^{\frac{m_0}{100}}}{2}\kappa^{-\frac{m_0}{2}} \end{aligned} \tag{158}$$

and for every $k \in \widetilde{\Phi}$ we have

$$\mathrm{Card}\left(\mathbb{R}\left(k\varrho_\sigma \kappa^{\frac{m_0}{2}}\right) \backslash G\right) \geq \kappa^{\frac{m_0}{20}}\kappa^{-\frac{m_0}{2}}.$$

So

$$\begin{aligned} \mathrm{Card}\left(G\left(\frac{H^{-2}\phi_1}{|H^{-2}\phi_1|}, \frac{H^{-2}\phi_2}{|H^{-2}\phi_2|}, \kappa^{\frac{m_0}{2}}\right) \backslash G\right) &\geq \mathrm{Card}\left(\widetilde{\Phi}\right)\kappa^{\frac{m_0}{20}}\kappa^{-\frac{m_0}{2}} \\ &\overset{(158)}{\geq} \frac{\kappa^{\frac{m_0}{100}}}{2}\kappa^{\frac{m_0}{20}}\kappa^{-m_0} \\ &\geq \frac{\kappa^{\frac{3m_0}{50}}}{2}\kappa^{-m_0} \end{aligned}$$

which contradicts (148), hence we have established (157).

Step 1.2. Let $S := W \cap P_{e_2^{\perp}}^{-1}\left(\left[\left(k_1 - \frac{1}{2}\right)\varrho_\sigma, \left(k_1 + \frac{1}{2}\right)\varrho_\sigma\right]\right)$ and we define function $E : S \to \mathbb{R}$ by

$$E(z) := \begin{cases} \mathrm{d}(Du(z), SO(2)) & \text{if } z \in P \in \mathbb{D} \\ \mathrm{d}(Du(z), SO(2)H) & \text{if } z \in P \in G\backslash\mathbb{D} \\ 2\zeta_2 \text{ if } z \in P \notin G. \end{cases}$$

We will show

$$\int_S E(z)\, \mathrm{d}L^2 z \leq 7\zeta_2 \kappa^{\frac{m_0}{20}}\kappa^{\frac{m_0}{2}}. \tag{159}$$

Proof of Step 1.2. To begin with note that if $P \cap S \neq \emptyset$ and $P \notin G$ then $P \in \mathbb{R}\left(j\varrho_\sigma \kappa^{\frac{m_0}{2}}\right) \backslash G$ for some $j \in \{(k_1-1), k_1, (k_1+1)\}$.

So

$$\{P : P \cap S \neq \emptyset,\ P \notin G\} \subset \bigcup_{j \in \{(k_1-1),k_1,(k_1+1)\}} \mathbb{R}\left(j\varrho_\sigma \kappa^{\frac{m_0}{2}}\right) \backslash G$$

and hence from (157)

$$\mathrm{Card}\left(\{P : P \cap S \neq \emptyset,\ P \notin G\}\right) \leq 3\kappa^{\frac{m_0}{20}} \kappa^{-\frac{m_0}{2}}. \tag{160}$$

Thus

$$\begin{aligned}\int_{S \backslash \left(\bigcup_{P \in G} P\right)} E(z)\, \mathrm{d}L^2 z &\leq 2\zeta_2 \kappa^{m_0} \mathrm{Card}\left(\{P : P \cap S \neq \emptyset,\ P \notin G\}\right) \\ &\leq 6\zeta_2 \kappa^{\frac{m_0}{20}} \kappa^{\frac{m_0}{2}}. \end{aligned} \tag{161}$$

On the other hand from the definition of G specifically from (149) we have

$$\begin{aligned}\int_{S \cap \left(\bigcup_{P \in G} P\right)} E(z)\, \mathrm{d}L^2 z &\leq \sum_{j \in \{(k_1-1),k_1,(k_1+1)\}} \sum_{P \in \mathbb{R}(\varrho_\sigma j) \cap G} \int_P \mathrm{d}\left(Du(z), SO(2) \cup SO(2)H\right) \mathrm{d}L^2 z \\ &\leq 3c\kappa^{\frac{m_0}{2}} \kappa^{\frac{m_0}{4}}. \end{aligned} \tag{162}$$

Hence putting (161), (162) together gives us (159) and this completes the proof of Step 1.2.

