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A Unified Approach to Reliability, Availability, Performability Analysis Based on Markov Processes with Rewards

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Abstract: This paper discusses a unified approach to reliability, availability and performability analysis of complex engineering systems. Theoretical basis of this approach is continuous-time discrete state Markov processes with rewards. From reliability modeling point of view complex systems are the systems with static and dynamic redundancy, imperfect fault coverage, various recovery strategies, multilevel operation and varying severity of failure states. We propose a unified method of calculating the reliability, availability and performability indices based on the definition of special forms of reward matrix. This method is proved to be effective in calculating both cumulative and instantaneous measures in steady-state and transient cases. We describe special analytical software which implements suggested method. We demonstrate the flexibility of the proposed method and software by analyzing multilevel process unit with protection and demand-based warm standby system.

Keywords: Availability, Markov process, Markov reward model, multilevel engineering systems, performability, reliability.

1. INTRODUCTION

At the present state of development of reliability theory the models of analysis can be subdivided into two categories (classes): the dynamic models and the static models. All changes of system states in the class of dynamic models are considered as the processes developing in time. In the class of the static models, states of the system are determined by the sets of states of system elements at the time moment t.

Markov [1–4], semi-Markov processes, asymptotic methods of the renewal theory and regenerative processes [5–7], Monte Carlo simulation techniques [8] are used within the framework of dynamic models. Dynamic models allow to calculate all the main dependability measures both for repairable and non-repairable systems. These measures are: instantaneous indices (e.g. availability at the time instant); interval indices (e.g. reliability during the time interval); time-independent stationary indicators (e.g. mean time between failures). Known drawbacks inherent in Markov models are the size and stiffness of transient solution of a system of Kolmogorov-Chapman equations (or ill-conditionality for stationary case). Possible ways to solve these problems are discussed in [9–11]. Monte Carlo simulation of modern high reliable systems may require large amount of simulations to obtain results with desired accuracy. These shortcomings can be eliminated with the help of special acceleration techniques. There is no limitation to type of distribution of waiting times between the changes in the class of Monte Carlo models. But the problem of creating a universal Monte Carlo

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model describing the complex reliability behavior of different kind of systems remains unsolved.

Static models use two main groups of methods – combinatorial-probabilistic [3] and logical-probabilistic [12]. Combinatorial-probabilistic methods use the basic formulas of combinatorics and probability theory (probability of the sum and product of events, the formula of total probability). These formulas are used mainly for serial-parallel and "m out of n" redundant schemes. Logical-probabilistic methods, which are based on the construction of a Boolean function relating the state of the system with the states of its elements. The resulting Boolean function is transformed to a form that allows to replace logical variables by corresponding probabilities. Classic failure trees and reliability block diagrams are the main methodology of logical-probabilistic methods [13, 14]. Classic static models for the case of repairable systems allow us to calculate only the differential (instantaneous) reliability indicators determined at the time instant t (e.g. availability, failure frequency, average efficiency at the time instant t).

In the last decade, combined approaches to reliability models construction are successfully developing. These approaches are based on the implementation of dynamic properties into static models. Dynamic fault trees are the most popular practical implementation of these approaches [15–19].

Modern complex engineering systems with high demands on reliability are characterized by various features, which include:

- multiple levels of operation efficiency (e.g., performance) and the graceful degradation of performance in the event of failures [20–23];
- a variety of redundancy implementation techniques (hardware, software, functional, temporal redundancy) [24–26], types of redundant schemes (parallel, standby, hybrid), modes of reserve components (cold and warm standby, shared) [3, 4];
- smart recovery strategies, restricted repair resources [27–29];
- imperfect built-in test equipment leading to the presence of undetected failures and false alarms [30, 31];
- implementation of special multiphase error handling procedures with classification on permanent and transient faults (mainly for fault tolerant computers) [32–34];
- the possibility of the occurrence of mutually exclusive failures that, with a certain multiplicity and sequence, can lead to different consequences at the system level [13,14].

Markov modeling allows taking into account the above-mentioned features of the reliability behavior of systems and calculating most part indicators of reliability, safety, efficiency. At the design stage, we don't have, as a rule, objective information about the distribution functions of random time variables of the model. Therefore, we believe that the assumption of exponentiality is entirely permissible. The problem of large size of the Markov model can be solved by decomposition and aggregation of the model parts and by automation of the model construction.

In this paper, we consider the Markov reward models as common tools for analyzing complex reliability behavior of technical systems. We propose the extension of Markov reward modeling for the reliability, abailability, performability (RAP) analysis of the systems at the design stages. We suggest the method of calculating reliability, availability and performability indicators based on a special definition of the reward matrix. This method is quite efficient and well suited for software implementation. It makes calculation both cumulative and point indices possible only by solving systems of differential or algebraic equations and does not require the usage of additional numerical integration operations.

The remainder of this paper is organized as follows. A brief summary of the mathematical foundations of Markov reward model is given in Section II. In Section III, firstly we give definitions of interval, point, steady-state RAP measures. Next, we present the method for for calculation of these measures in the framework of Markov reward model. Section IV presents analytical software implementing the proposed method. The flexibility of described techniques and software is illustrated by investigations of two practical examples in Section V. In Section VI, we give some concluding remarks.

2. MATHEMATICAL FOUNDATIONS

A system state at the time moment t can be described by stochastic process $z(t)$ of transitions in system state space Ω. We assume that the state space Ω is discrete, finite and of size N. Ω is defined by states of system components (failed or operational). We define a quality function Φ on $z(t)$. Changing system states leads to a change in the quality function Φ. $z(t)$ is stochastic process, therefore, $\Phi(z(t)) = \widetilde{\Phi}(t)$ are also stochastic functions. Due to the discrete nature of Ω, the functions $\widetilde{\Phi}(t)$ are the step functions, whose values are quality levels (for example, system performability) corresponding to the states. G are indices of system performability. G can be defined as a measure on the trajectories of the $\widetilde{\Phi}(t)$. General representation of G takes the form

$$G = M\{F[\Phi(z(t)]\}, \tag{2.1}$$

where M represents mathematical expectation; F is a functional, which is determined by the type of performability index.

Thus, the model of performability contains three components $\Omega, z(t), \Phi$. It is natural and obvious to present the model of performability by means of transition graph (state graph). The elements of the state space correspond to the vertices. Possible transitions in the state space correspond to the arcs of the graph. Φ is defined as a function of the system state s_i. This function takes the value of w_i corresponding to the performance level of the system in this state. That is

$$\Phi(s_i) = w_i. \tag{2.2}$$

In addition, this model can display effects occurring at the transition of the system from one state to another. This is accomplished by determining reward matrix

$$W = [w_{ij}]_n^n, \tag{2.3}$$

where w_{ij} – is an effect on the system that arises at the transition from state i to state j; $w_{ii} = w_i$.

Random transition process $z(t)$ is most simply set in the case where the value of time of stay in each state have an exponential distribution, i.e. when the process is a continuous-time Markov chain (CTMC).

The process $z(t)$ is determined by transition rates, which correspond to arcs of the Markov graph. Similarly to the case of matrix W, we can create a matrix of transition rates

$$\Lambda = [\lambda_{ij}]_n^n, \tag{2.4}$$

where λ_{ij} – is the rate of transition from state i to state j; $\lambda_{ii} = -\sum\limits_{j,j\neq i} \lambda_{ij}$.

2.1. Construction of RAP indices

Thus, the Markov model of performability is defined by two matrices Λ and W, i.e. $\{R, z(t), \Phi\} \sim \{\Lambda, W\}$. Model $\{\Lambda, W\}$ is called Markov reward model (MRM). Markov reward model describes the behavior of systems with exponential distribution of elements failure and repair time. States of the model represent the states of the system, which correspond to different sets of failed and operable items.

If criterion of system failure is specified, then expression (2.1) defines also the indices of reliability and availability.

Let failure criterion is $\{\widetilde{\Phi}(t) < \Phi_{cr}\} \Rightarrow$failure, where $\widetilde{\Phi}(t)$ - is the current value of the quality function; Φ_{cr} - is limit value of the quality, which is allowed by operating requirements. Then, by setting a particular type of the functional F the following indices can be determined from Equation (2.1):

- Reliability over the time interval $(0,t)R(t)$:

$$R(t) = P\{\widetilde{\Phi}(\tau) > \Phi_{cr}, \forall\tau \in (0,t)\} \text{ under } F[\cdot] = \begin{cases} 0, \text{if } \exists\tau \in (0,t):\ \widetilde{\Phi}(\tau) < \Phi_{cr} \\ 1, \text{otherwise} \end{cases} \tag{2.5}$$

- Mean time to first failure (MTTF) T_{FF}:

$$T_{FF} = \int_0^\infty R(t)dt \text{ under} F[\cdot] = min\{\tau\colon \widetilde{\Phi}(\tau) < \Phi_{cr}\} \tag{2.6}$$

- Availability $A(t)$ at the time instant t:

$$A(t) = P\{\widetilde{\Phi}(\tau) \geq \Phi_{cr}\} \text{ under } F[\cdot] = \begin{cases} 0, \widetilde{\Phi}(\tau) < \Phi_{cr} \\ 1, \widetilde{\Phi}(\tau) \geq \Phi_{cr} \end{cases} \tag{2.7}$$

- Probability of being at the i^{th} level of performance at the time instant t:

$$P(t) = P\{\widetilde{\Phi}(t) = \Phi_i\} \text{ under } F[\cdot] = \begin{cases} 1, \widetilde{\Phi}(t) = \Phi_i \\ 0, \text{otherwise} \end{cases} \tag{2.8}$$

- Average accumulated time spent at the i^{th} level of performance:

$$\text{under } F[\cdot] = \int_{t:\widetilde{\Phi}(t)=\Phi_i} dt \tag{2.9}$$

- Average number of transitions from the i^{th} to the j^{th} performance level during the time interval $(0,t)$:

$$\text{under } F[\cdot] = \sum_{\substack{l\,:\,\widetilde{\Phi}(t_{l-}) = \Phi_i \\ \widetilde{\Phi}(t_{l+}) = \Phi_j}} 1, \tag{2.10}$$

  where t_l - the time moment of transition from i^{th} level to j^{th} performance level.
- Time-averaged accumulated reward for the time interval $(0,t)$:

$$F[\cdot] = \frac{1}{t}\int_0^t \widetilde{\Phi}(\tau)d\tau \tag{2.11}$$

2.2. Description of MRM Basic Relations

The consept of reward in Markov reward model is a generalized concept. This can be any effect, loss, costs. According to the provisions of previous section we define the model state space as a discrete set $\{s_i\},\ i = \overline{1,N}$. Also we define reward matrix. The elements of the reward matrix are interpreted as follows: w_{ii} is a reward at time unit while the system is in state s_i (reward rate); w_{ij} is impulse reward received in the system under the transition from state i into state j; N - the number of states of the system. We define the total (or accumulated [20]) reward obtained by the system for a given time T as the functional $F[\Phi(z(t))]$:

$$\widetilde{G}(T) \;=\; \sum_{i=1}^{N} \tau_i w_{ii} + \sum_{i,j} n_{ij} w_{ij}, \tag{2.12}$$

where τ_i is the residence time in state s_i during the time interval $(0, T)$; n_{ij} is the number of system transitions from state i into state j during the time interval $(0, T)$.

