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A unified algorithm for solving split generalized mixed equilibrium problem, and for finding fixed point of nonspreading mapping in Hilbert spaces

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Abstract: The purpose of this paper is to study a split generalized mixed equilibrium problem and a fixed point problem for nonspreading mappings in real Hilbert spaces. We introduce a new iterative algorithm and prove its strong convergence for approximating a common solution of a split generalized mixed equilibrium problem and a fixed point problem for nonspreading mappings in real Hilbert spaces. Our algorithm is developed by combining a modified accelerated Mann algorithm and a viscosity approximation method to obtain a new faster iterative algorithm for finding a common solution of these problems in real Hilbert spaces. Also, our algorithm does not require any prior knowledge of the bounded linear operator norm. We further give a numerical example to show the efficiency and consistency of our algorithm. Our result improves and compliments many recent results previously obtained in this direction in the literature.

Keywords: split mixed equilibrium, nonspreading mapping, fixed point problem, accelerated algorithm, iterative method, viscosity approximation method

MSC: 65K15, 47J25, 65J15, 90C33

1 Introduction

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Let C be a nonempty, closed and convex subset of H, $\Theta : C \times C \to \mathbb{R}$ be a nonlinear bifunction, $h : C \to H$ be a nonlinear mapping, and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. The Generalized Mixed Equilibrium Problem (GMEP) is defined as finding a point $x \in C$ such that

$$\Theta(x, y) + \langle hx, y - x \rangle + \phi(y) - \phi(x) \geq 0, \quad \forall\, y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $GMEP(\Theta, h, \phi)$.
If $h = 0$, Problem (1) reduces to the Mixed Equilibrium Problem (MEP) which is to find a point $x \in C$ such that

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$$\Theta(x,y)+\phi(y)-\phi(x)\geq 0, \quad \forall\, y\in C. \tag{2}$$

The set of solutions of (2) is denoted by $MEP(\Theta,\phi)$.
In particular, if $\phi=0$ in (2), the MEP reduces to the classical equilibrium problem which was introduced by Blum and Oettli [1] and defined as finding a point $x\in C$ such that

$$\Theta(x,y)\geq 0, \quad \forall\, y\in C.$$

The GMEP is very general in the sense that it includes as special cases, minimization problems, variational inequality problems, fixed point problems, Nash equilibrium problems in noncooperative games and many others, see [2–4].
In 1994, Censor and Elfving [5] introduced the following Split Feasibility Problem (SFP) in finite-dimensional Hilbert spaces. Let H_1, H_2 be two Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2 respectively, and let $A:H_1\to H_2$ be a bounded linear operator. The SFP is formulated as finding a point $x^\dagger$ with the property

$$x^\dagger\in C \quad \text{and} \quad Ax^\dagger\in Q. \tag{3}$$

The SFP has been applied extensively in many areas of science and engineering such as in intensity-modulated radiation therapy, signal processing and image reconstruction. The SFP has received attention from many authors, and various iterative methods have been proposed for finding its solutions, see for instance [6–9].
Let H_1, H_2 be real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\Theta_1:C\times C\to\mathbb{R}$ and $\Theta_2:Q\times Q\to\mathbb{R}$ be nonlinear bifunctions, $h_1:C\to H_1$ and $h_2:Q\to H_2$ be nonlinear mappings, $\phi:C\to\mathbb{R}\cup\{+\infty\}$ and $\varphi:Q\to\mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous and convex functions, and let $A:H_1\to H_2$ be a bounded linear operator. In this paper, we study the following Split Generalized Mixed Equilibrium Problem (SGMEP). Find a point $x^\dagger\in C$ such that

$$\begin{cases}\Theta_1(x^\dagger,x)+\langle h_1(x^\dagger),x-x^\dagger\rangle+\phi(x)-\phi(x^\dagger)\geq 0, & \forall\, x\in C,\\ \text{where} & \\ y^\dagger=Ax^\dagger \quad \text{solves} \quad \Theta_2(y^\dagger,y)+\langle h_2(y^\dagger),y-y^\dagger\rangle+\varphi(y)-\varphi(y^\dagger)\geq 0, & \forall\, y\in Q.\end{cases} \tag{4}$$

The set of solutions of the SMEP is denoted by $\Omega:=\{x^\dagger\in GMEP(\Theta_1,h_1,\phi):Ax^\dagger\in GMEP(\Theta_2,h_2,\varphi)\}$.
We present the following examples to show that Ω is nonempty.

Example 1.1. Let $H_1=H_2=\mathbb{R}$, $C=[2,\infty)$ and $Q=(-\infty,-4]$. Let $A(x)=-2x$ for all $x\in\mathbb{R}$, so that A is a bounded linear operator. Let $\Theta_1:C\times C\to\mathbb{R}$ and $\Theta_2:Q\times Q\to\mathbb{R}$ be defined by $\Theta_1(x,y)=y-x$ and $\Theta_2(u,v)=3(u-v)$. Next, let $h_1:C\to\mathbb{R}$ and $h_2:Q\to\mathbb{R}$ be defined by $h_1(x)=x$ and $h_2(u)=2u$. Finally, let $\phi:C\to\mathbb{R}\cup\{+\infty\}$ and $\varphi:Q\to\mathbb{R}\cup\{+\infty\}$ be defined by $\phi(x)=\frac{x^2}{2}$ and $\varphi(u)=2u$. Clearly, $GMEP(\Theta_1,h,\phi)=\{2\}$ and $A(2)=-4\in GMEP(\Theta_2,h_2,\varphi)$. Thus, $\Omega=\{p\in GMEP(\Theta_1,h_1,\phi):Ap\in GMEP(\Theta_2,h_2,\varphi)\}\neq\emptyset$.

Example 1.2. Let $H_1=\mathbb{R}^2$ with the norm $||\bar{x}||=\sqrt{x_1^2+x_2^2}$ for $\bar{x}=(x_1,x_2)\in\mathbb{R}^2$, and let $H_2=\mathbb{R}$. Let $C:=\{\bar{x}=(x_1,x_2)\in\mathbb{R}^2:x_2-x_1\geq 1\}$ and $Q=[1,\infty)$. Define $\Theta_1(\bar{x},\bar{y})=y_2-y_1-x_2+x_1$, where $\bar{x}=(x_1,x_2)$ and $\bar{y}=(y_1,y_2)\in C$; then Θ_1 is a bifunction from $C\times C\to\mathbb{R}$. Let $h_1(\bar{x})=\phi(\bar{x})=x_2-x_1$, then $GMEP(\Theta_1,h_1,\phi)=\{\bar{q}=(q_1,q_2):q_2-q_1=1\}$. Also define $\Theta_2(u,v)=v-u$ for all $u,v\in Q$, so that Θ_2 is a bifunction from $Q\times Q$ to $\mathbb{R}$, and let $h_2(u)=2u$, $\varphi(u)=u$. For each $\bar{x}=(x_1,x_2)\in H_1$, let $A(\bar{x})=x_2-x_1$, so that A is bounded linear operator from H_1 into H_2. Clearly, when $\bar{q}\in GMEP(\Theta_1,h_1,\phi)$, we have $A\bar{q}=1\in GMEP(\Theta_2,h_2,\varphi)$. Thus $\Omega=\{\bar{q}\in GMEP(\Theta_1,h_1,\phi):A\bar{q}\in GMEP(\Theta_2,h_2,\varphi)\}\neq\emptyset$.

Remark 1.3. We note that SGMEP in Example 1.1 lies in two different subsets of the same space, while SGMEP in Example 1.2 lies in two different subsets of different spaces.

Let C be a subset of H, then a point $x\in C$ is called a fixed point of a nonlinear mapping $T:C\to C$ if $Tx=x$. We denote by $F(T)$, the set of all fixed points of T. A nonlinear mapping $T:C\to C$ is called

(i) L-Lipschitz if there exists $L > 0$ such that

$$||Tx - Ty|| \le L||x - y||, \quad \forall x, y \in H;$$

(ii) nonexpansive if for all $x, y \in C$, we have

$$||Tx - Ty|| \le ||x - y||;$$

(iii) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - p|| \le ||x - p||, \quad \forall x \in C \quad \text{and} \quad p \in F(T);$$

(iv) firmly nonexpansive if for all $x, y \in C$, we have

$$||Tx - Ty||^2 \le \langle Tx - Ty, x - y\rangle;$$

(v) β-inverse strongly monotone if there exists a constant $\beta > 0$, such that

$$\langle Tx - Ty, x - y\rangle \ge \beta||Tx - Ty||^2, \quad \forall\, x, y \in H;$$

(vi) nonspreading if for all $x, y \in C$, we have

$$2||Tx - Ty||^2 \le ||Tx - y||^2 + ||x - Ty||^2;$$

equivalently, T is nonspreading if for all $x, y \in C$,

$$||Tx - Ty||^2 \le ||x - y||^2 + 2\langle x - Tx, y - Ty\rangle.$$

Remark 1.4. We note that every β-inverse strongly monotone mapping T is $\frac{1}{\beta}$-Lipschitzian, and if $L \in [0, 1)$ in (i), then T is called a contraction mapping. It is clear that if $F(T) \neq \emptyset$, then a nonspreading mapping becomes a quasi-nonexpansive mapping. In addition, every nonexpansive mapping with a nonempty set of fixed points is quasi-nonexpansive.

