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A UNIFIED APPROACH TO IMPROVED L^p HARDY INEQUALITIES WITH BEST CONSTANTS

G. BARBATIS, S. FILIPPAS, AND A. TERTIKAS

ABSTRACT. We present a unified approach to improved L^p Hardy inequalities in $\mathbf{R}^N$. We consider Hardy potentials that involve either the distance from a point, or the distance from the boundary, or even the intermediate case where the distance is taken from a surface of codimension $1 < k < N$. In our main result, we add to the right hand side of the classical Hardy inequality a weighted L^p norm with optimal weight and best constant. We also prove non-homogeneous improved Hardy inequalities, where the right hand side involves weighted L^q norms, $q \neq p$.

1. INTRODUCTION

The classical Hardy inequality asserts that for any $p > 1$

$$\int_{\mathbf{R}^N} |\nabla u|^p dx \geq \left|\frac{N-p}{p}\right|^p \int_{\mathbf{R}^N} \frac{|u|^p}{|x|^p} dx, \ \ u \in C_c^\infty(\mathbf{R}^N \setminus \{0\}), \tag{1.1}$$

with $|\frac{N-p}{p}|^p$ the best constant; see for example [HLP], [OK], [DH]. The best constant remains the same if $\mathbf{R}^N$ is replaced by a domain $\Omega \subset \mathbf{R}^N$ containing the origin. Moreover, if $\Omega \subset \mathbf{R}^N$ is a convex domain, possibly unbounded, with smooth boundary, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, the Hardy inequality

$$\int_{\Omega} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{d^p} dx, \ \ u \in C_c^\infty(\Omega), \tag{1.2}$$

has recently been established, with $(\frac{p-1}{p})^p$ the best constant; cf. [MS], [MMP]. See [OK] for a comprehensive account of Hardy inequalities and [D] for a review of recent results.

Recently improved versions of (1.1) and (1.2) have been obtained. In [BV] it is shown that for a bounded domain $\Omega \subset \mathbf{R}^N$

$$\int_{\Omega} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} dx \geq \Lambda_2 (\frac{\omega_N}{|\Omega|})^{2/N} \int_{\Omega} u^2 dx, \quad u \in C_c^\infty(\Omega), \tag{1.3}$$

where $\Lambda_2 = 5.783...$ is the square of the first zero of the Bessel function J_0. It was shown in [FT] that the constant $\Lambda_2(\omega_N/|\Omega|)^{2/N}$ is not optimal unless Ω is a ball centered at the origin. In [GGM] estimate (1.3) was generalized for $1 < p < N$. It

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was shown that

$$(1.4)\qquad \int_\Omega |\nabla u|^p dx - \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \ge C_{N,p} (\frac{\omega_N}{|\Omega|})^{p/N} \int_\Omega |u|^p dx,$$

for an explicitly given constant $C_{N,p} > 0$ that satisfies $C_{N,2} = \Lambda_2$.

In another direction, in [VZ], Hilbert space methods were used to derive the following improved Hardy-Poincaré inequality:

$$(1.5)\quad \int_\Omega |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \ge c \left(\int_\Omega |\nabla u|^q dx\right)^{2/q}, \quad u \in C_c^\infty(\Omega),$$

for a bounded domain Ω containing the origin and for any $1 \le q < 2$.

Analogous results have been obtained in the case of Hardy inequalities with distance from the boundary. In particular it was proved in [BM] that for bounded and convex domains

$$(1.6)\qquad \int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \ge \frac{1}{4L^2}\int_\Omega u^2 dx\ , \quad u \in C_c^\infty(\Omega),$$

and

$$(1.7)\qquad \int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{u^2}{d^2} dx \ge \frac{1}{4}\int_\Omega \frac{u^2}{d^2(1-\log(d/L))^2} dx\ , \quad u \in C_c^\infty(\Omega),$$

where $L = \mathrm{diam}(\Omega)$.

Hardy inequalities as well as their improved versions have various applications in the theory of partial differential equations and nonlinear analysis. They have been useful in the study of the stability of solutions of semilinear elliptic and parabolic equations [PV], [BV], [V] as well as in the existence and asymptotic behavior of the heat equation with singular potentials (cf. [BC], [CM], [VZ]; see also [GP] for the p-heat equation). They have also been used to investigate the stability of eigenvalues in elliptic problems [D, FHT].

In this work we present a general approach to improved Hardy inequalities valid for any $p > 1$ and for different choices of the distance function $d(x)$: besides the two cases above — distance from a point and distance from the boundary — we consider the more general case where $d(x)$ is the distance of $x \in \Omega$ from a piecewise smooth surface K of codimension k, $1 \le k \le N$. In the case $k = N$ we adopt the convention that K is a point.

In our approach the following geometric assumption on K and Ω is crucial: if $d(x) = \mathrm{dist}(x, K)$, then the following inequality should hold in the weak sense:

$$(\mathbf{C})\qquad p \ne k, \qquad \Delta_p d^{\frac{p-k}{p-1}} \le 0, \qquad \text{in } \Omega \setminus K.$$

Here Δ_p denotes the usual p-Laplace operator, $\Delta_p w = \mathrm{div}(|\nabla w|^{p-2}\nabla w)$. This condition is analyzed in detail in Section 2. Here we simply note that (C) is always satisfied when $k = N$ and $d(x)$ measures the distance from a point as well as when $k = 1$, Ω is convex and $d(x)$ is the distance from $K = \partial\Omega$. Condition (C) can be interpreted as a higher-codimension analogue of the usual convexity condition that appears in Hardy's inequality when $k = 1$ and $K = \partial\Omega$; cf. (1.2).

In order to describe our results we introduce the function

$$X(t) = -1/\log t, \quad t \in (0,1).$$

Our first theorem is the following:

Theorem A (Improved Hardy Inequality). *Let Ω be a domain in $\mathbf{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1, \dots, N$. Suppose* $\sup_{x\in\Omega} d(x,K) < \infty$ *and condition* (C) *is satisfied. Then*

(1) *There exists a positive constant $D_0 = D_0(k,p) \geq \sup_{x\in\Omega} d(x,K)$ such that for any $D \geq D_0$ and all $u \in W_0^{1,p}(\Omega \setminus K)$*

$$\int_\Omega |\nabla u|^p dx - \left|\frac{k-p}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geq \frac{p-1}{2p}\left|\frac{k-p}{p}\right|^{p-2} \int_\Omega \frac{|u|^p}{d^p} X^2(d/D) dx. \tag{1.8}$$

If in addition $2 \leq p < k$, then we can take $D_0 = \sup_{x\in\Omega} d(x,K)$.

(2) *Both constants appearing in* (1.8) *as well as the exponent two in X^2 are optimal in either of the following cases:*

(a) $k = N$ *and* $K = \{0\} \subset \Omega$;

(b) $k = 1$ *and* $K = \partial\Omega$;

(c) $2 \leq k \leq N-1$ *and* $\Omega \cap K \neq \emptyset$.

The optimality of the constants and the exponent is meant in the following sense:

$$\left|\frac{p-k}{p}\right|^p = \inf\Big\{\int_\Omega |\nabla u|^p dx, \quad \int_\Omega \frac{|u|^p}{d^p} dx = 1\Big\}.$$

Further, if $\gamma < 2$, then, no matter how large D is, there is no $c > 0$ such that

$$\int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geq c \int_\Omega \frac{|u|^p}{d^p} X^\gamma(d/D) dx;$$

and finally, for any $D \geq D_0$,

$$\frac{p-1}{2p}\left|\frac{p-k}{p}\right|^{p-2} = \inf\Big\{\int_\Omega |\nabla u|^p dx - \left|\frac{k-p}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx, \ \int_\Omega \frac{|u|^p}{d^p} X^2(d/D) dx = 1\Big\}.$$

A few remarks are in order:

1. The assumption $D \geq D_0$ is only necessary in order to obtain the precise constant $(p-1)/(2p)|(k-p)/p|^{p-2}$. We can take any $D > \sup_{x\in\Omega} d(x,K)$ at the expense of having a smaller constant $c = c(p,k,D)$ in the right hand side of (1.8).

2. The logarithmic correction in the right hand side is independent of $p > 1$. Also it is worth pointing out that the constant of the improved Hardy inequality depends only on p and k and not on K, the dimension N, or Ω. This is in contrast to the improved Hardy inequalities which involve the unweighted L^p norm in the right hand side (see e.g. (1.3), (1.6)).

3. A simple density argument shows that if $p < k$, then $W_0^{1,p}(\Omega \setminus K) = W_0^{1,p}(\Omega)$.

4. We only assume that $\operatorname{dist}(x,K)$ is bounded on Ω, not that Ω itself is bounded.

In the case $p = 2$ and $k = 1$, part (1) of Theorem A has been obtained in [BM] by a different method. We are aware of very few results in the literature for $1 < k < N$, concerning even the simple Hardy inequality with best constant; for the case $p = 2$ see [D, DM], and [M], Section 2.1.6.

We present two different approaches to the improved Hardy inequality. The first is based on a suitable change of variables [BM, BV, GGM, M]. While this method does not yield the optimal constant in the right hand side of (1.8), it has the advantage that it easily leads to nonhomogeneous improved Hardy inequalities. We note that in this method the arguments used for $1 < p < 2$ differ from those used for $p \geq 2$. The second approach is based on the careful choice of a suitable vector field and an elementary integral inequality and is the one that gives the sharp constants. It is remarkable that condition (C) comes up naturally in both approaches.

It is well known that for $k = N$ (distance from a point) there is no Hardy inequality if $p = N$. More generally there is no Hardy inequality if $p = k$, $1 \leq k \leq N$. For that case we provide a substitute for Hardy inequality with optimal weight and best constant; see Theorems 4.2 and 5.4.

We next consider nonhomogeneous improved Hardy inequalities which involve L^q norms with $q \neq p$, in the right hand side. In this direction we have the following:

Theorem B (Improved Hardy-Poincaré Inequality). *Let Ω be a domain in $\mathbf{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1, \ldots, N$. Suppose that $\sup_{x\in\Omega} \operatorname{dist}(x, K) < \infty$ and condition* (C) *is satisfied. Then*

(1) *For any $D > \sup_{x\in\Omega} \operatorname{dist}(x, K)$, $1 \leq q < p$ and $\beta > 1 + q/p$ there exists a positive constant $c > 0$ such that for all $u \in W_0^{1,p}(\Omega \setminus K)$*

$$\int_\Omega |\nabla u|^p dx - \left|\frac{k-p}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geq c \left(\int_\Omega |\nabla u|^q d^{k(-1+q/p)} X^\beta(d/D) dx \right)^{p/q}. \tag{1.9}$$

(2) *The estimate is sharp in the sense that the exponent of X in the right hand side of* (1.9) *cannot be smaller than $1 + q/p$, in either of the cases* (a), (b), (c) *of part* (2) *of Theorem* A.

For $p = 2$ and $k = N$ this strengthens inequality (1.5).

