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by

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A Type of Multi-level Correction Method for Eigenvalue Problems by Nonconforming Finite Element Methods*

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Abstract

In this paper, a type of multi-level correction scheme is proposed to solve eigenvalue problems by the nonconforming finite element method. With this new scheme, the accuracy of eigenpair approximations can be improved after each correction step which only needs to solve a source problem on finer finite element space and an eigenvalue problem on the coarsest finite element space. This correction scheme can improve the efficiency of solving eigenvalue problems by the nonconforming finite element method. Furthermore, as same as the direct eigenvalue solving by the nonconforming finite element method, this multi-level correction method can also produce the lower-bound approximations of the eigenvalues.

Keywords. Eigenvalue problem, multigrid, multi-level correction, finite element method.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

The purpose of this paper is to propose a type of multi-level correction scheme based on the nonconforming finite element discretization to solve eigenvalue problems. The two-grid method for solving eigenvalue problems has been proposed and analyzed by Xu and Zhou in [24]. The idea of the two-grid comes from [22, 23] for nonsymmetric or indefinite problems and nonlinear elliptic equations. Since then, there have existed many numerical

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methods for solving eigenvalue problems based on the idea of two-grid method (see, e.g., [1, 6, 10, 20, 27, 28] and the reference cited therein).

In this paper, we present a type of multi-level correction scheme for solving eigenvalue problems by nonconforming finite element methods. With the proposed method solving eigenvalue problem will not be much more difficult than the solution of the corresponding source problem. The correction method for eigenvalue problems in this paper is based on a series of finite element spaces with different approximation properties which are related to the multilevel method (c.f. [21]).

The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [2, 3], Chatelin [5], Yang and Chen [26] and references cited therein. Here we adopt some basic results in these papers for our analysis. The corresponding error estimates of this type of multi-level correction scheme by nonconforming finite element methods which is introduced here will be analyzed. Based on the analysis, the method can reduce the error of the eigenpair approximations after each correction step. The multi-level correction procedure can be described as follows: (1) solve the eigenvalue problem in the coarsest nonconforming finite element space; (2) solve an additional source problem in an finer nonconforming finite element space using the previous obtained eigenvalue multiplying the corresponding eigenfunction as the load vector; (3) solve the eigenvalue problem again on the finite element space which is constructed by combining a very coarse conforming finite element space with the obtained eigenfunction approximation in step (2). Then go to step (2) for the next loop.

In order to describe our method clearly, we give the multi-level correction method for Laplace eigenvalue problem to illustrate the main idea in this paper (see section 5).

Find(λ, u) such that

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega,\\ \int_\Omega u^2 d\Omega &=& 1, & \end{array}\right. \tag{1.1}$$

where $\Omega \subset \mathcal{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and Δ denote the Laplace operator.

Let W_H denote a very coarse conforming linear finite element space defined on the coarsest mesh $\mathcal{T}_H$. Additionally, we also need to construct a series of finite element spaces $V_{h_1}, V_{h_2}, \cdots, V_{h_n}$ which are defined on the corresponding series of nested meshes $\mathcal{T}_{h_k}$ $(k = 1, 2, \cdots n)$ such that $\mathcal{T}_{h_1} = \mathcal{T}_H$ or is obtained by refining $\mathcal{T}_H$ with the regular way and each $\mathcal{T}_{h_{k+1}}$ is produced from $\mathcal{T}_{h_k}$ by refining in the regular way. Our multi-level correction algorithm to obtain the approximation for the eigenpair of (1.1) can be defined as follows (see Sections 3 and 4):

1. Solve an eigenvalue problem in the coarse space V_{h_1}:

   Find $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$ such that $\|u_{h_1}\|_0 = 1$ and

$$\sum_{K\in\mathcal{T}_{h_1}} \int_K \nabla u_{h_1} \nabla v_{h_1} dK = \lambda_{h_1} \int_\Omega u_{h_1} v_{h_1} d\Omega, \quad \forall v_{h_1} \in V_{h_1}.$$

2. Do $k = 1, \cdots, n-1$

   - Solve the following auxiliary source problem:

     Find $\widetilde{u}_{h_{k+1}} \in V_{h_{k+1}}$ such that

$$\sum_{K\in\mathcal{T}_{h_{k+1}}} \int_K \nabla \widetilde{u}_{h_{k+1}} \nabla v_{h_{k+1}} dK = \lambda_{h_k} \int_\Omega u_{h_k} v_{h_{k+1}} d\Omega, \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}.$$

   - Define a new finite element space $V_{H,h_{k+1}} = W_H + \mathrm{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:

     Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $\|u_{h_{k+1}}\|_0 = 1$ and

$$\sum_{K\in\mathcal{T}_{h_{k+1}}} \int_K \nabla u_{h_{k+1}} \nabla v_{H,h_{k+1}} dK = \lambda_{h_{k+1}} \int_\Omega u_{h_{k+1}} v_{H,h_{k+1}} d\Omega, \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}.$$

   end Do

If, for example, λ_{h_1} is the first eigenvalue of the problem (1.1) at the first step and Ω is a convex domain, we can establish the following results (see Sections 3 and 4 for details)

$$\Big(\sum_{K\in\mathcal{T}_{h_n}} \|\nabla(u-u_{h_n})\|_{0,K}^2\Big)^{1/2} = \mathcal{O}\Big(\sum_{k=1}^{n} h_k H^{n-k}\Big), \text{ and } |\lambda_{h_n} - \lambda| = \mathcal{O}\Big(\sum_{k=1}^{n} h_k^2 H^{2(n-k)}\Big).$$

These two estimates means that we can obtain asymptotic optimal errors by taking $H = \sqrt[n]{h_n}$ and $h_k = H^k$ $(k = 1, \cdots, n-1)$. This result is different with the two-grid method ($H = \sqrt{h_n}$ or $H = \sqrt[4]{h_n}$) (c.f. [24, 27]) and the method here dose not need to solve almost singular linear problems. Furthermore, the final eigenvalue approximation λ_{h_n} is the lower bound of the exact eigenvalue if the used nonconforming element can obtain the lower bound of the eigenvalue by the direct eigenvalue solving.

In this method, we replace solving eigenvalue problem in the finest nonconforming finite element space by solving a series of boundary value problems in the corresponding series of nonconforming finite element spaces and a series of eigenvalue problems in the coarse conforming linear finite element space plus one dimensional eigenfunction space. As we know, there exists efficient preconditioner for solving boundary value problems efficiently. So this correction method can improve the efficiency of solving eigenvalue problems by nonconforming finite element methods.

An outline of the paper goes as follows. In Section 2, we introduce the nonconforming finite element method for the eigenvalue problem and the corresponding error estimates. A type of one correction step is given in Section 3. In Section 4, we propose a type of multi-level correction algorithm for solving the eigenvalue problem by the nonconforming finite element method. A lower-bound analysis of the eigenvalue approximations is given in Section 5. In Section 6, two numerical examples are presented to validate our theoretical analysis and some concluding remarks are given in the last section.

2 Discretization by nonconforming finite element method

In this section, we introduce some notation and error estimates of the nonconforming finite element approximation for eigenvalue problems. In this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at different occurrences. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes (c.f. [21]). In this paper, we set $V := H_0^1(\Omega)$ and $W := L^2(\Omega)$.

In our methodology description, we are concerned with the following Laplace eigenvalue problem:

Find $(\lambda, u) \in \mathcal{R} \times V$ such that $b(u,u) = 1$ and

$$a(u,v) \quad = \quad \lambda b(u,v), \quad \forall v \in V, \tag{2.1}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear forms defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v d\Omega, \qquad b(u,v) = \int_\Omega uvd\Omega.$$

For the eigenvalue λ, there exists the following Rayleigh quotient expression (see, e.g., [2, 3, 24])

$$\lambda = \frac{a(u,u)}{b(u,u)}. \tag{2.2}$$

From [3, 5], we know the eigenvalue problem (2.1) has an eigenvalue sequence $\{\lambda_j\}:$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity. In this paper, we assume the eigenfunction u of (2.1) has the regularity $u \in H^{1+\gamma}(\Omega)$, where $0 < \gamma \leq 1$ depends on the maximum interior angle of $\partial\Omega$ (c.f. [9]).

Let $\mathcal{T}_h$ be a quasi-uniform decomposition of $\overline{\Omega}$ into triangles (c.f. [4, 7]). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Let $\mathcal{E}_h$ denote the edge set of $\mathcal{T}_h$ and $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$, where $\mathcal{E}_h^i$ denotes the interior edge set and $\mathcal{E}_h^b$ denotes the edge set lying on the boundary $\partial\Omega$. The finite element space V_h is the corresponding nonconforming finite element space on the partition, i.e. $V_h \not\subseteq V$.