Accumulated reward $\widetilde{G}(T)$ is a random variable, since τ_i and n_{ij} are random values. Average accumulated reward is defined as the mathematical expectation of $\widetilde{G}(T)$

$$Y(T) \;=\; M\{\widetilde{G}(T)\}. \tag{2.13}$$

According to (2.1), $Y(T)$ is an estimate of the performability index value. If we consider that members of (2.12) τ_i and n_{ij} depend only on the specific implementation of process $z(t)$, it is obvious that the type of index is completely determined by the type of matrix W. The system of Howard differential equations describes the behavior of the average accumulated reward:

$$dy_i(t)/dt \;=\; w_{ii} + \sum_{j,j\neq i} \lambda_{ij} w_{ij} + \sum_{j} \lambda_{ij} y_j(t);\; i, j = \overline{1, N}, y_i(0) = 0, \tag{2.14}$$

where $y_i(t)$ is the average accumulated reward for the time interval $(0, t)$, given that the initial state is $s_i (z(0) = s_i)$; λ_{ij} - the rate of transition from state s_i into state s_j.

The matrix form of equation (2.14) is

$$dY(t)/dt \;=\; \Lambda Y(t) + R;\; Y(0) = 0, \tag{2.15}$$

where $\Lambda \;=\; \|\lambda_{ij}\|$ - matrix of transition rates; R is a column vector of constant terms:

$$r_i \;=\; w_{ii} + \sum_{j,j\neq i} \lambda_{ij} w_{ij};\; i, j = \overline{1, N}. \tag{2.16}$$

We assume that for absorbing states s_r of the Markov process value $w_{rr} = 0$. The absorbing state is a state that once entered, can not be left. This assumption is quite natural, since the absorbing states are usually identified with the non-operational states of the system.

3. METHOD OF CALCULATING RAP INDICES BASED ON MARKOV REWARD MODEL

The expression (2.1) of expected reward will determine the RAP indices in accordance with (2.5) – (2.11), if the value of the reward for a particular process realization (2.12) coincides with the functional F[.] in (2.5) – (2.11). It requires to select the appropriate reward matrix W and optionally to correct the transition rate matrix Λ. Correction of matrix Λ is necessary in case when the index reflect operation of the system till some event e, for example, till the system failure. Since all performability indices in the model are interpreted through the reward, it is obvious that reward should be dismissed after event e has occurred. Formally, this can be achieved by identification of the event e with a transition to an absorbing state. The reward associated with this state is assumed to be zero. The correction of matrix Λ is that we treat the states corresponding to e as absorbing states. To do this correction, it is enough to equate to zero the elements of rows of matrix Λ corresponding to these states. To carry out further calculations, we partition the set of model states Ω into Ω_g, the set of operational system states, and Ω_f, the set of failed states.

All RAP measures can be subdivided into three groups:

1. Interval measures
   These metrics are dependent on time and are determined on finite time interval $[0, T]$, for example, reliability $R(0, T)$.
2. Point measures
   These metrics are dependent on time and are determined in time point t, for example, availability $A(t)$.
3. Steady-state measures
   These metrics are independent on time and are determined on an unlimited time interval $(0, \infty)$, for example, MTTF.

Calculation of steady-state measures is reduced to finding the stationary solutions of the system (2.15), i.e., to solving a system of linear algebraic equations:

$$\Lambda Y + R = 0. \tag{3.1}$$

We mention that all indices calculated by the expressions (2.15) and (3.1) are vectors. The i^{th} element of this vector is the estimated value of the index for the initial state s_i.

3.1. Unreliability

System unreliability over the time interval $(0, t)(Q(t))$ can be calculated if we define elements of matrix W as $w_{ij} = 0 \,\forall\, i, j \in \Omega_g; w_{ij} = 1 \,\forall\, i \in \Omega_g \; and \; j \in \Omega_f$:

$$w_{ij} = \begin{cases} 1, \text{if } i \in \Omega_g, \; j \in \Omega_f \\ 0, \text{otherwise} \end{cases} . \tag{3.2}$$

Besides we must treat states in Ω_f as absorbing states. Namely we should to delete in Markov graph all arcs leading from states in Ω_f to states in Ω_g.

The calculation is performed by (2.15).

Complementary index is reliability: $R(t) = 1 - Q(t)$.

3.2. Mean Time to First Failure

Mean time to first failure (T_{FF}) can be calculated if we define diagonal elements of matrix W corresponding to the operational states as $w_{ii} = 1 \,\forall\, i \in \Omega_g$.

$$w_{ij} = \begin{cases} 1, \text{if } i = j, \; i, j \in \Omega_g \\ 0, \text{otherwise} \end{cases} . \tag{3.3}$$

Besides we must treat states in Ω_f as absorbing states.

Calculation is performed by (3.1). For the given type of reward matrix expression (3.1) is reduced to

$$\Lambda^* T_{FF} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}, \tag{3.4}$$

where $\Lambda^* - k \times k$ matrix derived from matrix Λ by deleting rows and columns corresponding to nonoperational states; k - total number of operational states.

3.3. Standard Deviation of Random Time to First Failure

Standard deviation (σ) and variance (σ^2) of mean time to first failure can be calculated in accordance with the expression $\sigma = \sqrt{M(t^2) - M^2(t)} = \sqrt{T_{FF}^{(2)} - T_{FF}^2}$.

To calculate vector $T_{FF}^{(2)}$, we have to find the solution of algebraic equations

$$\Lambda^* T_{FF}^{(2)} = \begin{pmatrix} -2T_{FF1} \\ \vdots \\ -2T_{FFk} \end{pmatrix}, \tag{3.5}$$

where T_{FFi} is mean time to first failure given the process starts from the i^{th} state.

3.4. Average accumulated time spent in the operational states

Average accumulated time spent in the operational states during the time interval $(0, t)$ $(T_{\sum}(t))$ can be calculated if we define diagonal elements of matrix W corresponding to the operational states as $w_{ii} = 1 \,\forall\, i \in \Omega_g$.

$$w_{ij} = \begin{cases} 1, \text{if } i = j;\ i, j \in \Omega_g \\ 0, \text{otherwise} \end{cases}. \tag{3.6}$$

Here, we should not artificially create absorbing states.

Calculation is performed by the Equation (2.15).

We can also use this index for analysis of multilevel systems. In this case we should partition the set of system's states Ω into m sets $\Omega_1, \dots \Omega_r, \cdots, \Omega_m$, where Ω_r - the set of operational system states with r^{th} performance level. Average accumulated time spent in Ω_r during the time interval $(0, t)$ $(T_{\sum}^r)$ can be calculated at $w_{ii} = 1 \,\forall\, i \in \Omega_r$.

3.5. Availability

Availability $A(t)$ is the probability that the system is operational at the time instant t. $A(t)$ is the point measure. Taking into account that $A(t) = dT_{\Sigma}(t)/dt$ we can calculate $A(t)$ after calculation of $T_{\Sigma}(t)$ as follows:

$$A(t) = \Lambda T_{\Sigma} + R. \tag{3.7}$$

The formation of reward matrix is carried out in accordance with (3.6).

Unavailability $U(t) = 1 - A(t)$.

3.6. Time-averaged availability

General expression of time-averaged availability over the time interval $(0, t)$ $(A_a v(t))$ has the form

$$A_{av}(t) = \frac{1}{t} \int_0^t A(\tau)\, d\tau. \tag{3.8}$$

For calculation of this indicator, it is necessary to take the reward matrix, in which the diagonal elements of the operational system states, are equal to $1/t$ and all other elements are zero:

$$w_{ij} = \begin{cases} 1/t, \text{if } i = j;\ i, j \in \Omega_g \\ 0, \text{otherwise} \end{cases}. \tag{3.9}$$

Artificial creation of absorbing states is not required. Calculations are carried out by (2.15).

3.7. Average failures number

Average (expected) failures number during the time interval $(0, t)(N(t))$ can be calculated if we define elements of matrix W columns corresponding to the failed states as $w_{ij} = 1 \, \forall \, i \neq j$. Values of other elements of matrix W are zero:

$$w_{ij} = \begin{cases} 1, \text{if } i \in \Omega_g, \ j \in \Omega_f \\ 0, \text{otherwise} \end{cases} . \tag{3.10}$$

$N(0, t)$ is failure measure for repairable systems so we should not artificially create absorbing states. Calculation is performed by (2.15).