We next consider improved Hardy-Sobolev inequalities. Let $K = \{x \in \mathbf{R}^N \mid x_1 = x_2 = \ldots = x_k = 0\}$, $1 \leq k \leq N-1$. Then, in [M] the following inequality is established (see Corollary 3, Section 2.1.6) for any $2 < q \leq \frac{2N}{N-2}$:

$$\int_\Omega |\nabla u|^2 dx - \left|\frac{k-2}{2}\right|^2 \int_\Omega \frac{u^2}{d^2} dx \geq c \left(\int_\Omega |u|^q d^{-q-N+Nq/2} dx \right)^{2/q}, \tag{1.10}$$

for any $u \in C_c^\infty(\mathbf{R}^N \setminus K)$. The question was posed in [M] whether an analogue result holds for $p \neq 2$.

For $k = N$, that is, $K = \{0\} \subset \Omega$, a bounded domain in $\mathbf{R}^N$, an analogous inequality is shown in [BV], valid for any $2 \leq q < \frac{2N}{N-2}$:

$$\int_\Omega |\nabla u|^2 dx - \left|\frac{N-2}{2}\right|^2 \int_\Omega \frac{u^2}{|x|^2} dx \geq c \left(\int_\Omega |u|^q dx \right)^{2/q}.$$

Our result reads:

Theorem C (Improved Hardy-Sobolev Inequality). (1) *Let $K = \{x \in \mathbf{R}^N \mid x_1 = x_2 = \ldots = x_k = 0\}$, $1 \leq k \leq N-1$. Assume that $2 \leq p < N$ and $p < q \leq Np/(N-p)$. Then there exists a constant $c > 0$ such that for all $u \in W_0^{1,p}(\mathbf{R}^N \setminus K)$*

$$\int_{\mathbf{R}^N} |\nabla u|^p dx - \left|\frac{k-p}{p}\right|^p \int_{\mathbf{R}^N} \frac{|u|^p}{d^p} dx \geq c \left(\int_{\mathbf{R}^N} |u|^q d^{-q-N+Nq/p} dx \right)^{p/q}. \tag{1.11}$$

(2) *Let $k = N$, that is, $K = \{0\} \subset \Omega$, a bounded domain in $\mathbf{R}^N$. Assume that $1 < p < N$ and $p \leq q < Np/(N-p)$. Then for any $D > \sup_\Omega d(x)$ there exists a*

constant $c>0$ such that for all $u \in W_0^{1,p}(\Omega)$

(1.12)

$$\int_\Omega |\nabla u|^p dx - \left|\frac{k-p}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geq c \left(\int_\Omega |u|^q d^{-q-N+Nq/p} X^{1+q/p}(d/D) dx \right)^{p/q}.$$

Inequality (1.12) is optimal in the sense that $X^{1+q/p}$ cannot be replaced by a smaller power of X.

A simple scaling argument shows that the exponent of d in (1.11) is optimal. Hence it comes as a remarkable fact that the case $k=N$ is different from the case $k<N$. It is an open question whether (1.12) remains true in the critical case $q=Np/(N-p)$. One can see that for $q=Np/(N-p)$ one cannot have an inequality (1.12) without the presence of the logarithmic correction. In fact one cannot even have the weak $L^{Np/(N-p)}$ norm in the right hand side; see Proposition 6.3. On the other hand inequality (1.12) is true in the critical case if we replace $X^{1+q/p}$ by $X^{2q/p}$. This last result is contained in Theorem 6.4, where an inequality weaker than (1.11) and (1.12) is shown for the general case where $k \leq N$ and K is nonaffine.

The structure of the paper is as follows: in Section 2 we discuss the geometric assumptions on Ω and K; in particular we provide specific examples for which condition (C) is satisfied. Section 3 contains our first approach to improved Hardy inequalities, whereas Section 4 is devoted to the vector field approach which yields the best constants. In Section 5 we prove the optimality of the constants involved in Theorem A. Finally in Section 6, we use the results of Section 3 to obtain nonhomogeneous inequalities.

2. The geometry of K and Ω

In this section we shall introduce the main geometric assumptions concerning K and Ω, and we will fix some notational conventions. Throughout this work Ω is a domain in $\mathbf{R}^N$ and K is a piecewise smooth closed surface of codimension $k=2,3,\dots,N-1$. We also allow for the two extreme cases $k=1$ or N, with the following convention: If $k=N$ then K is reduced to a point, say the origin. If $k=1$, then we take K to be the boundary of Ω, that is, $K=\partial\Omega$.

In all cases we define the distance function $d(x)$ by

$$d(x) = \operatorname{dist}(x,K), \qquad x \in \Omega.$$

Hence for $k=N$ we have $d(x)=|x|$, whereas for $k=1$, $d(x)$ is the distance from the boundary of Ω. Let us note that $d(x)$ is a Lipschitz continuous function with $|\nabla d|=1$ a.e.

We now come to our main geometric assumption on K and Ω, expressed in terms of the distance function d. We introduce the following geometric condition:

$$(\mathbf{C}) \qquad p \neq k \quad \text{and} \quad \Delta_p d^{\frac{p-k}{p-1}} \leq 0 \quad \text{on } \Omega \setminus K.$$

Simple calculations give

$$\Delta_p d^{\frac{p-k}{p-1}} = \frac{p-k}{p-1}\left|\frac{p-k}{p-1}\right|^{p-2} d^{-k}(d\Delta d + (1-k)|\nabla d|^2),$$

so that, since $|\nabla d|=1$ a.e., an equivalent formulation of (C) is

$$(p-k)(d\Delta d + 1 - k) \leq 0 \quad \text{on} \quad \Omega \setminus K.$$

The precise meaning of the above condition is the following: we consider the linear functional

$$\begin{aligned} A[\phi] \;&:=\; -\int_\Omega |\nabla d|^2 \phi dx - \int_\Omega d\,\nabla d\cdot\nabla\phi\,dx + (1-k)\int_\Omega \phi dx \\ &=\; -\int_\Omega d\,\nabla d\cdot\nabla\phi\,dx - k\int_\Omega \phi dx, \quad \phi\in C_c^1(\Omega\setminus K), \end{aligned}$$

and require that for all nonnegative $\phi \in C_c^1(\Omega\setminus K)$ we have $(p-k)A[\phi]\le 0$. In this context, and in order to simplify our notation, we shall use the expression

$$\int_\Omega (d\Delta d + 1 - k)\phi\,dx, \quad \phi\in C_c^1(\Omega\setminus K),$$

to denote the functional $A[\phi]$. This allows us to perform formal integrations by parts as if Δd were a locally integrable function in Ω. Taking for instance $\phi=\psi/d$ in the definition above we obtain the relation

$$\int_\Omega \psi\Delta d\,dx = -\int_\Omega \nabla\psi\cdot\nabla d\,dx, \qquad \psi\in C_c^1(\Omega\setminus K).$$

This also justifies the following convention: assuming that (C) is satisfied, we define:

$$\int_\Omega |d\Delta d + 1 - k|\,\phi\,dx = \begin{cases} A[\phi] & \text{if } p<k, \\ -A[\phi] & \text{if } p>k; \end{cases}$$

this is a positive functional on $C_c^1(\Omega\setminus K)$ and it is then easily seen that

$$\left|\int_\Omega (d\Delta d + 1 - k)\,\phi\,dx\right| \le \int_\Omega |d\Delta d + 1 - k|\,|\phi|\,dx, \quad \phi\in C_c^1(\Omega\setminus K).$$

We next present some examples in which condition (C) is satisfied. The first two concern the cases $k=1$ and $k=N$, which are the most popular in the literature. Then we consider the intermediate cases $2\le k\le N-1$. One then is led to rather special assumptions on K and Ω. This is not due to lack of pairs (K,Ω) that satisfy (C); indeed, it is easy to see that given any K one can always find an Ω such that (C) is satisfied: simply take Ω to be any domain contained in the set

$$\{x\in\mathbf{R}^N \,:\, d\Delta d + 1 - k \ge 0 \;\;(\text{or} \le 0)\}.$$

An analytical description of such sets Ω is possible only after extra assumptions on K.

Example 1. Let $k=N$ so that $K=\{0\}$. Then $d(x)=|x|$ and $\Delta d^{2-N}=0$ away from $x=0$, hence condition (C) is satisfied for any $1<p<\infty$ and any $\Omega\subset\mathbf{R}^N$.

Example 2. Suppose that $k=1$, so that $K=\partial\Omega$. Then (C) is satisfied for all $1<p<\infty$ provided we make the additional assumption that Ω is *convex.* To see this we first claim that $d(x)$, $x\in\Omega$, is a concave function. Indeed, let $0<\lambda<1$, and x, y, $z=\lambda x+(1-\lambda)y$ be three points contained in Ω. Let $z_0\in\partial\Omega$ be a point that realizes the distance for z, that is, $d(z)=|z-z_0|$. We denote by T_{z_0} the hyperplane that contains z_0 and is orthogonal to the vector $z-z_0$. We also let x_0 and y_0 be the orthogonal projections of x and y onto T_{z_0}, respectively. It then follows by the convexity of Ω and a simple similarity argument that

$$d(z)=|z-z_0|=\lambda|x-x_0|+(1-\lambda)|y-y_0|\ge\lambda d(x)+(1-\lambda)d(y),$$

and the claim is proved. Since $d(x)$ is concave, we conclude from Theorem 6.3.2 of [EG] that Δd is nonpositive in the weak sense; more precisely there exists a nonnegative Radon measure $d\mu$ on Ω satisfying

$$\int_\Omega \nabla\psi \cdot \nabla d \, dx = \int_\Omega \psi d\mu, \quad \psi \in C_c^1(\Omega). \tag{2.1}$$

In particular, taking as test function $\psi = \phi d$, we see that $A[\phi] \le 0$, that is, condition (C) is satisfied.

Let us now consider the intermediate cases $2 \le k \le N-1$.

Example 3. If K is affine, $K \equiv \mathbf{R}^{N-k}$, then condition (C) is satisfied for all $1 < p < \infty$ without any restriction on Ω.

Indeed, changing coordinates if necessary, we see by a direct computation that

$$\Delta_p d^{\frac{p-k}{p-1}} = 0, \quad x \in \mathbf{R}^N \setminus K.$$

Further, if $p > k$ and K is the union of affine sets,

$$K = \bigcup_{i \in I} K_i,$$

then (C) is also satisfied, again with no restriction on Ω. To see this consider the functions $d_i(x) = \mathrm{dist}(x, K_i)$, $i \in I$. We have seen that $d_i^{(p-k)/(p-1)}$ is p-harmonic. But

$$d^{\frac{p-k}{p-1}}(x) = \inf_{i \in I} d_i^{\frac{p-k}{p-1}}(x), \quad x \in \Omega \setminus K,$$

and hence $d^{\frac{p-k}{p-1}}$ is p-super-harmonic by the comparison principle for the p-Laplacian; see [HKM]. Alternatively, observing that

$$\Delta_p d^{\frac{p-k}{p-1}} = \frac{p-k}{p-1}\left|\frac{p-k}{p-1}\right|^{p-2} \frac{1}{2-k}\Delta d^{2-k},$$

we may use the corresponding principle for the Laplacian. (When $k = 2$ we replace $\frac{1}{2-k}\Delta d^{2-k}$ by $\Delta \log d$.)