Approximating fixed point solutions of nonexpansive mappings has a variety of applications since many problems can be seen as a fixed point problem of nonexpansive mappings. A significant body of work on iteration methods for fixed point problems has accumulated in the literature (see for example [10–23] and the references therein). Specifically, the Mann algorithm [15], which can be expressed as follows. For each $n \ge 0$,

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Tx_n, \tag{5}$$

is often used to approximate a fixed point of a nonexpansive mapping. The iterative sequence $\{x_n\}$ converges weakly to a fixed point of T provided that $\{\lambda_n\} \subset [0, 1]$ satisfies

$$\sum_{n=1}^{\infty} \lambda_n(1 - \lambda_n) = +\infty.$$

In 2000, Moudafi [24] introduced the viscosity approximation method for approximating fixed points of nonexpansive mappings. Let f be a contraction on H, and starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)Tx_n, \quad n \ge 0, \tag{6}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$. Xu [25] proved that if $\{\lambda_n\}$ satisfies certain conditions, the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution $x \in F(T)$ of the variational inequality

$$\langle (I - f)x^{\dagger}, x - x^{\dagger}\rangle \ge 0, \quad \forall x \in F(T).$$

Based on the heavy ball methods of the two-order time dynamical system, Polyak [26] first proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates.

Recently, a lot of researchers have constructed some fast iterative algorithms by using inertial extrapolation which includes an inertial proximal method [27, 28], an inertial forward-backward method [29], inertial proximal ADMM [30] and the fast iterative shrinkage thresholding algorithm FISTA [31, 32]. Using the technique of inertial extrapolation, Mainge [33] introduced in 2008 the following inertial Mann algorithm. For each $n \geq 1$, compute

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T y_n. \end{cases} \tag{7}$$

Mainge [33] showed that the iterative sequence $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

(A1) $\theta_n \in [0, \alpha)$ for each $n \geq 1$, where $\alpha \in [0, 1)$,
(A2) $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}||^2 < +\infty$,
(A3) $0 < \inf \lambda_n \leq \sup \lambda_n < 1$.

To satisfy the summation condition (A2) of the sequence $\{x_n\}$, one needs to first calculate θ_n at each step (see [28]). In 2015, Bot and Csetnek [34] removed the condition (A2) and substituted (A1) and (A3) with the following conditions:

(B1) for each $n \geq 1$, $\{\theta_n\} \subset [0, \infty)$ is nondecreasing with $\theta_1 = 0$ and $0 \leq \alpha < 1$,
(B2) for each $n \geq 1$,

$$\delta > \frac{\alpha^2(1 + \alpha) + \alpha\sigma}{1 - \alpha^2}, \quad 0 \leq \lambda \leq \lambda_n \leq \frac{\delta - \alpha[\alpha(1 + \alpha) + \alpha\delta + \sigma]}{\delta[1 + \alpha(1 + \alpha) + \alpha\delta + \sigma]},$$

where $\lambda, \sigma, \delta > 0$.

By combining the Picard algorithm [35] with the conjugate gradient methods [36], the authors in [37] accelerated the Mann algorithm and obtained the following faster algorithm: For each $n \geq 0$, we compute

$$\begin{cases} d_{n+1} = \frac{1}{\lambda}(T(x_n) - x_n) + \beta_n d_n, \\ y_n = x_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\alpha_n x_n + (1 - \mu\alpha_n)y_n, \end{cases} \tag{8}$$

where $\mu \in (0, 1]$ and $\lambda > 0$. They proved that the iterative sequence $\{x_n\}$ converges weakly to a fixed point of T provided that the nonnegative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(BB1) $\sum_{n=0}^{\infty} \mu\alpha_n(1 - \mu\alpha_n) = \infty$,
(BB2) $\sum_{n=0}^{\infty} \beta_n < \infty$.

Moreover, the sequence $\{x_n\}$ satisfies the following condition:

$$\{T(x_n) - x_n\} \text{ is bounded.}$$

Also, they gave some numerical examples to show that the accelerated Mann algorithm is more efficient than the normal Mann algorithm.

In 2016, Suantai *et. al.* [38] studied the Split Equilibrium Problem which is define as follows: First, we find a point $x^* \in C$ such that

$$\Theta_1(x^*, x) \geq 0, \ \forall x \in C \quad \text{and such that} \quad y^* = Ax^* \in Q \quad \text{solves} \quad \Theta_2(y^*, y) \geq 0, \ \forall y \in C, \tag{9}$$

where $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifuncions. The set of solutions of (9) is denoted by $SEP(\Theta_1, \Theta_2)$. The authors in [38] proposed the following iterative algorithm to solve the problem of finding a common element in $SEP(\Theta_1, \Theta_2)$ and a fixed point of a nonspreading multi-valued mapping $S : C \to K(C)$. Given $x_1 \in C$, let $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{\Theta_1}(I - \gamma A^*(I - T_{r_n}^{\Theta_2})A)x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{10}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \in (0,\infty)$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A, and A^* is the adjoint of A. Further, they proved that under certain conditions, the sequence $\{x_n\}$ converges weakly to an element of $F(S) \cap SEP(\Theta_1, \Theta_2)$.
More recently, S.H. Rizvi [39] studied the following Split Mixed Equilibrium Problem (*SMEP*) in real Hilbert spaces. Find a point $x^* \in C$ such that

$$\begin{cases} \Theta_1(x^*, x) + \langle h_1 x^*, x - x^* \rangle \geq 0, \qquad \forall x \in C, \\ \text{where} \\ y^* = Ax^* \quad \text{solves} \quad \Theta_2(y^*, y) + \langle h_2 y^*, y - y^* \rangle \geq 0, \qquad \forall y \in Q, \end{cases} \tag{11}$$

where $h_1 : C \to C$ and $h_2 : Q \to Q$ are θ_1, θ_2-inverse strongly monotone mappings respectively with $\theta = \min(\theta_1, \theta_2)$. The set of solutions of (11) is denoted by $SMEP(\Theta_1, \Theta_2, h_1, h_2)$. Observe that when $\phi = \varphi = 0$ in (4), we obtain (11). Thus, Problem (4) is more general than Problem (11). Rizvi [39] introduced the following algorithm for solving (11), as well as fixed point problems for a nonexpansive mapping S in real Hilbert spaces. Let

$$\begin{cases} x_0 = x \in C, \\ y_n = T_{r_n}^{\Theta_1}(x_n - r_n \phi x_n), \\ v_n = T_{r_n}^{\Theta_2}(I - r_n \psi) A y_n, \\ z_n = P_C(y_n + \delta A^*(v_n - Ay_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) z_n], \quad n \geq 0, \end{cases} \tag{12}$$

where P_C is the metric projection from H onto C, $\{r_n\} \subset (0, 2\theta)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Rizvi [39] also proved that under some mild conditions on α_n, β_n and r_n, the sequence $\{x_n\}$ converges strongly to a solution in $SMEP(\Theta_1, \Theta_2, h_1, h_2) \cap F(S)$.
Motivated by the above works, it is our aim in this paper to study the SGMEP (4) and introduce a new iterative algorithm for approximating a common solution of (4) and a fixed point problem for nonspreading mappings in real Hilbert spaces. Our algorithm is developed by modifying the accelerated Mann algorithm (8), combined with a modified viscosity approximation method of (6) to obtain a new faster iterative algorithm for finding a common solution of (4) and a fixed point of nonspreading mappings in real Hilbert spaces. Further, our algorithm does not require any prior knowledge of the operator norm. We note here that norms of bounded linear operators are rarely known explicitly (see [40]). Our result is interesting and compliments many recent results previously obtained in this direction in the literature.

2 Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H. We denote the strong and weak convergence of a sequence $\{x_n\} \subseteq H$ to a point $p \in H$ by $x_n \to p$ and $x_n \rightharpoonup p$, respectively.

For each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \ \forall y \in C.$$

The mapping P_C is called the metric projection from H onto C. It is well known that P_C has the following characteristics:

1. $\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2$, for every $x, y \in H$;
2. for $x \in H$ and $z \in C$, $z = P_C x \Leftrightarrow$
$$\langle x - z, z - y \rangle \geq 0, \ \forall y \in C;$$
3. for $x \in H$ and $y \in C$,
$$||y - P_C(x)||^2 + ||x - P_C(x)||^2 \leq ||x - y||^2.$$

The following lemmas are useful in establishing our main result.

Lemma 2.1. *In a real Hilbert space H, the following inequalities hold:*

1. $||x-y||^2 \leq ||x||^2 - ||y||^2 - 2\langle x-y, y\rangle$, *for all* $x, y \in H$;
2. $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in H$;
3. $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$, *for all* $x, y \in H$ *and* $\alpha \in [0, 1]$.

