A UNIFIED APPROACH TO SINGULAR PROBLEMS ARISING IN THE MEMBRANE THEORY*

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Abstract. We consider the singular boundary value problem

$$(t^n u'(t))' + t^n f(t, u(t)) = 0, \quad \lim_{t\to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A,$$

where $f(t,x)$ is a given continuous function defined on the set $(0,1]\times(0,\infty)$ which can have a time singularity at $t=0$ and a space singularity at $x=0$. Moreover, $n\in\mathbb{N}$, $n\geqslant 2$, and a_0, a_1, A are real constants such that $a_0\in(0,\infty)$, whereas $a_1, A\in[0,\infty)$. The main aim of this paper is to discuss the existence of solutions to the above problem and apply the general results to cover certain classes of singular problems arising in the theory of shallow membrane caps, where we are especially interested in characterizing positive solutions. We illustrate the analytical findings by numerical simulations based on polynomial collocation.

Keywords: singular mixed boundary value problem, positive solution, shallow membrane, collocation method, lower and upper functions

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1. Introduction

We investigate the solvability of the singular mixed boundary value problem

$$(t^n u'(t))' + t^n f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1a}$$

$$\lim_{t\to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A, \tag{1.1b}$$

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where $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 \in (0,\infty)$, a_1, $A \in [0,\infty)$ and we denote $\lim_{t\to 1-} u'(t)$ by $u'(1-)$.

For the given function $f(t,x)$ we make the following assumption:

A1: The data function $f(t,x)$ is continuous on $(0,1]\times(0,\infty)$ and can have a time singularity at $t=0$ and a space singularity at $x=0$.

Definition 1.1. A function $f(t,x)$ has a *time singularity* at $t=0$, if there exists $x \in (0,\infty)$ such that

$$\int_0^{\varepsilon} |f(t,x)|\,\mathrm{d}t = \infty, \quad \varepsilon \in (0,1).$$

A function $f(t,x)$ has a *space singularity* at $x=0$, if

$$\limsup_{x\to 0+} |f(t,x)| = \infty, \quad t \in (0,1).$$

We focus our attention on the existence of positive solutions of problem (1.1) which are characterized in the following definition.

Definition 1.2. A function u is called a *positive solution* of problem (1.1) if u satisfies the following conditions:

(i) $u \in C[0,1]\cap C^2(0,1)$,

(ii) $u(t) > 0$ for $t \in (0,1)$,

(iii) u satisfies equation (1.1a) and boundary conditions (1.1b).

We want to prove a general existence theorem for problem (1.1) which will enable a unified approach to the existence and localization of positive solutions for certain classes of singular problems, such as

$$(t^3u'(t))' + t^3\left(\frac{1}{8u^2(t)} - \frac{\mu}{u(t)} - \frac{\lambda^2}{2}t^{2\gamma-4}\right) = 0, \tag{1.2a}$$

$$\lim_{t\to 0+} t^3u'(t) = 0, \quad a_0u(1) + a_1u'(1-) = A. \tag{1.2b}$$

With $\mu \geqslant 0$, $\lambda > 0$, $\gamma > 1$ problem (1.2) is a special case of (1.1). Boundary value problems (1.2) arise in the theory of shallow membrane caps and are investigated in [14], [15], [16], and [21]. Equation

$$u''(t) + \frac{3}{t}u'(t) + \frac{q(t)}{u^2(t)} = 0, \tag{1.3}$$

where q is continuous on $[0,1]$ and positive on $(0,1)$, augmented by boundary conditions (1.1b) was studied in [2]. It describes the behavior of symmetric circular membranes and can be easily transformed to the special case of (1.1). Finally, the problem posed on a semi-infinite interval,

$$z''(s) + \frac{1}{s^3}\left(\frac{\lambda^2}{8s^{\gamma-2}} - \frac{1}{32z^2(s)} + \frac{\mu}{4z(s)}\right) = 0, \quad 1 < s < \infty, \tag{1.4a}$$

$$\lim_{s\to\infty} |z(s)| < \infty, \quad b_0 z(1) - b_1 z'(1-) = A, \tag{1.4b}$$

also arises in the membrane theory and for $A > 0$ it was discussed in [1] and [8]. It can be written in the form (1.2), where $a_0 = b_0$, $a_1 = 2b_1$, by using the substitution

$$s = \frac{1}{t^2}, \quad z(s) = z\left(\frac{1}{t^2}\right) =: u(t). \tag{1.5}$$

2. Existence theorems for problem (1.1)

Our analytical approach is based on the lower and upper functions method which is here extended to the general singular problem of the form (1.1). In the sequel, we shall use the following definitions:

Definition 2.1. A function σ is called a *lower function* of equation (1.1a), if σ satisfies the following requirements:
(i) $\sigma \in C[0,1] \cap C^2(0,1)$,
(ii) $(t^n\sigma'(t))' + t^n f(t,\sigma(t)) \geqslant 0, \quad t \in (0,1)$.

If the inequality in (ii) is reversed, σ is called an *upper function* of equation (1.1a). If σ satisfies (i), (ii) and
(iii) $\lim\limits_{t\to 0+} t^n\sigma'(t) \geqslant 0, \quad a_0\sigma(1) + a_1\sigma'(1-) \leqslant A$,
then σ is called a *lower function* of the boundary value problem (1.1). If the inequalities in (ii) and (iii) are reversed, then σ is called an *upper function* of the boundary value problem (1.1).

In general, $\sigma'(t)$ can become unbounded at the endpoints of the integration interval, $t = 0$ and $t = 1$. For more general definitions of lower and upper functions, see e.g. [12], [17] or [22].