Definition 2.1. Let $E \subset \mathbf{R}^N$ be an affine set of codimension $k-1$, let $V \subset E$ be a convex domain (i.e. connected and open in the topology of E), and let $K = \partial_E V$. The cylinder $V \times E^\perp$ is called the *inner canal* of K; the cylinder $(E \setminus \overline{V}) \times E^\perp$ is called the *outer canal* of K. (See also [S].)

Example 4. (i) If $p > k$ and Ω is contained in the inner canal of K, then (C) is satisfied; (ii) If $p < k$ and Ω is contained in the outer canal of K, then (C) is satisfied.

To see (i) let $\{T_y \,|\, y \in \partial_E V\}$ be the family of hyperplanes in E which are tangent to K (if K is not smooth, we take the supporting hyperplanes instead). If Ω is contained in the inner canal of Ω, then

$$d(x) = \inf_{y \in \partial_E V} \mathrm{dist}(x, T_y), \quad x \in \Omega,$$

and we are back in the situation of Example 3.

To prove (ii) we use a different argument. We write any $x \in \mathbf{R}^N$ as $x = (y, z)$ with $y \in E \equiv \mathbf{R}^{N-k+1}$ and $z \in \mathbf{R}^{k-1}$; that is, the projection of x onto E is the point $(y, 0)$. We then have

$$d^2(x) = \tilde{d}^2(y) + |z|^2, \qquad \tilde{d}(y) = \mathrm{dist}((y,0), K). \tag{2.2}$$

Differentiating twice with respect to z_i, $i = 1, 2, \ldots, k-1$, and summing up over i we obtain

$$|\nabla_z d|^2 + d\Delta_z d = k - 1. \tag{2.3}$$

Differentiating (2.2) with respect to y_i, $i = 1, 2, \ldots, N-k+1$, we obtain in a similar way

$$|\nabla_y d|^2 + d\Delta_y d = |\nabla_y \tilde{d}|^2 + \tilde{d}\Delta_y \tilde{d} = 1 + \tilde{d}\Delta_y \tilde{d}. \tag{2.4}$$

Adding (2.3) and (2.4) we conclude that

$$d\Delta d + (1-k)|\nabla d|^2 = \tilde{d}\Delta_y \tilde{d}.$$

Since $\tilde{d}$ is the distance function in $E \equiv \mathbf{R}^{N-k+1}$ and $V \subset E$ is a convex domain, we have, as in Example 2, that $\Delta_y \tilde{d} \geq 0$ if $y \in V^c$. Hence (C) is satisfied in this case.

We point out that if a domain Ω satisfies $\overline{\Omega} \cap K \neq \emptyset$ (so that d^{-1} is singular in Ω), then for it to be contained in either the inner or the outer canal of K it is necessary that $K \cap \partial\Omega \neq \emptyset$.

Our fifth example combines ideas from the last two.

Example 5. Assume that $p > k$ and that Ω is contained in the inner canal of $L = \partial V$. Let K be a polytope contained in V and having its vertices on L. Then condition (C) is satisfied. To see this let F_i, $i = 1, \ldots, L$, be the faces of K. Our assumption on K and Ω imply that the distance of any $x \in \Omega$ from a face F_i is realized at a point $y \in F_i$ which is on the interior of F_i, that is, the distance is not realized at vertices, edges, etc. Hence

$$\Delta_p d_i^{\frac{p-k}{p-1}} = 0, \quad x \in \Omega \setminus F_i, \quad d_i(x) = \mathrm{dist}(x, F_i),$$

and the comparison argument of Example 3 goes through.

3. The improved Hardy inequality

In this section we give a first proof of the improved Hardy inequality and also obtain some inequalities which will be of use in Section 6. We start with some elementary pointwise inequalities.

Lemma 3.1. *For any $1 < p < \infty$ there exists a constant $c = c(p) > 0$ such that for all $a, b \in \mathbf{R}^N$ we have:*

(i) *if $1 < p < 2$, then*

$$|a-b|^p - |a|^p \geq c\frac{|b|^2}{(|a|+|b|)^{2-p}} - p|a|^{p-2}a \cdot b;$$

(ii) *if $p \geq 2$, then*

(a) $|a-b|^p - |a|^p \geq c|a|^{p-2}|b|^2 - p|a|^{p-2}a \cdot b\,;$

(b) $|a-b|^p - |a|^p \geq c|b|^p - p|a|^{p-2}a \cdot b\,.$

Proof. Parts (i) and (ii)(b) are contained in Lemma 4.2 of [L]. Hence, we only prove (ii)(a).

If $|b| \geq \frac{1}{2}|a|$ the inequality follows from (ii)(b). Suppose now that $|b| < \frac{1}{2}|a|$; then $|a-\xi b| \geq \frac{1}{2}|a|$ for all $\xi \in (0,1)$. Hence, taking the Taylor expansion of $f(t) = |a-bt|^p$ around $t=0$ we have,

$$\begin{aligned} |a-b|^p &= |a|^p - p|a|^{p-2}a\cdot b + \frac{p(p-2)}{2}|a-\xi b|^{p-4}((a-\xi b)\cdot b)^2 + \\ &\quad +p|a-\xi b|^{p-2}|b|^2 \qquad\qquad (\text{some } \xi \in (0,1)) \\ &\geq |a|^p - p|a|^{p-2}a\cdot b + \frac{p}{2^{p-2}}|a|^{p-2}|b|^2. \quad \square \end{aligned}$$

We next prove an auxiliary inequality that will be used in the sequel. Let us first recall that

$$X(s) = -\frac{1}{\log s}, \qquad s \in (0,1).$$

Note that if $D > \sup_{x\in\Omega} d(x)$, then

$$0 \leq X(d(x)/D) \leq M, \qquad x \in \Omega,$$

for a suitable positive constant $M = M(D)$. Furthermore, we shall often use the relation

$$\frac{d}{dr}X^\beta = \beta\frac{X^{\beta+1}}{r}, \tag{3.1}$$

as well as its integral version

$$\int_{s_1}^{s_2} r^{-1}X^{\beta+1}(r)dr = \frac{1}{\beta}[X^\beta(s_2) - X^\beta(s_1)]. \tag{3.2}$$

We next prove the following

Lemma 3.2. *Let $p>1$ and $\alpha \in \mathbf{R}$. Then for any $D \geq \sup_{x\in\Omega} d(x,K)$ we have*

$$\begin{aligned} \left(\frac{|\alpha-1|}{p}\right)^p \int_\Omega |v|^p d^{-k} X^\alpha(d/D)dx &\leq \int_\Omega |\nabla v|^p d^{p-k} X^{\alpha-p}(d/D)dx \\ &+ \left(\frac{|\alpha-1|}{p}\right)^{p-1} \int_\Omega |v|^p d^{-k}\,|d\Delta d + 1 - k|\,X^{\alpha-1}(d/D)dx \end{aligned} \tag{3.3}$$

for all $v \in C_c^\infty(\Omega\setminus K)$.

Proof. We prove (3.3) for $D=1$, the general case following by scaling. For $\alpha=1$ (3.3) is trivial, so we assume $\alpha \neq 1$. Recalling (3.1) we have

$$\begin{aligned} &\int_\Omega |v|^p d^{-k} X^\alpha(d)dx \\ &= \frac{1}{\alpha-1}\int_\Omega |v|^p d^{1-k}\nabla d\cdot\nabla X^{\alpha-1}(d)dx \\ &= -\frac{p}{\alpha-1}\int_\Omega |v|^{p-2}vX^{\alpha-1}(d)d^{1-k}\nabla v\cdot\nabla d\,dx \\ &\quad -\frac{1}{\alpha-1}\int_\Omega |v|^p d^{-k}(d\Delta d + (1-k)|\nabla d|^2)X^{\alpha-1}(d)\,dx \\ &\leq \frac{p}{|\alpha-1|}\left(\int_\Omega |\nabla v|^p d^{p-k}X^{\alpha-p}(d)dx\right)^{1/p}\left(\int_\Omega |v|^p d^{-k}X^\alpha(d)dx\right)^{(p-1)/p} \\ &\quad +\frac{1}{|\alpha-1|}\int_\Omega |v|^p d^{-k}\,|d\Delta d + 1 - k|\,X^{\alpha-1}(d)\,dx. \end{aligned}$$

Hence we have an estimate of the form

$$B \le \theta B^{(p-1)/p}\Gamma^{1/p} + A, \quad \theta = \frac{p}{|\alpha - 1|}.$$

Combining this with the relation

$$B^{(p-1)/p}\Gamma^{1/p} \le \frac{\epsilon(p-1)}{p}B + \frac{\epsilon^{-(p-1)}}{p}\Gamma,$$

and taking $\epsilon = \theta^{-1}$, we obtain $\theta^{-p}B \le \Gamma + p\theta^{-p}A$, which is the required inequality. □

Throughout the paper we will use the notation

$$I[u] = \int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p}dx, \quad u \in W_0^{1,p}(\Omega \setminus K), \tag{3.4}$$

and

$$H = \frac{k-p}{p}.$$

Our starting point is the following lower estimate on $I[u]$.

Lemma 3.3. *Let $u \in W_0^{1,p}(\Omega \setminus K)$ be given and set $v(x) = u(x)d^H(x)$. There exists a constant $c = c(p) > 0$ such that:* (i) *if $1 < p < 2$, then*

$$I[u] \ge c\int_\Omega \frac{|\nabla v|^2 d^{2-k}}{(|Hv| + |d\nabla v|)^{2-p}}dx + H|H|^{p-2}\int_\Omega |v|^p d^{-k}(d\Delta d + 1 - k)dx; \tag{3.5}$$

(ii) *if $2 \le p < \infty$, then*

$$I[u] \ge c|H|^{p-2}\int_\Omega |\nabla v|^2|v|^{p-2}d^{2-k}\,dx + H|H|^{p-2}\int_\Omega |v|^p d^{-k}(d\Delta d + 1 - k)dx, \tag{3.6}$$

$$I[u] \ge c\int_\Omega |\nabla v|^p d^{p-k}dx + H|H|^{p-2}\int_\Omega |v|^p d^{-k}(d\Delta d + 1 - k)dx. \tag{3.7}$$

Proof. It is straightforward to see that

$$|\nabla u|^p - |H|^p\frac{|u|^p}{d^p} = d^{-k}\left(|Hv\nabla d - d\nabla v|^p - |Hv|^p\right);$$

to estimate the right hand side we use the corresponding inequalities of Lemma 3.1 with $a = Hv\nabla d$ and $b = d\nabla v$. The expression $-p\int_\Omega d^{-k}|a|^{p-2}a\cdot b$ appears in all three cases and is equal to $H|H|^{p-2}\int_\Omega |v|^p d^{-k}(d\Delta d + 1 - k)dx$ as can be seen by an integration by parts. The stated estimates then follow at once. □

It should be noted that if condition (C) is satisfied, then the common term that appears in the right hand side of the three inequalities of the last lemma is equal to

$$|H|^{p-1}\int_\Omega |v|^p d^{-k}\,|d\Delta d + 1 - k|\,dx,$$

and, in particular, is nonnegative.

We next prove the improved Hardy inequality for $1 < p < 2$.