Lemma 2.2. [41] *Let C be a nonempty closed and convex subset of a Hilbert space H, $S : C \to C$ a nonspreading mapping, and $F(S) \neq \emptyset$. Then $I - S$ is demiclosed at 0, i.e. for any sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z \in F(S)$.*

For solving the GMEP we make the following assumptions:

Assumption 2.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. We make the following assumptions on the bifunction $\Theta : C \times C \to \mathbb{R}$:*

- *L1.* $\Theta(x, x) = 0$, *for all* $x \in C$;
- *L2.* Θ *is monotone, i.e.* $\Theta(x, y) + \Theta(y, x) \leq 0$, *for all* $x, y \in C$;
- *L3.* *for each* $x, y, z \in C$, $\lim_{t\downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y)$;
- *L4.* *for each* $x \in C$, $y \mapsto \Theta(x, y)$ *is convex and lower semicontinuous.*

Lemma 2.4. [42] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction which satisfies Assumption 2.3, $h : C \to H_1$ be a nonlinear mapping, and let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. For $r > 0$ and $x \in H_1$, define a resolvent function*

$$T_r^{\Theta}(x) = \{z \in C : \Theta(z, y) + \langle h(z), y - z\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\},$$

for all $x \in H$. Then the following conclusions hold:

- *(i)* *for each* $x \in H$, $T_r^{\Theta}(x) \neq \emptyset$;
- *(ii)* T_r^{Θ} *is single-valued;*
- *(iii)* T_r^{Θ} *is firmly nonexpansive, i.e. for any* $x, y \in H$,

$$||T_r^{\Theta}x - T_r^{\Theta}y||^2 \leq \langle T_r^{\Theta}x - T_r^{\Theta}y, x - y\rangle;$$

- *(iv)* $F(T_r^{\Theta}) = GMEP(\Theta_1, h, \phi)$;
- *(v)* $GMEP(\Theta, h, \phi)$ *is closed and convex.*

Lemma 2.5. [43] *Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \nu_n)a_n + \nu_n\delta_n, \, n \geq 0,$$

where $\{\nu_n\}_{n=1}^{\infty}$ is a sequence in (0,1) and $\{\delta_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with:

- *(i)* $\sum_{n=0}^{\infty} \nu_n = \infty$;
- *(ii)* $\limsup_{n\to\infty} \delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we introduce a new iterative algorithm with a choice of stepsize which does not depend on the operator norm $||A||$.

Algorithm 3.1. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.3. Let $h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be θ_1, θ_2-inverse strongly monotone operators, respectively, such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0,1)$ and D be a bounded operator with coefficient $\bar{\gamma} \in (0,1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial point $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0,1]$, $\beta_n \in [0,1]$, $w_n \in (0,1)$, $r_n \in (0, 2\theta)$ and $\lambda > 0$. Assume that the nth iterate has been constructed, and set $m_1 = \frac{\gamma_1 A^*(T_{r_1}^{\Theta_2} - I)Ax_1}{\lambda}$. We then compute the $(n+1)$th iterate via the formula*

$$\begin{cases} m_{n+1} = \dfrac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n, \\ y_n = x_n + \lambda m_{n+1}, \\ z_n = T_{r_n}^{\Theta_1}(I - r_n h_1)y_n, \\ x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)[(1 - w_n)z_n + w_n S z_n], \end{cases} \tag{13}$$

for $n \geq 1$, where A^* is the adjoint operator of A. Further, we choose the stepsize γ_n such that, if

$$n \in O := \{n : (I - T_{r_n}^{\Theta_2})Ax_n \neq 0\},$$

then

$$\gamma_n \in \left(0, \frac{2\|(I - T_{r_n}^{\Theta_2})x_n\|^2}{\|A^*(I - T_{r_n}^{\Theta_2})Ax_n\|^2}\right), \quad \forall n \in O. \tag{14}$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value).

Remark 3.2. Note that in (14), the choice of stepsize γ_n is independent of the norm $\|A\|$. The value of γ does not influence the considered algorithm but was introduced just for the sake of clarity. Furthermore, we will see from Lemma 3.3 that γ_n is well defined.

Lemma 3.3. *Assume that $\Omega := \{q \in GMEP(\Theta_1, h_1, \phi) : Aq \in GMEP(\Theta_2, h_2, \varphi)\}$ is nonempty. Then γ_n defined by* (14) *is well defined.*

Proof. We need to show that $\|A^*(I - T_{r_n}^{\Theta_2})Ax_n\| > 0$. Take $x \in \Omega$, then $T_{r_n}^{\Theta_1}x = x$ and $T_{r_n}^{\Theta_2}Ax = Ax$, and observe the following:

$$\begin{aligned} \|(I - T_{r_n}^{\Theta_2})Ax_n\|^2 &= \langle (I - T_{r_n}^{\Theta_2})Ax_n, (I - T_{r_n}^{\Theta_2})Ax_n \rangle \\ &= \langle (I - T_{r_n}^{\Theta_2})Ax_n, Ax_n - Ax + T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n \rangle \\ &= \langle (I - T_{r_n}^{\Theta_2})Ax_n, Ax_n - Ax \rangle + \langle (I - T_{r_n}^{\Theta_2})Ax_n, T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n \rangle \\ &= \langle A^*(I - T_{r_n}^{\Theta_2})Ax_n, x_n - x \rangle + \langle (I - T_{r_n}^{\Theta_2})Ax_n, T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n \rangle \\ &\leq \|A^*(I - T_{r_n}^{\Theta_2})Ax_n\| \cdot \|x_n - x\| + \|(I - T_{r_n}^{\Theta_2})Ax_n\| \cdot \|T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n\|. \end{aligned}$$

Consequently, for $n \in O$, that is $\|(I - T_{r_n}^{\Theta_2}Ax_n)Ax_n\| > 0$, we get $\|A^*(I - T_{r_n}^{\Theta_2}Ax_n)Ax_n\| \cdot \|x_n - x\| > 0$ and $\|(I - T_{r_n}^{\Theta_2})Ax_n\| \cdot \|T_{r_n}^{\Theta_2}Ax - T_{r_n}^{\Theta_2}Ax_n\| > 0$. Since $\|A^*(I - T_{r_n}^{\Theta_2})Ax_n\| \cdot \|x_n - x\| > 0$, we obtain that $\|A^*(I - T_{r_n}^{\Theta_2})Ax_n\| \neq 0$. This implies that γ_n is well defined. □

We now make the following assumptions:

Assumption 3.4. *The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in Algorithm 3.1 satisfy the following:*

- *(C1)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- *(C2)* $\sum_{n=0}^{\infty} \beta_n < \infty$,
- *(C3)* $\beta_n \leq \alpha_n^4$.

Furthermore, $\{x_n\}$ *satisfies*

(C4) $\{(T_{r_n}^{\Theta_2} - I)Ax_n\}$ *is bounded.*

Before giving the convergence analysis of Algorithm 3.1, we first show the following result.

Lemma 3.5. *Suppose that* $\Gamma := \Omega \cap F(S) \neq \emptyset$ *and* $\{x_n\}$ *is generated by* (13). *Also, let Assumption 3.4 be satisfied and suppose* r_n *satisfies the following condition:*

(C5) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2\theta.$

Then, $\{m_n\}$ *and* $\{x_n\}$ *are bounded, and consequently* $\{y_n\}$ *is bounded.*

Proof. It follows from (C2) that $\lim_{n\to\infty}\beta_n = 0$ and so there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq \frac{1}{2}$ for all $n \geq n_0$. Define a number $N_1 := \max\left\{\max_{1\leq k\leq n_0} ||m_k||, \frac{2}{\lambda}\sup_{n\geq 1} ||\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n||\right\}$. Then (C4) implies that $N_1 < \infty$. Assume that $||m_n|| \leq N_1$ for some $n \geq n_0$, then the triangle inequality ensures that

$$\begin{aligned} ||m_{n+1}|| &= \left|\left|\frac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n\right|\right| \\ &\leq \frac{1}{\lambda}||\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n|| + \beta_n||m_n|| \leq N_1, \end{aligned} \tag{15}$$

which means that $||m_{n+1}|| \leq N_1$ for all $n \geq 0$, hence $\{m_n\}$ is bounded.
Also, the definition of $\{y_n\}$ implies that

$$\begin{aligned} y_n &= x_n + \lambda\Big(\frac{1}{\lambda}(\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n) + \beta_n m_n\Big) \\ &= x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n + \lambda\beta_n m_n. \end{aligned}$$

Let $p \in \Gamma$, then

$$\begin{aligned} ||y_n - p|| &= ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n + \lambda\beta_n m_n - p|| \\ &\leq ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p|| + \lambda\beta_n||m_n||. \end{aligned} \tag{16}$$

Observe that

$$\begin{aligned} ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p||^2 &= ||x_n - p||^2 - 2\gamma_n\langle x_n - p, A^*(I - T_{r_n}^{\Theta_2})Ax_n\rangle + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &= ||x_n - p||^2 - 2\gamma_n\langle Ax_n - Ap, (I - T_{r_n}^{\Theta_2})Ax_n\rangle + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &= ||x_n - p||^2 - 2\gamma_n\langle T_{r_n}^{\Theta_2}Ax_n - Ap, (I - T_{r_n}^{\Theta_2})Ax_n\rangle - 2\gamma_n||(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &\quad + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2. \end{aligned} \tag{17}$$