3.8. Failure frequency

Failure frequency $(\omega(t))$ is a differential index with respect to $N(t)$. After calculating of vector $N(t)$ for each initial state, we put it into the right side of (2.15). The resulting vector $dY(t)/dt$ will be the vector $(\omega(t))$:

$$\omega(t) \ = \ \Lambda N(t) + \begin{pmatrix} \lambda_\Sigma^1 \\ \vdots \\ \lambda_\Sigma^N \end{pmatrix}, \tag{3.11}$$

where λ_Σ^i is total transition rate from i^{th} operational state to failure states. For the failure state, it is zero.

3.9. Average accumulated reward

Average accumulated reward over the time interval $(0, t)(E_T(t))$ integrates reward (loss) of the system during the time interval $(0, t)$ proportionally to the time of stay in the states and the quantity of transitions between the states. Reward matrix W has the most general form. The elements w_{ii}, w_{ij} are real rewards (losses) measured in terms of units of system performance. Calculations are carried out by (2.15).

3.10. Average reward

Average reward at the time instant $t(E_{av}(t))$ is a point measure. This indicator characterizes the multi-level systems. Therefore, system states set is subdivided into classes, in accordance with level of system performance. States of the i^{th} class are characterized by performance (E_i) per time unit (for example, throughput). The first step of $E_{av}(t)$ calculation is performed by the Equation (2.15). As a result of the first calculation step, we get the vector of the average accumulated reward $E_T(t)$. $E_{av}(t)$ is a differential index with respect to $E_T(t)$. So the second calculation step is to put the vector $E_T(t)$ into right side of equations (2.15):

$$E_{av}(t) \ = \ \Lambda E_T(t) + R. \tag{3.12}$$

3.11. Performability ratio

This index is the ratio of the average accumulated reward to the nominal reward during the time interval $(0, t)$:

$$K_p(t) \ = \ \frac{E_T(t)}{E_n(t)}, \tag{3.13}$$

where $E_n(t)$ - nominal (maximum) accumulated reward generated by the system in the absence of failures. $E_n(t) = E_{max}t$. Usually $E_{max} = E_1$.

4. ANALYTICAL SOFTWARE BASED ON MARKOV REWARD MODEL

The above uniform approach to calculating performance, availability and reliability measures was used by us in the development of analytical software. The software is operated under Windows OS and is written in $c\sharp$ programing language. It has an advanced graphical user interface (Fig.4.1) and uses OLE automation for generation reports in MS Word formats.

This software allows you to take into account all features of reliability behavior of complex technical systems, mentioned in sectoion 1:

- multiple levels of operation performance and the graceful degradation of performance in the event of failures;
- different types of redundancy implementation techniques;
- variety of redundant schemes and backup components load levels;
- recovery strategies and restrictions on repair resources;
- imperfect built-in test;
- possibility of transient faults occurrence;
- common cause failures;
- and so on.

The user of this software can calculate all the RAP indicators described in Section 3:

- Reliability (Unreliability) for a given time interval (0, t) ;
- Mean and Standard Deviation of random time to first failure;
- Average and Accumulated Time spent in the selected subset of states during the time interval (0, t);
- Availability (Unavailability) at given time instant t;
- Average Failures Number for a given time interval (0, t);
- Failure Frequency at given time instant t;
- Average Reward at given time instant t;
- Average Accumulated Reward for a given time interval (0, t);
- Performability Ratio for a given time interval (0, t).

The software execution process is divided into three steps: creation of the MRM, setting up of the MRM, numerical solution of the system of Howard equations. The composition and interrelationships of these processes are shown in Fig.4.2-Fig.4.4.

MRM creation is executed by the user in interactive mode with the help of built-in graphical editor. The created model can be saved in an external XML file. A saved model can be reloaded from the XML file (Fig.4.2).

The model's setting up is performed in the Matrix Configuration Module in accordance with equations of section 3. Setting up operations are performed automatically after the user chooses the metric for calculation (Fig.4.3).

Calculation of values of user-selected RAP indices is made in the numerical solution module (Fig.4.4). In this module, numerical procedures for solving systems of differential and algebraic Howard equations are realized. Different approaches and methods of computational mathematics with regard to their software implementation outlined in [35 - 37].

Detailed consideration of numerical evaluation of Markov models transient behavior was presented in [9], [10]. Three numerical techniques for finding the transient solution of large Markov models were examined. They are - uniformization, an explicit differential equation solution method (Runge-Kutta) and special stable implicit method (TR-BDF2). For the software implementation of numerical solution of the MRM we used efficient method based on evaluation of matrix exponential at small step. This method had been proposed and described in [38].

For a stationary case, a numerical solution is obtained by solving the system of algebraic equations by the Gauss method or the Rotation method [35,36]. The Rotation method works well for bad conditioned systems, which are often generated by RAP analysis models.

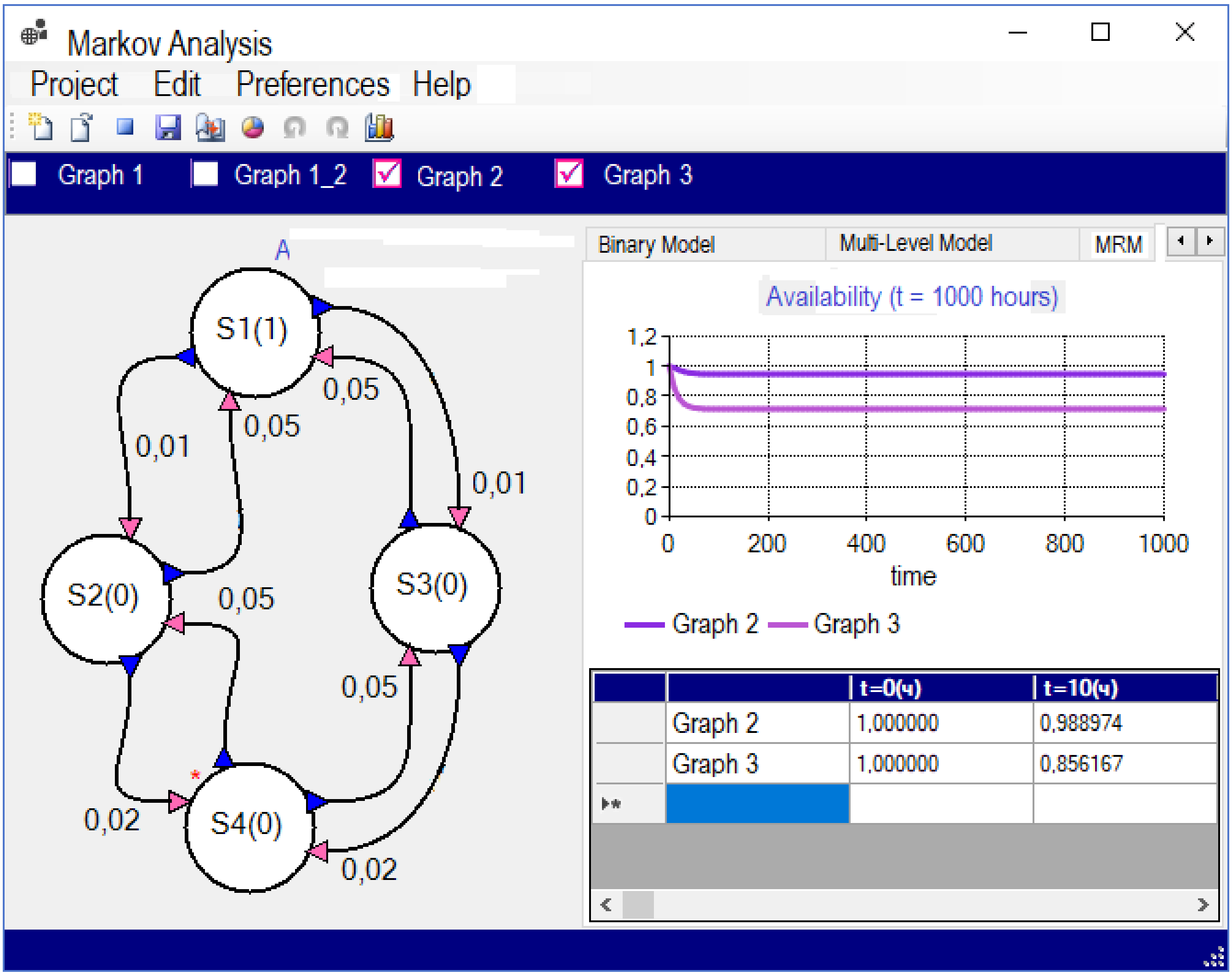

Fig. 4.1. Screenshot of main form of the analytical software.