Step 1.3. Now since $k_1 \in \Phi$ (see definition (155)) its clear that $L^1\left(P_{e_2^\perp}^{-1}\left(k_1\varrho_\sigma\kappa^{\frac{m_0}{2}}\right) \cap \{P : P \in \mathbb{D}\}\right) \geq \frac{\kappa^{\frac{m_0}{100}}\sigma^3}{4}$. Now from Figure 10 its easy to see that for any $x_0 \in \left[\left(k_1 - \frac{1}{2}\right)\varrho_\sigma\kappa^{\frac{m_0}{2}}, \left(k_1 + \frac{1}{2}\right)\varrho_\sigma\kappa^{\frac{m_0}{2}}\right]$ we have

$$\begin{aligned} L^1\left(P_{e_2^\perp}^{-1}(x_0) \cap \{P : P \in \mathbb{D}\}\right) &\geq \frac{1}{2} L^1\left(P_{e_2^\perp}^{-1}\left(k_1\varrho_\sigma\kappa^{\frac{m_0}{2}}\right) \cap \{P : P \in \mathbb{D}\}\right) \\ &\geq \frac{\kappa^{\frac{m_0}{100}}\sigma^3}{8}. \end{aligned} \tag{163}$$

Now by a Fubini type argument using (159) there must exists $x_1 \in \left[\left(k_1 - \frac{1}{2}\right)\varrho_\sigma\kappa^{\frac{m_0}{2}}, \left(k_1 + \frac{1}{2}\right)\varrho_\sigma\kappa^{\frac{m_0}{2}}\right]$ such that

$$\int_{P_{e_2^\perp}^{-1}(x_1) \cap W} E(z)\, \mathrm{d}L^1 z \leq c\kappa^{\frac{m_0}{20}}. \tag{164}$$

Now we must be able to find $P_{w_0}, P_{w_1} \in K_{i_0} \cup K_{i_1} \cup K_{i_2} \cup K_{i_3}$ with $P_{e_2^\perp}^{-1}(x_1) \cap P_{w_0} \neq \emptyset$, $P_{e_2^\perp}^{-1}(x_1) \cap P_{w_1} \neq \emptyset$. Without loss of generality assume $P_{w_0} \in K_{i_0}$ and $P_{w_1} \in K_{i_3}$. See Figure 10.

Let $z_0 \in P_{e_2^\perp}^{-1}(x_1) \cap P_{w_0}$ and $z_1 \in P_{e_2^\perp}^{-1}(x_1) \cap P_{w_1}$. We will show

$$|u(z_0) - u(z_1)| \leq \sigma^{-1}|z_0 - z_1| - \left(\sigma^{-1} - 1\right)\frac{\kappa^{\frac{m_0}{100}}\sigma^3}{8} + c\kappa^{\frac{m_0}{20}}. \tag{165}$$

Proof of Step 1.3.

$$\begin{aligned}\left|u\left(z_1\right)-u\left(z_0\right)\right| &= \int_{z_0}^{z_1} Du\left(z\right)\cdot e_2 \mathrm{d}L^1 z\\ &\leq \int_{[z_0,z_1]\cap\{P:P\in\mathbb{D}\}} \left|Du\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z + \int_{[z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}} \left|Du\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z\\ &\quad + \int_{[z_0,z_1]\cap\{P:P\notin G\}} \left|Du\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z. \end{aligned} \tag{166}$$

We define a function

$$\Gamma_1 : [z_0, z_1]\cap\{P : P\in\mathbb{D}\}\to SO\left(2\right)$$

such that $\Gamma_1\left(x\right)\in SO\left(2\right)$ is the unique matrix such that $\mathrm{d}\left(Du\left(x\right), SO\left(2\right)\right) = \left|Du\left(x\right)-\Gamma_1\left(x\right)\right|$.

Define

$$\Gamma_2 : [z_0, z_1]\cap\{P : P\in G\backslash\mathbb{D}\}\to SO\left(2\right)H$$

such that $\Gamma_2\left(x\right)\in SO\left(2\right)H$ is the unique matrix such that $\mathrm{d}\left(Du\left(x\right), SO\left(2\right)H\right) = \left|Du\left(x\right)-\Gamma_2\left(x\right)\right|$.