For the next two theorems we need the following assumptions:

A2.1: σ_1 and σ_2 are a lower and an upper function of problem (1.1), respectively.

A2.2: $0 < \sigma_1(t) \leqslant \sigma_2(t)$ for $t \in (0,1)$.

A2.3: There exists $p < 2$ such that $\lim\limits_{t\to 0+} t^p h(t) < \infty$, where

$$h(t) = \sup\{|f(t,x)|\colon\ \sigma_1(t) \leqslant x \leqslant \sigma_2(t)\}.$$

Note that σ_1 and σ_2 can vanish at $t=0$ and $t=1$. Since $f(t,x)$ may exhibit singularities at $t=0$ and $x=0$, we easily see that h can become unbounded, i.e.

$$\limsup_{t\to 0+} h(t)=\infty, \quad \limsup_{t\to 1-} h(t)=\infty. \tag{2.1}$$

Theorem 2.2. *Assume that* A1 *and* A2.1–A2.3 *hold.*

(i) *Let h be bounded on* $[0,1]$. *Then problem* (1.1) *has a positive solution u such that $u\in C^1[0,1]$ and $u'(0)=0$. Moreover,*

$$\sigma_1(t)\leqslant u(t)\leqslant \sigma_2(t), \quad t\in[0,1]. \tag{2.2}$$

(ii) *Let h satisfy* (2.1). *Furthermore, let us assume that there exists a constant $\delta_1\in(0,1)$ such that*

$$(t^n\sigma_1'(t))'\geqslant 0, \quad (t^n\sigma_2'(t))'\leqslant 0,\ \ t\in(0,\delta_1), \tag{2.3}$$

$\sigma_1(1)=\sigma_2(1)$, *and there exist $\delta_2\in(0,1)$, $K\in\mathbb{R}$ such that*

$$(t^n\sigma_1'(t))'\geqslant K, \quad (t^n\sigma_2'(t))'\leqslant K,\ \ t\in(1-\delta_2,1). \tag{2.4}$$

Then problem (1.1) *with $A=0$ in* (1.1b) *has a positive solution u satisfying* (2.2).

P r o o f. (i) For h bounded on $[0,1]$, (i) follows by arguing as in the regular case, where f is continuous or satisfies the Carathéodory conditions on $[0,1]\times[0,\infty)$, see e.g. Theorem 2.3 in [21].

(ii) Let h satisfy (2.1) and let (2.3), (2.4), and $\sigma_1(1)=\sigma_2(1)$ hold. Now the proof is carried out in five steps.

Step 1. We first show that $A=0$: The condition $\limsup\limits_{t\to 1-} h(t)=\infty$ and A1 imply $\sigma_1(1)=0$. From $\sigma_1(1)=\sigma_2(1)$ also $\sigma_2(1)=0$ follows. If $a_1=0$, then Definition 2.1 (iii) yields $0=a_0\sigma_1(1)\leqslant A$ and $0=a_0\sigma_2(1)\geqslant A$. Therefore, $A=0$. If $a_1>0$, Definition 2.1 (iii) yields $\sigma_2'(1-)\geqslant A/a_1$. Due to A2.2, $\sigma_2(t)>0$ for $t\in(0,1)$ and hence, $\sigma_2'(1-)\leqslant 0$. Therefore, $A=0$.

Step 2. Approximate solutions u_k: Choose $k\in\mathbb{N}$, $1/k\leqslant\min\{\delta_1,\delta_2\}$, and define

$$f_k(t,x):=\begin{cases} 0, & t\in\left[0,\dfrac{1}{k}\right),\\[2mm] f(t,x), & t\in\left[\dfrac{1}{k},1-\dfrac{1}{k}\right],\\[2mm] -\dfrac{K}{t^n}, & t\in\left(1-\dfrac{1}{k},1\right]. \end{cases}$$

Consider the equation

$$(t^n u'(t))' + t^n f_k(t, u(t)) = 0. \tag{2.5}$$

We see that σ_1 and σ_2 are lower and upper functions of equation (2.5) subject to (1.1b) and

$$h_k(t) := \sup\{|f_k(t,x)|\colon\ \sigma_1(t) \leqslant x \leqslant \sigma_2(t)\}$$

is bounded on $[0,1]$. By Part (i) of the proof, problem (2.5), (1.1b) has a solution $u_k \in C^1[0,1] \cap C^2(0,1)$ satisfying $u_k'(0) = 0$ and

$$\sigma_1(t) \leqslant u_k(t) \leqslant \sigma_2(t), \quad t \in [0,1]. \tag{2.6}$$

Step 3. Properties of the function h: We now derive some useful properties of h which will be required in the next steps of the proof. Choose an interval $[0,b] \subset [0,1)$. Due to A1 and A2.2, the function $t^n h(t)$ is continuous on $(0,b]$. Since $p < 2 \leqslant n$, it follows from A2.3 that $\lim\limits_{t\to 0+} t^n h(t) = 0$ holds. Therefore,

$$\int_0^b s^n h(s)\,\mathrm{d}s =: M_b \in (0,\infty). \tag{2.7}$$

Thus, by de l'Hospital's rule and A2.3,

$$\begin{aligned}\lim_{t\to 0+} \frac{1}{t^{n-p+1}} &\int_0^t s^n h(s)\,\mathrm{d}s \\ &= \lim_{t\to 0+} \frac{t^n h(t)}{(n-p+1)t^{n-p}} = \frac{1}{n-p+1} \lim_{t\to 0+} t^p h(t) =: c_0 \in (0,\infty).\end{aligned}$$