Since $T_{r_n}^{\Theta_2}$ is firmly nonexpansive, then

$$||T_{r_n}^{\Theta_2}Ax_n - Ap||^2 \leq \langle T_{r_n}^{\Theta_2}Ax_n - Ap, Ax_n - Ap\rangle,$$

and so

$$\langle T_{r_n}^{\Theta_2}Ax_n - Ap, T_{r_n}^{\Theta_2}Ax_n - Ax_n\rangle \leq 0. \tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned} ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p||^2 &\leq ||x_n - p||^2 - 2\gamma_n||(I - T_{r_n}^{\Theta_2})Ax_n||^2 + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 \\ &= ||x_n - p||^2 - \gamma_n\Big[2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2\Big] \\ &\leq ||x_n - p||^2. \end{aligned} \tag{19}$$

Therefore, from (16) and (19), we get

$$||y_n - p|| \leq ||x_n - p|| + \lambda\beta_n N_1. \tag{20}$$

Again from (13), we use the fact that $T_{r_n}^{\Theta_1}$ is firmly nonexpansive to show that

$$\begin{aligned}||z_n - p||^2 &= ||T_{r_n}^{\Theta_1}(I - r_nh_1)y_n - T_{r_n}^{\Theta_1}(I - r_nh_1)p||^2\\ &\le ||(I - r_nh_1)y_n - (I - r_nh_1)p||^2\\ &= ||(y_n - p) - r_n(h_1y_n - h_1p)||^2\\ &= ||y_n - p||^2 - 2r_n\langle y_n - p, h_1y_n - h_1p\rangle + r_n^2||h_1y_n - h_1p||^2\\ &\le ||y_n - p||^2 - 2r_n\theta||h_1y_n - h_1p||^2 + r_n^2||h_1y_n - h_1p||^2\\ &= ||y_n - p||^2 - r_n(2\theta - r_n)||h_1y_n - h_1p||^2. \end{aligned} \tag{21}$$

By condition (C5), we obtain

$$||z_n - p||^2 \le ||y_n - p||^2. \tag{22}$$

Now define $U_n = (1 - w_n)I + w_nS$, and observe that

$$\begin{aligned}||U_nz_n - p|| &= ||(1 - w_n)(z_n - p) + w_n(Sz_n - p)||\\ &\le (1 - w_n)||z_n - p|| + w_n||Sz_n - p||\\ &\le (1 - w_n)||z_n - p|| + w_n||z_n - p||\\ &= ||z_n - p||.\end{aligned}$$

Therefore, from (13), (20) and (22), we have

$$\begin{aligned}||x_{n+1} - p|| &= ||\alpha_n(\xi f(x_n) - Dp) + (1 - \alpha_nD)(U_nz_n - p)||\\ &\le \alpha_n||\xi f(x_n) - Dp|| + (1 - \alpha_n\bar{\gamma})||U_nz_n - p||\\ &\le \alpha_n\left[||\xi(f(x_n) - f(p)) + (\xi f(p) - Dp)||\right] + (1 - \alpha_n\bar{\gamma})||z_n - p||\\ &\le \alpha_n\xi\beta||x_n - p|| + \alpha_n||\xi f(p) - Dp|| + (1 - \alpha\bar{\gamma})[||x_n - p|| + \lambda\beta_nN_1]\\ &= (1 - \alpha_n(\bar{\gamma} - \xi\beta))||x_n - p|| + \alpha_n||\xi f(p) - Dp|| + \lambda\beta_nN_1\\ &\le \max\left\{||x_n - p||, \frac{||\xi f(p) - Dp||}{\bar{\gamma} - \xi\beta} + \frac{\lambda N_1}{\bar{\gamma} - \xi\beta}\right\}\\ &\ \ \vdots\\ &\le \max\left\{||x_1 - p||, \frac{||\xi f(p) - Dp||}{\bar{\gamma} - \xi\beta} + \frac{\lambda N_1}{\bar{\gamma} - \xi\beta}\right\}.\end{aligned} \tag{23}$$

This implies that $\{x_n\}$ is bounded. It follows from (20) that $\{y_n\}$ is also bounded. □

Theorem 3.6. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.3. Let $h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be θ_1, θ_2-inverse strongly monotone mappings, respectively, such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0, 1)$, and let D a bounded operator with coefficient $\bar{\gamma} \in (0, 1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial value $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0, 1]$, $\beta_n \in [0, 1]$, $w_n \in (0, 1)$, $r_n \in (0, 2\theta)$ and $\lambda > 0$. Suppose $\Gamma := \Omega \cap F(S) \neq \emptyset$, Assumption 3.4, condition (C5) and the following are satisfied:*

(C6) $\liminf_{n\to\infty} r_n > 0$;
(C7) $0 < \liminf_{n\to\infty} w_n \le \limsup_{n\to\infty} w_n < 1$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.1 converge strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of the variational inequality

$$\langle (D - \xi f)z, z - x\rangle \le 0, \quad x \in \Gamma. \tag{24}$$

Proof. Let $p \in \Gamma$, then from Lemma 2.1(2) and (19), we have

$$\begin{aligned} ||y_n - p||^2 &= ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p + \lambda\beta_n m_n||^2 \\ &\leq ||x_n - \gamma_n A^*(I - T_{r_n}^{\Theta_2})Ax_n - p||^2 + 2\lambda\beta_n\langle y_n - p, m_n\rangle \\ &\leq ||x_n - p||^2 + \beta_n\rho_n, \end{aligned} \tag{25}$$

where $\rho_n := \{2\lambda\langle y_n - p, m_n\rangle\}$. Using Lemma 3.5, it follows that $\{\rho_n\}$ is bounded. Thus, there exists $N_2 > 0$ such that $\rho_n \leq N_2$ for all $n \geq 1$. Hence, it follows from condition (C3) that

$$||y_n - p||^2 \leq ||x_n - p||^2 + 2\alpha_n^4 N_2. \tag{26}$$

Furthermore, from (22) and (26), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n(\xi f(x_n) - Dp) + (1 - \alpha_n D)(U_n z_n - p)||^2 \\ &\leq ||(1 - \alpha_n D)(U_n z_n - p)||^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2||z_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2||y_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2\Big[||x_n - p||^2 + 2\alpha^4 N_2\Big] + 2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p|| \\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{27}$$

We now divide the remaining proof of the theorem into two cases.

Case I: Suppose there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - p||\}$ is monotonically decreasing for all $n \geq n_0$. Then $\{||x_n - p||\}$ converges as $n \to \infty$ and so

$$||x_n - p||^2 - ||x_{n+1} - p||^2 \to 0, \quad n \to \infty.$$

Note that, from (19), (25) and (26), we obtain

$$||y_n - p||^2 \leq ||x_n - p||^2 - \gamma_n\Big[2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2\Big] + 2\alpha_n^4 N_2. \tag{28}$$

Also from (27), we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq (1 - \alpha_n\bar{\gamma})^2||y_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\leq ||y_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle. \end{aligned} \tag{29}$$

Substituting (28) into (29), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq ||x_n - p||^2 - \gamma_n\Big[2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2\Big] \\ &\quad +2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle + 2\alpha_n^4 N_2. \end{aligned} \tag{30}$$

Putting $\Lambda_n := 2||I - T_{r_n}^{\Theta_2}Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2$, then since $\alpha_n \to 0$, as $n \to \infty$, it follows from (30) that

$$\begin{aligned} \gamma_n\Lambda_n &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle + 2\alpha_n^4 N_2 \to 0. \end{aligned} \tag{31}$$

From the condition on the stepsize given by (14), for a small $\epsilon > 0$, we know that

$$\gamma_n < \frac{2||(I - T_{r_n}^{\Theta_2})Ax_n||^2}{||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2} - \epsilon, \tag{32}$$

which implies

$$\gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 < 2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \epsilon||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2$$

and thus we have

$$\epsilon||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 < 2||(I - T_{r_n}^{\Theta_2})Ax_n||^2 - \gamma_n||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2.$$

This implies that

$$\epsilon||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 < \Lambda_n \to 0, \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} ||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 = 0. \tag{33}$$

Further, from (31) and (33), we get

$$\begin{aligned}0 < \epsilon||(I-T_{r_n}^{\Theta_2})Ax_n||^2 &\leq \gamma_n||(I - T_{r_n}^{\Theta_2})Ax_n||^2\\ &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + \gamma_n^2||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle\\ &+ 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle + 2\alpha_n^4 N_2 \to 0, \text{ as } n \to \infty,\end{aligned} \tag{34}$$

and hence

$$\lim_{n\to\infty} ||(I - T_{r_n}^{\Theta_2})Ax_n|| = 0. \tag{35}$$

Also from (27), we obtain

$$\begin{aligned}||x_{n+1} - p||^2 &\leq (1 - \alpha_n\bar{\gamma})^2||z_n - p||^2 + 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle\\ &\leq ||z_n - p||^2 + 2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p|| + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle.\end{aligned} \tag{36}$$