Proposition 3.4. *Let $1 < p < 2$. Given $u \in W_0^{1,p}(\Omega \backslash K)$ we set $v(x) = u(x)d^H(x)$, $H = (k-p)/p$. If condition* (C) *is satisfied, then for any $D > \sup_\Omega d(x)$ there exist constants $c_i = c_i(p,k,D) > 0$, $i = 1,2$, such that*

$$\begin{aligned} I[u] &\geq c_1 \left(\int_\Omega |\nabla v|^p d^{p-k} X^{2-p}(d/D)\, dx + \int_\Omega |v|^p d^{-k} \left| d\Delta d + 1 - k \right| \, dx \right) \\ &\geq c_2 \int_\Omega \frac{|u|^p}{d^p} X^2(d/D)\, dx. \end{aligned} \tag{3.8}$$

Proof. We may assume that $D = 1$, the general case following by scaling. To simplify the subsequent calculations we set

$$A_1 = \int_\Omega \frac{|\nabla v|^2 d^{2-k}}{(|Hv| + |d\nabla v|)^{2-p}} dx, \quad A_2 = \int_\Omega |v|^p d^{-k} X^2(d/D) dx$$
$$A_3 = \int_\Omega |\nabla v|^p d^{p-k} X^{2-p}(d/D) dx, \quad A_4 = \int_\Omega |v|^p d^{-k} |d\Delta d + 1 - k| dx.$$

Note that all A_i's are positive and homogeneous of degree p in v. Hölder's inequality and elementary estimates yield

$$\begin{aligned} A_3 &= \int_\Omega \frac{|\nabla v|^2 d^{p(2-k)/2}}{(|Hv| + |d\nabla v|)^{p(2-p)/2}} \cdot (|Hv| + |d\nabla v|)^{p(2-p)/2} \, d^{-k(2-p)/2} X^{2-p} dx \\ &\leq A_1^{p/2} \left(\int_\Omega (|Hv| + |d\nabla v|)^p \, d^{-k} X^2 dx \right)^{(2-p)/2} \\ &\leq c A_1^{p/2} \left(\int_\Omega |v|^p d^{-k} X^2 dx + \int_\Omega |\nabla v|^p d^{p-k} X^2 dx \right)^{(2-p)/2} \\ &\leq c A_1^{p/2} (A_2 + A_3)^{(2-p)/2}, \end{aligned}$$

that is,

$$A_1 \geq c \frac{A_3^{2/p}}{(A_2 + A_3)^{(2-p)/p}}. \tag{3.9}$$

Now, it follows from (3.5) that

$$I[u] \geq c(A_1 + A_4). \tag{3.10}$$

We also have from Lemma 3.2 (with $\alpha = 2$) that

$$A_2 \leq c(A_3 + A_4). \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) we obtain

$$\begin{aligned} I[u] &\geq c \left(\frac{A_3^{2/p}}{(A_2 + A_3)^{(2-p)/p}} + A_4 \right) \\ &\geq c \left(\frac{A_3^{2/p}}{(A_3 + A_4)^{(2-p)/p}} + A_4 \right) \\ &\geq c(A_3 + A_4), \end{aligned}$$

which is the first inequality in (3.8). Using (3.11) once more we have

$$I[u] \geq c(A_3 + A_4) \geq cA_2 = c\int_\Omega \frac{|u|^p}{d^p} X^2\, dx,$$

and the proof of (3.8) is complete. □

We now consider the complementary case $p \geq 2$.

Proposition 3.5. *Let $p \geq 2$. Given $u \in W_0^{1,p}(\Omega \setminus K)$ we set $v(x) = u(x)d^H$, $H = (k-p)/p$. If condition* (C) *is satisfied, then for any $D > \sup_\Omega d(x)$ there exists a constant $c = c(p,k,D) > 0$ such that*

$$I[u] \geq c\int_\Omega \frac{|u|^p}{d^p} X^2(d/D)\, dx. \tag{3.12}$$

Proof. We will use the additional change of variables $w = |v|^{p/2}$. It follows from (3.6) that

$$\begin{aligned} I[u] &\geq c\int_\Omega |\nabla v|^2 |v|^{p-2} d^{2-k}\, dx + c\int_\Omega |v|^p d^{-k}\, |d\Delta d + 1 - k|\, dx \\ &\geq c\int_\Omega |\nabla w|^2 d^{2-k}\, dx + c\int_\Omega |w|^2 d^{-k}\, |d\Delta d + 1 - k|\, X(d/D) dx \\ \text{(by (3.3))} \quad &\geq c\int_\Omega |w|^2 d^{-k} X^2(d/D) dx \\ &= c\int_\Omega \frac{|u|^p}{d^p} X^2(d/D)\, dx. \end{aligned}$$

□

4. The vector field approach

In this section we provide an alternative proof of the improved Hardy inequality, based on the appropriate use of a suitable vector field and elementary calculations. It is essential for this approach that all terms in the improved Hardy inequality are homogeneous with respect to u. It has the advantage that it allows us to compute explicit constants for the remainder term. In contrast, it does not work for nonhomogeneous inequalities. We retain the geometric assumptions introduced in Section 2. In the theorem that follows we consider the case $p \neq k$, while Theorem 4.2 below concerns the degenerate case $p = k$. The optimality of the estimates is proved in Section 5.

Let us recall the improved Hardy inequality, which we now write in the form

$$\int_\Omega |\nabla u|^p dx \geq |H|^p \int_\Omega \frac{|u|^p}{d^p} dx + B\int_\Omega \frac{|u|^p}{d^p} X^2(d/D) dx. \tag{4.1}$$

We then have

Theorem 4.1. *Assume that condition* (C) *is satisfied. Then, there exists a $D_0 = D_0(k,p) \geq \sup_\Omega d(x)$ such that for $D \geq D_0$, inequality* (4.1) *holds true with*

$$B = \frac{p-1}{2p} |H|^{p-2}.$$

If in addition $2 \leq p < k$, then we can take $D_0 = \sup_{x\in\Omega} d(x, K)$.

Proof. Let T be a vector field on Ω. For any $u \in C_c^\infty(\Omega \setminus K)$ we integrate by parts and use Hölder's inequality to obtain

$$\begin{aligned}\int_\Omega \mathrm{div}T\ |u|^p dx &= -p\int_\Omega (T\cdot\nabla u)|u|^{p-2}u dx\\ &\le p\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |T|^{\frac{p}{p-1}}|u|^p dx\right)^{\frac{p-1}{p}}\\ &\le \int_\Omega |\nabla u|^p dx + (p-1)\int_\Omega |T|^{\frac{p}{p-1}}|u|^p dx.\end{aligned}$$

We therefore arrive at

$$\int_\Omega |\nabla u|^p dx \ge \int_\Omega (\mathrm{div}T - (p-1)|T|^{\frac{p}{p-1}})|u|^p dx. \tag{4.2}$$

In view of this and (4.1), the improved Hardy inequality will be proved once we establish the following pointwise inequality:

$$\mathrm{div}T - (p-1)|T|^{\frac{p}{p-1}} \ge \frac{|H|^p}{d^p}\left(1+\frac{p-1}{2pH^2}X^2(d/D)\right), \quad x\in\Omega. \tag{4.3}$$

To proceed we now make a specific choice of T. We take

$$T(x) = H|H|^{p-2}\frac{\nabla d(x)}{d^{p-1}(x)}\left(1+\frac{p-1}{pH}X(d(x)/D) + aX^2(d(x)/D)\right),$$

where a is a free parameter to be chosen later. In any case a will be such that the quantity $1+\frac{p-1}{pH}X(d/D)+aX^2(d/D)$ is positive on Ω. Note that $T(x)$ is singular at $x\in K$, but since $u\in C_c^\infty(\Omega\setminus K)$, all previous calculations are legitimate. A simple computation shows that

$$\begin{aligned}\mathrm{div}T &= H|H|^{p-2}\frac{d\Delta d-(p-1)|\nabla d|^2}{d^p}\left(1+\frac{p-1}{pH}X + aX^2(d/D)\right)\\ &\quad +H|H|^{p-2}\frac{|\nabla d|^2}{d^p}\left(\frac{p-1}{pH}X^2(d/D)+2aX^3(d/D)\right)\\ &\ge H|H|^{p-2}\frac{k-p}{d^p}\left(1+\frac{p-1}{pH}X(d/D)+aX^2(d/D)\right)\\ &\quad +H|H|^{p-2}\frac{1}{d^p}\left(\frac{p-1}{pH}X^2(d/D)+2aX^3(d/D)\right),\end{aligned}$$

where in the last inequality we used (C) and the fact that $|\nabla d| = 1$. Thus, we have

$$\begin{aligned}&\mathrm{div}T-(p-1)|T|^{\frac{p}{p-1}}\\ &\qquad\ge H|H|^{p-2}\frac{(k-p)(1+\frac{p-1}{pH}X(d/D)+aX^2(d/D))}{d^p}\\ &\qquad\quad +H|H|^{p-2}\frac{(\frac{p-1}{pH}X^2(d/D)+2aX^3(d/D))}{d^p}\\ &\qquad\quad -(p-1)|H|^p\frac{(1+\frac{p-1}{pH}X(d/D)+aX^2(d/D))^{\frac{p}{p-1}}}{d^p}.\end{aligned}$$

It then follows that for (4.3) to hold, it is enough to establish the inequality

$$f(t)\ge 1+\frac{p-1}{2pH^2}t^2, \qquad t\in[0,M], \tag{4.4}$$

where $M = M(D) := \sup_{x\in\Omega} X(d(x)/D)$ and

$$f(t) := p(1 + \frac{p-1}{pH}t + at^2) + \frac{1}{H}(\frac{p-1}{pH}t^2 + 2at^3) - (p-1)(1 + \frac{p-1}{pH}t + at^2)^{\frac{p}{p-1}}.$$

From Taylor's formula we have that

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(\xi_t)t^2, \quad 0 \le \xi_t \le t \le M. \tag{4.5}$$

We have $f(0) = 1$. Moreover,

$$\begin{aligned} f'(t) &= \frac{p-1}{H} + 2apt + \frac{2(p-1)}{pH^2}t + \frac{6a}{H}t^2 \\ &\quad - p(1 + \frac{p-1}{pH}t + at^2)^{\frac{1}{p-1}}(\frac{p-1}{pH} + 2at), \\ f''(t) &= 2ap + \frac{2(p-1)}{pH^2} + \frac{12a}{H}t - 2ap(1 + \frac{p-1}{pH}t + at^2)^{\frac{1}{p-1}} \\ &\quad - \frac{p}{p-1}(1 + \frac{p-1}{pH}t + at^2)^{\frac{2-p}{p-1}}(\frac{p-1}{pH} + 2at)^2, \\ f'''(t) &= \frac{12a}{H} - \frac{6ap}{p-1}(1 + \frac{p-1}{pH}t + at^2)^{\frac{2-p}{p-1}}(\frac{p-1}{pH} + 2at) \\ &\quad - \frac{p(2-p)}{(p-1)^2}(1 + \frac{p-1}{pH}t + at^2)^{\frac{3-2p}{p-1}}(\frac{p-1}{pH} + 2at)^3, \end{aligned} \tag{4.6}$$

and in particular

$$\begin{aligned} f'(0) &= 0, \\ f''(0) &= \frac{p-1}{pH^2}, \\ f'''(0) &= \frac{6a}{H} - \frac{(2-p)(p-1)}{p^2H^3}. \end{aligned} \tag{4.7}$$

To proceed we distinguish various cases.