In the rest of this paper, we are concerned with two types of nonconforming finite elements: Crouzeix-Raviart (CR) (c.f. [8]) and Enriched Crouzeix-Raviart (ECR) (c.f. [11, 14]), for triangular partitions, respectively.

- CR element is defined on the triangular partition and

$$V_h := \Big\{ v \in L^2(\Omega) : v|_K \in \text{span}\{1, x, y\}, \int_\ell v|_{K_1} \mathrm{d}s = \int_\ell v|_{K_2} \mathrm{d}s, \\ \text{when } K_1 \cap K_2 = \ell \in \mathcal{E}_h^i \text{ and } \int_\ell v|_K \mathrm{d}s = 0, \text{ if } \ell \in \mathcal{E}_h^b \Big\}, \tag{2.3}$$

  where $K,\ K_1,\ K_2 \in \mathcal{T}_h$.

- ECR element is defined on the triangular partition and

$$V_h := \Big\{ v \in L^2(\Omega) : v|_K \in \text{span}\{1, x, y, x^2 + y^2\}, \int_\ell v|_{K_1} \mathrm{d}s = \int_\ell v|_{K_2} \mathrm{d}s, \\ \text{when } K_1 \cap K_2 = \ell \in \mathcal{E}_h^i, \text{ and } \int_\ell v|_K \mathrm{d}s = 0, \text{ if } \ell \in \mathcal{E}_h^b \Big\}, \tag{2.4}$$

  where $K,\ K_1,\ K_2 \in \mathcal{T}_h$.

Both the above nonconforming elements possess the following common properties:

1. The space of shape functions contains the complete polynomials of degree 1;
2. $v \in V_h$ is integrally continuous at the common edge F between the neighboring elements K_1 and K_2, i.e.,

$$\int_\ell v|_{K_1} ds = \int_\ell v|_{K_2} ds \ \text{ if } K_1 \cap K_2 = \ell \in \mathcal{E}_h^i;$$

3. $V^h \not\subset H_0^1(\Omega)$ and $V^h \subset L^2(\Omega)$.

The nonconforming finite element approximation for (2.1) is defined as follows:

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $b(u_h, u_h) = 1$ and

$$a_h(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h, \tag{2.5}$$

where the bilinear forms $a_h(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u_h \nabla v_h dK, \qquad b(u_h, v_h) = \int_\Omega u_h v_h d\Omega.$$

The bilinear form $a_h(\cdot,\cdot)$ is V_h-elliptic on $V + V_h$. Thus we define the norms $\|\cdot\|_{a,h}$ and $\|\cdot\|_b$ on $V_h + V$ by

$$\|v\|_{a,h}^2 = a_h(v, v), \quad \|v\|_b^2 = b(v, v) \qquad \text{for } v \in V + V_h.$$

For the eigenvalue problem (2.5), the Rayleigh quotient holds for the eigenvalue λ_h

$$\lambda_h = \frac{a_h(v_h, v_h)}{b(v_h, v_h)}. \tag{2.6}$$

Similarly, the discrete eigenvalue problem (2.5) has also an eigenvalue sequence $\{\lambda_{j,h}\}$ with

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{N_h,h},$$

and the corresponding discrete eigenfunction sequence $\{u_{j,h}\}$

$$u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h}$$

with the property $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \le i, j \le N_h$ (N_h is the dimension of V_h).

The interpolation operator $\Pi_h : V \longmapsto V_h$ corresponding to CR element can be defined in the same way (c.f. [8]):

$$\int_\ell (u - \Pi_h u) ds \;=\; 0, \qquad \forall \ell \in \mathcal{E}_h. \tag{2.7}$$

The interpolation operator $\Pi_h : V \longmapsto V_h$ corresponding to ECR element can be defined as follows (c.f. [11, 14]):

$$\int_\ell (u - \Pi_h u) ds \;=\; 0, \qquad \forall \ell \in \mathcal{E}_h, \tag{2.8}$$

$$\int_K (u - \Pi_h u) \mathrm{d}K \;=\; 0, \qquad \forall K \in \mathcal{T}_h. \tag{2.9}$$

Lemma 2.1. *([17]) For any $u \in V$, the interpolation defined in (2.7) or (2.8)-(2.9) has the following results*

$$a_h(u - \Pi_h u, v_h) = 0, \qquad \forall v_h \in V_h, \tag{2.10}$$

$$\|\Pi_h u\|_{a,h} \le \|u\|_a. \tag{2.11}$$

Furthermore, the interpolation operator has error estimates

$$\|u-\Pi_h u\|_b + h\|u-\Pi_h u\|_{a,h} \leq Ch^{1+\gamma}\|u\|_{1+\gamma}, \tag{2.12}$$

for any $u \in H^{1+\gamma}(\Omega)$.

In order to give the error estimates of the eigenpair approximation by finite element methods, we define the operator $T: W \longmapsto V$ by

$$a(Tf,v) = b(f,v), \quad \forall v \in V, \tag{2.13}$$

for any $f \in W$. As we know, the operator T is compact. Then the eigenvalue problem (2.1) can be written as

$$\lambda Tu = u. \tag{2.14}$$

We also define the corresponding discrete operator $T_h: W \longmapsto V_h$ by

$$a_h(T_h f, v_h) = b(f, v_h), \quad \forall v_h \in V_h, \tag{2.15}$$

for any $f \in W$. Similarly the discrete eigenvalue problem (2.5) can be written as

$$\lambda_h T_h u_h = u_h. \tag{2.16}$$

Let $M(\lambda_j)$ denote the eigenfunction set corresponding to the eigenvalue λ_j which is defined by

$$\begin{aligned} M(\lambda_j) = \big\{& w \in V : w \text{ is an eigenfunction of (2.1) corresponding to} \\ & \lambda_j \text{ and } \|w\|_b = 1\big\}. \end{aligned} \tag{2.17}$$

Now we state the convergence result of the eigenvalue problem by nonconforming finite element methods. For this aim, we define the following notation

$$\varepsilon_h(\lambda_j) = \|(T-T_h)|_{M(\lambda_j)}\|_{a,h}, \tag{2.18}$$

$$\rho_h(\lambda_j) = \|(T-T_h)|_{M(\lambda_j)}\|_b. \tag{2.19}$$

Lemma 2.2. *([18, 19, 26]) Suppose that* $\|T_h - T\|_b \to 0$ $(h\to 0)$. *Let* $(\lambda_{j,h}, u_{j,h}) \in \mathcal{R}\times V_h$ *be the* j*-th nonconforming finite element eigenpair approximation satisfying (2.5). Then* $\lambda_{j,h} \to \lambda_j$ *and there exist* $u_j \in M(\lambda_j)$ *such that*

$$\|u_j - u_{j,h}\|_{a,h} \leq C_j\big(\varepsilon_h(\lambda_j) + \rho_h(\lambda_j)\big), \tag{2.20}$$

$$\|u_j - u_{j,h}\|_b \leq C_j \rho_h(\lambda_j), \tag{2.21}$$

$$|\lambda_j - \lambda_{j,h}| \leq C_j \rho_h(\lambda_j), \tag{2.22}$$

where the constants C_j depending on the j-th eigenvalue λ_j.

Furthermore, for any $w_h \in V_h$ with $\|w_h\|_b \neq 0$, the following expansion holds

$$\frac{a_h(w_h,w_h)}{b(w_h,w_h)}-\lambda=\frac{a_h(w_h-u,w_h-u)}{b(w_h,w_h)}-\lambda\frac{b(w_h-u,w_h-u)}{b(w_h,w_h)}+2\frac{E_h(u,w_h)}{b(w_h,w_h)}, \tag{2.23}$$

where $E_h(u,w_h)=a_h(u,w_h)-b(\lambda u,w_h)$.

Now we state a lower bound of the convergence rate for the eigenfunction approximation by finite element methods which will be used in the analysis for the error estimates.

Lemma 2.3. *([15, Section 3]) If we solve the eigenvalue problem (2.1) by CR element or ECR element, the following lower bound of the convergence rate holds*

$$\|(T-T_h)|_{M(\lambda_j)}\|_{a,h} \quad \geq \quad C_j h, \tag{2.24}$$

where the constants C_j depending on the j-th eigenvalue λ_j.

For the aim of analyzing the error estimates, we need to get the result of the estimate $\|(T-T_h)|_{M(\lambda)}\|_b$ is a higher order term corresponding to $\|(T-T_h)|_{M(\lambda)}\|_{a,h}$.