So

$$\begin{aligned}\int_{[z_0,z_1]\cap\{P:P\in\mathbb{D}\}} \left|Du\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z &\leq \int_{[z_0,z_1]\cap\{P:P\in\mathbb{D}\}} \left|\Gamma_1\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z + \int_{[z_0,z_1]\cap\{P:P\in\mathbb{D}\}} E\left(z\right)\mathrm{d}L^1 z\\ &\leq L^1\left([z_0,z_1]\cap\{P:P\in\mathbb{D}\}\right) + \int_{[z_0,z_1]\cap\{P:P\in\mathbb{D}\}} E\left(z\right)\mathrm{d}L^1 z. \end{aligned} \tag{167}$$

Similarly

$$\begin{aligned}&\int_{[z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}} \left|Du\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z\\ &\quad\leq \int_{[z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}} \left|\Gamma_2\left(z\right)\cdot e_2\right| \mathrm{d}L^1 z + \int_{[z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}} E\left(z\right)\mathrm{d}L^1 z\\ &\quad\leq \left|He_2\right| L^1\left([z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}\right) + \int_{[z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}} E\left(z\right)\mathrm{d}L^1 z. \end{aligned} \tag{168}$$

So using (160), (163), (164), (166), (167), (168) we have

$$\begin{aligned}\left|u\left(z_0\right)-u\left(z_1\right)\right| &\overset{(166),(167),(168)}{\leq} L^1\left([z_0,z_1]\cap\{P:P\in\mathbb{D}\}\right) + \left|He_2\right| L^1\left([z_0,z_1]\cap\{P:P\in G\backslash\mathbb{D}\}\right)\\ &\qquad +2\zeta_2 L^1\left([z_0,z_1]\cap\{P:P\notin G\}\right) + \int_{[z_0,z_1]\cap\{P:P\in G\}} E\left(z\right)\mathrm{d}L^1 z\\ &= \left(1-\left|He_2\right|\right) L^1\left([z_0,z_1]\cap\{P:P\in\mathbb{D}\}\right) + \left|He_2\right| L^1\left([z_0,z_1]\cap\{P:P\in G\}\right)\\ &\qquad + \int_{z_0}^{z_1} E\left(z\right)\mathrm{d}L^1 z + 2\zeta_2 L^1\left([z_0,z_1]\cap\{P:P\notin G\}\right)\\ &\overset{(160),(163),(164)}{\leq} \left(1-\sigma^{-1}\right)\frac{\kappa^{\frac{m_0}{100}}\sigma^3}{8} + \sigma^{-1}\left|z_0-z_1\right| + c\kappa^{\frac{m_0}{20}}. \end{aligned}$$

Hence we have completed the proof of Step 1.3.

Step 1.4. We will show

$$\left|\left|u\left(z_1\right)-u\left(z_0\right)\right| - \left|z_1-z_0\right|\widetilde{R}H\cdot e_2\right| \leq c\kappa^{\frac{m_0}{64}}.$$

Proof of Step 1.4. Now recall $P_{w_0}, P_{w_1} \in K_{i_0} \cup K_{i_1} \cup K_{i_2} \cup K_{i_3}$. Assume without loss of generality $P_{w_1} \in K_{i_0}$, $P_{w_0} \in K_{i_3}$, see Figure 10.

Now from (151), (152) we know that

$$\int_{\widetilde{K_{i_2}} \cup \widetilde{K_{i_3}}} \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^2 z \leq c\kappa^{\frac{m_0}{64}} \kappa^{m_0} \mathrm{Card}\,(K_{i_2} \cup K_{i_3}) + c\kappa^{\frac{m_0}{32}} \kappa^{\frac{m_0}{2}} \\ \leq c\kappa^{\frac{m_0}{64}} \kappa^{\frac{m_0}{2}}.$$

So by a Fubini argument (using the fact that the width of $\widetilde{K_{i_1}}$ and $\widetilde{K_{i_3}}$ is bigger than $\frac{\sigma^3 \kappa^{\frac{m_0}{2}}}{4}$) we must be able to find points $y_0 \in P_{w_0}$, $y_1 \in P_{w_1}$ such that

$$\int_{(y_0 + \langle H^{-2} n_2 \rangle) \cap Q_1(0)} \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^1 z \leq c\kappa^{\frac{m_0}{64}}.$$