Substituting (21) into (36), and from (26), we have

$$\begin{aligned}||x_{n+1} - p||^2 &\leq ||y_n - p||^2 - r_n(2\theta - r_n)||h_1y_n - h_1p||^2 + 2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p||\\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle\\ &\leq ||x_n - p||^2 + 2\alpha_n^4 N_2 - r_n(2\theta - r_n)||h_1y_n - h_1p||^2 + 2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p||\\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle.\end{aligned} \tag{37}$$

Thus, we have

$$\begin{aligned}r_n(2\theta - r_n)||h_1y_n - h_1p||^2 &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p||\\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \to 0, \text{ as } n \to \infty.\end{aligned}$$

Since $\{r_n\} \subset (0, 2\theta)$, we conclude that

$$\lim_{n\to\infty} ||h_1y_n - h_1p||^2 = 0. \tag{38}$$

Further, observe that

$$\begin{aligned}||z_n - p||^2 &= ||T_{r_n}^{\Theta_1}(y_n - r_nh_1y_n) - T_{r_n}^{\Theta_1}(p - r_nh_1p)||^2\\ &\leq \langle z_n - p, (y_n - r_nh_1y_n) - (p - r_nh_1p)\rangle\\ &\leq \frac{1}{2}\Big\{||z_n - p||^2 + ||(y_n - r_nh_1y_n) - (p - r_nh_1p)||^2 - ||(z_n - p) - [(y_n - r_nh_1y_n) - (p - r_nh_1p)]||^2\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}||z_n - p||^2 &\leq ||(y_n - r_nh_1y_n) - (p - r_nh_1p)||^2 - ||(z_n - y_n) + r_n(h_1y_n - h_1p)||^2\\ &\leq ||y_n - p||^2 - ||z_n - y_n||^2 + 2r_n||z_n - y_n|| \cdot ||h_1y_n - h_1p||^2.\end{aligned} \tag{39}$$

From (36) and (39), we obtain

$$\begin{aligned}||x_{n+1} - p||^2 &\leq ||y_n - p||^2 - ||z_n - y_n||^2 + 2r_n||z_n - y_n|| \cdot ||h_1y_n - h_1p||^2 + 2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p||\\ &\quad +2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle\\ &\leq ||x_n - p||^2 + 2\alpha_n^4 N_2 - ||z_n - y_n||^2 + 2r_n||z_n - y_n|| \cdot |h_1y_n - h_1p||^2\\ &\quad +2\alpha_n\xi\beta||x_n - p|| \cdot ||x_{n+1} - p|| + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}||z_n - y_n||^2 &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n^4 N_2 + 2r_n||z_n - y_n|| \cdot ||h_1 y_n - h_1 p||^2 \\ &\quad +2\alpha_n \xi\beta||x_n - p|| \cdot ||x_{n+1} - p|| + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle.\end{aligned}$$

Since $\alpha_n \to 0$ as $n \to \infty$, and using (38), we obtain

$$\lim_{n\to\infty} ||z_n - y_n||^2 = 0. \tag{40}$$

Moreover

$$\begin{aligned}||U_n z_n - p||^2 &= ||(1 - w_n)z_n + w_n S z_n - p||^2 \\ &\leq (1 - w_n)||z_n - p||^2 + w_n||Sz_n - p||^2 - w_n(1 - w_n)||Sz_n - z_n||^2 \\ &\leq (1 - w_n)||z_n - p||^2 + w_n||z_n - p||^2 - w_n(1 - w_n)||Sz_n - z_n||^2 \\ &= ||z_n - p||^2 - w_n(1 - w_n)||Sz_n - z_n||^2 \\ &\leq ||x_n - p||^2 + 2\alpha_n^4 N_2 - w_n(1 - w_n)||Sz_n - z_n||^2.\end{aligned} \tag{41}$$

Note that from (27), we have

$$\begin{aligned}||x_{n+1} - p||^2 &\leq (1 - \alpha_n\bar{\gamma})^2||U_n z_n - p||^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\leq ||U_n z_n - p||^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle,\end{aligned} \tag{42}$$

then from (41) and (42), we get

$$\begin{aligned}w_n(1 - w_n)||Sz_n - z_n||^2 &\leq ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &\quad +2\alpha_n^4 N_2 \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

By condition (C7), we have

$$\lim_{n\to\infty} ||Sz_n - z_n|| = 0. \tag{43}$$

Also

$$||U_n z_n - z_n|| = w_n||Sz_n - z_n|| \to 0, \quad \text{as } n \to \infty. \tag{44}$$

It is clear from (3.1) that

$$||x_{n+1} - U_n z_n|| = \alpha_n||\xi f(x_n) - DU_n z_n|| \to 0, \quad \text{as } n \to \infty, \tag{45}$$

and

$$||y_n - x_n|| \to 0, \quad n \to \infty, \tag{46}$$

then, it follows from (40) and (46) that

$$||z_n - x_n|| \leq ||z_n - y_n|| + ||y_n - x_n|| \to 0, \quad \text{as } n \to \infty. \tag{47}$$

Furthermore, it follows from (44), (45) and (47) that

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - U_n z_n|| + ||U_n z_n - z_n|| + ||z_n - x_n|| \to 0, \text{ as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. It follows from (46) and (47) that $y_{n_j} \rightharpoonup \bar{x}$ and $z_{n_j} \rightharpoonup \bar{x}$, respectively. Since $\lim_{n\to\infty} ||Sz_n - z_n|| = 0$, and by Lemma 2.2, we have $\bar{x} \in F(S)$. Next, we show that $\bar{x} \in \Omega$. Since $z_n = T_{r_n}^{\Theta_1}(y_n - r_n h_1 y_n)$, then

$$\Theta_1(z_n, y) + \langle h_1 z_n, y - z_n\rangle + \phi(y) - \phi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from the monotonicity of Θ_1 that

$$\langle h_1 z_n, y - z_n\rangle + \phi(y) - \phi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - y_n\rangle \geq \Theta_1(y, z_n).$$

Replacing n by n_j, we get

$$\langle h_1 z_{n_j}, y - z_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - z_{n_j}, z_{n_j} - y_{n_j}\rangle \geq \Theta_1(y, z_{n_j}) + \phi(z_{n_j}) - \phi(y). \tag{48}$$

Further, for any $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)\bar{x}$. Since $\bar{x} \in C$ and $y \in C$, then $y_t \in C$. So from (48), we have

$$\begin{aligned}\langle y_t - z_{n_j}, h_1 y_t\rangle \;&\geq\; \langle y_t - z_{n_j}, h_1 y_t\rangle - \langle y_t - z_{n_j}, h_1 y_{n_j}\rangle - \left\langle y_t - z_{n_j}, \frac{z_{n_j} - y_{n_j}}{r_{n_j}}\right\rangle + \Theta_1(y_t, z_{n_j})\\ &\quad + \phi(z_{n_j}) - \phi(y_t)\\ &= \langle y_t - z_{n_j}, h_1 y_t - h_1 z_{n_j}\rangle + \langle y_t - z_{n_j}, h_1 z_{n_j} - h_1 y_{n_j}\rangle - \left\langle y_t - z_{n_j}, \frac{z_{n_j} - y_{n_j}}{r_{n_j}}\right\rangle + \Theta_1(y_t, z_{n_j})\\ &\quad + \phi(z_{n_j}) - \phi(y_t).\end{aligned} \tag{49}$$

From the Lipschitz continuity of h_1 and $\lim_{n\to\infty} ||z_n - y_n|| = 0$, we obtain $||h_1 z_{n_j} - h_1 y_{n_j}|| \to 0$, as $n \to \infty$. Also since h_1 is monotone, we have $\langle y_t - z_{n_j}, h_1 y_t - h_1 z_{n_j}\rangle \geq 0$. Therefore, by L4 and the weak lower semicontinuity of ϕ, taking the limit of (49) as $j \to \infty$, we have

$$\langle y_t - \bar{x}, h_1 y_t\rangle \geq \Theta_1(y_t, \bar{x}) + \phi(\bar{x}) - \phi(y_t). \tag{50}$$

Hence, from L1 and (50), we get

$$\begin{aligned} 0 &= \Theta_1(y_t, y_t) + \phi(y_t) - \phi(y_t)\\ &\leq t\Theta_1(y_t, y) + (1-t)\Theta_1(y_t, \bar{x}) + t\phi(y) + (1-t)\phi(\bar{x}) - \phi(y_t)\\ &= t(\Theta_1(y_t, y) + \phi(y) - \phi(y_t)) + (1-t)(\Theta_1(y_t, \bar{x}) + \phi(\bar{x}) - \phi(y_t))\\ &\leq t(\Theta_1(y_t, y) + \phi(y) - \phi(y_t)) + (1-t)\langle y_t - \bar{x}, h_1 y_t\rangle\\ &\leq t(\Theta_1(y_t, y) + \phi(y) - \phi(y_t)) + (1-t)t\langle y - \bar{x}, h_1 y_t\rangle,\end{aligned}$$

which implies that

$$\Theta_1(y_t, y) + (1-t)\langle y - \bar{x}, h_1 y_t\rangle + \phi(y) - \phi(y_t) \geq 0.$$