Theorem 2.1. *([13]) Assume the nonconforming finite element owns a type of interpolation operator Π_h satisfying the orthogonal property (2.10). We have the following error estimate*

$$\|(T-T_h)|_{M(\lambda)}\|_b \quad \lesssim \quad h^\gamma \|(T-T_h)|_{M(\lambda)}\|_{a,h}. \tag{2.25}$$

Proof. For any $f\in M(\lambda)$, let $u=Tf$ and $u_h=T_hf$. For any $\psi\in W$, we have

$$\begin{aligned}
\|u-u_h\|_b &= b(u-u_h,\psi)\\
&= a_h(u,\varphi_\psi)-a_h(u_h,\varphi_h)\\
&= a_h(u-u_h,\varphi_\psi-\varphi_h)+a_h(u_h,\varphi_\psi-\varphi_h)+a_h(u-u_h,\varphi_h)\\
&= a_h(u-u_h,\varphi_\psi-\varphi_h)-[a_h(u-u_h,\varphi_\psi)-b(u-u_h,\psi)]\\
&\quad -[a_h(u,\varphi_\psi-\varphi_h)-b(f,\varphi_\psi-\varphi_h)],
\end{aligned} \tag{2.26}$$

where $\varphi_\psi=T\psi$, $\varphi_h=T_h\psi$ and (c.f. [9])

$$\|\varphi_\psi\|_{1+\gamma} \quad \lesssim \quad \|\psi\|_b. \tag{2.27}$$

For the first term in the right hand side of (2.26), we have the estimate

$$\big|a(u-u_h,\varphi_\psi-\varphi_h)\big| \quad \lesssim \quad h^\gamma\|u-u_h\|_{a,h}. \tag{2.28}$$

From the standard error estimate theory of the nonconforming finite element method, the following estimate holds

$$\begin{aligned}
& \big|a_h(u-u_h,\varphi_\psi)-b(u-u_h,\psi)\big| \\
= & \Big|\sum_{K\in\mathcal{T}_h}\int_K \nabla(u-u_h)\nabla\varphi_\psi dK-\int_\Omega (u-u_h)\psi d\Omega\Big| \\
= & \Big|\sum_{K\in\mathcal{T}_h}\int_{\partial K}(u-u_h)\partial_\nu\varphi_\psi ds\Big| \\
\lesssim & \ h^\gamma\|u-u_h\|_{a,h}\|\varphi_\psi\|_{1+\gamma}.
\end{aligned} \tag{2.29}$$

From the orthogonal property (2.10), we have the estimate for the third term in the right hand side of (2.26)

$$\begin{aligned}
& \big|a_h(u,\varphi_\psi-\varphi_h)-b(f,\varphi_\psi-\varphi_h)\big| \\
\lesssim & \big|a_h(u,\Pi_h\varphi_\psi-\varphi_h)-b(f,\Pi_h\varphi_\psi-\varphi_h)\big|+\big|a_h(u,\varphi_\psi-\Pi_h\varphi_\psi)\big| \\
& +\big|b(f,\varphi_\psi-\Pi_h\varphi_\psi)\big| \\
= & \big|a_h(u-u_h,\Pi_h\varphi_\psi-\varphi_h)\big|+\big|a_h(u-u_h,\varphi_\psi-\Pi_h\varphi_\psi)\big| \\
& +\big|b(f,\varphi_\psi-\Pi_h\varphi_\psi)\big| \\
\lesssim & C\|u-u_h\|_{a,h}\big(\|\Pi_h\varphi_\psi-\varphi_h\|_{a,h}+\|\varphi_\psi-\Pi_h\varphi_\psi\|_{a,h}\big) \\
& +\|f\|_b\|\varphi_\psi-\Pi_h\varphi_\psi\|_b \\
\lesssim & h^\gamma\|\varphi_\psi\|_{1+\gamma}\|u-u_h\|_{a,h}+h^{1+\gamma}\|f\|_b\|\varphi_\psi\|_{1+\gamma} \\
\lesssim & h^\gamma\|\varphi_\psi\|_{1+\gamma}\|u-u_h\|_{a,h}+h^{1+\gamma}\|f\|_b.
\end{aligned} \tag{2.30}$$

Combining (2.26), (2.28), (2.29) and (2.30), we have

$$\begin{aligned}
\|u-u_h\|_b &= \sup_{0\neq\psi\in W}\frac{b(u-u_h,\psi)}{\|\psi\|_b} \\
&\lesssim h^\gamma\sup_{0\neq\psi\in W}\frac{\|\varphi_\psi\|_{1+\gamma}\|u-u_h\|_{a,h}}{\|\psi\|_b}+h^{1+\gamma}\|f\|_b \\
&\lesssim h^\gamma\|u-u_h\|_{a,h}+Ch^{1+\gamma}\|f\|_b.
\end{aligned} \tag{2.31}$$

The desired inequality (2.25) can be obtained by combining (2.24), (2.31) and $\|f\|_b=1$ and we complete the proof. $\square$

3 One correction step

In this section, we present a type of correction step to improve the accuracy of the current eigenvalue and eigenfunction approximations. This correction method contains solving some auxiliary source problems in the finer nonconforming finite element space and an eigenvalue problem on an coarse finite element space. For generality, we set the multiplicity of our desired eigenvalue is q. It means $\lambda_i=\cdots=\lambda_{i+q-1}$. We use $(\lambda_{i,h},u_{i,h}),\cdots,(\lambda_{i+q-1,h},u_{i+q-1,h})$ to denote the eigenpair approximations for the eigenvalues $\lambda_i=\cdots=\lambda_{i+q-1}$ and their corresponding eigenfunction space $M(\lambda)$. Let

$$M_h(\lambda_i)=\mathrm{span}\big\{u_{i,h},\cdots,u_{i+q-1,h}\big\}. \tag{3.1}$$

For two linear spaces A and B, we define

$$\widehat{\Theta}(A,B)=\sup_{w\in A,\|w\|_{a,h}=1}\inf_{v\in B}\|w-v\|_{a,h},\quad \widehat{\Phi}(A,B)=\sup_{w\in A,\|w\|_b=1}\inf_{v\in B}\|w-v\|_b.$$

We define the gaps between $M(\lambda_i)$ and $M_h(\lambda_i)$ in $\|\cdot\|_{a,h}$ as

$$\Theta(M(\lambda_i),M_h(\lambda_i))=\max\big\{\widehat{\Theta}(M(\lambda_i),M_h(\lambda_i)),\widehat{\Theta}(M_h(\lambda_i),M(\lambda_i))\big\}, \tag{3.2}$$

and in $\|\cdot\|_b$ as

$$\Phi(M(\lambda_i),M_h(\lambda_i))=\max\big\{\widehat{\Phi}(M(\lambda_i),M_h(\lambda_i)),\widehat{\Phi}(M_h(\lambda_i),M(\lambda_i))\big\}. \tag{3.3}$$

Assume we have obtained the eigenpair approximations $(\lambda_{j,h_k},u_{j,h_k})\in\mathcal{R}\times V_{h_k}$ for $j=i,\cdots,i+q-1$. Now we introduce a type of correction step to improve the accuracy of the current eigenpair approximation $\{(\lambda_{j,h_k},u_{j,h_k})\}_{j=i}^{i+q-1}$. Let $V_{h_{k+1}}\not\subset V$ be the nonconforming finite element space based on the finer mesh $\mathcal{T}_{h_{k+1}}$ which is produced by refining $\mathcal{T}_{h_k}$ in the regular way. In order to do the correction step, we also define the conforming linear finite element space W_H on the coarsest mesh $\mathcal{T}_H$ and I_H denote the corresponding Lagrange type interpolation operator. Now we define the following correction step.

Algorithm 3.1. *One Correction Step*

1. *For* $j=i,\cdots,i+q-1$ *Do*

   Define the following auxiliary source problem:

   Find $\tilde{u}_{j,h_{k+1}}\in V_{h_{k+1}}$ *such that*

$$a_{h_{k+1}}(\widetilde{u}_{j,h_{k+1}},v_{h_{k+1}}) \;=\; \lambda_{j,h_k}b(u_{j,h_k},v_{h_{k+1}}),\quad \forall v_{h_{k+1}}\in V_{h_{k+1}}. \tag{3.4}$$

   Solve this equation to obtain a new eigenfunction approximation $\widetilde{u}_{j,h_{k+1}}\in V_{h_{k+1}}$.