And

$$\int_{(y_1 + \langle H^{-2} n_1 \rangle) \cap Q_1(0)} \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^1 z \leq c\kappa^{\frac{m_0}{64}}.$$

Let $\tilde{y} := \left\{ y_0 + \langle H^{-2} n_2 \rangle \right\} \cap \left\{ y_1 + \langle H^{-2} n_1 \rangle \right\}$. So

$$\begin{aligned} &\Bigg| \left(\int_{y_0}^{\tilde{y}} Du(z) \cdot \frac{H^{-2} n_2}{|H^{-2} n_2|} \mathrm{d}L^1 z + \int_{\tilde{y}}^{y_1} Du(z) \cdot \frac{H^{-2} n_1}{|H^{-2} n_1|} \mathrm{d}L^1 z \right) \\ &\qquad - \left(\int_{y_0}^{\tilde{y}} \widetilde{R}H \cdot \frac{H^{-2} n_2}{|H^{-2} n_2|} \mathrm{d}L^1 z + \int_{\tilde{y}}^{y_1} \widetilde{R}H \cdot \frac{H^{-2} n_1}{|H^{-2} n_1|} \mathrm{d}L^1 z \right) \Bigg| \\ &\leq \int_{y_0}^{\tilde{y}} \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^1 z + \int_{\tilde{y}}^{y_1} \left| Du(z) - \widetilde{R}H \right| \mathrm{d}L^1 z \\ &\leq c\kappa^{\frac{m_0}{64}}. \end{aligned} \tag{169}$$

As

$$\int_{y_0}^{\tilde{y}} \widetilde{R}H \cdot \frac{H^{-2} n_1}{|H^{-2} n_1|} \mathrm{d}L^1 z + \int_{\tilde{y}}^{y_1} \widetilde{R}H \cdot \frac{H^{-2} n_2}{|H^{-2} n_2|} \mathrm{d}L^1 z = \int_{y_0}^{y_1} \widetilde{R}H \cdot e_2 \\ = |y_1 - y_0| \, \widetilde{R}H \cdot e_2.$$

And as

$$\left(\int_{y_0}^{\tilde{y}} Du(z) \cdot \frac{H^{-2} n_1}{|H^{-2} n_1|} \mathrm{d}L^1 z + \int_{\tilde{y}}^{y_1} Du(z) \cdot \frac{H^{-2} n_2}{|H^{-2} n_2|} \mathrm{d}L^1 z \right) = u(y_1) - u(y_0).$$

So (169) becomes

$$\left| |u(y_1) - u(y_0)| - |y_1 - y_0| \, \widetilde{R}H \cdot e_2 \right| \leq c\kappa^{\frac{m_0}{64}}. \tag{170}$$

By Lipschitzness this implies

$$\left| |u(z_1) - u(z_0)| - |z_1 - z_0| \, \widetilde{R}H \cdot e_2 \right| \leq c\kappa^{\frac{m_0}{64}}. \tag{171}$$

This completes the proof of Step 1.4.

Proof of Part 1 continued.

So in particular, from (171)

$$|u(z_1)-u(z_0)| \geq \sigma^{-1}|z_1-z_0| - c\kappa^{\frac{m_0}{64}}. \tag{172}$$

Putting this together with (165) we have

$$\begin{aligned}\sigma^{-1}|z_1-z_0| - c\kappa^{\frac{m_0}{64}} \leq &-\left(\sigma^{-1}-1\right)\frac{\kappa^{\frac{m_0}{100}}\sigma^3}{8}\\ &+\sigma^{-1}|z_0-z_1| + c\kappa^{\frac{m_0}{20}}.\end{aligned}$$

This implies

$$\frac{\left(\sigma^{-1}-1\right)}{8}\kappa^{\frac{m_0}{100}}\sigma^3 \leq c\kappa^{\frac{m_0}{64}} + c\kappa^{\frac{m_0}{20}}$$

which is a contradiction for small enough κ. Hence we have shown Part 1.

Part 2. We will complete the proof of Theorem 2.1.