(a) $1 < p < 2 \le k$. In this case $H > 0$. We now choose a so that $f'''(0) > 0$, that is, $a > \frac{2-p}{6(p-1)} > 0$. Hence f'' is an increasing function in some interval of the form $(0, M_0)$. Consequently, for $t \in (0, M_0)$

$$f''(\xi_t) \ge f''(0) = \frac{p-1}{pH^2}.$$

It then follows from (4.5) that

$$f(t) \ge 1 + \frac{p-1}{2pH^2}t^2, \quad t \in [0, M_0].$$

Hence (4.4) has been proved in this case.

(b) $2 \le p < k$. We still have $H > 0$. We now choose $a = 0$. It is clear that $f'''(0) > 0$. Moreover, we compute

$$f'''(t) = \frac{(p-1)(p-2)}{p^2H^3}(1 + \frac{p-1}{pH}t)^{\frac{3-2p}{p-1}} > 0, \qquad \text{for all} \quad t > 0.$$

We then repeat the argument of case (a), taking $M_0 = +\infty$.

(c) $k = 1 < p < 2$. We now have $H < 0$. We then choose a such that $0 < a < (2-p)(p-1)/(6p^2H^2)$, so that $f'''(0) > 0$ and the previous argument goes through.

(d) $p \geq 2$, $p > k$. Again $H < 0$. We now take $a < (2-p)(p-1)/(6p^2H^2) < 0$ and proceed as before.

It is clear that we can choose an M_0 (small enough) that works simultaneously in all cases, and at the same time $(1 + \frac{p-1}{pH}X + aX^2) > 0$, for $0 < X < M_0$. We can even estimate this M_0 using (4.6), if needed. Since $X(d/D) = -\log^{-1}(d/D)$, the condition $X \leq M_0$ is equivalent to $D \geq D_0 := e^{1/M_0}\sup_{x\in\Omega} d(x)$. The proof of the theorem is now complete. $\square$

Remark. The assumption $\sup_{x\in\Omega} d(x) < +\infty$ is only needed in order to obtain the improved Hardy inequality. For the plain Hardy inequality one can choose the vector field $T(x) = H|H|^{p-2}\frac{\nabla d(x)}{d^{p-1}(x)}$, in which case the boundedness of $d(x)$ is not required.

Clearly the usual Hardy inequality does not hold when $p = k$. In our next result we give a substitute for the Hardy inequality in that case. The analogue of condition (C) is now

$$(\mathbf{C}') \qquad p = k, \qquad d\Delta d + 1 - k \geq 0.$$

In Theorem 5.4 we shall prove that estimate (4.8) below is sharp. Our result reads

Theorem 4.2. *Let $p = k$ and assume that $d(\cdot)$ is bounded in Ω. If* (C′) *is satisfied, then for any $D \geq \sup_\Omega d(x)$*

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} X^p(d/D)dx, \quad u \in W_0^{1,p}(\Omega\setminus K). \tag{4.8}$$

Proof. We define the vector field

$$T(x) = \left(\frac{p-1}{p}\right)^{p-1} \frac{X^{p-1}(d(x)/D)}{d^{p-1}(x)}\nabla d(x), \quad x \in \Omega,$$

and use inequality (4.2). We have

$$\begin{aligned}
\mathrm{div}T &= (\frac{p-1}{p})^{p-1} d^{-p} X^{p-1}(d/D)\Big((p-1)X(d/D) - p + 1 + d\Delta d\Big) \\
&\geq (\frac{p-1}{p})^{p-1}(p-1)d^{-p}X^p(d/D),
\end{aligned}$$

and hence

$$\mathrm{div}T - (p-1)|T|^{\frac{p}{p-1}} \geq (\frac{p-1}{p})^p d^{-p} X^p(d/D),$$

which yields (4.8). $\square$

5. Best constants for improved Hardy inequalities

In this section we will prove the optimality of the constants appearing in the improved Hardy inequalities we derived in Section 4. This will be done by deriving optimal bounds for all constants appearing in improved Hardy inequalities of the type we consider in this work. More precisely, recalling that $H = (k-p)/p$, we have the following theorem.

Theorem 5.1. *Let Ω be a domain in $\mathbf{R}^N$.* (i) *If $2 \le k \le N-1$, then we take K to be a piecewise smooth surface of codimension k and assume $K \cap \Omega \neq \emptyset$;* (ii) *if $k = N$, then we take $K = \{0\} \subset \Omega$;* (iii) *if $k = 1$, then we take $K = \partial\Omega$. Suppose that for some constants $A > 0$, $B \ge 0$, $\gamma > 0$ and $D \ge \sup_\Omega d(x)$, the following inequality holds true for all $u \in C_c^\infty(\Omega \setminus K)$:*

$$\int_\Omega |\nabla u|^p dx \ge A \int_\Omega \frac{|u|^p}{d^p} dx + B \int_\Omega \frac{|u|^p}{d^p} X^\gamma(d/D) dx. \tag{5.1}$$

Then:

(i) $A \le |H|^p$.

(ii) *If $A = |H|^p$, and $B > 0$, then $\gamma \ge 2$.*

(iii) *If $A = |H|^p$ and $\gamma = 2$, then $B \le \frac{p-1}{2p}|H|^{p-2}$.*

To prove this theorem we will use a minimizing sequence for the improved Hardy inequality. Without any loss of generality we may assume that $0 \in K \cap \Omega$ if $k \neq 1$, while $0 \in \partial\Omega$ if $k = 1$. All our analysis will be local, say, in a fixed ball of radius δ (denoted by B_δ) centered at the origin, for some fixed small δ. We next introduce the function

$$w_\epsilon(x) = d^{-H+\epsilon}(x) X^{-\theta}(d(x)/D), \quad 1/p < \theta < 2/p. \tag{5.2}$$

In order to localize it we also define a suitable nonnegative test function $\phi \in C_c^2(B_\delta)$ such that $\phi(x) = 1$ for $x \in B_{\delta/2}$. We then set

$$U_\epsilon(x) = \phi(x) w_\epsilon(x), \quad \mathrm{supp} U_\epsilon \subset B_\delta. \tag{5.3}$$

The proof we present works for any $k = 1, 2, \ldots, N$. We note however that for $k = N$ (distance from a point) the subsequent calculations are substantially simplified, whereas for $k = 1$ (distance from the boundary) one should replace B_δ by $B_\delta \cap \Omega$. This last change entails some minor modifications, the arguments otherwise being the same.

Throughout the rest of this section we denote by C, $c(p)$, etc., various positive constants, not necessarily the same in each occurrence, which may depend on δ, p or k but are independent of ϵ.

We begin by presenting some lemmas that contain all technical estimates that we need for the proof of the theorem. For $\beta \in \mathbf{R}$ and $\epsilon > 0$ small we define

$$J_\beta(\epsilon) = \int_\Omega \phi^p d^{-k+\epsilon p} X^{-\beta}(d/D) dx. \tag{5.4}$$

Lemma 5.2. *For ϵ small we have*

$$\begin{aligned}
&\text{(i)} && c\epsilon^{-1-\beta} \le J_\beta(\epsilon) \le C\epsilon^{-1-\beta}, && \beta > -1; \\
&\text{(ii)} && J_\beta(\epsilon) = \frac{p\epsilon}{\beta+1} J_{\beta+1}(\epsilon) + O_\epsilon(1), && \beta > -1; \\
&\text{(iii)} && J_\beta(\epsilon) = O_\epsilon(1), && \beta < -1.
\end{aligned}$$

Proof. Since $|\nabla d| = 1$, we have

$$J_\beta(\epsilon) = \int_0^\delta \int_{d=r} \phi^p r^{-k+\epsilon p} X^{-\beta}(r/D) dS\, dr.$$

Hence using the fact that $0 \le \phi \le 1$ and $\int_{\{d=r\} \cap B_\delta} dS < cr^{k-1}$, we obtain

$$J_\beta(\epsilon) \le c \int_0^\delta r^{-1+\epsilon p} X^{-\beta}(r/D) dr.$$

Recalling (3.2) we see that for $\beta < -1$ the integral above has a finite limit as $\epsilon \to 0$, hence (iii) follows. To show (i) we use the change of variables $r = Ds^{1/\epsilon}$ to obtain that

$$J_\beta(\epsilon) \le \epsilon^{-1-\beta} D^{\epsilon p} \int_0^{(\delta/D)^\epsilon} s^{p-1} X^{-\beta}(s) ds,$$

and the upper estimate of (i) follows. For the lower estimate we use the fact that $\phi = 1$ for $d \le \delta/2$ and argue similarly.

To prove (ii) we recall (3.1) to write

$$(\beta+1) J_\beta(\epsilon) = -\int_\Omega \phi^p d^{1-k+\epsilon p} \nabla d \cdot \nabla X^{-\beta-1}(d/D) dx.$$

We now perform an integration by parts and note that no boundary terms appear. Indeed, if $k = 1$, then the factor $d^{1-k+\epsilon p} = d^{\epsilon p}$ guarantees that the integrand vanishes on K. If $2 \le k \le N$, then we approximate Ω by $\Omega_\eta := \{x \in \Omega : d(x) > \eta\}$, $\eta > 0$ small. This yields the boundary term

$$-\int_{d=\eta} \phi^p d^{1-k+\epsilon p} X^{-\beta-1}(d/D) \nabla d \cdot \vec{n}\, dS,$$

which vanishes as $\eta \to 0$. Hence in any case we have

$$\begin{aligned}
(\beta+1) J_\beta(\epsilon) &= \int_\Omega \operatorname{div}(\phi^p d^{1-k+\epsilon p} \nabla d) X^{-\beta-1}(d/D) dx \\
&= p \int_\Omega \phi^{p-1} d^{1-k+\epsilon p} X^{-\beta-1}(d/D) \nabla\phi \cdot \nabla d\, dx \\
&\quad + (1-k+\epsilon p) \int_\Omega d^{-k+\epsilon p} X^{-\beta-1}(d/D) dx \\
&\quad + \int_\Omega \phi^p d^{1-k+\epsilon} \Delta d X^{-\beta-1}(d/D) dx.
\end{aligned}$$

The first integral is easily seen to be of order $O_\epsilon(1)$. The other two integrals combine to give

$$\epsilon p J_{\beta+1}(\epsilon) + \int_\Omega \phi^p d^{-k+\epsilon p} X^{-\beta-1}(d/D)(d\Delta d + 1 - k) dx. \tag{5.5}$$

But it is a direct consequence of [AS, Theorem 3.2] that

$$d\Delta d + 1 - k = O(d) \text{ as } d(x) \to 0; \tag{5.6}$$

this implies that the integral in (5.5) is of order $O_\epsilon(1)$, and the result follows. □

We next estimate the quantity

$$I[U_\epsilon] = \int_\Omega |\nabla U_\epsilon|^p dx - |H|^p \int_\Omega \frac{|U_\epsilon|^p}{d^p} dx.$$