This yields the existence of $\varepsilon \in (0,1)$ such that

$$\frac{1}{t^n} \int_0^t s^n h(s)\,\mathrm{d}s \leqslant (c_0+1)\frac{1}{t^{p-1}}, \quad t \in [0,\varepsilon].$$

Moreover, by (2.7),

$$\frac{1}{t^n} \int_0^t s^n h(s)\,\mathrm{d}s \leqslant \frac{1}{\varepsilon^n} \int_0^b s^n h(s)\,\mathrm{d}s = \frac{M_b}{\varepsilon^n} \quad \text{for } t \in [\varepsilon, b].$$

Finally, imply the last two inequalities

$$\int_0^b \frac{1}{t^n} \int_0^t s^n h(s)\,\mathrm{d}s\,\mathrm{d}t < \infty. \tag{2.8}$$

Step 4. Properties of the sequence $\{u_k\}$: Consider the sequence of equations (2.5) subject to (1.1b) with $k \in \mathbb{N}$, $1/k \leqslant \min\{\delta_1, \delta_2\}$, where δ_1 and δ_2 are specified by (2.3) and (2.4), respectively. From Step 2 we obtain the corresponding sequence $\{u_k\}$ of their solutions which are approximations for u. Let us first discuss the convergence properties of $\{u_k\}$. Choose an interval $[0,b] \subset [0,1)$. Then there exists an index $k_1 \in \mathbb{N}$, $1/k_1 \leqslant \min\{\delta_1, \delta_2\}$, such that

$$[0,b] \subset \Big[0, 1 - \frac{1}{k}\Big], \quad k \geqslant k_1.$$

Due to boundary conditions (1.1b) and equation (2.5) we have

$$t^n u_k'(t) + \int_0^t s^n f_k(s, u_k(s))\,\mathrm{d}s = 0, \quad t \in [0,b], \ \ k \geqslant k_1. \tag{2.9}$$

The inequality

$$|f_k(t, u_k(t))| \leqslant h(t), \quad t \in \Big[0, 1 - \frac{1}{k}\Big], \ \ k \geqslant k_1, \tag{2.10}$$

condition (2.7) and equality (2.9) yield

$$|t^n u_k'(t)| \leqslant \int_0^t s^n h(s)\,\mathrm{d}s \leqslant M_b, \quad t \in [0,b], \ \ k \geqslant k_1. \tag{2.11}$$

According to (2.6) and (2.11) the sequences $\{u_k\}$ and $\{t^n u_k'\}$ are bounded on $[0,b]$. Moreover, by (2.7) and (2.8), for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $t_1, t_2 \in [0,b]$ with $|t_1 - t_2| < \delta$ and any $k \geqslant k_1$ we have

$$|t_1^n u_k'(t_1) - t_2^n u_k'(t_2)| \leqslant \left| \int_{t_1}^{t_2} s^n h(s)\,\mathrm{d}s \right| < \varepsilon$$

and

$$|u_k(t_1) - u_k(t_2)| \leqslant \left| \int_{t_1}^{t_2} \frac{1}{t^n} \int_0^t s^n h(s)\,\mathrm{d}s\,\mathrm{d}t \right| < \varepsilon$$

holds. Hence, the sequences $\{u_k\}$ and $\{t^n u_k'\}$ are equicontinuous on $[0,b]$. The Arzelà-Ascoli theorem now implies that there exists a subsequence $\{u_\ell\} \subset \{u_k\}$ such that

$$\lim_{\ell\to\infty} u_\ell = u, \quad \lim_{\ell\to\infty} t^n u_\ell' = t^n u'$$

uniformly on $[0,b]$. Finally, by the diagonalization principle, we find a subsequence[1] $\{u_k\}$ satisfying

$$\lim_{k\to\infty} u_k = u, \quad \lim_{k\to\infty} t^n u_k' = t^n u' \tag{2.12}$$

locally uniformly on $[0,1)$.

[1] For simplicity, the previous notation $\{u_k\}$ for this subsequence is used.

Step 5. Properties of the function u: We now prove that the limit function u is a positive solution of problem (1.1) satisfying (2.2). Due to (2.6) and (2.12) we have

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t), \quad t \in [0,1), \ u \in C[0,1), \tag{2.13a}$$

$$t^n u'(t) \in C[0,1), \quad \lim_{t \to 0+} t^n u'(t) = 0. \tag{2.13b}$$

Choose $t \in (0,1)$. Then there exists $k_t \geqslant k_1$ such that

$$f(t, u_k(t)) = f_k(t, u_k(t)), \quad k \geqslant k_t,$$

and hence, by A1 and (2.12),

$$\lim_{k \to \infty} f_k(t, u_k(t)) = \lim_{k \to \infty} f(t, u_k(t)) = f(t, u(t)).$$

Consequently, the sequence $\{f_k(t, u_k(t))\}$ is pointwise converging on $(0,1)$. Furthermore, for an arbitrary interval $[0,b] \subset [0,1)$ we have, by (2.10),

$$|t^n f_k(t, u_k(t))| \leqslant t^n h(t), \quad t \in [0,b], \ k \geqslant k_1.$$

Therefore, due to (2.7), we can use the Lebesgue dominated convergence theorem for the sequence of equalities (2.9). Having in mind that $b \in (0,1)$ is arbitrary and letting $k \to \infty$, we conclude that

$$t^n u'(t) + \int_0^t s^n f(s, u(s)) \, \mathrm{d}s = 0, \quad t \in [0,1).$$

Thus $u \in C^2(0,1)$ and u satisfies equation (1.1a) for $t \in (0,1)$. By Step 1, we have $\sigma_1(1) = \sigma_2(1) = A = 0$ and consequently, by (2.13a), $\lim_{t \to 1-} u(t) = 0$ follows. For $u(1) = 0$, we can see that $u \in C[0,1]$ is a positive solution of problem (1.1), which completes the proof. $\square$