2. *Define a new finite element space* $V_{H,h_{k+1}}=W_H+\mathrm{span}\{\widetilde{u}_{i,h_{k+1}},\cdots,\widetilde{u}_{i+q-1,h_{k+1}}\}$ *and solve the following eigenvalue problem:*

   Find $(\lambda_{j,h_{k+1}},u_{j,h_{k+1}})\in\mathcal{R}\times V_{H,h_{k+1}}$ *such that* $b(u_{j,h_{k+1}},u_{j,h_{k+1}})=1$ *and*

$$a_h(u_{j,h_{k+1}},v_{H,h_{k+1}})=\lambda_{j,h_{k+1}}b(u_{j,h_{k+1}},v_{H,h_{k+1}}),\quad \forall v_{H,h_{k+1}}\in V_{H,h_{k+1}}. \tag{3.5}$$

Define $\{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}\}_{j=i}^{i+q-1} = Correction(W_H, \{\lambda_{j,h_k}, u_{j,h_k}\}_{j=i}^{i+q-1}, V_{h_{k+1}})$.

Lemma 3.1. *([13]) Assume $T_{H,h_{k+1}}$ denotes the discrete operator defined by (2.15) on the finite element space $V_{H,h_{k+1}} = W_H + \mathrm{span}\{\widetilde{u}_{i,h_{k+1}}, \cdots, \widetilde{u}_{i+q-1,h_{k+1}}\}$. We have the following error estimates*

$$\begin{aligned} \|(T - T_{H,h_{k+1}})\|_{a,h} &\lesssim H^\gamma, &(3.6)\\ \|(T - T_{H,h_{k+1}})|_{M(\lambda_i)}\|_{a,h} &\lesssim \sup_{w\in M(\lambda_i)} \inf_{v\in V_{H,h_{k+1}}} \|w - v\|_{a,h} + h_{k+1}^\gamma, &(3.7)\\ \|(T - T_{H,h_{k+1}})|_{M(\lambda_i)}\|_b &\lesssim H^\gamma \|(T - T_{H,h_{k+1}})|_{M(\lambda_i)}\|_{a,h}. &(3.8)\end{aligned}$$

Proof. For the simplicity of notation, we set $h := h_{k+1}$ and $\lambda := \lambda_i$ in this proof. First we prove (3.6). For any $f \in W$ and $\|f\|_b = 1$, by the standard nonconforming finite element error estimate theory, we have

$$\begin{aligned} \|(T - T_{H,h})f\|_{a,h} &\lesssim \inf_{v_{H,h}\in V_{H,h}} \|Tf - v_{H,h}\|_{a,h}\\ &\quad + \sup_{0\neq w_{H,h}\in V_{H,h}} \frac{|b(f, w_{H,h}) - a_h(Tf, w_{H,h})|}{\|w_{H,h}\|_{a,h}}\\ &\lesssim H^\gamma \|Tf\|_{1+\gamma} + \sup_{0\neq w_{H,h}\in V_{H,h}} \frac{\left|\sum_{K\in\mathcal{T}_h} \int_{\partial K} \partial_\nu (Tf) w_{H,h} ds\right|}{\|w_{H,h}\|_{a,h}}\\ &\lesssim \big(H^\gamma + h^\gamma\big)\|Tf\|_{1+\gamma}\\ &\lesssim \big(H^\gamma + h^\gamma\big)\|f\|_b. \end{aligned} \tag{3.9}$$

This is the desired result (3.6). Similarly, (3.7) can be obtained by (3.9) and the following estimates

$$\begin{aligned} &\|(T - T_{H,h})|_{M(\lambda)}\|_{a,h} = \sup_{f\in M(\lambda), \|f\|_{a,h}=1} \|(T - T_{H,h})f\|_{a,h}\\ \lesssim &\sup_{f\in M(\lambda), \|f\|_{a,h}=1} \Big(\inf_{v_{H,h}\in V_{H,h}} \|Tf - v_{H,h}\|_{a,h}\\ &\qquad + \sup_{0\neq w_{H,h}\in V_{H,h}} \frac{|b(f, w_{H,h}) - a_h(Tf, w_{H,h})|}{\|w_{H,h}\|_{a,h}}\Big), \end{aligned}$$

and

$$\sup_{f\in M(\lambda), \|f\|_{a,h}=1} \inf_{v_{H,h}\in V_{H,h}} \|Tf - v_{H,h}\|_{a,h} \lesssim \sup_{w\in M(\lambda)} \inf_{v\in V_{H,h_{k+1}}} \|w - v\|_{a,h}.$$

Now, let us prove (3.8). For any $f \in M(\lambda)$, let $u = Tf$ and $u_{H,h} = T_{H,h}f$. Similarly to (2.26), for any $\psi \in W$, we have

$$\|u - u_{H,h}\|_b = a_h(u - u_{H,h}, \varphi_\psi - \varphi_{H,h}) - [a_h(u - u_{H,h}, \varphi_\psi) - b(u - u_{H,h}, \psi)]$$

$$-[a_h(u,\varphi_\psi-\varphi_{H,h})-b(f,\varphi_\psi-\varphi_{H,h})], \tag{3.10}$$

where $\varphi_\psi=T\psi$, $\varphi_{H,h}=T_{H,h}\psi$ and

$$\|\varphi_\psi\|_{1+\gamma} \lesssim \|\psi\|_b. \tag{3.11}$$

For the first term in the right hand side of (3.10), we have the estimate from (3.6)

$$\big|a(u-u_{H,h},\varphi_\psi-\varphi_{H,h})\big| \lesssim H^\gamma\|u-u_{H,h}\|_{a,h}. \tag{3.12}$$

From the standard error estimate theory of the nonconforming finite element method, the following estimate holds

$$\begin{aligned}
&\big|a_h(u-u_{H,h},\varphi_\psi)-b(u-u_{H,h},\psi)\big|\\
=&\ \Big|\sum_{K\in\mathcal{T}_h}\int_K\nabla(u-u_{H,h})\nabla\varphi_\psi dK-\int_\Omega(u-u_{H,h})\psi d\Omega\Big|\\
=&\ \Big|\sum_{K\in\mathcal{T}_h}\int_{\partial K}(u-u_{H,h})\partial_\nu\varphi_\psi ds\Big|\\
\lesssim&\ h^\gamma\|u-u_{H,h}\|_{a,h}\|\varphi_\psi\|_{1+\gamma}.
\end{aligned} \tag{3.13}$$

From (3.6) and $\|\varphi_\psi-I_H\varphi_\psi\|_{a,h}\lesssim H^\gamma\|\varphi_\psi\|_{1+\gamma}$, we have the following estimates for the third term in the right hand side of (3.10)

$$\begin{aligned}
&\big|a_h(u,\varphi_\psi-\varphi_{H,h})-b(f,\varphi_\psi-\varphi_{H,h})\big|\\
\leq&\ \big|a_h(u,I_H\varphi_\psi-\varphi_{H,h})-b(f,I_H\varphi_\psi-\varphi_{H,h})\big|\\
&+\big|a_h(u,\varphi_\psi-I_H\varphi_\psi)-b(f,I_H\varphi_\psi-\varphi_\psi)\big|\\
=&\ \big|a_h(u-u_{H,h},I_H\varphi_\psi-\varphi_{H,h})\big|\\
\lesssim&\ \|u-u_{H,h}\|_{a,h}\|I_H\varphi_\psi-\varphi_{H,h}\|_{a,h}\\
\lesssim&\ H^\gamma\|\varphi_\psi\|_{1+\gamma}\|u-u_{H,h}\|_{a,h},
\end{aligned} \tag{3.14}$$

where we used $a_h(u,\varphi_\psi-I_H\varphi_\psi)-b(f,I_H\varphi_\psi-\varphi_\psi)=0$.

Combining (3.10), (3.11), (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
\|u-u_{H,h}\|_b &= \sup_{0\neq\psi\in W}\frac{b(u-u_{H,h},\psi)}{\|\psi\|_b}\\
&\lesssim H^\gamma\sup_{0\neq\psi\in W}\frac{\|\varphi_\psi\|_{1+\gamma}\|u-u_{H,h}\|_{a,h}}{\|\psi\|_b}\\
&\lesssim H^\gamma\|u-u_{H,h}\|_{a,h}.
\end{aligned} \tag{3.15}$$

The is the desired inequality (3.8) and we complete the proof. □

Theorem 3.1. *Assume the given eigenpairs $\{\lambda_{j,h_k}, u_{j,h_k}\}_{j=i}^{i+q-1}$ in One Correction Step 3.1 have the following error estimates*

$$
\begin{aligned}
\Theta(M(\lambda_i), M_{h_k}(\lambda_i)) &\lesssim \varepsilon_{h_k}(\lambda_i), && (3.16)\\
\Phi(M(\lambda_i), M_{h_k}(\lambda_i)) &\lesssim H^\gamma \Theta(M(\lambda_i), M_{h_k}(\lambda_i)), && (3.17)\\
|\lambda_i - \lambda_{j,h_k}| &\lesssim H^\gamma \Theta(M(\lambda_i), M_{h_k}(\lambda_i)), && (3.18)
\end{aligned}
$$

for $j = i, \cdots, i+q-1$. Then after one correction step, the resultant eigenpair approximation $\{\lambda_{h_{k+1}}, u_{h_{k+1}}\}_{j=i}^{i+q-1}$ have the following error estimates

$$
\begin{aligned}
\Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) &\lesssim \varepsilon_{h_{k+1}}(\lambda_i), && (3.19)\\
\Phi(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) &\lesssim H^\gamma \Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)), && (3.20)\\
|\lambda_i - \lambda_{j,h_{k+1}}| &\lesssim H^\gamma \Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)), && (3.21)
\end{aligned}
$$

where $\varepsilon_{h_{k+1}}(\lambda) := H^\gamma \varepsilon_{h_k}(\lambda) + h_{k+1}^\gamma$ and $j = i, \cdots, i+q-1$.

