Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 44, 2161 - 2170

A Type of Ricci Flow on Riemannian Manifolds[1]

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Abstract

In this note we define multitime Ricci flow, studied it on Einstein, quasi Einstein and Ricci-recurrent quasi Einstein manifolds. We have also obtained some inequalities regarding scalar curvature on closed manifold under multitime Ricci flow applying weak maximum and minimum principle.

Mathematics Subject Classification: 53C44, 53C21, 58J35, 78A25, 37J20

Keywords: Multitime Ricci flow; Weak maximum principle; Ricci-recurrent space

1 Introduction

Multitime geometric flow was first introduced by C.Udriste. Actually geometric dynamics techniques are used by them to study problems of an evolution in electro-magnetism and market-equilibrium [12]. Two-time, three-time geometric dynamics induced by two and three magnetic fields respectively have been studied by them.
On the other hand Ricci flow is a parabolic PDE which is a mean of evolving a smooth Riemannian metric g on a smooth closed manifold M satisfying

$$\frac{\partial g}{\partial t} = -2\ Ric(g) \tag{1.1}$$

where $Ric(g)$ is the Ricci curvature tensor [11].

[1]Supported by the grant of CSIR Sanction No 09/096(0612)/2010-EMR-I.

It is introduced by R.S.Hamilton [4] in the year 1982 and proved its existence. Later much simpler proof has been given by DeTurck [11]. This concept was developed to answer Thurston's geometric conjecture which says that each closed three-manifold admits a geometric decomposition.
Hamilton himself and many other researchers like Cao [1], Yau [13], B.Chow, P.Lu, L.Ni [3], G.Perelman [9], [10], J.W.Morgan and G.Tian [7] developed the theory of Ricci flow.
We know celestial bodies are always changing their configuration due to motion of the moving particles. So at certain place at time t^1 and at time t^2, the Ricci curvature may be different.
So we define multitime Ricci flow in next section and study the evolution of scalar curvature on closed manifold and the behavior of this flow on Einstein and quasi Einstein manifolds.

2 Multitime Ricci flow

We define multitime Ricci flow as follows.

Definition 2.1. Multitime Ricci flow is a mean of processing the fundamental metric tensor g^α by allowing to evolve under the PDE

$$\frac{\partial g^\alpha}{\partial t^\alpha} = -2\ Ric^\alpha(g), \qquad \alpha = 1, 2, ..., p \tag{2.1}$$

where $Ric^\alpha(g)$ is the Ricci curvature at some point at time t^α.
In Einstein manifold when

$$Ric^\alpha(g) \ = \ \lambda\ g^\alpha, \qquad \lambda : M \to \Re(\text{set of reals}) \tag{2.2}$$

we have the following two cases.
<u>*Case1 :*</u> When at $t^\alpha = 0$, $g^\alpha = g_0$ for all α, then (2.1) has solution

$$g^\alpha = (1 - 2\lambda t^\alpha)g_0 \tag{2.3}$$

<u>*Case2 :*</u> When at $t^\alpha = 0$, $g^\alpha = g_0^\alpha$, then we have from (2.1)

$$g^\alpha = -2\lambda \int g_0^\alpha\ dt^\alpha + c^\alpha \tag{2.4}$$

where c^α depends on t^α. So we state

Theorem 2.1. *In an Einstein manifold the evolution of multitime Ricci flow follows (2.3) and (2.4) according as g^α is constant and varies with respect to α at initial time.*

Example 2.1.(*Example of multitime Ricci flow*) For unit sphere (S^n, g^α), we have $Ric^\alpha(g) = (n-1)\ g^\alpha$

and so (2.1) has solution

$$g^\alpha = (1 - 2(n-1)t^\alpha)g_0 \tag{2.5}$$

when at $t^\alpha = 0$, $g^\alpha = g_0$ for all α, i.e. initial metric does not depend on α and

$$g^\alpha = -2(n-1)\int g_0^\alpha\ dt^\alpha + c^\alpha \tag{2.6}$$

when at $t^\alpha = 0$, $g^\alpha = g_0^\alpha$, i.e. initial metric and constant of integration c^α depends on α.

Its physical interpretation can be seen for comets, dying star, whose structural deformation depends on time factor. So the constant of integration (i.e. c^α) depends on chosen time component.

Again a non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be a quasi Einstein manifold [2] if its Ricci tensor *Ric* is not identically zero and satisfies the condition

$$Ric = ag + bw \otimes w, \qquad b \neq 0 \tag{2.7}$$

where $w = g(., \rho)$ is a non-zero 1-form, ρ being a unit vector field and a, b are scalars, called associated scalars. w is called associated 1-form and ρ is called the generator of the manifold.