Theorem 2.3. *Assume that* A1 *and* A2.1–A2.3 *hold.*

(i) *Let h be bounded at $t = 0$ and let us assume that $\limsup_{t \to 1-} h(t) = \infty$ and condition* (2.4) *hold. Then problem* (1.1) *with $A = 0$ in* (1.1b) *has a positive solution $u \in C^1[0,1)$ which satisfies estimate* (2.2) *and $u'(0) = 0$.*

(ii) *Let h be bounded at $t = 1$ and let $\limsup_{t \to 0+} h(t) = \infty$ and condition* (2.3) *hold. Then problem* (1.1) *has a positive solution $u \in C^1(0,1]$ which satisfies estimate* (2.2).

P r o o f. We use arguments similar to those from the proof of Theorem 2.2.

(i) Since h is bounded at $t = 0$, we define

$$f_k(t,x) := \begin{cases} f(t,x), & t \in \left[0, 1 - \dfrac{1}{k}\right], \\ -\dfrac{K}{t^n}, & t \in \left(1 - \dfrac{1}{k}, 1\right], \end{cases}$$

where $k \in \mathbb{N}$, $1/k \leqslant \delta_2$, and δ_2, K are given by (2.4). As in Steps 2–4, we construct the sequence $\{u_k\}$ of solutions of equations (2.5) subject to (1.1b) which satisfy (2.6) and (2.12). By Step 5, the limit function u is a positive solution of problem (1.1) satisfying (2.2). Since h is bounded at $t = 0$, we have

$$\sup\left\{|h(t)| \colon \ t \in \left[0, \frac{1}{2}\right]\right\} =: M < \infty$$

and therefore,

$$|u'(t)| \leqslant \frac{1}{t^n} \int_0^t s^n h(s)\,\mathrm{d}s \leqslant \frac{M}{n+1}\, t, \quad t \in \left[0, \frac{1}{2}\right].$$

For $u'(0) = 0$, $u \in C^1[0,1)$ follows.

(ii) Since h is bounded at $t = 1$, we set

$$f_k(t,x) := \begin{cases} 0, & t \in \left[0, \dfrac{1}{k}\right), \\ f(t,x), & t \in \left[\dfrac{1}{k}, 1\right], \end{cases}$$

where $k \in \mathbb{N}$, $1/k \leqslant \delta_1$, and δ_1 is specified by (2.3). As in Step 2 we derive the sequence $\{u_k\}$ of solutions of equations (2.5) subject to (1.1b) and satisfying (2.6). Moreover, similarly to Step 3, we obtain

$$\int_0^1 s^n h(s)\,\mathrm{d}s < \infty, \quad \int_0^1 \frac{1}{t^n} \int_0^t s^n h(s)\,\mathrm{d}s\,\mathrm{d}t < \infty,$$

and we deduce, as in Step 4, that

$$\lim_{k\to\infty} u_k = u, \quad \lim_{k\to\infty} t^n u_k' = t^n u'$$

holds uniformly on $[0,1]$. Therefore, $u \in C[0,1] \cap C^1(0,1]$ and u satisfies (1.1b) and (2.2). By the Lebesgue dominated convergence theorem, as in Step 5, we conclude that $u \in C^2(0,1)$ satisfies equation (1.1a) for $t \in (0,1)$ and the result follows. $\square$

Note that the existence of nonnegative solutions for mixed problems, where f may be singular just at $x = 0$, was also proved in [3].

In this section we use Theorems 2.2 and 2.3 to prove the solvability of singular membrane problems. We study the boundary value problem

$$(t^n u'(t))' + t^n\Big(\frac{a}{u^{2m}(t)} - \frac{b}{u^m(t)} - ct^{2r}\Big) = 0, \tag{3.1a}$$

$$\lim_{t\to 0+} t^n u'(t) = 0, \quad a_0 u(1) + a_1 u'(1-) = A, \tag{3.1b}$$

where $a \in (0,\infty)$, $b, c \in [0,\infty)$, $r \in (-1,\infty)$, $m, n \in \mathbb{N}$, and $n \geqslant 2$. Problem (3.1) covers the membrane problem (1.2) and, after substitution (1.5), also the infinite interval problem (1.4).

In order to be able to utilize the results formulated in Theorems 2.2 and 2.3, it is necessary to show how to find proper lower and upper functions of the above problem. We begin with lower and upper functions of equation (3.1a), the choice of which depends on the parameters a, b, c, r, n, and m.

Lemma 3.1. *Assume that* $c_1 \in (0, x_1^{-1/m}]$, *where* $x_1 = \frac{1}{2}(b + \sqrt{b^2 + 4ac})/a$. *For* $t \in [0,1]$, *we define*

$$\sigma_1(t) := \begin{cases} c_1, & r \geqslant 0, \\ c_1 t^{-r/m}, & r \in (-1, 0). \end{cases} \tag{3.2}$$

Then σ_1 *is a lower function of equation* (3.1a).