Proof. From (3.16) and (3.17), we know there exist an orthogonal basis $\{u_j\}_{j=i}^{i+q-1}$ of $M(\lambda_i)$ such that

$$
\begin{aligned}
\|u_j - u_{j,h_k}\|_{a,h} &\lesssim \varepsilon_{h_k}(\lambda_i), && (3.22)\\
\|u_j - u_{j,h_k}\|_b &\lesssim H^\gamma \|u_j - u_{j,h_k}\|_{a,h}, && (3.23)\\
|\lambda_i - \lambda_{j,h_k}| &\lesssim H^\gamma \|u_j - u_{j,h_k}\|_{a,h}. && (3.24)
\end{aligned}
$$

From problems (2.1) and (3.4), and (2.12), (3.22), (3.23), and (3.24), the following estimate holds for $j = i, \cdots, i+q-1$

$$
\begin{aligned}
& \|\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j\|_{a,h}^2 \lesssim a(\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j, \widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j)\\
= \;& b(\lambda_{j,h_k} u_{j,h_k} - \lambda_i u_j, \widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j) - \sum_{\ell \in \mathcal{E}_h^i} \int_\ell \partial_\nu u(\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j) ds\\
\lesssim \;& \big(\|\lambda_{j,h_k} u_{j,h_k} - \lambda_i u_j\|_b + h_{k+1}^\gamma \|u_j\|_{1+\gamma}\big) \|\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j\|_{a,h}\\
\lesssim \;& \big(|\lambda_{j,h_k} - \lambda_i| \|u_{j,h_k}\|_b + \lambda_i \|u_j\|_{1+\gamma} \|u_{j,h_k} - u_j\|_b + h_{k+1}^\gamma \|u_j\|_{1+\gamma}\big)\\
& \qquad \times \|\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j\|_{a,h}\\
\lesssim \;& \big(H^\gamma \varepsilon_{h_k}(\lambda_i) + h_{k+1}^\gamma \|u_j\|_{1+\gamma}\big) \|\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j\|_{a,h}.
\end{aligned}
$$

Then we have

$$
\|\widetilde{u}_{j,h_{k+1}} - \Pi_{h_{k+1}} u_j\|_{a,h} \lesssim H^\gamma \varepsilon_{h_k}(\lambda_i) + h_{k+1}^\gamma, \quad j = i, \cdots, i+q-1. \tag{3.25}
$$

Combining (3.25) and the error estimate of the finite element interpolation

$$
\|u_j - \Pi_{h_{k+1}} u_j\|_{a,h} \lesssim h_{k+1}^\gamma,
$$

we have

$$\|\widetilde{u}_{j,h_{k+1}} - u_j\|_{a,h} \lesssim H^\gamma \varepsilon_{h_k}(\lambda_i) + h_{k+1}^\gamma, \quad j = i, \cdots, i+q-1. \tag{3.26}$$

Now we come to estimate the error of the eigenpair solutions $\{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}\}_{j=i}^{i+q-1}$ of (3.5). Based on Lemma 2.2, (3.6)-(3.8) and (3.26), the following estimates hold

$$\begin{aligned}
\Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) &\lesssim \|(T - T_{H,h_{k+1}})|_{M(\lambda_i)}\|_{a,h} + \|(T - T_{H,h_{k+1}})|_{M(\lambda_i)}\|_b \\
&\lesssim \sup_{w \in M(\lambda_i)} \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}} \|w - v_{H,h_{k+1}}\|_{a,h} + h_{k+1}^\gamma \\
&\lesssim H^\gamma \varepsilon_{h_k}(\lambda_i) + h_{k+1}^\gamma,
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
\Phi(M(\lambda_i), M_{h_{k+1}}(\lambda_i)) &\lesssim \|(T - T_{H,h_{k+1}})|_{M(\lambda_i)}\|_b \\
&\lesssim H^\gamma \Theta(M(\lambda_i), M_{h_{k+1}}(\lambda_i)).
\end{aligned} \tag{3.28}$$

From (3.26), (3.27), and (3.28), we can obtain (3.19) and (3.20). The estimate (3.21) can be derived by Lemma 2.2 and (3.20). ☐

4 Multi-level correction scheme

In this section, we introduce a type of multi-level correction scheme based on the *One Correction Step* 3.1. This type of correction method can improve the accuracy after each correction step which is different from the two-grid methods in [24, 10, 27]. As described in Section 3, we are willing to obtain the approximations of the eigenpairs corresponding to the eigenvalue λ_i which has multiplicity of q.

Algorithm 4.1. *Multi-level Correction Scheme*

1. *Construct a coarse nonconforming finite element space V_{h_1} on $\mathcal{T}_{h_1}$ and solve the following eigenvalue problem:*

   Find $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$ such that $b(u_{h_1}, u_{h_1}) = 1$ and

   $$a_{h_1}(u_{h_1}, v_{h_1}) = \lambda_{h_1} b(u_{h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}. \tag{4.1}$$

   Choose q eigenpairs $\{\lambda_{j,h_1}, u_{j,h_j}\}_{j=i}^{i+q-1}$ which approximate the desired eigenvalue λ_i and its eigenspace to do the following correction steps.

2. *Construct a series of finer finite element spaces $V_{h_2}, \cdots, V_{h_n}$ on the sequence of nested meshes $\mathcal{T}_{h_2}, \cdots, \mathcal{T}_{h_n}$.*

3. *Do* $k = 1, \cdots, n-1$

   Obtain new eigenpair approximations $\{\lambda_{h_{k+1}}, u_{h_{k+1}}\}_{j=i}^{i+q-1} \in \mathcal{R}\times V_{h_{k+1}}$ *by a correction step*

$$\{\lambda_{h_{k+1}}, u_{h_{k+1}}\}_{j=i}^{i+q-1} = Correction(W_H, \{\lambda_{h_k}, u_{h_k}\}_{j=i}^{i+q-1}, V_{h_{k+1}}). \tag{4.2}$$

   end Do

Finally, we obtain eigenpair approximations $(\lambda_{j,h_n}, u_{j,h_n}) \in \mathcal{R}\times V_{h_n}$ *for* $j = i, \cdots, i+q-1$.

Theorem 4.1. *The resultant eigenpair approximations* $\{\lambda_{j,h_n}, u_{j,h_n}\}_{j=i}^{i+q-1}$ *obtained by Algorithm 4.1 have the following error estimates*

$$\Theta(M(\lambda_i), M_{h_n}(\lambda_i)) \lesssim \sum_{k=1}^{n} h_k^{\gamma} H^{\gamma(n-k)}, \tag{4.3}$$

$$\Phi(M(\lambda_i), M_{h_n}(\lambda_i)) \lesssim \sum_{k=1}^{n} h_k^{\gamma} H^{\gamma(n-k+1)}, \tag{4.4}$$

$$|\lambda_{j,h_n} - \lambda_i| \lesssim \sum_{k=1}^{n} h_k^{2\gamma} H^{2\gamma(n-k)} + h_n^{2\gamma}, \tag{4.5}$$

where $j = i, \cdots, i+q-1$.