In quasi Einstein manifold, from (2.1) and (2.7) we have the following two cases.

Case1 : When at $t^\alpha = 0$, $g^\alpha = g_0$, $g^\alpha(X, \rho) = h_{01}$, $g^\alpha(Y, \rho) = h_{02}$ for all α, then

$$g^\alpha = (a't^\alpha + 1)g_0 + b't^\alpha h_{01}h_{02} \tag{2.8}$$

Case2 : When at $t^\alpha = 0$, $g^\alpha = g_0^\alpha$, $g^\alpha(X, \rho) = h_{01}^\alpha$, $g^\alpha(Y, \rho) = h_{02}^\alpha$, then from (2.1) we have the solution

$$g^\alpha = a'\int g_0^\alpha dt^\alpha + b'\int h_{01}^\alpha h_{02}^\alpha dt^\alpha + c_1^\alpha \tag{2.9}$$

where c_1^α depends on α. For both the cases $a' = -2a, b' = -2b$.

Hence we can state

Theorem 2.2. *In a quasi Einstein manifold the evolution of multitime Ricci flow follows (2.8) and (2.9) according as g^α is constant and varies with respect to α at initial time.*

A Riemannian manifold is said to be Ricci-recurrent [8] if the Ricci tensor is non-zero and satisfies the condition

$$\nabla_X(Ric(g)) = A(X)Ric(g)$$

where A is a non-zero 1-form.
In a Ricci-recurrent quasi-Einstein manifold we have

$$a + b = 0 \tag{2.10}$$

So (2.7) takes the form

$$Ric^\alpha = a(g^\alpha - w^\alpha \otimes w^\alpha)$$

and hence we have the following two solutions.

$$g^\alpha = a'(t^\alpha g_0 + g_0 - h_{01}h_{02}) \tag{2.11}$$

for the first case and

$$g^\alpha = a'(\int g_0^\alpha dt^\alpha - \int h_{01}^\alpha h_{02}^\alpha dt^\alpha) + c_1^\alpha \tag{2.12}$$

corresponding to the second case. This leads to

Theorem 2.3. *In a Ricci-recurrent quasi Einstein manifold the evolution of multitime Ricci flow follows (2.11) and (2.12) according as g^α is constant and varies with respect to α at initial time.*

3 Evolution of scalar curvature on closed manifold with respect to multitime Ricci flow

From [11] we know the weak maximum principle and its corresponding application on Ricci flow. Here we frame the corresponding weak maximum principle for multitime Ricci flow.

Theorem 3.1. (Weak maximum principle for scalars) *Suppose $g(t^\alpha)$ is a family of metrics on a closed manifold M^n and $u_\alpha \epsilon\ M^n \times [0,T) \to \Re$ satisfies*

$$\frac{\partial u_\alpha}{\partial t^\alpha} \leq \triangle_{g(t^\alpha)} u_\alpha + < X(t^\alpha), \nabla u_\alpha > + F(u_\alpha) \tag{3.1}$$

where $X(t^\alpha)$ is a time-dependent vector field and F is a Lipschitz function. If $u_\alpha \leq p_\alpha$ at $t^\alpha = 0$ for some $p_\alpha \epsilon \Re$, then $u_\alpha(x,t^\alpha) \leq \varphi_\alpha(t^\alpha)$ for all $x \epsilon\ M^n$ and $t^\alpha \epsilon [0,T], 0 < T < \infty$, where $\varphi_\alpha(t^\alpha)$ is the solution to the ODE

$$\frac{d\varphi_\alpha}{dt^\alpha} = F(\varphi_\alpha) \qquad with \qquad \varphi_\alpha(0) = p_\alpha \tag{3.2}$$

Corollary 3.1. (Weak minimum principle) *Replacing all three instances of $\leq$ by $\geq$, theorem 3.1 also holds good.*

With the help of theorem 3.1 and corollary 3.1 we prove

Theorem 3.2. *Suppose $g(t^\alpha)$ is a multitime Ricci flow on a closed manifold M for $0 \leq t^\alpha \leq T$. If at time $t^\alpha = 0$, $R_\alpha \geq p_\alpha \epsilon \Re$, then for all times t^α*

$$R_\alpha \geq \frac{p_\alpha}{1 - 2(\frac{p_\alpha}{n})t^\alpha}. \tag{3.3}$$

Proof: We know

$$\frac{\partial R_\alpha}{\partial t^\alpha} \geq \triangle R_\alpha + \frac{2}{n} R_\alpha^2$$

Hence considering $u_\alpha = R_\alpha,\ X(t^\alpha) = 0$ and $F_\alpha(r, t^\alpha) = \frac{2}{n} r^2$ and applying the weak minimum principle, the required result follows. In this case

$$\varphi_\alpha(t^\alpha) = \frac{p_\alpha}{1 - 2(\frac{p_\alpha}{n})t^\alpha}.$$