P r o o f. Since $c_1^{-m} \geqslant x_1$ and x_1 is a positive solution of the equation $ax^2 - bx - c = 0$, we have

$$\frac{a}{c_1^{2m}} - \frac{b}{c_1^m} - c \geqslant 0. \tag{3.3}$$

Let $r \geqslant 0$. Then $\sigma_1(t) \equiv c_1$ and, by (3.3),

$$(t^n \sigma_1'(t))' + t^n\Big(\frac{a}{\sigma_1^{2m}(t)} - \frac{b}{\sigma_1^m(t)} - ct^{2r}\Big) \geqslant t^n\Big(\frac{a}{c_1^{2m}} - \frac{b}{c_1^m} - c\Big) \geqslant 0, \quad t \in (0,1).$$

Let $r \in (-1,0)$. Then $\sigma_1(t) = c_1 t^{-r/m}$ and, by (3.3),

$$(t^n \sigma_1'(t))' + t^n\Big(\frac{a}{\sigma_1^{2m}(t)} - \frac{b}{\sigma_1^m(t)} - ct^{2r}\Big) \geqslant t^{n+2r}\Big(\frac{a}{c_1^{2m}} - \frac{b}{c_1^m} - c\Big) \geqslant 0, \quad t \in (0,1).$$

This means σ_1 satisfies conditions (i) and (ii) of Definition 2.1. $\square$

Lemma 3.2. *Let us assume that* $c_2 \in [x_1^{-1/m}, \infty)$, *where* x_1 *is defined in Lemma* 3.1. *For* $t \in [0,1]$ *define*

$$\sigma_2(t) := \begin{cases} c_2 + \dfrac{c}{n}(1-t), & r > 0, \\ c_2, & r \in (-1, 0]. \end{cases} \tag{3.4}$$

Then σ_2 *is an upper function of equation* (3.1a).

P r o o f. Let $r \in (-1,0]$. Then $\sigma_2(t) \equiv c_2$. Since $0 < c_2^{-m} \leqslant x_1$, we have

$$(t^n \sigma_2'(t))' + t^n \Big(\frac{a}{\sigma_2^{2m}(t)} - \frac{b}{\sigma_2^m(t)} - ct^{2r} \Big) \leqslant t^n \Big(\frac{a}{c_2^{2m}} - \frac{b}{c_2^m} - c \Big) \leqslant 0, \quad t \in (0,1).$$

Let $r > 0$. Then $\sigma_2(t) = c_2 + \frac{c}{n}(1-t)$ and

$$(t^n \sigma_2'(t))' + t^n \Big(\frac{a}{\sigma_2^{2m}(t)} - \frac{b}{\sigma_2^m(t)} - ct^{2r} \Big) \leqslant t^{n-1}(-c + t\psi(t)), \quad t \in (0,1),$$

where

$$\psi(t) = \frac{a}{[c_2 + \frac{c}{n}(1-t)]^{2m}} - \frac{b}{[c_2 + \frac{c}{n}(1-t)]^m}.$$

If $\psi(t)$ is positive for some $t \in (0,1)$, we can conclude

$$-c + t\psi(t) \leqslant -c + \frac{a}{c_2^{2m}} - \frac{b}{c_2^m} \leqslant 0$$

and thus, by Definition 2.1, the function σ_2 is an upper function of (3.1a). $\square$

We now specify the c_1 and c_2 in σ_1 and σ_2 from Lemmas 3.1 and 3.2, respectively, in order to satisfy condition A2.2 and Definition 2.1 (iii). For σ_2 we take Definition 2.1 (iii) with the reversed inequalities.

Lemma 3.3. *Let* $A > 0$ *and* x_1 *be as in Lemma* 3.1. *Set* $r^- := \max\{0, -r\}$ *and*

$$c_1 := \min\Big\{ \frac{Am}{a_0 m + a_1 r^-}, x_1^{-1/m} \Big\}, \quad c_2 := \max\Big\{ \frac{1}{a_0}\Big(A + \frac{a_1 c}{n}\Big), x_1^{-1/m} \Big\}.$$

Then σ_1 *and* σ_2 *given by* (3.2) *and* (3.4) *are, respectively, lower and upper functions of problem* (3.1) *and satisfy* A2.2.

P r o o f. By Lemmas 3.1 and 3.2, σ_1 and σ_2 are lower and upper functions of equation (3.1a). We see that A2.2 holds and (3.2), (3.4) yield

$$\lim_{t\to 0+} t^n \sigma_1'(t) = 0, \quad \lim_{t\to 0+} t^n \sigma_2'(t) = 0.$$

Finally,

$$a_0\sigma_1(1) + a_1\sigma_1'(1-) = \begin{cases} a_0c_1 \leqslant A, & r \geqslant 0, \\ c_1\Big(a_0 - a_1\dfrac{r}{m}\Big) \leqslant A, & r \in (-1, 0), \end{cases}$$

$$a_0\sigma_2(1) + a_1\sigma_2'(1-) = \begin{cases} a_0c_2 \geqslant A, & r \in (-1, 0], \\ a_0c_2 - a_1\dfrac{c}{n} \geqslant A, & r > 0. \end{cases}$$

□

Lemma 3.3 deals with the case $A > 0$. In the next two lemmas we will discuss the case $A = 0$, where constant lower and upper functions do not exist.

Lemma 3.4. *Let* $A = 0$ *and* $a_1 > 0$. *Set* $k := 1 + a_1/(a_0 - a_1(r/m))$ *and for* $t \in [0, 1]$ *define*

$$\sigma_1(t) := \begin{cases} \nu\Big(1 - t^2 + 2\dfrac{a_1}{a_0}\Big), & r \geqslant 0, \\ \nu t^{-r/m}(k - t), & r \in (-1, 0), \end{cases} \qquad \sigma_2(t) := \beta\Big(1 - t^2 + 2\frac{a_1}{a_0}\Big). \tag{3.5}$$

Then there exist constants ν^*, $\beta^* \in (0, \infty)$ *such that for each* $\nu \in (0, \nu^*)$ *and* $\beta \geqslant \beta^*$, *the functions* σ_1 *and* σ_2 *are a lower function and an upper function of problem* (3.1) *satisfying* A2.2.