As we have noted before. $\mathbb{B}$ is made up of a union of G-lines in direction $H^{-2}n_1$. Denote them $K_{s_1}, K_{s_2}, \ldots K_{s_{n_4}}$ where $n_4 \geq \frac{\sigma^3}{8}\kappa^{-\frac{m_0}{2}}$. Formally $\left\{K_{s_1}, K_{s_2}, \ldots K_{s_{n_4}}\right\} := \left\{K_i : \widetilde{K_i} \cap W \neq \emptyset\right\}$. Let

$$W_1 := \left\{K_i : \mathrm{Card}\left(K_i \cap \mathbb{D}\right) \leq \kappa^{\frac{m_0}{200}}\kappa^{-\frac{m_0}{2}},\ K_i \cap W \neq \emptyset\right\}. \tag{173}$$

Note that from (153) we have

$$\mathrm{Card}\left(\left\{K_1, \ldots K_{n_4}\right\}\backslash W_1\right) \leq 5\kappa^{\frac{m_0}{200}}\kappa^{-\frac{m_0}{2}}. \tag{174}$$

Let

$$W_2 := \left\{K_i \in W_1 : \mathrm{Card}\left(K_i \backslash G\right) \leq \kappa^{\frac{m_0}{16}}\kappa^{-\frac{m_0}{2}}\right\} \tag{175}$$

So from (148) we know $\mathrm{Card}\left(W_1\backslash W_2\right)\kappa^{\frac{m_0}{16}}\kappa^{-\frac{m_0}{2}} \leq c\kappa^{\frac{m_0}{4}}\kappa^{-m_0}$ so

$$\mathrm{Card}\left(W_1\backslash W_2\right) \leq c\kappa^{\frac{3m_0}{16}}\kappa^{-\frac{m_0}{2}}. \tag{176}$$

Let $\{q_1, q_2, \ldots q_{n_5}\} \in \mathbb{N}$ be such that $W_2 := \left\{K_{q_1}, K_{q_2}, \ldots K_{q_{n_5}}\right\}$. Note that we of course have $n_5 \leq \kappa^{-\frac{m_0}{2}}$. Note that from (174), (176)

$$\mathrm{Card}\left(\left\{K_{s_1}, K_{s_2}, \ldots K_{s_{n_4}}\right\} \backslash \left\{K_{q_1}, K_{q_2}, \ldots K_{q_{n_5}}\right\}\right) \leq 6\kappa^{\frac{m_0}{200}}\kappa^{-\frac{m_0}{2}}. \tag{177}$$

Now for any G-line $K_{q_i} \in W_2$ let $P_{q_i}^{(1)}$ be the "first" parallelopiped in $K_{q_i} \cap \mathbb{B}$ (*i.e.* the parallelopiped such that $C\left(P_{q_i}^{(1)}\right) \cdot H^{-2}n_1 \leq C(P) \cdot H^{-2}n_1$ for any $P \in K_{q_i} \cap \mathbb{B}$). Let $P_{q_i}^{(2)}$ be the similarly defined "last" parallelopiped. Note that by (150) we have

$$\left|C\left(P_{q_i}^{(1)}\right) - C\left(P_{q_i}^{(2)}\right)\right| > \frac{\sigma^3}{4}\cdot \tag{178}$$

Let $x_1 := C\left(P_{q_i}^{(1)}\right)$ and $x_2 := C\left(P_{q_i}^{(2)}\right)$. By arguing as we did to establish (170) in Part 1 we can show that there exists $R_2 \in SO(2)$ independent of i such that

$$\left|\left(u(x_2)-u(x_1)\right) - R_2 H\left(x_2-x_1\right)\right| \leq c\kappa^{\frac{m_0}{64}}. \tag{179}$$

Let $R_1 \in SO(2)$ be such that

$$R_1\left(H^{-1}n_1\right) = \frac{u(x_2)-u(x_1)}{|u(x_2)-u(x_1)|}\cdot \tag{180}$$

Now

$$\left|\left|u\left(x_2\right)-u\left(x_1\right)\right|-\left|R_2 H\left(x_2-x_1\right)\right|\right| \leq \left|\left(u\left(x_2\right)-u\left(x_1\right)\right)-R_2 H\left(x_2-x_1\right)\right| \leq c\kappa^{\frac{m_0}{64}}. \quad (181)$$

As $\frac{x_2-x_1}{|x_2-x_1|}=\frac{H^{-2}n_1}{|H^{-2}n_1|}$

$$\left|R_2 H\left(x_2-x_1\right)\right| = \left|x_2-x_1\right|\left|H\left(\frac{H^{-2}n_1}{|H^{-2}n_1|}\right)\right| = \frac{|x_2-x_1|}{|H^{-2}n_1|}. \quad (182)$$