Proof. From Theorem 3.1, we have

$$\varepsilon_{h_{k+1}}(\lambda_i) \lesssim H^{\gamma}\varepsilon_{h_k}(\lambda_i) + h_{k+1}^{\gamma}, \quad \text{for } 1 \le k \le n-1. \tag{4.6}$$

Then by recursive relation, we can obtain

$$\begin{aligned}
\varepsilon_{h_n}(\lambda_i) &\lesssim H^{\gamma}\varepsilon_{h_{n-1}}(\lambda_i) + h_n^{\gamma} \\
&\lesssim H^{2\gamma}\varepsilon_{h_{n-2}}(\lambda_i) + H^{\gamma}\varepsilon_{h_{n-1}}(\lambda_i) + h_n^{\gamma} \\
&\lesssim \sum_{k=1}^{n} H^{\gamma(n-k)} h_k^{\gamma}.
\end{aligned} \tag{4.7}$$

This is the estimate (4.3) and we can obtain (4.4) similarly by Theorem 3.1. From (2.23) and the property of the conforming linear interpolation I_{h_n}, we have

$$\begin{aligned}
&|\lambda_{j,h_n} - \lambda_i| = \Big|\frac{a_h(u_{j,h_n}, u_{j,h_n})}{b(u_{j,h_n}, u_{j,h_n})} - \lambda_i\Big| \\
\lesssim\ & \|u_j - u_{j,h_n}\|_{a,h}^2 + \big|E_h(u_j, u_{j,h_n})\big| \\
\lesssim\ & \|u_j - u_{j,h_n}\|_{a,h}^2 + \big|E_h(u_j, u_{j,h_n} - I_{h_n}u_j)\big| \\
\lesssim\ & \|u_j - u_{j,h_n}\|_{a,h}^2 + h_n^{\gamma}\|u_j\|_{1+\gamma}\|u_{j,h_n} - I_{h_n}u_j\|_{a,h} \\
\lesssim\ & \|u_j - u_{j,h_n}\|_{a,h}^2 + h_n^{\gamma}\|u_j\|_{1+\gamma}\big(\|u_{j,h_n} - u\|_{a,h} + \|u_j - I_{h_n}u_j\|_{a,h}\big) \\
\lesssim\ & \|u_j - u_{j,h_n}\|_{a,h}^2 + h_n^{2\gamma}\|u_j\|_{1+\gamma}^2,
\end{aligned} \tag{4.8}$$

where we used $E_h(u_j, I_{h_n}u_j) = 0$. Then the desired estimate (4.5) can be derived by (4.3), (4.4) and (4.8). $\square$

5 lower-bound analysis

In the numerical implementation, we find the multi-level correction method can also obtain the lower bounds of the exact eigenvalues (see Section 6). This pheonmena comes from that the eigenfunction approximations by the multi-level correction and the one by direct eigenvalue solving have some type of "superclose" property. In this section, we give the lower-bound analysis of the multi-level correction method. For the simplicity, we only consider the simple eigenvalue cases and the results also hold for the multiple eigenvalue cases.

Let $(\bar{\lambda}_h, \bar{u}_h)$ denote the eigenpair approximation by the direct eigenvalue solving which is defined as follows:

Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathcal{R} \times V_h$ such that $b(\bar{u}_h, \bar{u}_h) = 1$ and

$$a_h(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in V_h. \tag{5.1}$$

Lemma 5.1. *For the eigenvalue approximations λ_h and $\bar{\lambda}_h$, the following expansion holds*

$$\lambda_h - \bar{\lambda}_h = \frac{a_h(\bar{u}_h - u_h, \bar{u}_h - u_h) - \bar{\lambda}_h b(\bar{u}_h - u_h, \bar{u}_h - u_h)}{b(u_h, u_h)}. \tag{5.2}$$

Proof. First from (2.5), (5.1) and $u_h \in V_h$, the following equalities hold

$$\begin{aligned}
& a_h(\bar{u}_h - u_h, \bar{u}_h - u_h) - \bar{\lambda}_h b(\bar{u}_h - u_h, \bar{u}_h - u_h) \\
= \; & a_h(\bar{u}_h, \bar{u}_h) + a_h(u_h, u_h) - 2a_h(\bar{u}_h, u_h) - \bar{\lambda}_h b(\bar{u}_h, \bar{u}_h) \\
& -\bar{\lambda}_h b(u_h, u_h) + 2\bar{\lambda}_h b(\bar{u}_h, u_h) \\
= \; & \bar{\lambda}_h b(\bar{u}_h, \bar{u}_h) + a_h(u_h, u_h) - 2\bar{\lambda}_h b(\bar{u}_h, u_h) - \bar{\lambda}_h b(\bar{u}_h, \bar{u}_h) \\
& -\bar{\lambda}_h b(u_h, u_h) + 2\bar{\lambda}_h b(\bar{u}_h, u_h) \\
= \; & a_h(u_h, u_h) - \bar{\lambda}_h b(u_h, u_h).
\end{aligned} \tag{5.3}$$

From (2.5) and (5.3), we have

$$\begin{aligned}
\lambda_h - \bar{\lambda}_h &= \frac{a_h(u_h, u_h) - \bar{\lambda}_h b(u_h, u_h)}{b(u_h, u_h)} \\
&= \frac{a_h(\bar{u}_h - u_h, \bar{u}_h - u_h) - \bar{\lambda}_h b(\bar{u}_h - u_h, \bar{u}_h - u_h)}{b(u_h, u_h)}.
\end{aligned} \tag{5.4}$$

This is the desired result (5.2) and we complete the proof. □

Theorem 5.1. *Let $(\bar{\lambda}_{h_n}, \bar{u}_{h_n})$ denote the eigenpair approximation of (5.1). Then we have the following superclose properties*

$$\|u_{h_n} - \bar{u}_{h_n}\|_{a,\Omega} \lesssim \sum_{k=2}^{n} H^{\gamma(n-k)} (\|\bar{u}_{h_{k-1}} - \bar{u}_{h_k}\|_b + |\bar{\lambda}_{h_{k-1}} - \bar{\lambda}_{h_k}|), \tag{5.5}$$

$$\|u_{h_n} - \bar{u}_{h_n}\|_b \lesssim H^\gamma \|u_{h_n} - \bar{u}_{h_n}\|_{a,\Omega}, \tag{5.6}$$

$$|\lambda_h - \bar{\lambda}_h| \lesssim \|u_{h_n} - \bar{u}_{h_n}\|_{a,\Omega}^2. \tag{5.7}$$

Assume the series of the meshes satisfies the following estimates

$$\sum_{k=1}^{n} H^{\gamma(n-k)} h_k^\gamma \lesssim h_n, \tag{5.8}$$

and the eigenvalue approximation $\bar{\lambda}_h$ *has the lower-bound property:* $\bar{\lambda}_h < \lambda$. *Then the eigenvalue* λ_h *also has the lower-bound property:*

$$\lambda_h < \lambda. \tag{5.9}$$

Proof. We prove (5.5)-(5.7) by induction. Since we solve the eigenvalue problem directly by the nonconforming element, the following equalities hold

$$u_{h_1} = \bar{u}_{h_1}, \quad \lambda_{h_1} = \bar{\lambda}_{h_1}.$$

So (5.5)-(5.7) holds for $n = 1$. Assume the results (5.5)-(5.7) hold for $n = k$. Now we come to prove (5.5)-(5.7) also hold for $n = k+1$. From (3.4) and (5.1), the following estimates hold

$$\begin{aligned}
& a_h(\widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) \\
= \ & \lambda_{h_k} b(u_{h_k}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) - \bar{\lambda}_{h_{k+1}} b(\bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) \\
= \ & \lambda_{h_k} b(u_{h_k} - \bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) + (\lambda_{h_k} - \bar{\lambda}_{h_{k+1}}) b(\bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) \\
= \ & \lambda_{h_k} b(u_{h_k} - \bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) + \lambda_{h_k} b(\bar{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) \\
& + (\lambda_{h_k} - \bar{\lambda}_{h_k}) b(\bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) + (\bar{\lambda}_{h_k} - \bar{\lambda}_{h_{k+1}}) b(\bar{u}_{h_{k+1}}, \widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}) \\
\lesssim \ & (H^\gamma \|u_{h_k} - \bar{u}_{h_k}\|_{a,\Omega} + \|\bar{u}_{h_k} - \bar{u}_{h_{k+1}}\|_b + |\bar{\lambda}_{h_k} - \bar{\lambda}_{h_{k+1}}|) \|\widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}\|_{a,h}.
\end{aligned}$$

This means we have

$$\begin{aligned}
\|\widetilde{u}_{h_{k+1}} - \bar{u}_{h_{k+1}}\|_{a,h} & \lesssim H^\gamma \|u_{h_k} - \bar{u}_{h_k}\|_{a,\Omega} + \|\bar{u}_{h_k} - \bar{u}_{h_{k+1}}\|_b + |\bar{\lambda}_{h_k} - \bar{\lambda}_{h_{k+1}}| \\
& \lesssim \sum_{m=2}^{k+1} H^{k-j} (\|\bar{u}_{h_{m-1}} - \bar{u}_{h_m}\|_b + |\bar{\lambda}_{h_{m-1}} - \bar{\lambda}_{h_m}|).
\end{aligned} \tag{5.10}$$