Proof. By direct calculations we can see that σ_1 and σ_2 satisfy

$$\lim_{t\to 0+} t^n\sigma_i'(t) = 0, \quad a_0\sigma_i(1) + a_1\sigma_i'(1-) = 0, \quad i = 1, 2.$$

Let $r \geqslant 0$. Then $\sigma_1(t) = \nu(1 - t^2 + 2a_1/a_0)$ and

$$(t^n\sigma_1'(t))' + t^n\Big(\frac{a}{\sigma_1^{2m}(t)} - \frac{b}{\sigma_1^m(t)} - ct^{2r}\Big) \geqslant t^n\varphi_1(t, \nu), \quad t \in (0, 1),$$

where

$$\varphi_1(t, \nu) = -2\nu(n + 1) + \frac{a}{[\nu(1 - t^2 + 2a_1/a_0)]^{2m}} - \frac{b}{[\nu(1 - t^2 + 2a_1/a_0)]^m} - c.$$

Since $\lim_{\nu\to 0+} \varphi_1(t, \nu) = \infty$ uniformly on $[0, 1]$, we can find $\nu^* > 0$ such that for each $\nu \in (0, \nu^*]$, the inequality $\varphi_1(t, \nu) \geqslant 0$ holds for $t \in [0, 1]$.

Let $r \in (-1, 0)$. Then $\sigma_1(t) = \nu t^{-r/m}(k - t)$ and

$$(t^n\sigma_1'(t))' + t^n\Big(\frac{a}{\sigma_1^{2m}(t)} - \frac{b}{\sigma_1^m(t)} - ct^{2r}\Big) \geqslant t^{n-r/m-2}\psi_1(t, \nu), \quad t \in (0, 1),$$

where

$$\psi_1(t,\nu) = \nu\ell(t) + t^{2r+r/m+2}h(t,\nu), \quad h(t,\nu) = \frac{a}{[\nu(k-t)]^{2m}} - \frac{b}{[\nu(k-t)]^{m}} - c,$$

and

$$\ell(t) = -\frac{rk}{m}\Big(n - \frac{r}{m} - 1\Big) - \Big(n - \frac{r}{m}\Big)\Big(1 - \frac{r}{m}\Big)t. \tag{3.6}$$

Choose $c_1 > 0$ as in Lemma 3.1 and let $\nu_1 \in (0, c_1/k]$. Then, by (3.3), $h(t,\nu) \geqslant 0$ for $\nu \in (0,\nu_1]$, $t \in [0,1]$. We now denote the unique zero of $\ell(t)$ by t_0 and have $\ell(t) \geqslant 0$ for $t \in [0,t_0]$. Consequently,

$$\psi_1(t,\nu) \geqslant 0, \quad \nu \in (0,\nu_1],\ \ t \in [0,t_0].$$

Furthermore,

$$\lim_{\nu\to 0+} \psi_1(t,\nu) = \infty$$

uniformly for $t \in [t_0,1]$. Therefore, we can find $\nu^* \in (0,\nu_1]$ such that for each $\nu \in (0,\nu^*]$, the inequality $\psi_1(t,\nu) \geqslant 0$ holds for $t \in [0,1]$.

Let us now consider $\sigma_2(t) = \beta(1 - t^2 + 2a_1/a_0)$. We have

$$(t^n\sigma_2'(t))' + t^n\Big(\frac{a}{\sigma_2^{2m}(t)} - \frac{b}{\sigma_2^{m}(t)} - ct^{2r}\Big) \leqslant t^n\varphi(\beta), \quad t \in (0,1),$$

where

$$\varphi(\beta) = -2(n+1)\beta + a\Big(2\beta\frac{a_1}{a_0}\Big)^{-2m}.$$

Since $\lim\limits_{\beta\to\infty}\varphi(\beta) = -\infty$, there exists $\beta^* > \nu^*$ such that for each $\beta \geqslant \beta^*$ we have $\varphi(\beta) > 0$. $\square$

Lemma 3.5. *Let $A = 0$ and $a_1 = 0$. For $t \in [0,1]$ let us define*

$$\sigma_1(t) := \begin{cases} \nu(1-t^2), & r \geqslant 0, \\ \nu t^{-r/m}(1-t), & r \in (-1,0), \end{cases} \qquad \sigma_2(t) := \beta(1-t^2)^{1/2m}. \tag{3.7}$$

Then there exist constants ν^*, $\beta^* \in (0,\infty)$ such that for each $\nu \in (0,\nu^*)$ and $\beta \geqslant \beta^*$, the functions σ_1 and σ_2 are a lower function and an upper function of problem (3.1) *satisfying* A2.2.

Proof. We can easily check that σ_1 and σ_2 satisfy

$$\lim_{t\to 0+} t^n\sigma_i'(t) = 0, \quad \sigma_i(1) = 0,\ \ i = 1,2.$$

Let $r \geqslant 0$. Then $\sigma_1(t) = \nu(1-t^2)$ and

$$(t^n\sigma_1'(t))' + t^n\Big(\frac{a}{\sigma_1^{2m}(t)} - \frac{b}{\sigma_1^m(t)} - ct^{2r}\Big) \geqslant t^n\varphi(t,\nu), \quad t\in(0,1),$$

where

$$\varphi(t,\nu) = -2\nu(n+1) + \frac{a}{[\nu(1-t^2)]^{2m}} - \frac{b}{[\nu(1-t^2)]^m} - c.$$

Since $\lim\limits_{\nu\to 0+} \varphi(t,\nu) = \infty$ uniformly on $[0,1]$, we can find $\nu^* > 0$ such that for each $\nu \in (0,\nu^*]$, the inequality $\varphi(t,\nu) \geqslant 0$ holds for $t \in [0,1]$.