Since from (69) we have $H^{-2}n_1\cdot n_1=1$ so $\frac{H^{-2}n_1}{|H^{-2}n_1|}\cdot n_1=\frac{1}{|H^{-2}n_1|}$ so since $\frac{x_2-x_1}{|x_2-x_1|}=\frac{H^{-2}n_1}{|H^{-2}n_1|}$ using this on (182) we have

$$\left|R_2 H\left(x_2-x_1\right)\right|=\left|x_2-x_1\right|\frac{H^{-2}n_1}{|H^{-2}n_1|}\cdot n_1=\left|\left(x_2-x_1\right)\cdot n_1\right|. \quad (183)$$

Applying this to (181) gives

$$\left|\left|u\left(x_2\right)-u\left(x_1\right)\right|-\left|\left(x_2-x_1\right)\cdot n_1\right|\right|<c\kappa^{\frac{m_0}{64}}. \quad (184)$$

Using inequalities (179), (180) and the fact that $\frac{x_2-x_1}{|x_2-x_1|}=\frac{H^{-2}n_1}{|H^{-2}n_1|}$

$$\left|\left|u\left(x_2\right)-u\left(x_1\right)\right|R_1\left(H^{-1}n_1\right)-\left|x_1-x_2\right|R_2\left(\frac{H^{-1}n_1}{|H^{-2}n_1|}\right)\right| \overset{(180)}{=} \left|\left(u\left(x_2\right)-u\left(x_1\right)\right)-R_2 H\left(x_2-x_1\right)\right| \overset{(179)}{\leq} c\kappa^{\frac{m_0}{64}}. \quad (185)$$

And as we have see $\left|H^{-2}n_1\right|^{-1}=\frac{H^{-2}n_1}{|H^{-2}n_1|}\cdot n_1$ so $\frac{|x_1-x_2|}{|H^{-2}n_1|}=\left|\left(x_2-x_1\right)\cdot n_1\right|$ and so using this in (184) and inserting it into (185) we have

$$\left|\left|x_2-x_1\right|R_1\left(\frac{H^{-1}n_1}{|H^{-1}n_1|}\right)-\left|x_2-x_1\right|R_2\left(\frac{H^{-1}n_1}{|H^{-1}n_1|}\right)\right|<c\kappa^{\frac{m_0}{64}} \quad (186)$$

from (178) we know $|x_1-x_2|>\frac{\sigma^3}{4}$ and so (186) implies

$$\left|R_1-R_2\right|<c\kappa^{\frac{m_0}{64}}. \quad (187)$$

By definition of W_1 and W_2 (173) and (175) we know

$$\mathrm{Card}\left(K_{q_i}\backslash\left(G\backslash\mathbb{D}\right)\right) \leq \kappa^{\frac{m_0}{16}}\kappa^{-\frac{m_0}{2}}+\kappa^{\frac{m_0}{200}}\kappa^{-\frac{m_0}{2}} \leq 2\kappa^{\frac{m_0}{200}}\kappa^{-\frac{m_0}{2}}. \quad (188)$$

So setting $p_0=\frac{m_0}{200}$, $q_0=\frac{m_0}{128}$ we see (188) and (178), (179), (187) gives us the necessary conditions to apply Lemma 7.4. So by Lemma 7.4 we have the existence of a set $M_i\subset K_{q_i}\cap\left(G\backslash\mathbb{D}\right)$ such that

$$\mathrm{Card}\left(K_{q_i}\backslash M_i\right)\leq c\kappa^{\frac{m_0}{400}}\kappa^{-\frac{m_0}{2}} \quad (189)$$

and every $P \in M_i$ has the property

$$\int_P |Du(z) - R_2 H| \, \mathrm{d}L^2 z \le c\kappa^{\frac{m_0}{800}} \kappa^{m_0}. \tag{190}$$