Letting $t \to 0$, we have

$$\Theta_1(\bar{x}, y) + \langle y - \bar{x}, h_1\bar{x}\rangle + \phi(y) - \phi(\bar{x}) \geq 0, \quad y \in C,$$

which implies that $\bar{x} \in GMEP(\Theta_1, h, \phi)$.
Since A is a bounded linear operator, $Ax_{n_j} \rightharpoonup A\bar{x}$. It follows from (35) that

$$T_{r_{n_j}}^{\Theta_2} Ax_{n_j} \rightharpoonup A\bar{x}, \text{ as } j \to \infty.$$

By the definition of $T_{r_{n_j}}^{\Theta_2} Ax_{n_j}$, we have

$$\begin{aligned}&\Theta_2(T_{r_{n_j}}^{\Theta_2} Ax_{n_j}, g) + \langle h_2(T_{r_{n_j}}^{\Theta_2} Ax_{n_j}), g - T_{r_{n_j}}^{\Theta_2} Ax_{n_j}\rangle + \varphi(g) - \varphi(T_{r_{n_j}}^{\Theta_2} Ax_{n_j})\\ &+ \frac{1}{r_{n_j}}\langle y - T_{r_{n_j}}^{\Theta_2} Ax_{n_j}, T_{r_{n_j}}^{\Theta_2} Ax_{n_j} - Ax_{n_j}\rangle \geq 0, \qquad \forall g \in Q \quad \text{and} \quad y \in H_2.\end{aligned} \tag{51}$$

Since Θ_2 is upper semicontinuous in the first argument, taking limsup of the above inequality as $j \to \infty$, we get

$$\Theta_2(A\bar{x}, g) + \langle h_2(A\bar{x}), g - A\bar{x}\rangle + \varphi(g) - \varphi(A\bar{x}) \geq 0, \quad \forall g \in Q,$$

which implies $A\bar{x} \in GMEP(\Theta_2, h_2, \varphi)$ and thus $\bar{x} \in \Omega$. Therefore $\bar{x} \in \Gamma = \Omega \cap F(S)$.

We now show that $\{x_n\}$ converges strongly to $z = P_\Gamma(I-D+\xi f)(z)$ which is the unique solution of the variational inequality (24). To do this, we first prove that $\limsup_{n\to\infty}\langle(D-\xi f)z, z-x_n\rangle \le 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup\langle(D-\xi f)z, z-x_n\rangle = \lim_{j\to\infty}\langle(D-\xi f)z, z-x_{n_j}\rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$, we get

$$\begin{aligned}\limsup\langle(D-\xi f)z, z-x_n\rangle &= \lim_{j\to\infty}\langle(D-\xi f)z, z-x_{n_j}\rangle\\ &= \langle(D-\xi f)z, z-\bar{x}\rangle \le 0.\end{aligned}$$

Now from (27), we have

$$\begin{aligned}||x_{n+1}-z||^2 &\le (1-\alpha_n\bar{\gamma})^2\left[||x_n-z||^2+2\alpha^4N_2\right]+2\alpha_n\xi\beta||x_n-z||\cdot||x_{n+1}-z||+2\alpha_n\langle\xi f(z)-Dz, x_{n+1}-z\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2||x_n-z||^2+\alpha_n\xi\beta(||x_n-z||^2+||x_{n+1}-z||^2)+2\alpha_n\langle\xi f(z)-Dz, x_{n+1}-z\rangle+2\alpha_n^4N_2\\ &\le (1-\alpha_n\bar{\gamma})||x_n-z||^2+\alpha_n\xi\beta(||x_n-z||^2+||x_{n+1}-z||^2)+2\alpha_n\langle\xi f(z)-Dz, x_{n+1}-z\rangle+2\alpha_n^4N_2\\ &\le \left(1-\frac{\alpha_n(\bar{\gamma}-\xi\beta)}{1-\alpha_n\xi\beta}\right)||x_n-z||^2+\frac{2\alpha_n}{1-\alpha_n\xi\beta}\left(\langle\xi f(z)-Dz, x_{n+1}-z\rangle+\alpha_n^3N_2\right)\\ &= (1-\nu_n)||x_n-z||+\nu_n\delta_n,\end{aligned}\tag{52}$$

where

$$\nu_n = \frac{\alpha_n(\bar{\gamma}-\xi\beta)}{1-\alpha_n\xi\beta} \quad\text{and}\quad \delta_n = \frac{2}{\bar{\gamma}-\xi\beta}[\langle\xi f(z)-Dz, x_{n+1}-z\rangle+\alpha_n^3N_2].$$

It is easy to verify that $\sum_{n=0}^{\infty}\nu_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Therefore, from Lemma 2.5, we get $||x_n-z|| \to 0$, as $n\to\infty$ and hence $\{x_n\}$ converges strongly to z. From (46) and (47), it is easy to see that $\{y_n\}$ and $\{z_n\}$ converge strongly z.

Case II: Assume that $\{||x_n-p||\}$ is not monotonically decreasing. For all $n \ge n_0$ (for some n_0 large enough), let $\tau:\mathbb{N}\to\mathbb{N}$ be defined by

$$\tau(n) = \max\{k\in\mathbb{N} : k\le n : \tau_k \le \tau_{k+1}\}.$$

Clearly, τ is nondecreasing since $\tau(n)\to\infty$ as $n\to\infty$ and

$$0 \le ||x_{\tau(n)}-p|| \le ||x_{\tau(n)+1}-p||,\quad \forall n\ge n_0.$$

Following a similar argument as in Case I, we have $||(I-T^{\Theta_2}_{r_{\tau(n)}})Ax_{\tau(n)}|| \to 0$, $||Sz_{\tau(n)}-z_{\tau(n)}|| \to 0$, and $||x_{\tau(n)+1}-x_{\tau(n)}|| \to 0$. Also, we obtain

$$\limsup_{n\to\infty}\langle(D-\xi f)p, p-x_{\tau(n)}\rangle \le 0.$$

Now, since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ denoted by $\{x_{\tau(n_j)}\}$ which converges weakly to $\bar{x}$. Suppose $\{x_{\tau(n_j)}\}$ is such that

$$\limsup_{n\to\infty}\langle\xi f(p)-Dp, x_{\tau(n)+1}-p\rangle = \lim_{j\to\infty}\langle\xi f(p)-Dp, x_{\tau(n_j)+1}-p\rangle.$$

Since $x_{\tau(n)} \rightharpoonup \bar{x}$, and from (24), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\xi f(p)-Dp, x_{\tau(n)+1}-p\rangle &= \lim_{j\to\infty}\langle\xi f(p)-Dp, x_{\tau(n_j)+1}-p\rangle\\ &= \langle\xi f(p)-Dp, \bar{x}-p\rangle \le 0.\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\langle\xi f(p)-Dp, x_{\tau(n)+1}-p\rangle \le 0.\tag{53}$$

Similarly, as in (52) we obtain

$$\begin{aligned} ||x_{\tau(n)+1} - p||^2 &\le (1-\alpha_{\tau(n)}\bar{\gamma})^2\Big[||x_{\tau(n)} - p||^2 + 2\alpha^4 N_2\Big] + 2\alpha_{\tau(n)}\xi\beta||x_{\tau(n)} - p|| \cdot ||x_{\tau(n)+1} - p|| \\ &\quad + 2\alpha_{\tau(n)}\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle \\ &\le \left(1 - \frac{\alpha_{\tau(n)}(\bar{\gamma} - \xi\beta)}{1 - \alpha_{\tau(n)}\xi\beta}\right)||x_{\tau(n)} - p||^2 + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\xi\beta}[\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle + \alpha_n^3 N_2]. \end{aligned} \tag{54}$$

Since $||x_{\tau(n)} - p||^2 \le ||x_{\tau(n)+1} - p||^2$, then from (54), we have

$$\begin{aligned} 0 &\le ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\le \left(1 - \frac{\alpha_{\tau(n)}(\bar{\gamma} - \xi\beta)}{1 - \alpha_{\tau(n)}\xi\beta}\right)||x_{\tau(n)} - p||^2 + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\xi\beta}[\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle + \alpha_n^3 N_2] - ||x_{\tau(n)} - p||^2. \end{aligned}$$

It follows that

$$\frac{\bar{\gamma} - \xi\beta}{1 - \alpha_{\tau(n)}\xi\beta}||x_{\tau(n)} - p||^2 \le \frac{2}{1 - \alpha_{\tau(n)}\xi\beta}[\langle \xi f(p) - Dp, x_{\tau(n)+1} - p\rangle + \alpha_n^3 N_2]. \tag{55}$$

Since $\alpha_{\tau(n)} \to 0$, as $n \to \infty$ and from (53), we have

$$\lim_{n\to\infty} ||x_{\tau(n)} - p|| = 0. \tag{56}$$

As a consequence, we obtain for all $n \ge n_0$,

$$0 \le ||x_n - p||^2 \le \max\{||x_{\tau(n)} - p||^2, ||x_{\tau(n)+1} - p||^2\} = ||x_{\tau(n)+1} - p||^2.$$