From this theorem we can state

Corollary 3.2. *Suppose $g(t^\alpha)$ is a multitime Ricci flow on a closed manifold M for $0 \leq t^\alpha \leq T$. If $R_\alpha \geq p_\alpha \epsilon \Re$ at time $t^\alpha = 0$, then $R_\alpha \geq p_\alpha$ for all times $0 \leq t^\alpha \leq T$.*

This corollary implies

Corollary 3.3. *Multitime Ricci flow preserves positive (or weakly positive) scalar curvature.*

One can easily prove

Corollary 3.4. *Let $g(t^\alpha)$ be a multitime Ricci flow on a closed manifold M for $0 \le t^\alpha < T$. If at time $t^\alpha = 0$, $R_\alpha \ge p_\alpha > 0$, then $T \le \frac{n}{2p_\alpha}$.*

Corollary 3.5. *Let $g(t^\alpha)$ be a multitime Ricci flow on a closed manifold M where $0 < t^\alpha \le T$. Then for all t^α, $R_\alpha \ge -\frac{n}{2t^\alpha}$.*

Next we recall the formula

$$\frac{dV_\alpha}{dt^\alpha} = -\int R_\alpha dV_\alpha \tag{3.4}$$

for the evolution of the volume. Hence using corollary 3.3, we can state

Corollary 3.6. *Suppose $g(t^\alpha)$ is a multitime Ricci flow on a closed manifold M for $0 \le t^\alpha \le T$. If at time $t^\alpha = 0$ the scalar curvature $R_\alpha \ge 0$, then the volume $V(t^\alpha)$ is (weakly) decreasing.*

Corollary 3.7. *Suppose $g(t^\alpha)$ is a multitime Ricci flow on a closed manifold M for $t^\alpha \epsilon\ [0,T]$ with $p_\alpha = \ inf R_\alpha <\ 0$ at time $t^\alpha = 0$. Then*

$$\frac{V(t^\alpha)}{(1+2\frac{(-p_\alpha)}{n}t^\alpha)^{\frac{n}{2}}} \tag{3.5}$$

is weakly decreasing and in particular

$$V(t^\alpha) \le V(0)(1+2\frac{(-p_\alpha)}{n}t^\alpha)^{\frac{n}{2}}. \tag{3.6}$$

Proof: Proceeding as corollary 3.2.7 of [11], we can obtain the required result.

4 Evolution of Riemannian curvature tensor under multitime Ricci flow

In this section we estimate Riemannian curvature tensor Rm by the following theorem.

Theorem 4.1. *Suppose $g(t^\alpha)$ is a multitime Ricci flow on a closed manifold M for $0 \le t^\alpha \le T$, and at time $t^\alpha = 0$, $|Rm| \le M_\alpha$. Then for all $0 < t^\alpha \le T$*

$$|Rm| \le \frac{M_\alpha}{1-\frac{1}{2}A_\alpha M_\alpha t^\alpha} \tag{4.1}$$

where A_α is a constant corresponding to t^α.

Proof: Under multitime Ricci flow we have

$$\frac{\partial}{\partial t^{\alpha}}|Rm|^2 \leq \triangle|Rm|^2 - 2|\nabla Rm|^2 + A_{\alpha}|Rm|^3 \tag{4.2}$$

where A_{α} is a constant.
After weakening it becomes

$$\frac{\partial}{\partial t^{\alpha}}|Rm|^2 \leq \triangle|Rm|^2 + A_{\alpha}|Rm|^3 \tag{4.3}$$

Now if we apply the weak maximum principle, theorem 3.1, considering $u_{\alpha} = |Rm|^2,\ X_{\alpha} = 0,\ F_{\alpha}(r, t^{\alpha}) = A_{\alpha}r^{\frac{3}{2}},\ p_{\alpha} = M_{\alpha}^2$ and

$$\varphi_{\alpha}(t^{\alpha}) = \frac{1}{(M_{\alpha}^{-1} - \frac{1}{2}A_{\alpha}t^{\alpha})^2}$$

which is a solution of

$$\varphi'_{\alpha}(t^{\alpha}) = A_{\alpha}\varphi_{\alpha}(t^{\alpha})^{\frac{3}{2}} \quad with \quad \varphi_{\alpha}(0) = M_{\alpha}^2,$$

then the required result follows.

Finally we have the derivative estimates over the whole manifold under multitime Ricci flow.