Lemma 5.3. *As $\epsilon \to 0$,*

$$\text{(i)} \qquad I[U_\epsilon] \le \frac{\theta(p-1)}{2} |H|^{p-2} J_{p\theta-2}(\epsilon) + O_\epsilon(1); \tag{5.7}$$

$$\text{(ii)} \qquad \int_{B_\delta} |\nabla U_\epsilon|^p dx \le |H|^p J_{p\theta}(\epsilon) + O_\epsilon(\epsilon^{1-p\theta}). \tag{5.8}$$

Proof. We have $\nabla U_\epsilon = \phi\,\nabla w_\epsilon + \nabla\phi\, w_\epsilon$ and hence, using the elementary inequality

$$|a+b|^p \le |a|^p + c_p(|a|^{p-1}|b| + |b|^p), a, b \in \mathbf{R}^N, \quad p>1, \tag{5.9}$$

we obtain

$$\begin{aligned} \int_\Omega |\nabla U_\epsilon|^p dx \quad &\le \quad \int_{B_\delta} \phi^p d^{-k+\epsilon p} X^{-p\theta}(d/D)\,|H - (\epsilon - \theta X(d/D))|^p\,dx \\ &\quad + c_p \int_{B_\delta} |\nabla\phi||\phi|^{p-1}|\nabla w_\epsilon|^{p-1}|w_\epsilon|\,dx + c_p \int_{B_\delta} |\nabla \phi|^p |w_\epsilon|^p\,dx \\ &=: \quad I_A + I_2 + I_3. \end{aligned} \tag{5.10}$$

We claim that

$$I_2,\ I_3 = O_\epsilon(1) \quad \text{as } \epsilon \to 0. \tag{5.11}$$

Let us give the proof for I_2. Using the definition of w_ϵ and the fact that ϕ is a nice function we get

$$I_2 \le c\int_{B_\delta} d^{1-k+\epsilon p} X^{-p\theta}(d/D)\,|H - (\epsilon - \theta X(d/D))|^{p-1}\,dx.$$

Since $(\epsilon - \theta X(d/D))$ is small compared to H we have

$$I_2 \le c\int_{B_\delta} d^{1-k+\epsilon p} X^{-p\theta}(d/D)dx$$

and it is easily seen that the last integral has a finite limit as $\epsilon \to 0$. The integral I_3 is treated in the same way.

From (5.10), (5.11) and the definition of J_β we easily obtain

$$\begin{aligned} I[U_\epsilon] \quad &= \quad \int_{B_\delta} |\nabla U_\epsilon|^p dx - |H|^p J_{p\theta} \\ &\le \quad I_A - |H|^p J_{p\theta} + O_\epsilon(1) \\ &= \quad I_1 + O_\epsilon(1), \end{aligned} \tag{5.12}$$

where

$$I_1 := \int_{B_\delta} \phi^p d^{-k+\epsilon p} X^{-p\theta}(d/D)\Big(\big|H - \big(\epsilon - \theta X(d/D)\big)\big|^p - |H|^p\Big)dx.$$

We proceed by estimating I_1. Since $\eta := (\epsilon - \theta X(d/D))$ is small compared to H, we may use Taylor's expansion to obtain

$$|H-\eta|^p - |H|^p \le -p|H|^{p-2}H\eta + \frac{1}{2}p(p-1)|H|^{p-2}\eta^2 + C|\eta|^3.$$

Using this inequality we can bound I_1 by

$$I_1 \le I_{11} + I_{12} + I_{13}, \tag{5.13}$$

where

$$\begin{aligned} I_{11} \quad &= \quad -p|H|^{p-2}H\int_{B_\delta} \phi^p d^{-k+\epsilon p} X^{-p\theta}(d/D)(\epsilon - \theta X(d/D))dx, \\ I_{12} \quad &= \quad \frac{1}{2}p(p-1)|H|^{p-2}\int_{B_\delta} \phi^p d^{-k+\epsilon p} X^{-p\theta}(d/D)\big(\epsilon - \theta X(d/D)\big)^2 dx, \\ I_{13} \quad &= \quad C\int_{B_\delta} \phi^p d^{-k+\epsilon p} X^{-p\theta}(d/D)|\epsilon - \theta X(d/D)|^3 dx. \end{aligned}$$

We shall prove that

$$I_{11}, I_{13} = O_\epsilon(1), \quad \epsilon \to 0. \tag{5.14}$$

Indeed, an application of part (ii) of Lemma 5.2 (with $\beta = -1+p\theta$) gives $I_{11} = O_\epsilon(1)$ for small $\epsilon > 0$. Concerning I_{13} we clearly have

$$I_{13} \leq c\epsilon^3 J_{p\theta} + cJ_{p\theta-3}.$$

It follows from Lemma 5.2, parts (i) and (iii), and the fact that $1 < p\theta < 2$ that $I_{13} = O_\epsilon(1)$.

To calculate the term I_{12} we first expand the square in the integrand and then apply (ii) of Lemma 5.2 twice (with $\beta = -1+p\theta$ the first time and $\beta = -2+p\theta > -1$ the second time) to conclude that

$$I_{12} = \frac{\theta(p-1)}{2}|H|^{p-2}\int_{B_\delta} \phi^p d^{-k+\epsilon p} X(d/D)^{2-p\theta} dx + O_\epsilon(1), \qquad \epsilon \to 0. \tag{5.15}$$

From (5.12), (5.13), (5.14) and (5.15) we derive (5.7). The second inequality of the lemma follows from the first equality in (5.12), estimate (5.7) and Lemma 5.2. $\square$

We are now ready to give the proof of Theorem 5.1.

Proof of Theorem 5.1. It follows directly from part (i) of Lemma 5.2 that for any $\gamma \in \mathbf{R}$

$$R_\gamma[U_\epsilon] := \int_\Omega \frac{|U_\epsilon|^p}{d^p} X^\gamma(d/D) dx = J_{p\theta-\gamma}(\epsilon). \tag{5.16}$$

(i) Since inequality (5.1) holds for every $u \in W_0^p(\Omega \setminus K)$, we have

$$\begin{aligned} A &\leq \frac{\int_{B_\delta} |\nabla U_\epsilon|^p dx}{R_0[U_\epsilon]} \\ \text{(by (5.8))} \quad &\leq \frac{|H|^p(1+O_\epsilon(\epsilon))J_{p\theta}(\epsilon) + O_\epsilon(1)}{J_{p\theta}(\epsilon)}; \end{aligned}$$

letting $\epsilon \to 0$ and recalling that $J_{p\theta}(\epsilon) \to \infty$ we conclude that $A \leq |H|^p$.

(ii) Let $A = |H|^p$. Assuming that $\gamma < 2$ we will reach a contradiction. Since $p\theta - \gamma > -1$, arguing as in (i) we have that

$$\begin{aligned} 0 < B &\leq \frac{I[U_\epsilon]}{R_\gamma[U_\epsilon]} \\ \text{(by (5.7) and Lemma 5.2 (i))} \quad &\leq \frac{c\epsilon^{-p\theta+1}}{c\epsilon^{-1-p\theta+\gamma}} \\ &= c\epsilon^{2-\gamma} \to 0 \qquad \text{as } \epsilon \downarrow 0, \end{aligned}$$

which is a contradiction. Hence $\gamma \geq 2$.

(iii) If $A = |H|^p$ and $\gamma = 2$, then

$$\begin{aligned} B &\leq \frac{I[U_\epsilon]}{R_2[U_\epsilon]} \\ \text{(by (5.7))} \quad &\leq \frac{\frac{1}{2}\theta(p-1)|H|^{p-2}J_{p\theta-2}(\epsilon) + O_\epsilon(1)}{J_{p\theta-2}(\epsilon)}. \end{aligned}$$

The assumption $\theta > 1/p$ implies $J_{p\theta-2} \to \infty$ as $\epsilon \to 0$ by (i) of Lemma 5.2. Hence $B \leq \frac{\theta(p-1)}{2}|H|^{p-2}$; letting $\theta \to 1/p$ concludes the proof. $\square$

We close this section proving the optimality of the estimate in Theorem 4.2.

Theorem 5.4. *Let Ω be a domain in $\mathbf{R}^N$.* (i) *If $2 \le k \le N-1$, then we take K to be a piecewise smooth surface of codimension k and assume that $K \cap \Omega \ne \emptyset$;* (ii) *if $k = N$, then we take $K = \{0\} \subset \Omega$. Suppose that $p = k$ and that for some constants $B \ge 0$ and $\gamma > 0$ the following inequality holds true for all $u \in C_c^\infty(\Omega \setminus K)$:*

$$\int_\Omega |\nabla u|^p dx \ge B \int_\Omega \frac{|u|^p}{d^p} X^\gamma(d/D) dx. \tag{5.17}$$

We then have:

(i) *If $B > 0$, then $\gamma \ge p$.*

(ii) *If $\gamma = p$, then $B \le \left(\frac{p-1}{p}\right)^p$.*

Proof. The proof uses an argument similar to that of Theorem 5.1. Without any loss of generality we assume that $0 \in K \cap \Omega$ if $2 \le k \le N$ and $0 \in \partial\Omega = K$ if $k = 1$. As in the last theorem we let ϕ be a nonnegative, smooth cut-off function supported in $B_\delta = \{|x| < \delta\}$ and equal to one on $B_{\delta/2}$. For any $\epsilon > 0$ and $\theta > (p-1)/p$ define $w_\epsilon = d^\epsilon X^{-\theta}(d/D)$ and $U_\epsilon = \phi w_\epsilon$. Using (5.9) we have

$$\begin{aligned}
& \int_\Omega |\nabla U_\epsilon|^p dx \\
\le\ & \int_{B_\delta} \phi^p |\nabla w_\epsilon|^p dx + c_p \int_{B_\delta} \phi^{p-1} |\nabla w_\epsilon|^{p-1} w_\epsilon |\nabla \phi| dx + c_p \int_{B_\delta} w_\epsilon^p |\nabla \phi|^p dx \\
=:\ & I_A + I_2 + I_3.
\end{aligned}$$

Arguing as in the proof of the previous theorem we see that

$$I_2, I_3 = O_\epsilon(1), \qquad \epsilon \to 0.$$

Denoting by c_i^p the coefficients of the binomial expansion we have

$$\begin{aligned}
|\nabla w_\epsilon|^p &= d^{-k+\epsilon p} X^{-p\theta}(d/D) |\epsilon - \theta X(d/D)|^p \\
&\le d^{-k+\epsilon p} \theta X^{-p\theta}(d/D) (\epsilon + \theta X(d/D))^p \\
&= d^{-k+\epsilon p} X^{-p\theta}(d/D) \sum_{i=0}^p c_i^p \epsilon^{p-i} \theta^i X^i(d/D),
\end{aligned}$$

and hence

$$I_A \le \sum_{i=0}^p c_i^p \epsilon^{p-i} \theta^i J_{p\theta - i}(\epsilon),$$

where the functions $J_\beta(\epsilon) = \int_\Omega \phi^p d^{-p+\epsilon p} X^{-\beta}(d/D)$ are as in (5.4). Now it follows from (ii) of Lemma 5.2 and a simple induction argument that

$$\epsilon^{p-i} J_{p\theta-i} = (\theta - \frac{i}{p})(\theta - \frac{i+1}{p}) \cdots (\theta - \frac{p-1}{p}) J_{p\theta - p} + O_\epsilon(1), \quad i = 0, \ldots, p-1.$$

But the fact that $\theta > (p-1)/p$ implies that $J_{p\theta-p}(\epsilon) \to +\infty$ as $\epsilon \to 0$, by (i) of Lemma 5.2. It follows that

$$\begin{aligned} B &\leq \limsup_{\epsilon\to 0} \frac{\int_\Omega |\nabla U_\epsilon|^p dx}{\int_\Omega \frac{U_\epsilon^p}{d^p} X^p(d/D)dx} \\ &\leq \limsup_{\epsilon\to 0} \frac{\left(\theta^p \sum_{i=0}^{p-1} c_i^p \theta^i (\theta - \frac{i}{p})\cdots(\theta - \frac{p-1}{p})\right) J_{p\theta-p}(\epsilon) + O_\epsilon(1)}{J_{p\theta-p}(\epsilon)} \\ &= \theta^p + \sum_{i=0}^{p-1} c_i^p \theta^i (\theta - \frac{i}{p})\cdots(\theta - \frac{p-1}{p}). \end{aligned}$$

This last expression converges to $(\frac{p-1}{p})^p$ as $\theta \to (p-1)/p$; this completes the proof. □

6. Nonhomogeneous improved Hardy inequalities

As an application of the results in Section 3 we first prove Theorem B, the improved Hardy-Poincaré inequality.