Hence, $\lim_{n\to\infty} ||x_n - p|| = 0$. This implies that $\{x_n\}$ converges strongly to p. This complete the proof. □

We now give the following consequences of Theorem 3.6.
1. Consider the following split mixed equilibrium problem. Find $x^\star \in C$ such that

$$\begin{cases} \Theta_1(x^\star, x) + \phi(x) - \phi(x^\star) \ge 0, \quad \forall x \in C, \\ \text{with} \\ y^\star = Ax^\star \quad \text{which solves} \quad \Theta_2(y^\star, y) + \varphi(y^\star) - \varphi(y) \ge 0, \quad \forall y \in Q. \end{cases} \tag{57}$$

The set of solutions of (57) is denoted by $MEP(\Theta_1, \Theta_2, \phi, \varphi)$. In [44], the authors proved a weak convergence theorem for solving (57) and a fixed point problem of a nonlinear multi-valued mapping in real Hilbert spaces. Putting $h_1 = h_2 = 0$ in Theorem 3.6, we obtain a strong convergence result for approximating a common solution of (57) and a fixed point problem for nonspreading mappings without prior knowledge of the operator norm in real Hilbert spaces. Thus, the following result complements the result in [44].

Corollary 3.7. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions which satisfy Assumption 2.3. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0,1)$ and let D be a bounded linear operator with coefficient $\bar{\gamma} \in (0,1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial guess $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0,1]$, $\beta_n \in [0,1]$, $w_n \in (0,1)$, $r_n > 0$ and $\lambda > 0$. Assume that the nth iterate has been constructed, and set $m_1 = \frac{\gamma_1 A^*(T_{r_1}^{\Theta_2} - I)Ax_1}{\lambda}$. We then compute the $(n+1)$th iterate via the formula*

$$\begin{cases} m_{n+1} = \dfrac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n, \\ y_n = x_n + \lambda m_{n+1}, \\ z_n = T_{r_n}^{\Theta_1} y_n, \\ x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)[(1 - w_n)z_n + w_n S z_n], \end{cases} \tag{58}$$

for $n \geq 1$, where A^* is the adjoint operator of A. Further, we choose the stepsize γ_n such that, if

$$n \in O := \{n : (I - T_{r_n}^{\Theta_2})Ax_n \neq 0\},$$

then

$$\gamma_n \in \left(0, \frac{2||(I - T_{r_n}^{\Theta_2})x_n||^2}{||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2}\right), \quad \forall n \in O. \tag{59}$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value). Suppose $\Gamma := MEP(\Theta_1, \Theta_2, \phi, \varphi) \cap F(S) \neq \emptyset$, and that Assumption 3.4 and the following condition is satisfied:

$$\liminf_{n\to\infty} r_n > 0.$$

Then, $\{x_n\}$ converges strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of

$$\langle (D - \xi f)z, z - x\rangle \leq 0, \quad x \in \Gamma.$$

2. In [38], Suantai *et.al.* proved a weak convergence result for finding a common solution of Problem (9) and a fixed point problem of a $\frac{1}{2}$-nonspreading multi-valued mapping in real Hilbert space.
Putting $h_1 = h_2 = \phi = \varphi = 0$ in Theorem 3.6, we obtain a strong convergence result for approximating a common solution of (9) and a fixed point of a nonspreading mappings without prior knowledge of the operator norm. Thus, the following result complements the result of Suantai *et.al.* [38].

Corollary 3.8. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\Theta_1 : C\times C \to \mathbb{R}$ and $\Theta_2 : Q\times Q \to \mathbb{R}$ be bifunctions which satisfy Assumption 2.3. Let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0, 1)$, and let D be a bounded linear operator with coefficient $\bar{\gamma} \in (0, 1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial guess $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0, 1]$, $\beta_n \in [0, 1]$, $w_n \in (0, 1)$, $r_n > 0$ and $\lambda > 0$. Assume that the nth iterate has been constructed, and set $m_1 = \frac{\gamma_1 A^*(T_{r_1}^{\Theta_2} - I)Ax_1}{\lambda}$. We then compute the $(n + 1)$th iterate via the formula*

$$\begin{cases} m_{n+1} = \dfrac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n, \\ y_n = x_n + \lambda m_{n+1}, \\ z_n = T_{r_n}^{\Theta_1} y_n, \\ x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)[(1 - w_n)z_n + w_n S z_n], \end{cases} \tag{60}$$

for $n \geq 1$, where A^* is the adjoint operator of A. Further, we choose the stepsize γ_n such that, if

$$n \in O := \{n : (I - T_{r_n}^{\Theta_2})Ax_n \neq 0\},$$

then

$$\gamma_n \in \left(0, \frac{2||(I - T_{r_n}^{\Theta_2})x_n||^2}{||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2}\right), \quad \forall n \in O. \tag{61}$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value). Suppose $\Gamma := SEP(\Theta_1, \Theta_2) \cap F(S) \neq \emptyset$, Assumption 3.4 is satisfied, and further suppose that the following condition is satisfied:

$$\liminf_{n\to\infty} r_n > 0.$$

Then, $\{x_n\}$ converges strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of

$$\langle (D - \xi f)z, z - x\rangle \leq 0, \quad x \in \Gamma.$$

3. Let $S : C \to C$ be a nonexpansive mapping in Theorem 3.6, then we have the following result:

Corollary 3.9. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions which satisfy Assumption 2.3. Let $h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be θ_1, θ_2-inverse strongly monotone mappings respectively such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0, 1)$, and let D be a bounded linear operator with coefficient $\bar{\gamma} \in (0, 1)$ such that $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Choose an initial guess $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0, 1]$, $\beta_n \in [0, 1]$, $w_n \in (0, 1)$, $r_n \in (0, 2\theta)$ and $\lambda > 0$. Assume that the nth iterate has been constructed, and set $m_1 = \frac{\gamma_1 A^*(T_{r_1}^{\Theta_2} - I)Ax_1}{\lambda}$. We then compute the $(n + 1)$th iterate via the formula*

$$\begin{cases} m_{n+1} = \dfrac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n, \\ y_n = x_n + \lambda m_{n+1}, \\ z_n = T_{r_n}^{\Theta_1}(I - r_n h_1)y_n, \\ x_{n+1} = \alpha_n \xi f(x_n) + (1 - \alpha_n D)[(1 - w_n)z_n + w_n S z_n], \end{cases} \tag{62}$$

for $n \geq 1$, where A^* is the adjoint operator of A. Further, we choose the stepsize γ_n such that, if

$$n \in O := \{n : (I - T_{r_n}^{\Theta_2})Ax_n \neq 0\},$$

then

$$\gamma_n \in \left(0, \frac{2\|(I - T_{r_n}^{\Theta_2})x_n\|^2}{\|A^*(I - T_{r_n}^{\Theta_2})Ax_n\|^2}\right), \quad \forall n \in O. \tag{63}$$

Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value). Suppose $\Gamma := \Omega \cap F(S) \neq \emptyset$, Assumption 3.4 and the following conditions are satisfied:

(i) $\liminf_{n\to\infty} r_n > 0$;
(ii) $0 < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$.

Then $\{x_n\}$ converges strongly to a point z, where $z = P_\Gamma(I - D + \xi f)(z)$ is a unique solution of

$$\langle (D - \xi f)z, z - x\rangle \leq 0, \quad x \in \Gamma.$$

4. Putting $\xi = 1$ and $D = I$ where I is an identity mapping in Theorem 3.6, we have the following result:

Corollary 3.10. *Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\Theta_1 : C \times C \to \mathbb{R}$ and $\Theta_2 : Q \times Q \to \mathbb{R}$ be bifunctions which satisfy Assumption 2.3. Let $h_1 : C \to H_1$ and $h_2 : Q \to H_2$ be θ_1, θ_2-inverse strongly monotone mappings respectively such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper, lowersemicontinuous and convex functions, and let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\beta \in (0, 1)$. Choose an initial guess $x_1 \in H_1$ arbitrarily and let $\alpha_n \in [0, 1]$, $\beta_n \in [0, 1]$, $w_n \in (0, 1)$, $r_n \in (0, 2\theta)$ and $\lambda > 0$. Assume that the nth iterate has been constructed, and set $m_1 = \frac{\gamma_1 A^*(T_{r_1}^{\Theta_2} - I)Ax_1}{\lambda}$. We then compute the $(n + 1)$th iterate via the formula*

$$\begin{cases} m_{n+1} = \dfrac{\gamma_n A^*(T_{r_n}^{\Theta_2} - I)Ax_n}{\lambda} + \beta_n m_n, \\ y_n = x_n + \lambda m_{n+1}, \\ z_n = T_{r_n}^{\Theta_1}(I - r_n h_1)y_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[(1 - w_n)z_n + w_n S z_n], \end{cases} \tag{64}$$

for $n \geq 1$, where A^* is the adjoint operator of A. Further, we choose the stepsize γ_n such that, if

$$n \in O := \{n : (I - T_{r_n}^{\Theta_2})Ax_n \neq 0\},$$

then

$$\gamma_n \in \left(0, \frac{2||(I - T_{r_n}^{\Theta_2})x_n||^2}{||A^*(I - T_{r_n}^{\Theta_2})Ax_n||^2}\right), \quad \forall n \in O. \tag{65}$$

Otherwise, $\gamma_n = \gamma$ *(*γ *being any nonnegative value). Suppose* $\Gamma := \Omega \cap F(S) \neq \emptyset$*, Assumption 3.4 and the following conditions are satisfied:*

(i) $\liminf_{n\to\infty} r_n > 0$*;*
(ii) $0 < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$*;*

then, $\{x_n\}$ *converges strongly to a point z, where* $z = P_\Gamma(f)z$ *is a unique solution of*

$$\langle (I - f)z, z - x\rangle \leq 0, \quad x \in \Gamma.$$

Remark 3.11. The condition that $\{(I - T_{r_n}^{\Theta_2})Ax_n\}$ is bounded is satisfied if the set of solutions Ω of SMEP (4) is bounded. If Ω is not bounded, then we need to verify the condition that $\{(I - T_{r_n}^{\Theta_2})Ax_n\}$ is bounded before applying our algorithm.