Let $r \in (-1,0)$. Then $\sigma_1(t) = \nu t^{-r/m}(1-t)$ and similarly to the proof of Lemma 3.4 we conclude that for each sufficiently small positive ν the function σ_1 is a lower function of problem (3.1).

Now, consider $\sigma_2(t) = \beta(1-t^2)^{1/2m}$. We have

$$(t^n\sigma_2'(t))' + t^n\Big(\frac{a}{\sigma_2^{2m}(t)} - \frac{b}{\sigma_2^m(t)} - ct^{2r}\Big) \leqslant t^n(1-t^2)^{1/2m-2}\varphi_2(t,\beta), \quad t\in(0,1),$$

where

$$\varphi_2(t,\beta) = -\frac{2\beta}{m}\Big(1-\frac{1}{2m}\Big) + a\beta^{-2m}(1-t^2)^{1-1/2m}.$$

Since $\lim\limits_{\beta\to\infty} \varphi_2(t,\beta) = -\infty$ uniformly on $[0,1]$, we can find a $\beta^* > \nu^*$ such that for each $\beta \geqslant \beta^*$, $\varphi_2(t,\beta) \leqslant 0$ on $(0,1)$ holds. Therefore, σ_2 is an upper function of (3.1) and A2.2 is satisfied. □

Having derived lower and upper functions of problem (3.1) for all values of its parameters, we can prove the existence of a positive solution u to this problem and describe how u' behaves at the singular points $t=0$ and $t=1$.

Theorem 3.6. *Problem* (3.1) *has a positive solution u such that*

$$\begin{cases} u(0) > 0, \quad u'(0+) = 0, \;\; r > -\dfrac{1}{2}, \\ u(0) > 0, \quad u'(0+) = \dfrac{c}{n}, \;\; r = -\dfrac{1}{2}, \\ u(0) \geqslant 0, \quad u'(0+) = \infty, \;\; r < -\dfrac{1}{2}, \end{cases} \tag{3.8}$$

and

$$\begin{cases} u'(1-) \in \mathbb{R}, & A > 0, \\ u'(1-) \in \mathbb{R}, & A = 0, \;\; a_1 > 0, \\ u'(1-) = -\infty, & A = 0, \;\; a_1 = 0. \end{cases} \tag{3.9}$$

Proof. Lower and upper functions σ_1 and σ_2 of problem (3.1) satisfying A2.2 are given according to Lemmas 3.3, 3.4, and 3.5. The function

$$f(t,x) = \frac{a}{x^{2m}} - \frac{b}{x^m} - ct^{2r}$$

satisfies A1. Consider the function h from A2.3. Then we have

$$0 \leqslant h(t) \leqslant \frac{a}{\sigma_1^{2m}(t)} + \frac{b}{\sigma_1^m(t)} + ct^{2r}, \quad t \in (0,1). \tag{3.10}$$

Case 1. We assume that $A > 0$ or $A = 0$, $a_1 > 0$. We first find c_1 by Lemma 3.3, and then choose $\nu \in (0, \nu^*)$ in (3.5) such that $\nu k \leqslant c_1$.

Let $r \geqslant 0$. Then σ_1 is positive on $[0,1]$ and (3.10) implies that h is bounded on $[0,1]$ and $\lim\limits_{t\to 0+} th(t) = 0$. Thus, h satisfies condition A2.3 with $p = 1$ and, by Theorem 2.2 (i), problem (3.1) has a positive solution $u \in C^1[0,1]$ satisfying $u'(0) = 0$ and (2.2). Since $\sigma_1(0) > 0$, the inequality $u(0) > 0$ follows.

Let $r \in (-1, 0)$. Then (3.10) yields

$$0 \leqslant h(t) \leqslant t^{2r}\Big(\frac{a}{[\nu(k-1)]^{2m}} + \frac{b}{[\nu(k-1)]^m} + c\Big), \quad t \in (0,1). \tag{3.11}$$

Also,

$$h(t) \geqslant \frac{a}{\sigma_1^{2m}(t)} - \frac{b}{\sigma_1^m(t)} - ct^{2r} \geqslant t^{2r}\Big(\frac{a}{c_1^{2m}} - \frac{b}{c_1^m} - c\Big) > 0, \quad t \in (0,1).$$

By (3.11) and the last inequality we have

$$\limsup_{t\to 0+} h(t) = \infty, \quad \limsup_{t\to 1-} h(t) < \infty.$$

Due to (3.11), for $p = -2r$ we can show A2.3, since

$$\lim_{t\to 0+} t^p h(t) \leqslant \frac{a}{[\nu(k-1)]^{2m}} + \frac{b}{[\nu(k-1)]^m} + c < \infty.$$

Now we prove (2.3). If $A > 0$, we use Lemma 3.3 and have $\sigma_1(t) = c_1 t^{-r/m}$, $\sigma_2(t) \equiv c_2$. Hence,

$$(t^n\sigma_1'(t))' = c_1\Big(-\frac{r}{m}\Big)\Big(n - 1 - \frac{r}{m}\Big)t^{n-2-r/m} \geqslant 0, \quad (t^n\sigma_2'(t))' = 0, \quad t \in (0,1).$$

For $A = 0$ and $a_1 > 0$ we use Lemma 3.4 and have $\sigma_1(t) = \nu t^{-r/m}(k - t)$, $\sigma_2(t) = \beta(1 - t^2 + 2a_1/a_0)$. Hence,

$$(t^n\sigma_1'(t))' = \nu t^{n-2-r/m}\ell(t) \geqslant 0, \quad (t^n\sigma_2'(t))' = -2\beta(n+1)t^n \leqslant 0, \quad t \in (0,\delta_1),$$

where $\ell(t)$ is given by (3.6) and $\delta_1 = t_0$ is its unique zero. Therefore, condition (2.3) holds. Consequently, by Theorem 2.3 (ii), problem (3.1) has a positive solution $u \in C^1(0,1]$ satisfying (2.2).