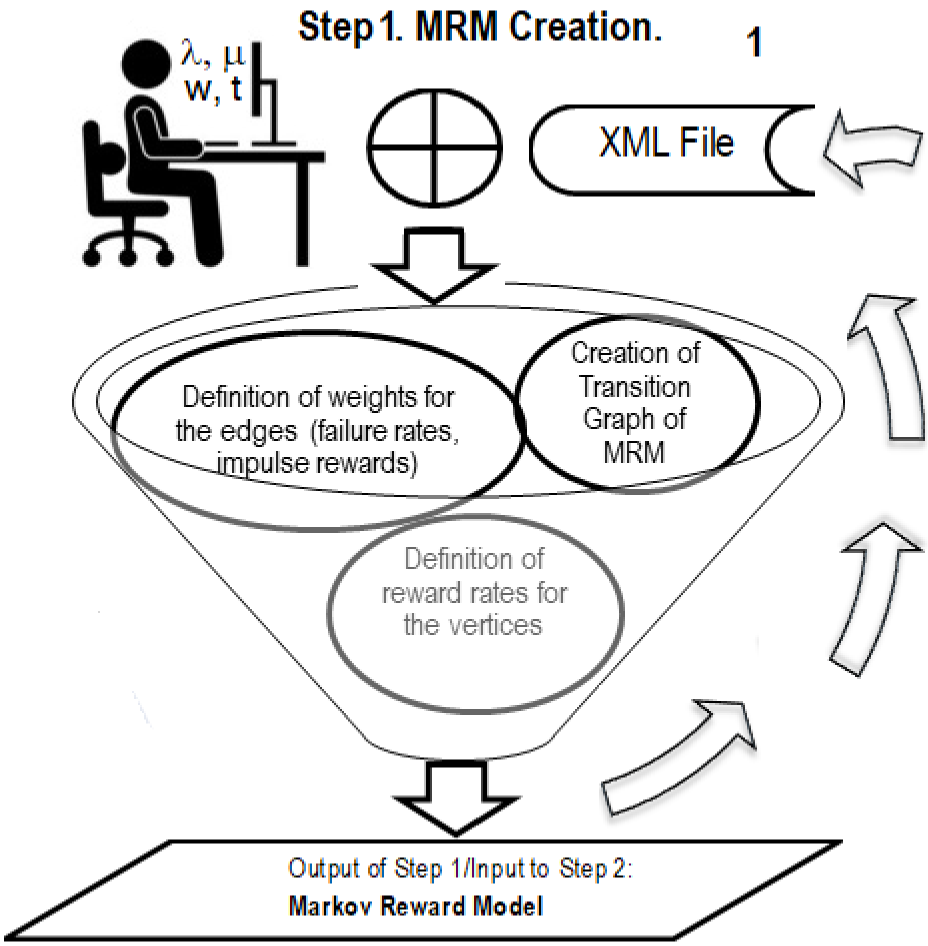

Fig. 4.2. Execution process of the analytical software (Step 1).

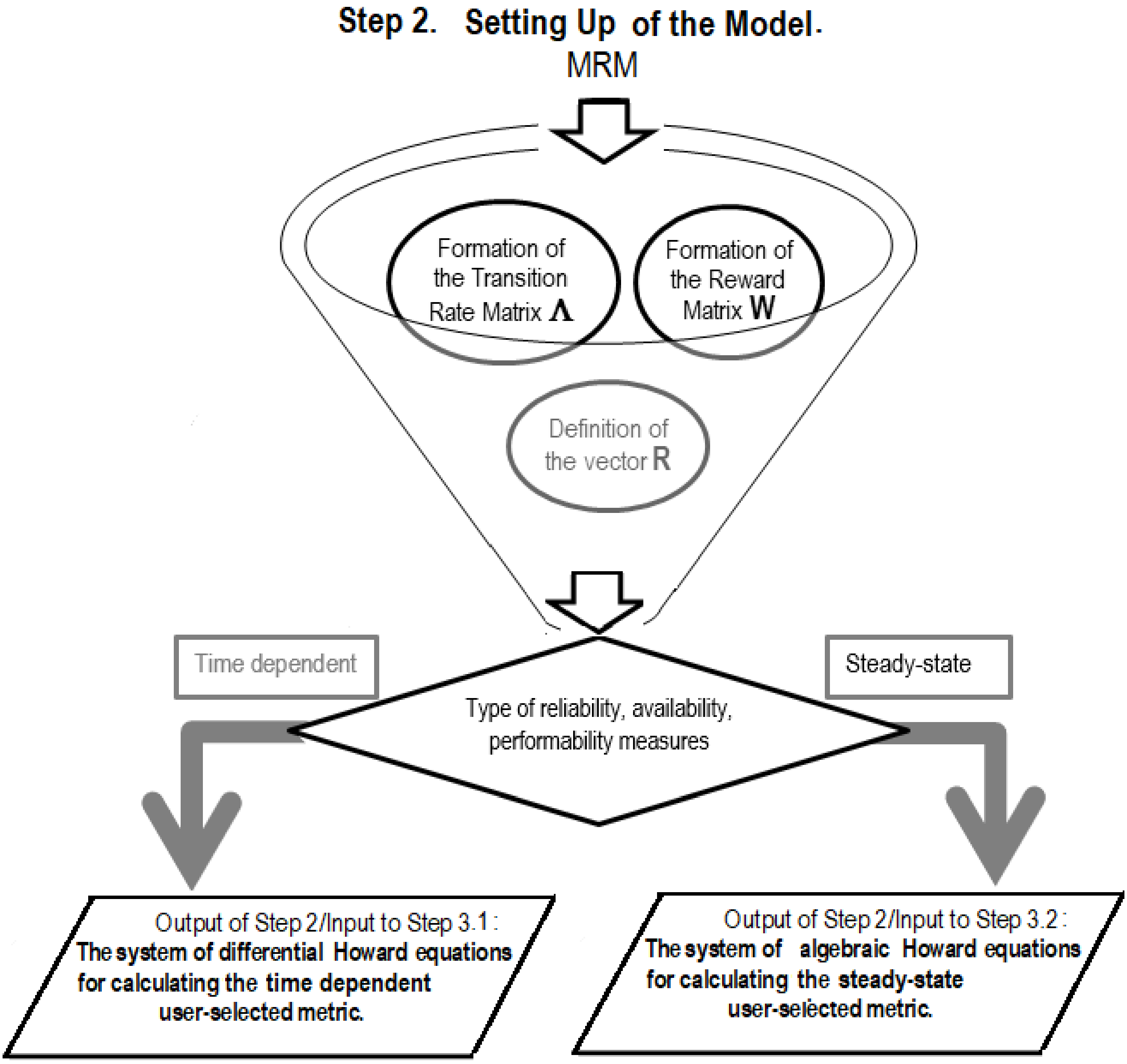

Fig. 4.3. Execution process of the analytical software (Step 2).

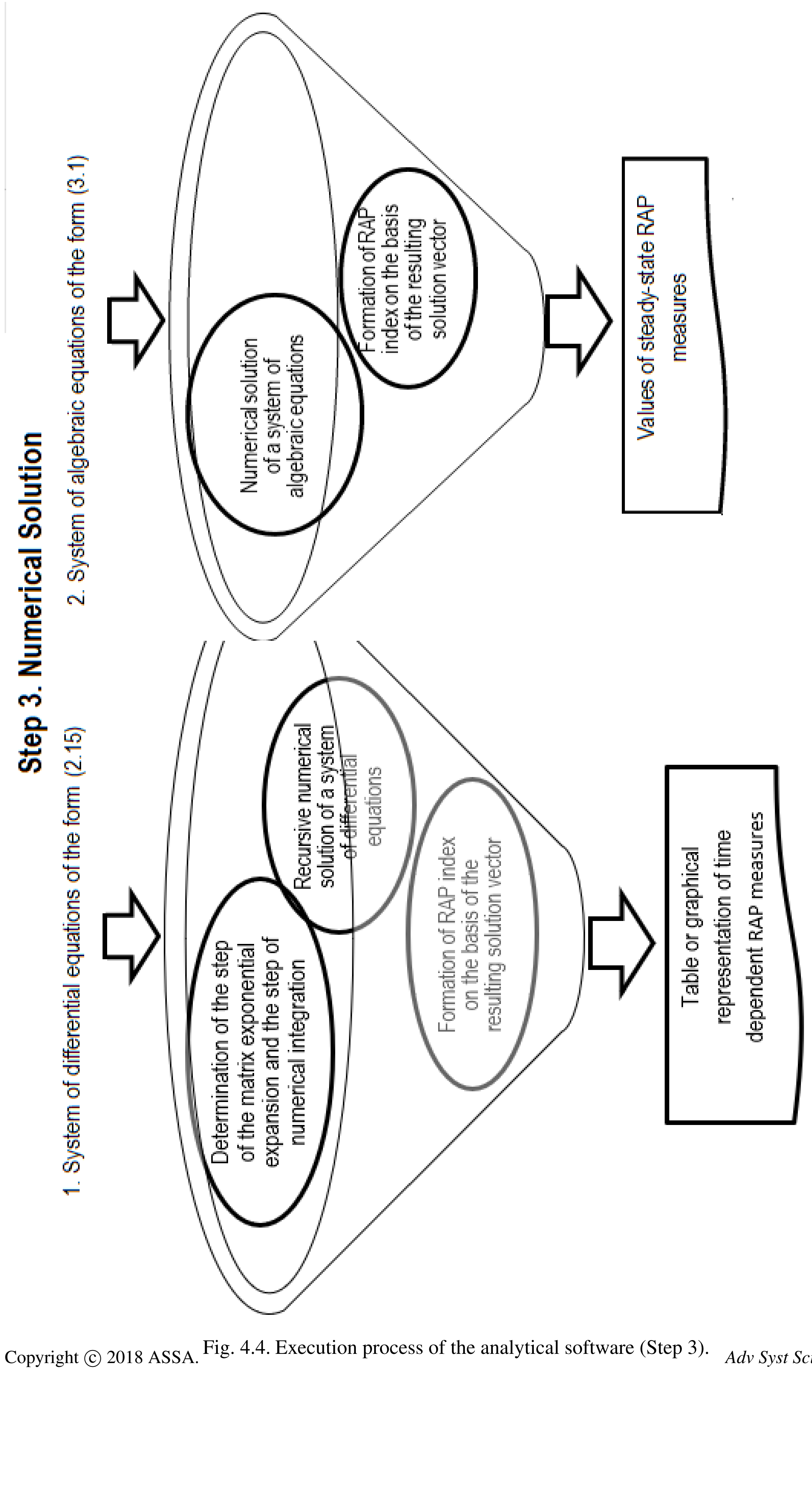

Fig. 4.4. Execution process of the analytical software (Step 3).

5. CASE STUDIES

To demonstrate the effectiveness of the software implemented MRM, we present two examples of reliability, availability, performability evaluation of complex systems. In first example we investigate technological object with protection system. The second one is demand-based warm standby system (DB-WSS), described in article [23].

5.1. Markov reward model of multilevel process unit

We consider process unit consisting of technological object (TO) with protection system (PS). The protection system includes a diagnostic unit (DU) and the actuator (A) (Fig.5.5).