Theorem 4.2. *Suppose $g(t^{\alpha})$ is a multitime Ricci flow on a closed manifold M^n for $t^{\alpha}\epsilon(0, \frac{1}{\bar{M}_{\alpha}}], \bar{M}_{\alpha} > 0$. For all $k\epsilon\aleph$ (set of natural numbers) there exists $A_{\alpha} = A_{\alpha}(n, k)$ such that if $|Rm| \leq \bar{M}_{\alpha}$ throughout $M \times [0, \frac{1}{\bar{M}_{\alpha}}]$, then for all $t^{\alpha}\epsilon(0, \frac{1}{\bar{M}_{\alpha}}]$,*

$$|\nabla^k Rm| \leq \frac{A_{\alpha}\bar{M}_{\alpha}}{(t^{\alpha})^{\frac{k}{2}}}. \tag{4.4}$$

Proof: We will prove the result for $k = 1$ and the higher derivative estimates can be done in a similar way by induction.
First we note the following three significant results.

$$\frac{\partial}{\partial t^{\alpha}}Rm = \triangle Rm + Rm * Rm \tag{4.5}$$

$$\nabla(\triangle Rm) + Rm * \nabla Rm \tag{4.6}$$

$$\nabla\frac{\partial}{\partial t^{\alpha}}Rm = \frac{\partial}{\partial t^{\alpha}}\nabla Rm + Rm * \nabla Rm. \tag{4.7}$$

Hence we have from (4.5)

$$\frac{\partial}{\partial t^{\alpha}}\nabla Rm = \triangle(\nabla Rm) + Rm * \nabla Rm. \tag{4.8}$$

Again we have

$$\frac{\partial}{\partial t^{\alpha}}|\nabla Rm|^2 \leq \triangle|\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + A_{\alpha}|Rm||\nabla Rm|^2 \tag{4.9}$$

$$\leq \triangle|\nabla Rm|^2 + A_{\alpha}|Rm||\nabla Rm|^2$$

If we consider $u_{\alpha}(x,t^{\alpha}) = t^{\alpha}|\nabla Rm|^2 + q_{\alpha}|Rm|^2$ where q_{α} is to be chosen later, then with the help of (4.3) and (4.9) we have

$$\frac{\partial u_{\alpha}}{\partial t^{\alpha}} \leq \nabla u_{\alpha} + |\nabla Rm|^2(1 + A_{\alpha}t^{\alpha}|Rm| - 2q_{\alpha}) + A_{\alpha}q_{\alpha}|Rm|^3$$

By hypothesis $\quad |Rm| \leq \bar{M}_{\alpha} \quad$ *and* $\quad t^{\alpha} \leq \frac{1}{\bar{M}_{\alpha}}$

Thus we obtain

$$\frac{\partial u_{\alpha}}{\partial t^{\alpha}} \leq \nabla u_{\alpha} + |\nabla Rm|^2(1 + A_{\alpha} - 2q_{\alpha}) + A_{\alpha}q_{\alpha}\bar{M}_{\alpha}^{\;3}$$

for $A_{\alpha} = A_{\alpha}(n)$.
If we consider q_{α} to be sufficiently large (say $q_{\alpha} = \frac{1}{2}(1 + A_{\alpha})$) we get

$$\frac{\partial u_{\alpha}}{\partial t^{\alpha}} \leq \nabla u_{\alpha} + A_{\alpha}(n)\bar{M}_{\alpha}^{\;3}.$$

Also $\quad u_{\alpha}(.,0) = q_{\alpha}|Rm|^2 \leq q_{\alpha}\bar{M}_{\alpha}^{\;2}.$

So applying weak maximum principle with

$$\varphi_{\alpha}(t^{\alpha}) = q_{\alpha}\bar{M}_{\alpha}^{\;2} + A_{\alpha}t^{\alpha}\bar{M}_{\alpha}^{\;3}$$

which is a solution of

$$\varphi_{\alpha}'(t^{\alpha}) = A_{\alpha}\bar{M}_{\alpha}^{\;3} \qquad \textit{and} \qquad \varphi_{\alpha}(0) = q_{\alpha}\bar{M}_{\alpha}^{\;2}$$

we have

$$u_{\alpha}(.,t^{\alpha}) \leq q_{\alpha}\bar{M}_{\alpha}^{\;2} + A_{\alpha}t^{\alpha}\bar{M}_{\alpha}^{\;3} \leq A_{\alpha}\bar{M}_{\alpha}^{\;2}$$

which shows that

$$t^{\alpha}|\nabla Rm|^{2} \leq u_{\alpha}(.,t^{\alpha}) \leq A_{\alpha}\bar{M}_{\alpha}{}^{2}$$

and hence $|\nabla Rm| \leq \frac{A_{\alpha}\bar{M}_{\alpha}}{(t^{\alpha})^{\frac{1}{2}}}$

for some $A_{\alpha} = A_{\alpha}(n)$.

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Received: May, 2011