Proof of Theorem B. We shall prove that

$$I[u] \geq c\left(\int_\Omega |\nabla u|^q d^{k(-1+q/p)} X^\beta(d/D)dx\right)^{p/q}, \quad u \in W_0^{1,p}(\Omega \setminus K). \tag{6.1}$$

Letting $v = ud^{(k-p)/p}$ we have

$$\int_\Omega |\nabla u|^q d^{k(-1+q/p)} X^\beta dx \leq c(q,k)\left(\int_\Omega |\nabla v|^q d^{q-k} X^\beta dx + \int_\Omega |v|^q d^{-k} X^\beta dx\right). \tag{6.2}$$

To proceed we will estimate the two integrals in the right hand side of (6.2). We first consider the case $1 < p < 2$. Applying Hölder's inequality we obtain

$$\int_\Omega |\nabla v|^q d^{q-k} X^\beta dx \leq \left(\int_\Omega |\nabla v|^p d^{p-k} X^{2-p} dx\right)^{q/p} \left(\int_\Omega d^{-k} X^\theta dx\right)^{(p-q)/p},$$

with $\theta = (\beta p - 2q + qp)/(p-q)$. We next show that the last integral above is finite. The integrand has a singularity as $d \to 0$. However for d near zero the integral behaves like

$$\int_0^\epsilon \int_{d=t} \frac{d^{-k} X^\theta}{|\nabla d|} dSdt \leq c \int_0^\epsilon \int_{d=t} d^{-k} X^\theta dSdt \leq c \int_0^\epsilon t^{-1} X^\theta(t)dt.$$

The last integral is finite iff $\theta > 1$ (cf (3.2)), a condition that is easily seen to be satisfied under our assumptions on p, q, β. Hence we end up with

$$\left(\int_\Omega |\nabla v|^q d^{q-k} X^\beta dx\right)^{p/q} \leq c \int_\Omega |\nabla v|^p d^{p-k} X^{2-p} dx \leq cI[u], \tag{6.3}$$

where in the last inequality we used (3.8). Applying in a similar fashion Hölder's inequality and then the improved Hardy inequality (3.8), we estimate the last integral in (6.2),

$$\left(\int_\Omega |v|^q d^{-k} X^\beta dx\right)^{p/q} \leq c \int_\Omega |v|^p d^{-k} X^2 dx = \int_\Omega \frac{|u|^p}{d^p} X^2 dx \leq cI[u], \tag{6.4}$$

and (6.1) follows.

Consider now the case $p \geq 2$. The proof is quite similar. In particular estimate (6.4) remains valid, whereas the analogue of (6.3) is

$$\left(\int_\Omega |\nabla v|^q d^{q-k} X^\beta dx\right)^{p/q} \leq c \int_\Omega |\nabla v|^p d^{p-k} dx \leq c I[u],$$

where in the last inequality we used (3.7) and condition (C). The proof of (6.1) is now complete.

To prove the sharpness of the estimate we consider the functions U_ϵ of Section 5 (see (5.3)). We have already seen that they satisfy $I[U_\epsilon] \leq c\epsilon^{1-p\theta}$. Moreover, simple calculations show that for $\beta > 0$ we have

$$\int_\Omega |\nabla U_\epsilon|^q d^{k(-1+q/p)} X^\beta(d/D) dx \geq c\epsilon^{\beta-\theta q-1},$$

for all $\epsilon > 0$ small enough. Hence (6.1) cannot be true if $\beta < 1 + q/p$. $\square$

We now turn our attention to the improved Hardy-Sobolev inequalities. By this we mean lower estimates on $I[u]$ in terms of weighted L^q norms of the function u, $q > p$. It will be seen that there is a difference in the form the estimates take, depending on whether $k = N$ or $k < N$. We first consider the case of affine K, $K \equiv \mathbf{R}^{N-k}$, and take $\Omega = \mathbf{R}^N$. More precisely, we write points in $\mathbf{R}^N$ as $x = (y,z)$, $y \in \mathbf{R}^{N-k}$, $z \in \mathbf{R}^k$. Under this representation we take

$$K = \{(y,0) \,|\, y \in \mathbf{R}^{N-k}\}$$

so that $d(y,z) = |z|$.

Our next two propositions yield Theorem C.

Proposition 6.1. *Assume that $k < N$ and that condition* (C) *is satisfied. Then for any $2 \leq p < N$ and any $p < q \leq Np/(N-p)$ there exists $c > 0$ such that*

$$I[u] \geq c\left(\int_\Omega |u|^q |z|^{-q-N+Nq/p} dx\right)^{p/q}, \quad u \in W_0^{1,p}(\mathbf{R}^N \setminus K). \tag{6.5}$$

Proof. Let $v(y,z) = u(y,z)|z|^{(k-p)/p}$. It follows from (3.7) and condition (C) that

$$I[u] \geq c \int_{\mathbf{R}^N} |\nabla v|^p |z|^{p-k} dz dy.$$

Moreover, Corollary 2 in Section 2.1.6 of [M] gives

$$\begin{aligned}\int_{\mathbf{R}^N} &|\nabla v|^p |z|^{p-k} dz dy \\ &\geq c\left(\int_{\mathbf{R}^N} |v|^q |z|^{-N+(N-k)q/p} dz dy\right)^{p/q}, \qquad v \in C_c^\infty(\mathbf{R}^N \setminus K).\end{aligned} \tag{6.6}$$

Combining the last two inequalities we obtain (6.5). $\square$

Estimate (6.6) is not valid when $k = N$ and K reduces to the single point $0 \in \Omega$. Indeed, it is remarkable that (6.5) fails in this case. In our next proposition we use decreasing rearrangement techniques to obtain a modified version of Proposition 6.1 which involves a logarithmic correction $X^{1+q/p}$ in the right hand side; we then show that the exponent $1 + q/p$ is optimal.

Proposition 6.2. *Let $1 < p < N$. Let $\Omega \subset \mathbf{R}^N$ be bounded containing the origin and $D > \sup_{x\in\Omega} d(x)$. For any $p < q < Np/(N-p)$, there exists $c = c(p,q,D,\Omega) > 0$ such that*

$$I[u] \geq c\left(\int_\Omega |u|^q |x|^{-q-N+Nq/p} X^{1+q/p}(|x|/D)dx\right)^{p/q}, \quad u \in W_0^{1,p}(\Omega). \tag{6.7}$$

Moreover one cannot replace $X^{1+q/p}$ by a lower power of X.

Proof. We may assume that $D = 1$. Let $u \in C_c^\infty(\Omega)$ be given and let u^* denote its radially symmetric decreasing rearrangement on the ball Ω^* having the same volume as Ω and centered at the origin. It is a standard property of decreasing rearrangements that

$$I[u] \geq I[u^*].$$

Define

$$f(r) = r^{-q-N+Nq/p} X^{1+q/p}(r)$$

and note that this decreases near $r = 0$. Let $f^* : \Omega^* \to [0,\infty)$ be the symmetric decreasing rearrangement of $f(|\cdot|) : \Omega \to [0,\infty)$. Using Lemma 4.1 of [FT] one sees that $f^*(r) \leq f(r)$, near $r = 0$. Hence, using also the standard relations $\int_\Omega fg \leq \int_{\Omega^*} f^* g^*$ $(f, g \geq 0)$, and $(|u|^q)^* = |u^*|^q$, we conclude that it is enough to establish (6.7) in the case where Ω is the unit ball and $u = u(r)$ is a radially symmetric decreasing function of $r = |x|$. Assume first that $1 < p < 2$ and set $v(r) = u(r) r^{(N-p)/p}$. Using first (3.8) (with $d = r$, $k = N$) and then Lemma 7.1 (with $\alpha = 2 - p$) we have

$$\begin{aligned}
I[u] &\geq c\int_0^1 |v'|^p r^{p-1} X^{2-p} dr \\
&\geq c\left(\int_0^1 |v|^q r^{-1} X^{1+q/p} dr\right)^{p/q} \\
&= c\left(\int_0^1 |u|^q r^{-q-1+Nq/p} X^{1+q/p} dr\right)^{p/q} \\
&= c\left(\int_\Omega |u|^q |x|^{-q-N+Nq/p} X^{1+q/p}(|x|) dx\right)^{p/q}.
\end{aligned}$$

Suppose now that $p \geq 2$. Let $w = |v|^{p/2}$ with v as above. Using first (3.6) — with $d = r$, $k = N$ — and then Lemma 7.1 (see Appendix) — with $\alpha = 0$ and $2q/p$ in the place of q — we have

$$\begin{aligned}
I[u] &\geq c\int_0^1 |v'|^2 |v|^{p-2} r dr \\
&= c\int_0^1 |w'|^2 r dr \\
&\geq c\left(\int_0^1 |w|^{2q/p} r^{-1} X^{1+q/p} dr\right)^{p/q} \\
&= c\left(\int_\Omega |u|^q |x|^{-q-N+Nq/p} X^{1+q/p}(|x|) dx\right)^{p/q},
\end{aligned}$$

as required.

To prove that the exponent $1+q/p$ is optimal we consider once again the functions U_ϵ of Section 5, $U_\epsilon(x)=\phi(x)|x|^{\epsilon-(N-p)/p}X^{-\theta}(|x|/D)$, $\epsilon>0$, $\theta>1/p$, ϕ a cut-off. An argument similar to that used in Section 5 shows the optimality of the exponent $1+q/p$. We omit the details. $\square$

Given that the estimate $I[u]\geq c\|u\|^p_{L^{Np/(N-p)}}$ is not valid when $K=\{0\}\subset\Omega$, one may ask whether the next best thing is true, i.e. whether

$$I[u]\geq c\|u\|_{L^{Np/(N-p),\infty}},\quad u\in W^{1,p}(\Omega),$$

where in the right hand side we have the weak $L^{Np/(N-p)}$ norm of u,

$$\|u\|_{L^{q,\infty}}=\sup_{E\subset\Omega}|E|^{\frac{1}{q}-1}\int_E|u|dx,\qquad 1<q<\infty.$$

This question was risen in a different context in [BL], where the improved Sobolev inequalities are considered. In that paper the authors obtain lower estimates on

$$J[u]:=\int_\Omega|\nabla u|^2dx-c_*\left(\int_\Omega|u|^{2N/(N-2)}dx\right)^{(N-2)/N},\quad u\in W_0^{1,2}(\Omega),$$

where c_* is the best Sobolev constant. It is shown in [BL] that $J[u]\geq c\|u\|_q^2$, $c>0$, when $q<N/(N-2)$. This of course fails for the critical value $q=N/(N-2)$, but it is shown instead that

$$J[u]\geq c\|u\|_{L^{N/(N-2),\infty}},\quad u\in W_0^{1,2}(\Omega).$$

In our case there is no room even for such a weak norm as the following proposition shows.