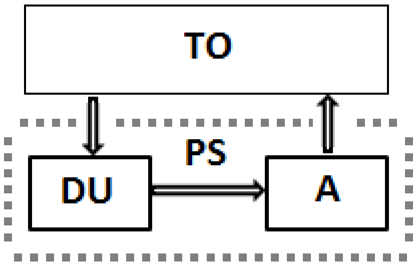

Fig. 5.5. Technological object with protection.

Protection systems are designed to generate control actions to the protected object in order to prevent transition from process equipment failures to accident. Control actions may be different, such as change of the operating mode, reduction in productivity, emergency shutdown of faulty elements and elements belonging to the same processing chain. These actions prevent the development of accident. Here, we consider the case of technological process shutdown. However, proposed approach is also suitable for the control actions that lead to poor performance or mode change.

The main function of protection system is performed sporadically at the time instants of the occurrence of the object failures, so that the protection system operates in standby temporary mode. Let us define rewards of operational states and losses of failed states. In this example, all possible system states can be subdivided into four classes. The first class includes such technical states in which the object is functioning normally and earns specific rewards per unit of time of stay in these states (e.g., these states correspond to the nominal performance of technological object). Let us note that this group includes states with latent failures such as failure of diagnostic unit. The second class consists of the states of accident-free shutdown of the technological object. The losses associated with these states are only related to the downtime of the object. The reward in these states is either zero or negative, if the idle leads to additional losses per unit time. The third and fourth classes of states are catastrophic failures of the object. The transition to this class of states from the states of the first class brings a one-time damage (negative reward) associated with the occurrence of an accident (death of people, equipment breakdowns, emissions into the atmosphere, etc.). In this model, we consider two types of catastrophic failures differing in the severity of the consequences – accident I and accident II. Thus, the normal functioning of the technological object is accompanied by a linear increase in the accumulated reward in proportion to the time spent in the first class states. Idle time leads to the preservation of the achieved level of accumulated reward (at zero values of the reward rate in each state) or to its descent (at negative values of the reward rates) in proportion to the time spent in states of second, third and fourth classes. When impulse rewards associated with the transitions between the states are not zero, there is an abrupt change (more often a decrease) in the accumulated reward. Negative impulse rewards are due to the costs of recovery from failures or accidents, purchase of equipment, payment of fines or insurance, etc.

We suppose that all accidents occur only due to failures of the technological equipment. Let protection system in the event of critical situation instantly shut down the technological process by making the necessary equipment control (for example, switching off). Let the protection system processes critical situation with coverage $\beta (0 \le \beta \le 1)$. Failures of the protection system which occur during interval of normal TO operation can lead to different consequences. Let us single out PS failures of two types: latent failures (no operation) and explicit failures (false alarm). Latent failures of PS are manifested in the form of absence of protection actions in the event of critical situation which entails an accident. False alarms lead to the undesired protection actions in the absence of TO failures.

Parameters of the Markov reward model of the technological object with protection system are:

w_{ij} - impulse losses (negative impulse reward) under the transition from state i into state j;
w_{ii} - reward rate (negative or positive) in state i;
β - coverage probability for TO, determined as conditional probability of processing of TO catastrophic failure by a protection system, provided that the failure has occurred;
α_{DU} - fraction of the protection system latent failures such as non operation of DU;
α_A - fraction of the protection system latent failures such as non triggering of actuator;
$1-\alpha_{DU}$ - fraction of the protection system explicit failures such as DU false alarm;
$1-\alpha_A$ - fraction of the protection system explicit failures such as actuator false triggering;
η_I - fraction of technological object catastrophic failures of the first kind (accident I);
η_{II} - fraction of technological object catastrophic failures of the second kind (accident II);
$\lambda_{TO}, \lambda_{DU}, \lambda_A$ – failure rates of technological object, diagnostic unit, actuator respectively;
μ - repair rate of technological object after shutdown;
μ_a - repair rate of technological object after falling into accident II state.

The transition graph of Markov reward model of the technological object with protection is shown in 5.6.

We partition the set of the MRM states into four subsets:

- normal operation (OP) (states 1,3,4)
- accident-free shutdown (SD) (states 2,6)
- accident I (AI) (state 5)
- accident II (AII) (state 7)

Transition rates matrix Λ of the model is

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} & 0 & \lambda_{17} \\ \lambda_{21} & \lambda_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{32} & \lambda_{33} & 0 & \lambda_{35} & 0 & \lambda_{37} \\ 0 & 0 & 0 & \lambda_{44} & \lambda_{45} & \lambda_{46} & \lambda_{47} \\ \lambda_{51} & 0 & 0 & 0 & \lambda_{55} & 0 & 0 \\ \lambda_{61} & 0 & 0 & 0 & 0 & \lambda_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{77} \end{bmatrix}. \tag{5.1}$$

The diagonal elements of the matrix λ_{ii} are given by $-\sum\limits_{j,j\neq i} \lambda_{ij}$.

Transition rates between states of the graph are determined based on the model parameters as follows:

$$\begin{aligned}
\lambda_{12} &= [1-(1-\beta)(\eta_I+\eta_{II})]\lambda_{TO} + (1-\alpha_{DU})\lambda_{DU} + (1-\alpha_A)\lambda_A; \\
\lambda_{13} &= \alpha_{DU}\lambda_{DU}; \\
\lambda_{14} &= \alpha_A\lambda_A; \\
\lambda_{15} &= (1-\beta)\eta_I\lambda_{TO}; \\
\lambda_{17} &= (1-\beta)\eta_{II}\lambda_{TO}; \\
\lambda_{21} &= \lambda_{61} = \mu; \\
\lambda_{32} &= (1-\eta_I-\eta_{II})\lambda_{TO} + (1-\alpha_A)\lambda_A;
\end{aligned}$$

$$\begin{aligned}
\lambda_{35} &= \lambda_{45} = \eta_I \lambda_{TO}; \\
\lambda_{37} &= \lambda_{47} = \eta_{II} \lambda_{TO}; \\
\lambda_{46} &= (1 - \eta_I - \eta_{II}) \lambda_{TO}; \\
\lambda_{51} &= \mu_a;
\end{aligned}$$

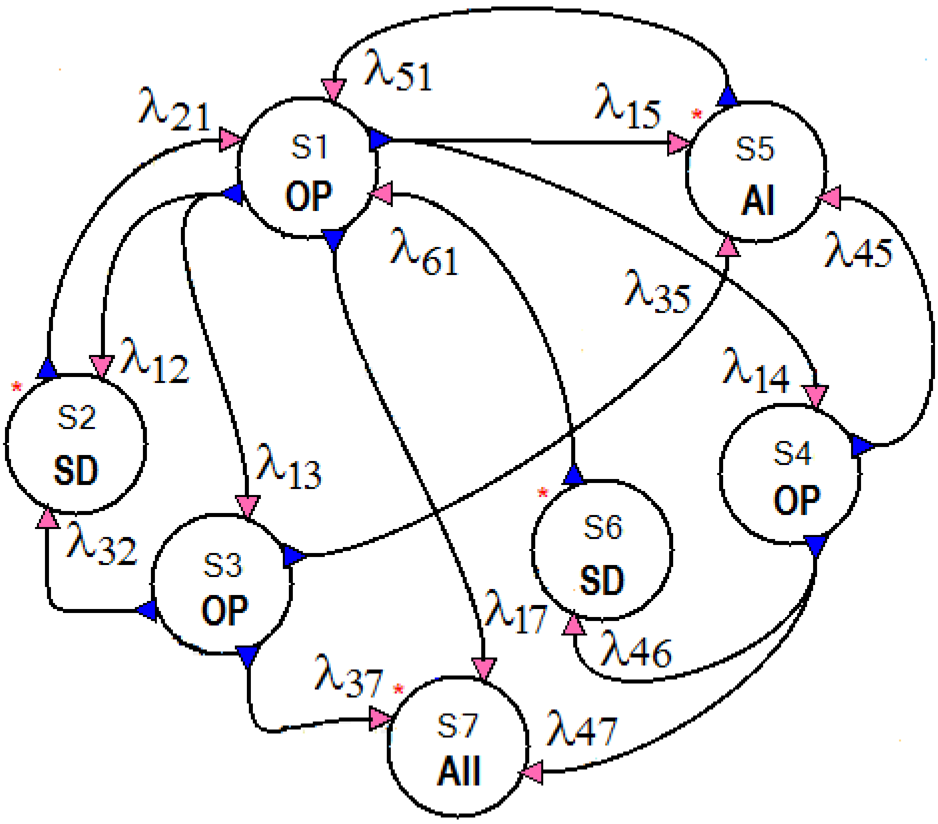

Fig. 5.6. Markov graph of TO with protection.