Since $V_{H,h_{k+1}} \subset V_{h_{k+1}}$, we can regard the discrete eigenvalue problem (3.5) as a conforming finite element discretization of the discrete eigenvalue problem (5.1) in the space $V_{h_{k+1}}$. So we can use the standard error estimate results of the conforming finite element method for the eigenvalue problem. So the following estimates hold

$$\|u_{h_{k+1}} - \bar{u}_{h_{k+1}}\|_{a,h} \lesssim \inf_{v_{H,h_{k+1}} \in V_{H,h_{k+1}}} \|\bar{u}_{h_{k+a}} - v_{H,h_{k+1}}\|_{a,h} \lesssim \|\bar{u}_{h_{k+1}} - \widetilde{u}_{h_{k+1}}\|_{a,h}$$

$$
\begin{aligned}
&\lesssim H^{\gamma}\|u_{h_k}-\bar{u}_{h_k}\|_{a,\Omega}+\|\bar{u}_{h_k}-\bar{u}_{h_{k+1}}\|_b+|\bar{\lambda}_{h_k}-\bar{\lambda}_{h_{k+1}}| \\
&\lesssim \sum_{m=2}^{k+1} H^{k+1-m}(\|\bar{u}_{h_{m-1}}-\bar{u}_{h_m}\|_b+|\bar{\lambda}_{h_{m-1}}-\bar{\lambda}_{h_m}|), \qquad (5.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_{h_{k+1}}-\bar{u}_{h_{k+1}}\|_b \\
\lesssim\ & \sup_{f\in W,\|f\|_b=1}\ \inf_{v_{H,h_{k+1}}\in V_{H,h_{k+1}}}\|T_{h_{k+1}}f-v_{H,h_{k+1}}\|_{a,h}\|u_{h_{k+1}}-\bar{u}_{h_{k+1}}\|_{a,h} \\
\lesssim\ & H^{\gamma}\|\bar{u}_{h_{k+1}}-\widetilde{u}_{h_{k+1}}\|_{a,h}. \qquad (5.12)
\end{aligned}
$$

From Lemma 5.1, we have the following estimate for the eigenvalue approximations

$$
|\lambda_{h_{k+1}}-\bar{\lambda}_{h_{k+1}}| \ \lesssim\ \|\bar{u}_{h_{k+1}}-\widetilde{u}_{h_{k+1}}\|_{a,h}^2. \qquad (5.13)
$$

These three estimates (5.11)-(5.13) means the results (5.5)-(5.7) also hold for $n=k+1$. So we obtain the results (5.5)-(5.7) hold for any integer n.

The assumption of the eigenfunction by the multi-correction method owning the optimal error estimate (5.8) leads to that $\|u_h-\bar{u}_h\|_{a,h}$ is higher order term corresponding to $\|u-\bar{u}_h\|_{a,h}$. So λ_h and $\bar{\lambda}_h$ have the same lower-bound property and the desired result (5.9) has been obtained. □

Remark 5.1. *From (2.21) and (5.5)-(5.7), we have the following estimates*

$$
\begin{aligned}
\|u_{h_n}-\bar{u}_{h_n}\|_{a,\Omega} &\lesssim \sum_{k=2}^{n} H^{\gamma(n-k)}h_{k-1}^{2\gamma}, \qquad (5.14)\\
\|u_{h_n}-\bar{u}_{h_n}\|_{a,\Omega} &\lesssim \sum_{k=2}^{n} H^{\gamma(n-k+1)}h_{k-1}^{2\gamma}, \qquad (5.15)\\
|\lambda_h-\bar{\lambda}_h| &\lesssim \sum_{k=2}^{n} H^{2\gamma(n-k)}h_{k-1}^{4\gamma}. \qquad (5.16)
\end{aligned}
$$

Compared with (4.3)-(4.5), (5.14)-(5.16) are higher-order terms. Always, we call this phenomena as "superclose" property (see [16]).

6 Numerical results

In this section, we give two numerical examples to illustrate the efficiency of the multi-level correction algorithm proposed in this paper. For simplicity, we only give the numerical results by the CR element.

6.1 Model eigenvalue problem

In this example, we solve the model eigenvalue problem (1.1) on the unit square $\Omega = (0,1)\times(0,1)$ with $\gamma = 1$. Here, we adopt the meshes which are produced by regular refinement from the initial mesh generated by Delaunay method to investigate the convergence behaviors. We checked the numerical results for two regular refinement ways with $h_{k+1} = h_k/2$ and $h_{k+1} = h_k/4$ $(k = 1,\cdots,n-1)$, respectively. Furthermore, we choose $\mathcal{T}_H = \mathcal{T}_{h_1}$ with $H = 1/4$. From Theorem 4.1, we have the following error estimates for these two refinement ways

$$\|u_{h_n} - u\|_{a,h} \lesssim h_n,\quad \|u_{h_n} - u\|_b \lesssim Hh_n,\quad |\lambda_{h_n} - \lambda| \lesssim h_n^2,$$

which means the multi-levle correction method also obtained the optimal convergence order.

Figure 1 shows the initial mesh. Figure 2 and 3 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 2\pi^2$ and the corresponding eigenfunction. Figure 3 gives the numerical results for the first 6 eigenvalues: $2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2$ and $10\pi^2$.

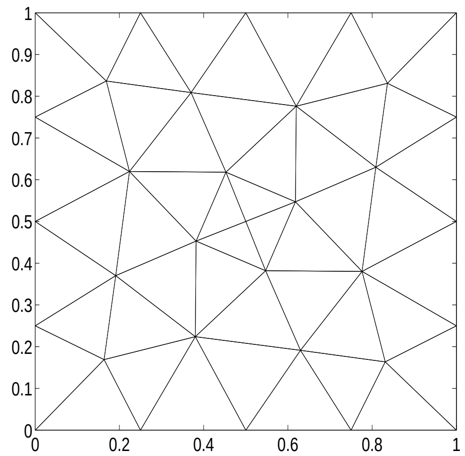

Figure 1: Initial mesh for Example 6.1

In order to show the efficiency more clearly, we compare the results by the multi-level correction method with those obtained by the direct eigenvalue solving. From Figures 2 and 3, the multi-level correction method can obtain almost the same results as the direct eigenvalue solving method but with smaller computational work. Furthermore, from Figures 2 and 3, the multi-level correction method can also obtain the lower-bound approximations of the eigenvalues and have the superclose property.

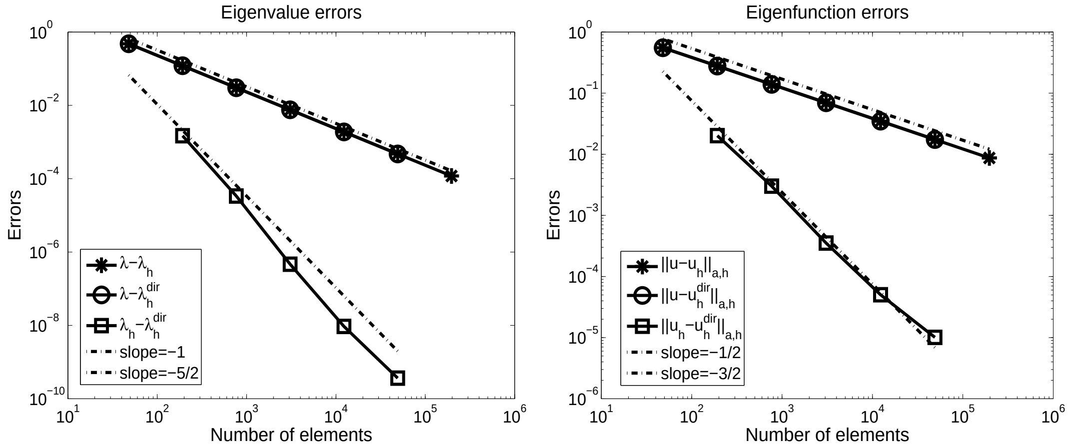

Figure 2: The errors for the eigenpair approximations by the multi-level correction algorithm for the first eigenvalue $2\pi^2$ and the corresponding eigenfunction with $h_{k+1} = h_k/2$, where (λ_h, u_h) is produced by the multi-level correction method and $(\lambda_h^{\rm dir}, u_h^{\rm dir})$ by the direct eigenvalue solving method

6.2 Eigenvalue problem on L-shape domain

In the second example, we consider the model eigenvalue problem on the L-shape domain $\Omega = (-1,1)\times(-1,1)\backslash[0,1)\times(-1,0]$. Since Ω has a reentrant corner, eigenfunctions with singularities are expected. The convergence order for eigenvalue approximation is less than 2 by the linear finite element method which is the order predicted by the theory for regular eigenfunctions.