Proposition 6.3. *Let $1<p<N$ and $K=\{0\}\subset\Omega$. There does not exist $c>0$ such that*

$$I[u]\geq c\|u\|_{L^{\frac{Np}{N-p},\infty}},\quad u\in W_0^{1,p}(\Omega). \tag{6.8}$$

Proof. We may assume for simplicity that $\{|x|<2\}\subset\Omega$. Let U_ϵ be the functions introduced in (5.3) for $D=1$ and assume that $1/p<\theta<1/(p-1)$. We claim that

$$\|U_\epsilon\|_{L^{\frac{Np}{N-p},\infty}(B_1)}\geq c\epsilon^{-\theta},\qquad \text{small } \epsilon>0. \tag{6.9}$$

Let B_ρ denote the ball of radius ρ centered at the origin. We then have that

$$\|U_\epsilon\|_{L^{\frac{Np}{N-p},\infty}(B_1)}\geq\sup_{0<\rho<1}|B_\rho|^{-\frac{Np-N+p}{Np}}\int_{B_\rho}|U_\epsilon|dx.$$

Also, using the explicit expression of U_ϵ and integrating once by parts we get

$$\begin{aligned}\int_{B_\rho}|U_\epsilon|dx &\geq C\int_0^\rho r^{N-\frac{N}{p}+\epsilon}X^{-\theta}(r)dr\\ &= C\left(\rho^{N-\frac{N}{p}+\epsilon+1}X^{-\theta}(\rho)+\int_0^\rho r^{N-\frac{N}{p}+\epsilon}X^{-\theta+1}(r)dr\right)\\ &\geq C\rho^{N-\frac{N}{p}+\epsilon+1}X^{-\theta}(\rho).\end{aligned}$$

Hence, we arrive at

$$\|U_\epsilon\|_{L^{\frac{Np}{N-p},\infty}(B_1)}\geq C\sup_{0<\rho<1}\rho^\epsilon(-\log\rho)^\theta.$$

It is easy to check that $\max_{0<\rho<1}\rho^\epsilon(-\log\rho)^\theta=(\theta/e)^\theta\epsilon^{-\theta}$ and (6.9) follows.

On the other hand, we have seen in Section 5 that for small ϵ

$$I[U_\epsilon] \le c\epsilon^{1-p\theta}. \tag{6.10}$$

Combining (6.9) and (6.10) we obtain the stated result. □

We close this section presenting an improved Hardy-Sobolev inequality that is valid for all $k \le N$ without assuming that K is affine. The estimate obtained is weaker than that of Theorem C.

Theorem 6.4. *Let $k \le N$ and $1 < p < N$. Let $D > \sup_{x\in\Omega} d(x,K)$, and assume condition* (C) *is satisfied. For any $p < q \le Np/(N-p)$ there exists $c = c(p,q,D,\Omega,K) > 0$ such that*

$$I[u] \ge c\left(\int_\Omega |u|^q d^{-q-N+Nq/p} X^{2q/p}(d/D)dx\right)^{p/q}, \tag{6.11}$$

for all $u \in W_0^{1,p}(\Omega \setminus K)$.

Proof. We may assume as usual that $D = 1$. Once more we set $u = vd^{-H}$, $H = (k-p)/p$. Applying Lemma 7.2 for $\alpha = 2$ (see the Appendix) we have

$$\begin{aligned}\int_\Omega |u|^q d^{-q-N+Nq/p} X^{2q/p}dx &= \int_\Omega |v|^q d^{-N+(N-k)q/p} X^{2q/p}dx \\ &\le c\left(\int_\Omega |\nabla v|^p d^{p-k}X^2dx + \int_\Omega |v|^p d^{-k}X^2dx\right)^{q/p}.\end{aligned} \tag{6.12}$$

The last integral in (6.12) is easily estimated by the improved Hardy inequality

$$\int_\Omega |v|^p d^{-k}X^2dx = \int_\Omega \frac{|u|^p}{d^p}X^2\,dx \le cI[u]. \tag{6.13}$$

To estimate the other integral, suppose first that $1 < p < 2$. Using the fact that $X^2 \le cX^{2-p}$ and (3.8) we have

$$\int_\Omega |\nabla v|^p d^{p-k}X^2dx \le c\int_\Omega |\nabla v|^p d^{p-k}X^{2-p}dx \le cI[u], \tag{6.14}$$

and (6.11) follows from (6.12), (6.13), (6.14).

Consider now the case $p \ge 2$. Using the fact that $X \le 1$ and (3.7) we obtain in a similar fashion

$$\int_\Omega |\nabla v|^p d^{p-k}X^2dx \le \int_\Omega |\nabla v|^p d^{p-k}dx \le cI[u], \tag{6.15}$$

and (6.11) follows from (6.12), (6.13), (6.15). □

7. Appendix

Here we present the two auxiliary lemmas that were used in Section 6. The first is a one-dimensional Hardy type inequality.

Lemma 7.1. *Let $p \in (1,\infty)$ and $q \ge p$ be given. For any $\alpha > -(p-1)$ there exists $c > 0$ such that*

$$\int_0^1 |v'|^p r^{p-1}X^\alpha dr \ge c\left(\int_0^1 |v|^q r^{-1}X^{1+(\alpha+p-1)q/p}dr\right)^{p/q}, \tag{7.1}$$

for all $v \in C_c^\infty(0,1)$.

Proof. Apply [M, Theorem 3, p. 44] with $d\mu = r^{-1}X^{1+(\alpha+p-1)q/p}\chi_{[0,1]}dr$ and $d\nu = r^{p-1}X^{\alpha}\chi_{[0,1]}dr$. $\square$

The second lemma is a weighted Sobolev inequality.

Lemma 7.2. *Let $D > \sup_{x\in\Omega} d(x,K)$. Given $1 < p < N$, $p < q \le Np/(N-p)$ and $a \in \mathbf{R}$, there exists $c = C(p,q,D,\Omega) > 0$ such that for all $v \in C_c^\infty(\Omega\setminus K)$*

$$\begin{aligned}\int_\Omega |v|^q d^{-N+(N-k)q/p}X^{\alpha q/p}(d/D)dx \\ \le c\left(\int_\Omega |\nabla v|^p d^{p-k}X^\alpha(d/D)dx + \int_\Omega |v|^p d^{-k}X^\alpha(d/D)dx\right)^{q/p}.\end{aligned} \tag{7.2}$$

When d is the distance from the boundary $\partial\Omega$ (that is, $k=1$), the above result is given in [OK]; see Example 18.16 on p. 264 there. Since for the general case we have not found a reference, we present a proof.

Proof. Once again it suffices to consider the case $D = 1$, the general case following by scaling. We shall make use of the standard Sobolev inequality

$$\int_{B(r)} |v|^q dx \le cr^{N+q-Nq/p}\left(r^{-p}\int_{B(r)}|v|^p dx + \int_{B(r)}|\nabla v|^p dx\right)^{q/p}, \quad v \in W^{1,p}(B(r)), \tag{7.3}$$

where $B(r)$ is any ball of radius r and the constant is independent of r. Now, it follows from the Besicovich covering lemma (see [M]) that there exists a sequence (x_m) of points in Ω with the following properties: defining, say, $r_m = d(x_m)/10$, the balls

$$B_m := B(x_m, r_m), \quad m = 1,2,\dots,$$

satisfy

(i) $\Omega \subset \bigcup_m B_m$;
(ii) there exists a number M depending only on the dimension N such that each $x \in \Omega$ belongs to at most M of the B_m's.

It follows from the choice of the radii r_m that there exist constants c_i, c_i' such that

$$\begin{cases} c_1 r_m \le d(x) \le c_2 r_m, \\ c_1' X(r_m) \le X(d(x)) \le c_2' X(r_m) \end{cases} \qquad x \in B(x_m, r_m),\ m = 1,2,\dots. \tag{7.4}$$

This implies in particular that for any fixed $\theta, \eta \in \mathbf{R}$ we have

$$c_1'' r_m^\theta X^\eta(r_m)\int_{B_m}|u|^p dx \le \int_{B_m}|u|^p d^\theta X^\eta(d)dx \le c_2'' r_m^\theta X^\eta(r_m)\int_{B_m}|u|^p dx,$$

for all $m = 1, 2, \dots$ and $u \in W^{1,p}(B_m)$. Hence

$$\begin{aligned}
&\int_\Omega |v|^q d^{-N+(N-k)q/p} X^{\alpha q/p}(d)dx \\
\leq\ & \sum_{m=1}^{\infty} \int_{B_m} |v|^q d^{-N+(N-k)q/p} X^{\alpha q/p}(d)dx \\
\leq\ & c \sum_{m=1}^{\infty} r_m^{-N+(N-k)q/p} X^{\alpha q/p}(r_m) \int_{B_m} |v|^q dx \\
\leq\ & c \sum_{m=1}^{\infty} \left(\int_{B_m} |\nabla v|^p r_m^{p-k} X^{\alpha}(r_m)dx + \int_{B_m} |v|^p r_m^{-k} X^{\alpha}(r_m)dx \right)^{q/p} \\
\leq\ & c \sum_{m=1}^{\infty} \left(\int_{B_m} |\nabla v|^p d(x)^{p-k} X^{\alpha}(d(x))dx + \int_{B_m} |v|^p d(x)^{-k} X^{\alpha}(d(x))dx \right)^{q/p} \\
\leq\ & c \left(\sum_{m=1}^{\infty} \int_{B_m} |\nabla v|^p d(x)^{p-k} X^{\alpha}(d(x))dx + \sum_{m=1}^{\infty} \int_{B_m} |v|^p d(x)^{-k} X^{\alpha}(d(x))dx \right)^{q/p} \\
\leq\ & c \left(\int_{\Omega} |\nabla v|^p d(x)^{p-k} X^{\alpha}(d(x))dx + \int_{\Omega} |v|^p d(x)^{-k} X^{\alpha}(d(x))dx \right)^{q/p},
\end{aligned}$$

since by (ii) we have $\sum_m \int_{B_m} f \leq M \int_\Omega f$ for any nonnegative function f on Ω. □

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Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
E-mail address: `gbarbati@cc.uoi.gr`

Department of Applied Mathematics, University of Crete, 71409 Heraklion, Greece
E-mail address: `filippas@tem.uoc.gr`

Department of Mathematics, University of Crete, 71409 Heraklion, Greece and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Greece
E-mail address: `tertikas@math.uoc.gr`