By specifying the form of reward matrix, it is possible to obtain equations for calculating various indices. We want to determine the cumulative effect of operation of technological object taking into account the positive and negative impact of the protection system. Therefore, it is advisable to calculate the following indices:

1. $Q(t)$ - unreliability over $(0, t)$.
   $Q(t)$ can be calculated from equation (2.15).
   To calculate $Q(t)$, we must equate to zero $\lambda_{21}, \lambda_{51}, \lambda_{61}$ and define W as

$$W = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.2}$$

2. $T_\Sigma(t)$ -average accumulated time spent in the operational states over the time interval $(0, t)$.
   $T_\Sigma(t)$ is determined by (2.15), setting R as

$$R = [\ 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\ ]^T. \tag{5.3}$$

3. $A(t)$ - system availability at the time instant t.
   $A(t)$ is calculated by (3.7) after calculation of vector $T_\Sigma(t)$.
4. T_{FF} - mean time to first failure.
   T_{FF} is determined by (3.4), where matrix Λ^* is

$$\Lambda^* = \begin{bmatrix} \lambda_{11} & \lambda_{13} & \lambda_{14} \\ 0 & \lambda_{33} & 0 \\ 0 & 0 & \lambda_{44} \end{bmatrix}. \tag{5.4}$$

5. σ - standard deviation of random time to first failure.
   σ is obtained from Equation (3.5), where vector of free terms is $[\ -2T_{FF1} \quad -2T_{FF2} \quad -2T_{FF3}\ ]^T$.
6. T_A - mean time to first catastrophic failure (accident I or II).
   T_A is determined by Equation (3.4), where matrix Λ^* is 5×5 matrix derived from matrix (5.1) by deleting rows and columns corresponding to accident I and accident II states (states 5 and 7).
7. $N(t)$ - expected failures number.
   $N(t)$ can be calculated from (2.15) where matrix W takes the form (5.2).
8. $\omega(t)$ - failure frequency.
   $\omega(t)$ can be calculated from (3.11) after $N(t)$ calculation and setting free terms vector as $[\ \lambda_{12} + \lambda_{15} + \lambda_{17}, 0, \lambda_{32} + \lambda_{35} + \lambda_{37}, \lambda_{45} + \lambda_{46} + \lambda_{47}, 0, 0, 0\ ]^T$.
9. $N_A(t)$ - expected catastrophic failures number.
   $N_A(t)$ can be calculated from (2.15) where matrix W has unit elements only in the transitions to catastrophic failures (states 5 and 7). Similarly, we can find separately $N_{AI}(t)$ and $N_{AII}(t)$.
10. $T_{\Sigma SD}(t)$ –average accumulated time spent in the states of technological object shutdown over time interval $(0, t)$. $T_{\Sigma SD}(t)$ is determined by Equation (2.15), setting R as $[\ 0, 1, 0, 0, 0, 1, 0\ ]^T$.
11. $N_{SD}(t)$ - average number of transitions to states of shutdown.
    $N_{SD}(t)$ can be calculated from Equation (2.15) where matrix W has unit elements only in the transitions to shutdown states (states 2 and 6).
12. $E_T(t)$ - average accumulated reward over the time interval $(0, t)$.
    Calculation of $E_T(t)$ is performed by (2.15).
    Matrix of reward W is

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & 0 & w_{17} \\ w_{21} & w_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{32} & w_{33} & 0 & w_{35} & 0 & w_{37} \\ 0 & 0 & 0 & w_{44} & w_{45} & w_{46} & w_{47} \\ w_{51} & 0 & 0 & 0 & w_{55} & 0 & 0 \\ w_{61} & 0 & 0 & 0 & 0 & w_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_{77} \end{bmatrix}. \tag{5.5}$$

13. $E_{av}(t)$ - average reward at the time instant t.
    In accordance with (3.12) $E_{av}(t)$ is determined by solving the system of algebraic equations

$$\begin{bmatrix} E_{av1}(t) \\ \vdots \\ E_{av7}(t) \end{bmatrix} = \Lambda \begin{bmatrix} E_{T1}(t) \\ \vdots \\ E_{T7}(t) \end{bmatrix} + \begin{bmatrix} w_{11} \\ \vdots \\ w_{77}(t) \end{bmatrix} + \begin{bmatrix} \sum\limits_{j \neq 1} \lambda_{1j} w_{1j} \\ \vdots \\ \sum\limits_{j \neq 7} \lambda_{7j} w_{7j} \end{bmatrix}. \tag{5.6}$$

The numerical results of above indices calculation for different values of parameter β are summarized in table 5.1. All time-dependent metrics were calculated on one year period ($t = 8760 hours$). We note once again that the solution, obtained by the MRM, is a vector whose i^{th} component corresponds to the start of the Markov process from state i. The table and diagrams include the first components of solution vectors, i.e the system starts from completely operational state $S1$.

The functions of $E_T(t)$, $E_{av}(t)$ for different values of catastrophic failure coverage parameter β are shown in Fig.5.7 and Fig.5.8 respectively.

Calculation of indices and plotting are performed with the following system parameters:
$\lambda_{TO} = 0.0001(/hour); \lambda_{DU} = 0.00001(/hour); \lambda_A = 0.00005(/hour)$.
$\mu = 0.1(/hour); \mu_a = 0.004(/hour)$.
$\alpha_{DU} = 0.5; \alpha_A = 0.9$.
$\eta_I = 0.4; \eta_{II} = 0.05$.

Numerical values of impulse rewards are
$w_{15} = -105; w_{17} = -108; w_{12} = w_{13} = w_{14} = 0;$
$w_{21} = 0; w_{32} = 0; w_{35} = -105; w_{37} = -108;$
$w_{45} = -105; w_{46} = 0; w_{47} = -108;$
$w_{51} = w_{61} = 0.$

Numerical values of reward rates are
$w_{11} = 15(/hour); w_{33} = 10(/hour); w_{44} = 10(/hour);$
$w_{22} = w_{55} = w_{66} = w_{77} = 0(/hour).$

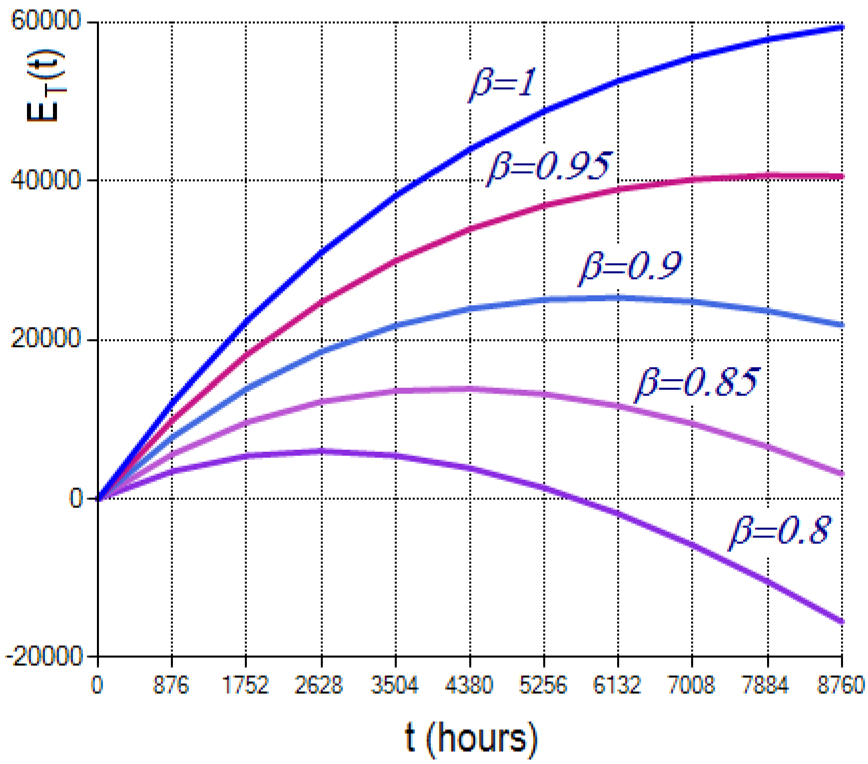

Fig. 5.7. TO average accumulated reward $E_T(t)$ versus time t for different coverage values β.

The first family of curves (Fig.5.7) shows the significant dependence of the average accumulated reward on the catastrophic failure coverage β. For the model parameters under consideration β should be greater than 0,9. The second family of curves(Fig.5.8) shows a sharp decline in the growth of the average accumulated reward in time for all coverage values. It occurs due to the high rate of PS latent failures. To prevent such a recession, it is advisable

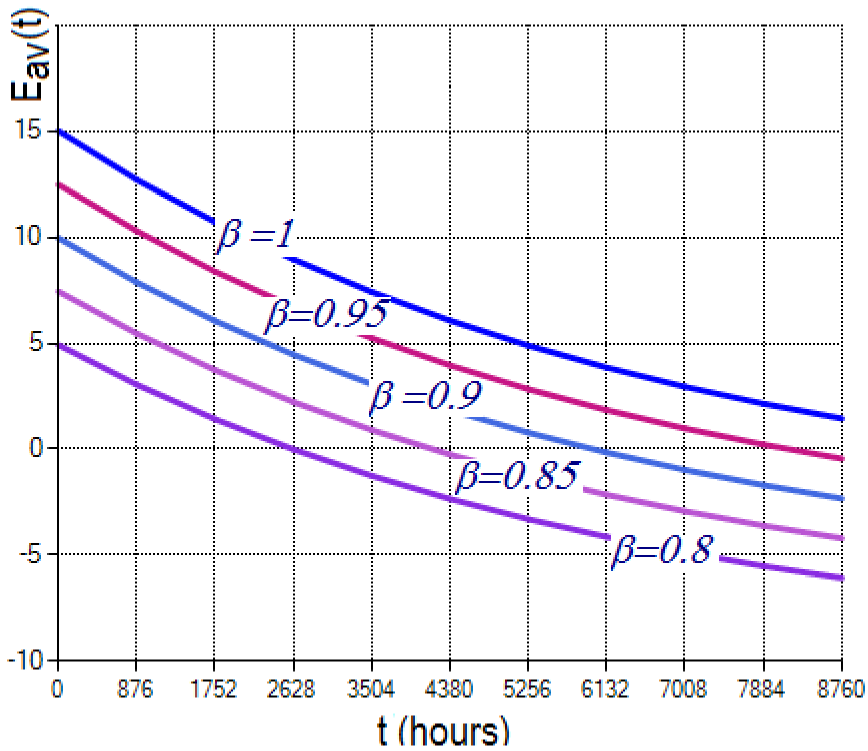

Fig. 5.8. TO average reward $E_{av}(t)$ versus time t for different coverage values β.

Table 5.1. Calculation results over one year period.