Recall (see (179)) R_2 is independent of i. Let

$$\Pi = \mathbb{B} \backslash \bigcup_{i=1}^{n_5} M_i.$$

So by (177) and (189) we have

$$\begin{aligned} \mathrm{Card}(\Pi) &\le \kappa^{-\frac{m_0}{2}} \mathrm{Card}\left(\left\{K_{s_1}, K_{s_2}, \ldots K_{s_{n_4}}\right\} \backslash \left\{K_{q_1}, K_{q_2}, \ldots K_{q_{n_5}}\right\}\right) + \sum_{k=1}^{n_5} \mathrm{Card}(K_{q_k} \backslash M_{q_k}) \\ &\overset{(177),(189)}{\le} 6\kappa^{\frac{m_0}{200}} \kappa^{-m_0} + c\kappa^{\frac{m_0}{400}} \kappa^{-m_0} \\ &\le c\kappa^{\frac{m_0}{400}} \kappa^{-m_0}. \end{aligned} \tag{191}$$

And note any $P \in \mathbb{B} \backslash \Pi$ satisfies inequality (190) and so using (191) we have

$$\begin{aligned} \int_W |Du(z) - R_2 H| \, \mathrm{d}L^2 z &\le 20\zeta_2 \kappa^{\frac{m_0}{2}} + \sum_{P \in \mathbb{B}} \int_P |Du(z) - R_2 H| \, \mathrm{d}L^2 \\ &\overset{(191)}{\le} 20\zeta_2 \kappa^{\frac{m_0}{2}} + \mathrm{Card}(\mathbb{B} \backslash \Pi) c\kappa^{\frac{m_0}{800}} \kappa^{m_0} + c\kappa^{\frac{m_0}{400}} \\ &\le c\kappa^{\frac{m_0}{800}}. \end{aligned}$$

This completes the proof of the theorem. □

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References

[1] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems. *Oxford Math. Monogr.* The Clarendon Press, Oxford University Press, New York (2000).

[2] J.M. Ball and R.D. James, Fine phase mixtures as minimisers of energy. *Arch. Rat. Mech. Anal.* **100** (1987) 13–52.

[3] J.M. Ball and R.D. James, Proposed experimental tests of a theory of fine microstructure and the two well problem. *Phil. Trans. Roy. Soc. London Ser. A* **338** (1992) 389–450.

[4] N. Chaudhuri and S. Müller, Rigidity Estimate for Two Incompatible Wells. *Calc. Var. Partial Differ. Equ.* **19** (2004) 379–390.

[5] M. Chipot and D. Kinderlehrer, Equilibrium configurations of crystals. *Arch. Rat. Mech. Anal.* **103** (1988) 237–277.

[6] M. Chipot and S. Müller, Sharp energy estimates for finite element approximations of non-convex problems. Variations of domain and free-boundary problems in solid mechanics (Paris, 1997). *Solid Mech. Appl.* **66** (1999) 317–325.

[7] S. Conti, D. Faraco and F. Maggi, A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions. *Arch. Rat. Mech. Anal.* **175** (2005) 287–300.

[8] S. Conti and B. Schweizer, *A sharp-interface limit for a two-well problem in geometrically linear elasticity.* MPI MIS Preprint Nr. 87/2003.

[9] S. Conti and B. Schweizer, *Rigidity and Gamma convergence for solid-solid phase transitions with $SO(2)$-invariance.* MPI MIS Preprint Nr. 69/2004.

[10] B. Dacorogna and P. Marcellini, General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. *Acta Math.* **178** (1997) 1–37.
[11] G. Friesecke, R.D. James and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. *Comm. Pure Appl. Math.* **55** (2002) 1461–1506.
[12] A. Lorent, An optimal scaling law for finite element approximations of a variational problem with non-trivial microstructure. *ESAIM: M2AN* **35** (2001) 921–934.
[13] A. Lorent, *The two well problem with surface energy.* MPI MIS Preprint No. 22/2004.
[14] A. Lorent, *On the scaling of the two well problem.* Forthcoming.
[15] S. Müller and V. Šverák, Attainment results for the two-well problem by convex integration, in *Geometric Analysis and the Calculus of Variations,* Stefan Hildebrandt, J. Jost Ed. International Press, Cambridge (1996) 239–251.
[16] S. Müller and V. Šverák, Convex integration with constraints and applications to phase transitions and partial differential equations. *J. Eur. Math. Soc.* **1** (1999) 393–422.
[17] O. Pantz, On the justification of the nonlinear inextensional plate model. *Arch. Ration. Mech. Anal.* **167** (2003) 179–209.