4 Numerical example

In this section, we provide a numerical result on the problem considered in Section 3.

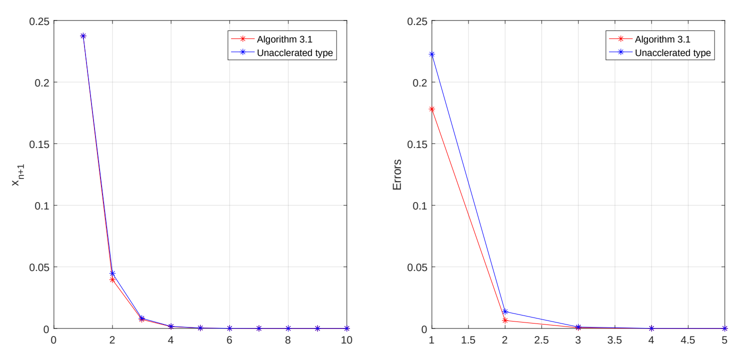

Figure 1: $x_1 = 0.2375$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 2]$. Define $\Theta_1 : C \times C \to \mathbb{R}$ by $\Theta_1(x, y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2$, $h_1 : C \to \mathbb{R}$ by $h_1(x) = x$ and $\phi : C \to \mathbb{R}$ by $\phi(x) = \frac{1}{2}x^2$. It is easy to see that

$$T_{r_n}^{\Theta_1}(z) = \frac{z}{3r_n + 1}, \quad \forall z \in \mathbb{R}.$$

Also, let $\Theta_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\Theta_2(u, v) = -3u^2 + 2uv + v^2$, $h_2 : Q \to \mathbb{R}$ be defined by $h_2(u) = 2u$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(u) = u^2$, then

$$T_{r_n}^{\Theta_2}(w) = \frac{w}{6r_n + 1}, \quad \forall w \in \mathbb{R}.$$

Let $A : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = 2x$ for all $x \in \mathbb{R}$. Then A is a bounded linear operator and $A^T(x) = 2x$ for all $x \in \mathbb{R}$. Clearly, $\Omega := \{p \in GMEP(\Theta_1, h_1, \phi) : Ap \in GMEP(\Theta_2, h_2, \varphi)\} = \{0\}$. This shows that Ω is bounded and thus, the sequence $\{(I - T_{r_n}^{\Theta_2})Ax_n\}$ is also bounded.

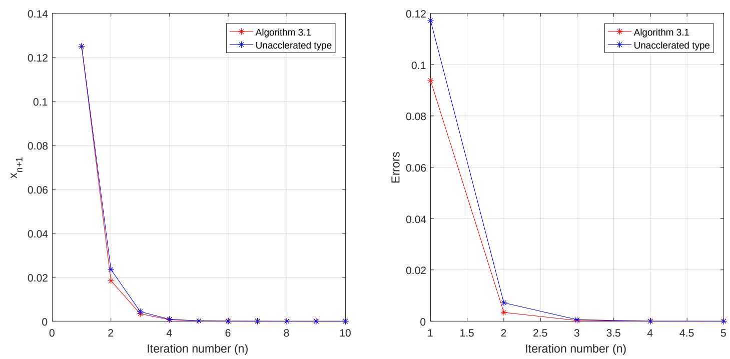

Figure 2: $x_1 = 0.1250$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

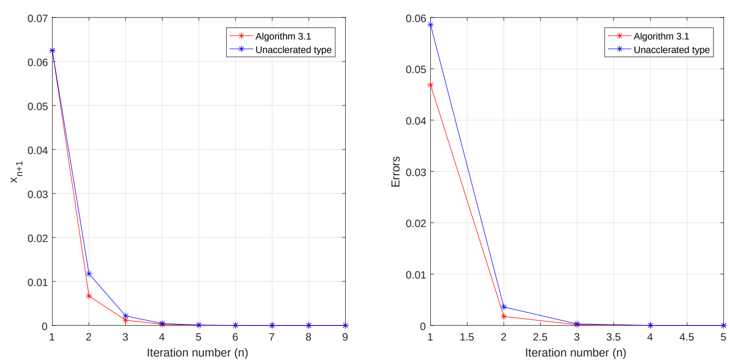

Figure 3: $x_1 = 0.0625$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

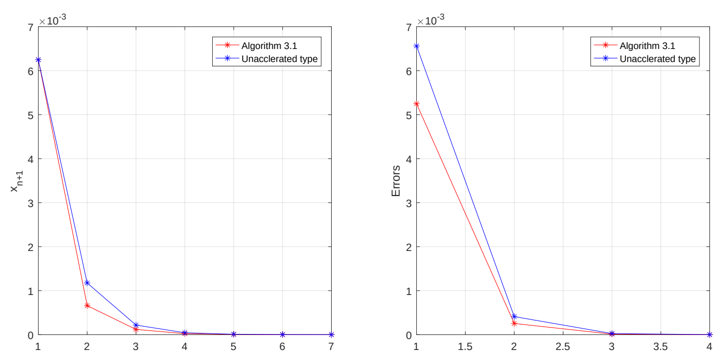

Figure 4: $x_1 = 0.007$, Left: accuracy against number of iterations; Right: errors against numbers of iterations.

Define $S : \mathbb{R} \to \mathbb{R}$ by

$$Sx = \begin{cases} x, & if \;\; x \in (-\infty, 1), \\ 1, & if \;\; x \in [1, +\infty). \end{cases} \tag{66}$$

It is easy to see that S is nonspreading and $\Gamma = \{0\}$. Take $\xi = 1$, $D = I$, where I is an identity mapping and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{2}$. Choose $\alpha_n = \dfrac{1}{n+1}$, $w_n = \dfrac{1}{5(1+\frac{1}{n})}$, $r_n = \dfrac{2}{n+1}$, $\beta_n = \dfrac{1}{2(n+1)^4}$ and $\lambda = 1.5$,

Table 1

Initial values	Algorithm 3.1	Unaccelerated alg.
$x_1 = 0.0005$	0.0011	0.0027
$x_1 = 0.01$	0.0020	0.0262
$x_1 = 0.1$	0.0026	0.0357
$x_1 = 10$	0.0038	0.1119

and set $m_1 = \frac{\gamma_1}{1.5}\left(\frac{-24r_n}{6r_n+1}\right)x_1$. Then Algorithm 3.1 gives the following:

$$\begin{cases} m_{n+1} = \frac{\gamma_n}{1.5}(\frac{-24r_n}{6r_n+1})x_n + \frac{m_n}{2(n+1)^4}, \\ y_n = x_n + 1.5m_{n+1}, \\ z_n = \frac{1}{3r_n+1}(\frac{n-1}{n+1})y_n, \\ x_{n+1} = \frac{1}{n+1}f(x_n) + \frac{n}{n+1}\left[\frac{4n+5}{5(n+1)}z_n + \frac{n}{5(n+1)}Sz_n\right], \quad n \geq 1. \end{cases} \tag{67}$$

We now make a different choice of the initial value x_1 and use $\epsilon < 10^{-6}$ for the stopping criterion.

Case 1: $x_1 = 0.2375$, Case 2: $x_1 = 0.1250$, Case 3: $x_1 = 0.0625$, Case 4: $x_1 = 0.007$.

We note that the choice of γ_n, as long as it is in the range, does not have any significant effect on either the number of iterations, nor the cpu time. Using Matlab version 2016b, we compare the computational result of Algorithm 3.1 with its unaccelerated form (i.e. taking $\beta_n = 0$) and plot the graphs of accuracy against number of iterations, and errors against number of iterations (see Figure 1-4 and Table 1) which are located after the references below. This shows that Algorithm 3.1 converges faster and is more efficient than its unaccelerated form (i.e. when $\beta_n = 0$).

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