	Catastrophic failure coverage β				
Index	0.8	0.85	0.9	0.95	1
$Q(t)$	0.612	0.612	0.612	0.612	0.612
$T_{\Sigma}(t)$	8670.6	8682.4	8694.3	8706.2	8718.1
$A(t)$	0.9814	0.9836	0.9858	0.988	0.99
T_{FF}	9360.1	9360.1	9360.1	9360.2	9360.1
σ	9555.6	9555.6	9555.6	9555.6	9555.6
T_A	4774.7	5143.4	5573.8	6082.8	6694.1
$N(t)$	0.942	0.943	0.944	0.946	0.947
$N_A(t)$	0.124	0.108	0.091	0.075	0.058
$N_{AI}(t)$	0.11	0.096	0.081	0.0663	0.0516
$N_{AII}(t)$	0.014	0.012	0.01	0.0083	0.0064
$T_{\Sigma SD}(t)$	8.16	8.34	8.52	8.7	8.88
$N_{SD}(t)$	0.817	0.835	0.853	0.871	0.889
$E_T(t)$	-15489	3146	21834	40575	59368
$E_{av}(t)$	-6.068	-4.202	-2.33	-0.443	1.45

to carry out preventive maintenance of the process unit (TO+PS), during which latent failures of the protection system are detected.

5.2. *Reliability and Performability Investigation of DB-WSS*

We take demand-based warm standby system (DB-WSS), described in Section IV.A of article [23] as the second object of study. This DB-WSS is composed of two components A_1 and A_3 (we keep the numbering made by the authors of [23]).

The components A_1 and A_3 can be in four degradation states, which differ in capacity. The components capacity levels are 5, 4, 2, 0 and 5,4,1,0, respectively. Component A_1 is online. Component A_3 can be online or in warm standby state depends on the system demand. We consider scenario B from [23], when demand determined as 5. So initial state of A_3 is warm standby. The transition time distributions for the degradation processes for each component are exponential distributions with parameters $\lambda_1 = 1/100, \lambda_3^w = 1/300, \lambda_3 = \lambda_3^o = 1/150$ (/day), where (w) indicates warm standby state and (o) – operating online state.

First, we will construct a Markov model completely corresponding to the degradation process described in [23]. Namely, we will construct a model of a binary-state system consisting of multistate components. The transition Markov graph of this DB-WSS is shown in Fig.5.9. The states S1 and S10 are operational states of the system that differ in the capacity of components. The state (j, k) corresponds to the j^{th} capacity level of element $A_1(C_j^{A1})$ and the k^{th} capacity level of element $A_3(C_k^{A3})$. State S11 is failed system state. $C_j^{A1} + C_k^{A3} \geq 5$ for operational states; $C_j^{A1} + C_k^{A3} < 5$ for failed state.

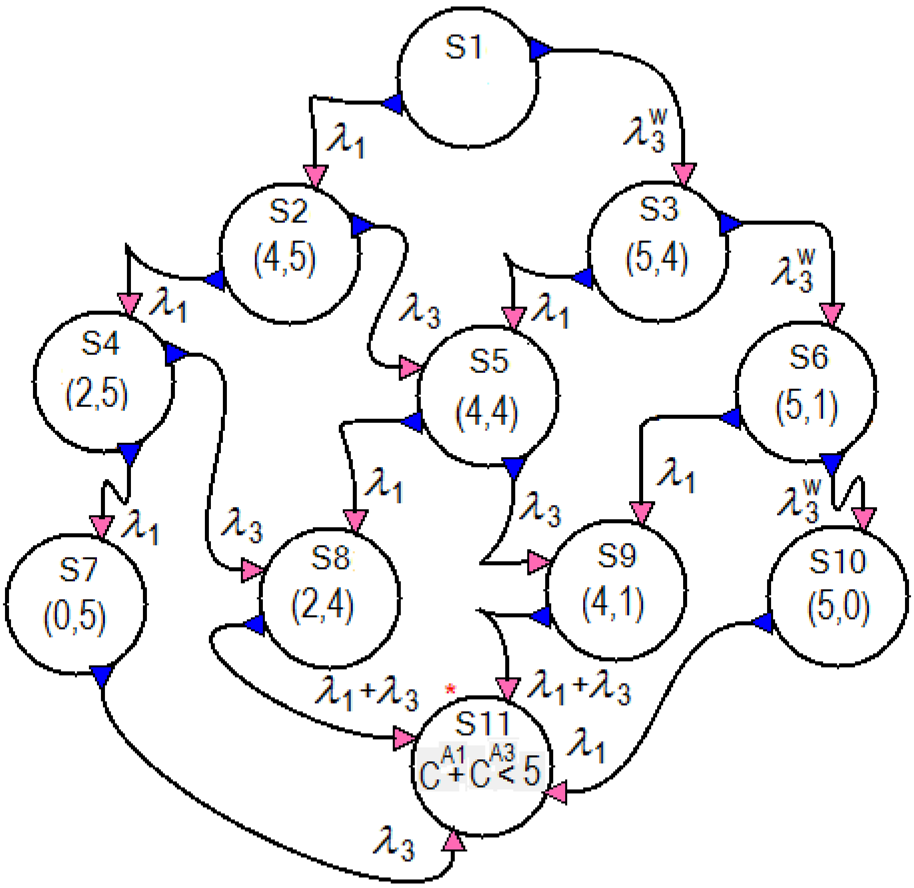

Fig. 5.9. Markov graph of binary-state DB-WSS.

For the first case, we calculate the following reliability indicators:

1. $R(t)$ reliability during $(0, t)$.
   Unreliability $Q(t)$ can be calculated from equation (2.15), where vector $R = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \lambda_3\ \ \lambda_1 + \lambda_3\ \ \lambda_1 + \lambda_3\ \ \lambda_1\ \ 0\ ]^T$.
   Reliability $R(t) = 1 - Q(t)$. Table 5.2 shows the results of our calculation of $R(t)$ and values of $R(t)$ taken from [23]. For the model under study (Fig.5.9), an exact analytical solution was obtained (see Appendix). The results of $R(t)$ calculations using (A.2), (A.3) are given in the last column of Table 5.2.
2. T_{FF} -mean time to first failure.

Table 5.2. Calculation results of system reliability.

	$R(t)$			
Time [day]	MRM	Method proposed in [23]	Monte Carlo from [23]	Exact analytical solution
50	0.99317	0.9917	0.9931	0.993170
100	0.93879	0.9333	0.9385	0.938794
200	0.67104	0.6610	0.6730	0.671043
300	0.38090	-	-	0.380899
400	0.18958	-	-	0.189584
500	0.08858	0.0864	0.0884	0.088584

The T_{FF} is determined in accordance with (3.4) T_{FF}=284.61 [day].

3. σ - standard deviation of random time to first failure.
The σ is obtained from (3.5), where vector of free terms is
$[\ -2T_1\ \ -2T_2\ \ -2T_3\ \ -2T_4\ \ -2T_5\ \ -2T_6\ \ -2T_7\ \ -2T_8\ \ -2T_9\ \ -2T_{10}\ ]^T$.
σ = 154.05 [day].

Next, we assume the possibility of a multi-level system operation. We assume that the states of the system with a capacity less than 5 can be subdivided into groups (levels) with capacity $\{4, 3, 2, 1, 0\}$. The model of the multistate system consisting of multistate components is shown in Fig.(5.10). The properties of the levels are given in Table 5.3.

Table 5.3. The groups properties and measures for multistate model

Levels	The group	Capacity	T_Σ^i
1	S1,S2,S3,S4,S5,S6,S7,S8,S9,S10	5	273.10
2	S11,S13	4	63.48
3	S12	3	17.75
4	S15	2	16.38
5	S14	1	52.43
6	S16	0	76.86

For the second case, we calculate the following reliability and performability indicators:

1. T_Σ^i - average total residence time of the system in the states of the i^{th} group during $(0, t)$.
The T_Σ^i is determined in accordance with Section 3.4 . For example, T_Σ^2 is calculated by (2.15) with vector $R = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ ]^T$.
Last column of Table 5.3 shows the results of calculation of T_Σ^i for $t = 500[day]$.
2. $K_p(t)$ - performability ratio.
The $K_p(t)$ is calculated in accordance with (3.13).
To calculate performability ratio, we should determine the reward matrix. For this system, the reward matrix will consist only of diagonal elements. We will define reward rates through the capacities, corresponding to the states of operation and degradation as $w_{ii} = C_i/(C_{max}t)$. For $C_{max} = 5$ and $t = 500$ [day] we have $w_1 = \cdots = w_{10} = 0.002; w_{11} = w_{13} = 0.0016; w_{12} = 0.0012; w_{14} = 0.0008; w_{15} = 0.0004; w_{16} = 0$. As a result of solving the system of equations (2.15), we obtain $K_p(500) = 0.70315$.

This performability index is useful in choosing the best project version of a multi-level system, provided that the reliability requirements are met. In this case, the reliability of the system during the time interval (0,500) [days] is extremely small (see last raw of Table 5.2). However, the value of the performability index is quite satisfactory, which confirms the correctness of the choice of the redundancy scheme.