We investigate the numerical results for the first eigenvalue. Since the exact eigenvalue is not known, we choose an adequately accurate approximation $\lambda = 9.6397238440219$ as the exact first eigenvalue for our numerical tests. We give the numerical results of the multi-level correction in which the sequence of meshes $\mathcal{T}_{h_1}$, $\mathcal{T}_{h_2}, \cdots, \mathcal{T}_{h_n}$ is produced by the adaptive refinement with the a posteriori error estimator given by the ZZ recovery method (see [29]). Also we choose $\mathcal{T}_H = \mathcal{T}_{h_1}$ with $H = 1/4$.

Figure 5 shows the initial mesh and the one after 12 adaptive iterations. Figure 6 gives the corresponding numerical results for the adaptive iterations. In order to show the accuracy of multi-level correction method more clearly, we compare the results with those obtained by the direct adaptive finite element method.

From Figure 6, we can find the multi-level correction method can also work on the adaptive family of meshes and obtain the optimal accuracy. Furthermore, the initial mesh is nothing to do with the finest one which is different from the two-gird method [24, 27]. We can also find the multi-level correction method can obtain the lower-bounds of the

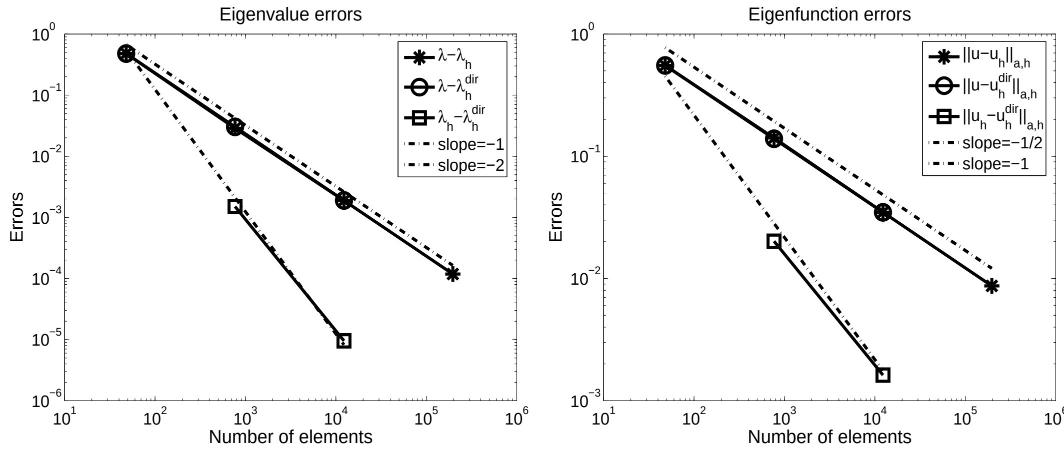

Figure 3: The errors for the eigenpair approximations by the multi-level correction algorithm for the first eigenvalue $2\pi^2$ and the corresponding eigenfunction with $h_{k+1} = h_k/4$, where (λ_h, u_h) is produced by the multi-level correction method and $(\lambda_h^{\rm dir}, u_h^{\rm dir})$ by the direct eigenvalue solving method

eigenvalues and have the superclose property.

7 Concluding remarks

In this paper, we give a type of multi-level correction scheme to solve the Laplace eigenalue problem by the nonconforming finite element method. In this scheme, the eigenvalue problem solving can be transformed to a series of boundary value problem solving and the eigenvalue problem solving in the very coarse space. We also derive a type of superclose property of the eigenpair approximations and the lower-bound results of the eigenvalue approximations by the multi-level correction algorithm.

Furthermore, our multi-level correction scheme can be coupled with the multigrid method to construct a type of multigrid and parallel method for eigenvalue problems by the nonconforming finite element method (see Example 5.1). It can also be combined with the adaptive refinement technique for the singular eigenfunction cases (see Example 5.2).

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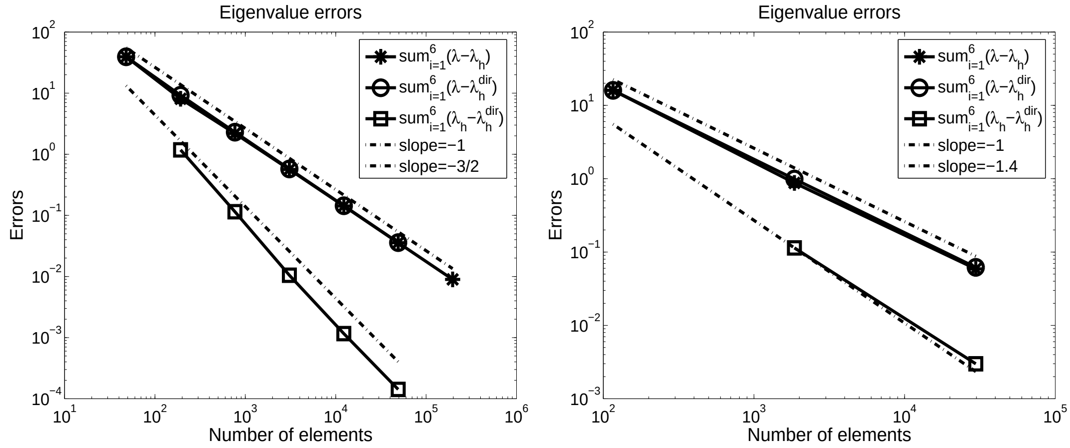

Figure 4: The errors for the eigenvalue approximations by multi-level correction algorithm for the first 6 eigenvalues with $h_{k+1} = h_k/2$, $H = 1/4$ (left) and $h_{k+1} = h_k/4$, $H = 1/6$ (right), where λ_h is produced by the multi-level correction method and $\lambda_h^{\rm dir}$ by the direct eigenvalue solving method

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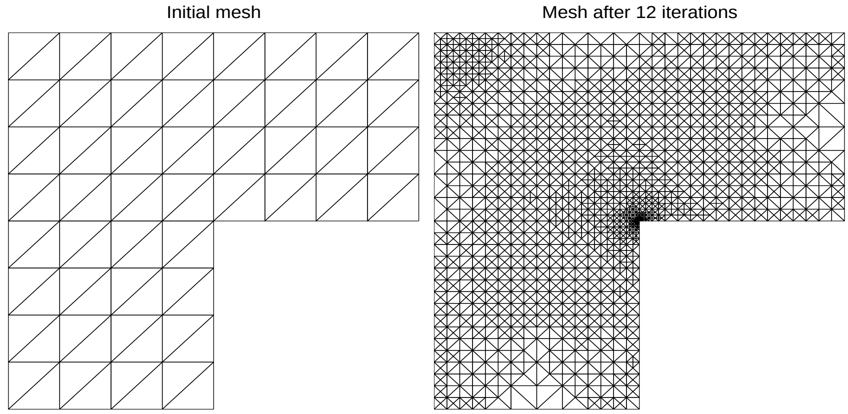

Figure 5: The initial mesh and the one after 12 adaptive iterations for Example 2

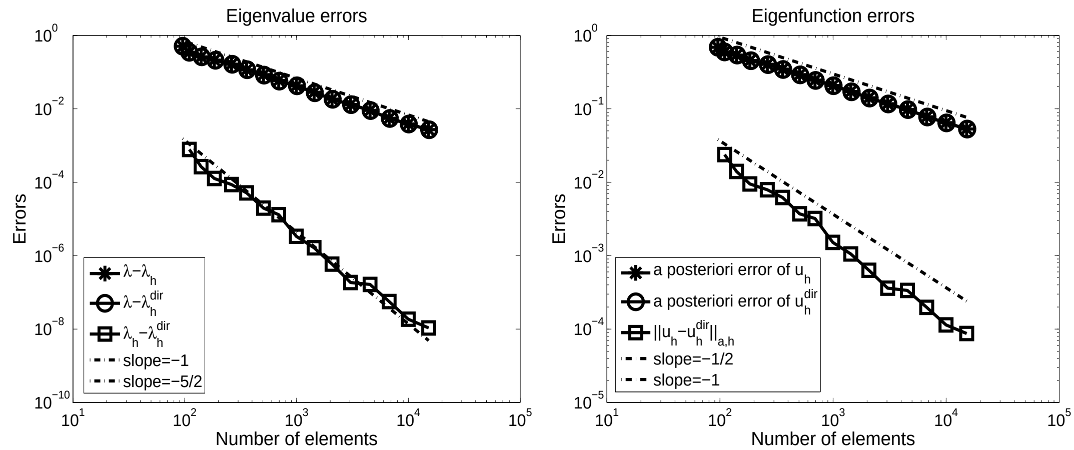

Figure 6: The errors of the smallest eigenvalue approximations and the a posteriori errors of the associated eigenfunction approximations by multi-level correction method and direct adaptive finite element method for Example 2

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