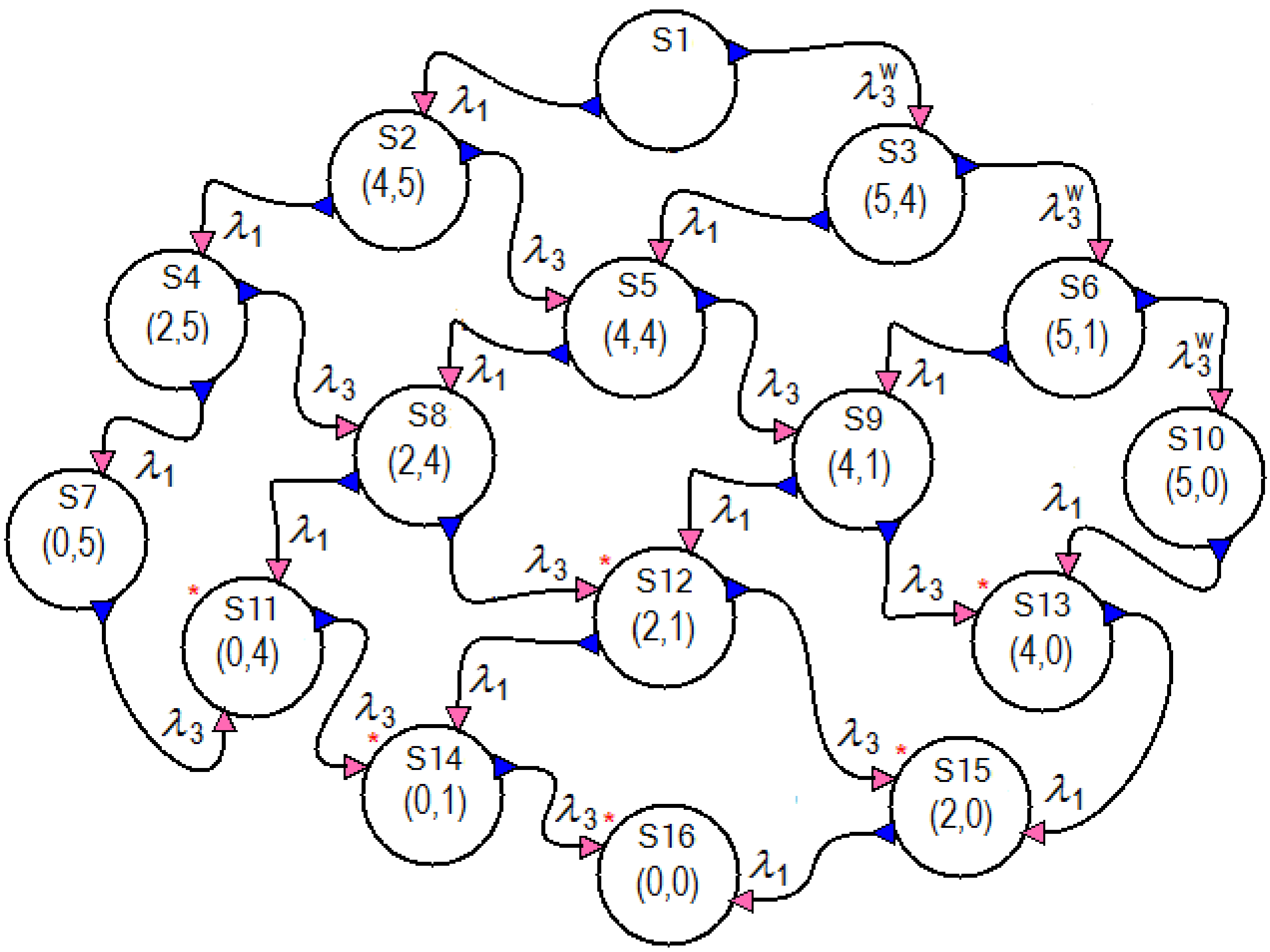

Fig. 5.10. Markov graph of multistate DB-WSS.

6. CONCLUSION

This paper presents a unified approach for the estimation of reliability, availability, performability of engineering systems. The approach uses Markov process with rewards to model transient and steady-state behavior of the systems. By using Markov reward models, we can derive various instantaneous and cumulative measures and estimate wide range of indices. We present straightforward calculation technique for evaluation of reliability, availability and performability indices based on special definition of reward matrix. The technique implementation requires only solution of system of differential equations describing the behavior of the average reward. Any additional calculations, such as numerical integration, are not required. Based on this approach a software for reliability, availability, performability analysis was created. The software is operated under Windows OS and is written in $c\sharp$ programing language. It has an advanced graphical user interface, uses effective numerical method for solving MRM and applies OLE automation for generation reports in MS Word formats. The efficiency of the proposed approach and accuracy of numerical solution have been shown by the case studies of the technological system with protection and demand-based warm standby system.

7. APPENDIX

In this section, we present analytical solution of the model of a binary-state system consisting of multistate components (Fig.5.9). The column vector $p(t)$ of the state probabilities is calculated by solving a system of Kolmogorov-Chapman equations:

$$\begin{aligned}
&p'_1(t) = -p_1(t) \cdot (\lambda_1 + \lambda_3^w);\\
&p'_2(t) = p_1(t) \cdot \lambda_1 - p_2(t) \cdot (\lambda_1 + \lambda_3);\\
&p'_3(t) = p_1(t) \cdot \lambda_3^w - p_3(t) \cdot (\lambda_1 + \lambda_3^w);\\
&p'_4(t) = p_2(t) \cdot \lambda_1 - p_4(t) \cdot (\lambda_1 + \lambda_3);\\
&p'_5(t) = p_2(t) \cdot \lambda_3 + p_3(t) \cdot \lambda_1 - p_5(t) \cdot (\lambda_1 + \lambda_3);\\
&p'_6(t) = p_3(t) \cdot \lambda_3^w - p_6(t) \cdot (\lambda_1 + \lambda_3^w);\\
&p'_7(t) = p_4(t) \cdot \lambda_1 - p_7(t) \cdot \lambda_3;\\
&p'_8(t) = p_4(t) \cdot \lambda_3 + p_5(t) \cdot \lambda_1 - p_8(t) \cdot (\lambda_1 + \lambda_3);\\
&p'_9(t) = p_5(t) \cdot \lambda_3 + p_6(t) \cdot \lambda_1 - p_9(t) \cdot (\lambda_1 + \lambda_3);\\
&p'_{10}(t) = p_6(t) \cdot \lambda_3^w - p_{10}(t) \cdot \lambda_1;
\end{aligned} \tag{7.1}$$

The solution is

$$
\begin{aligned}
&p_1(t) = -e^{-(\lambda_1+\lambda_3^w)t};\\
&p_2(t) = \frac{\lambda_1}{\lambda_3-\lambda_3^w}(e^{-(\lambda_1+\lambda_3^w)t} - e^{-(\lambda_1+\lambda_3)t});\\
&p_3(t) = \lambda_3^w t e^{-(\lambda_1+\lambda_3^w)t};\\
&p_4(t) = (\frac{\lambda_1}{\lambda_3-\lambda_3^w})^2 (e^{-(\lambda_1+\lambda_3^w)t} - e^{-(\lambda_1+\lambda_3)t}) - \frac{\lambda_1^2}{\lambda_3-\lambda_3^w} e^{-(\lambda_1+\lambda_3)t};\\
&p_5(t) = \frac{\lambda_1}{\lambda_3-\lambda_3^w}(e^{-(\lambda_1+\lambda_3^w)t} - e^{-(\lambda_1+\lambda_3)t}) - \frac{\lambda_1\lambda_3 t}{\lambda_3-\lambda_3^w} e^{-(\lambda_1+\lambda_3)t} + \frac{\lambda_1\lambda_3^w t}{\lambda_3-\lambda_3^w} e^{-(\lambda_1+\lambda_3^w)t};\\
&p_6(t) = \frac{(\lambda_3^w t)^2}{2} e^{-(\lambda_1+\lambda_3^w)t};\\
&p_7(t) = \frac{\lambda_1}{\lambda_1+\lambda_3^w-\lambda_3} e^{-\lambda_3 t} + \frac{\lambda_1^2+\lambda_1(\lambda_3-\lambda_3^w)}{(\lambda_3-\lambda_3^w)^2} e^{-(\lambda_1+\lambda_3)t} - \frac{\lambda_1^3}{(\lambda_3-\lambda_3^w)^2(\lambda_1+\lambda_3^w-\lambda_3)} e^{-(\lambda_1+\lambda_3^w)t}\\
&+\frac{\lambda_1^2 t}{\lambda_3-\lambda_3^w} e^{-(\lambda_1+\lambda_3)t};\\
&p_8(t) = \frac{2\lambda_1^2}{(\lambda_3^w-\lambda_3)^2} e^{-(\lambda_1+\lambda_3^w)t} + \frac{\lambda_1^2(\lambda_3^w-2\lambda_3)t}{(\lambda_3^w-\lambda_3)^2} e^{-(\lambda_1+\lambda_3)t} - \frac{(\lambda_1 t)^2\lambda_3}{\lambda_3-\lambda_3^w} e^{-(\lambda_1+\lambda_3)t}\\
&+\frac{\lambda_1^2\lambda_3^w t}{(\lambda_3^w-\lambda_3)^2} e^{-(\lambda_1+\lambda_3^w)t} - \frac{2\lambda_1^2}{(\lambda_3^w-\lambda_3)^2} e^{-(\lambda_1+\lambda_3)t};\\
&p_9(t) = \frac{\lambda_1\lambda_3^w t}{\lambda_3^w-\lambda_3} e^{-(\lambda_1+\lambda_3)t} - \frac{\lambda_1(\lambda_3 t)^2}{2(\lambda_3-\lambda_3^w)} e^{-(\lambda_1+\lambda_3)t} - \frac{\lambda_1}{\lambda_3^w-\lambda_3} e^{-(\lambda_1+\lambda_3^w)t} - \frac{\lambda_1\lambda_3^w t}{\lambda_3^w-\lambda_3} e^{-(\lambda_1+\lambda_3^w)t}\\
&-\frac{\lambda_1(\lambda_3^w t)^2}{2(\lambda_3^w-\lambda_3)} e^{-(\lambda_1+\lambda_3^w)t} + \frac{\lambda_1}{\lambda_3^w-\lambda_3} e^{-(\lambda_1+\lambda_3)t};\\
&p_{10}(t) = e^{-\lambda_1 t} - \frac{(\lambda_3^w t)^2 + 2\lambda_3^w t + 2}{2} e^{-(\lambda_1+\lambda_3^w)t}.
\end{aligned}
\tag{7.2}
$$

Sought reliability is found as

$$R(t) = \sum_{i=1}^{10} p_i(t). \tag{7.3}$$

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