It remains to prove (3.8) for $r \in (-1,0)$. Equation (3.1a) and condition (3.1b) result in

$$t^n u'(t) = -\int_0^t s^n\Big(\frac{a}{u^{2m}(s)} - \frac{b}{u^m(s)} - cs^{2r}\Big)\,\mathrm{d}s, \quad t \in (0,1), \tag{3.12}$$

and consequently, since $n \geqslant 2$ and $r > -1$,

$$\lim_{t\to 0+}\int_0^t s^n\Big(\frac{b}{u^m(s)} - \frac{a}{u^{2m}(s)}\Big)\,\mathrm{d}s = 0. \tag{3.13}$$

Assume $u(0) = 0$. Since $\sigma_1(0) = 0$ and $\lim\limits_{t\to 0+}\sigma_1'(t) = \infty$, inequality (2.2) implies

$$\lim_{t\to 0+} u'(t) = \infty. \tag{3.14}$$

On the other hand, the assumption $u(0) = 0$ guarantees the existence of $\delta > 0$ such that $u^m(t) \leqslant a/b$ for $t \in [0,\delta]$. Then, by (3.12),

$$u'(t) = \frac{1}{t^n}\int_0^t \frac{s^n}{u^{2m}(s)}(bu^m(s) - a)\,\mathrm{d}s + \frac{ct^{2r+1}}{n+2r+1} \leqslant \frac{ct^{2r+1}}{n+2r+1}, \quad t \in (0,\delta).$$

If $r \in [-\frac{1}{2},0)$, then $u'(t) \leqslant c/n$ on $(0,\delta)$, a contradiction to (3.14). This means that we have shown

$$r \geqslant -\frac{1}{2} \Longrightarrow u(0) > 0. \tag{3.15}$$

For $r \in [-\frac{1}{2},0)$, using (3.12), (3.13), (3.15) and de l'Hospital's rule we obtain

$$\lim_{t\to 0+} u'(t) = \lim_{t\to 0+}\frac{ct^{2r+1}}{n+2r+1} = \begin{cases} 0, & r \in \Big(-\dfrac{1}{2},0\Big), \\[2mm] \dfrac{c}{n}, & r = -\dfrac{1}{2}. \end{cases} \tag{3.16}$$

Let $r \in (-1,-\frac{1}{2})$. If $u(0) = 0$, then (3.14) holds. If $u(0) > 0$, then by (3.12), (3.13), and de l'Hospital's rule we deduce as before,

$$\lim_{t\to 0+} u'(t) = \lim_{t\to 0+}\frac{ct^{2r+1}}{n+2r+1} = \infty.$$

Case 2. Now, we consider the case $A = 0$, $a_1 = 0$.

Let $r \geqslant 0$, then by Lemma 3.5,

$$\sigma_1(t) = \nu(1-t^2), \quad \sigma_2(t) = \beta(1-t^2)^{1/2m},$$

where $0 < \nu < \beta$ with a sufficiently small ν and a sufficiently large β. For $t \in (0,1)$ we have

$$0 < \frac{1}{(1-t^2)^{2m}}\Big(\frac{a}{\nu^{2m}} - \frac{b}{\nu^m} - c\Big) \leqslant h(t) \leqslant \frac{1}{(1-t^2)^{2m}}\Big(\frac{a}{\nu^{2m}} + \frac{b}{\nu^m} + c\Big)$$

and consequently,

$$\limsup_{t\to 0+} h(t) < \infty, \quad \limsup_{t\to 1-} h(t) = \infty.$$

Hence, A2.3 holds. Moreover,

$$\sigma_1(1) = \sigma_2(1) = 0 = A, \quad (t^n\sigma_1'(t))' = -2\nu(n+1)t^n,$$

and

$$(t^n\sigma_2'(t))' = -\frac{\beta}{m}t^n(1-t^2)^{1/2m-2}\Big((n+1)(1-t^2) + 2t^2\Big(1-\frac{1}{m}\Big)\Big).$$

This means that there exists $\delta_2 \in (0,1)$ such that (2.4) is valid for $K = -2\nu(n+1)$. Therefore, by Theorem 2.3 (i), problem (3.1) has a positive solution $u \in C^1[0,1)$ satisfying $u'(0) = 0$ and (2.2). Since $\sigma_1(0) > 0$, we have $u(0) > 0$.

Let $r \in (-1,0)$. By Lemma 3.5,

$$\sigma_1(t) = \nu t^{-r/m}(1-t), \quad \sigma_2(t) = \beta(1-t^2)^{1/2m},$$

where $0 < \nu < \beta$ and ν is sufficiently small, while β is sufficiently large. Then for $t \in (0,1)$

$$0 < \frac{t^{2r}}{(1-t)^{2m}}\Big(\frac{a}{\nu^{2m}} - \frac{b}{\nu^m} - c\Big) \leqslant h(t) \leqslant \frac{t^{2r}}{(1-t)^{2m}}\Big(\frac{a}{\nu^{2m}} + \frac{b}{\nu^m} + c\Big).$$

Consequently,

$$\limsup_{t\to 0+} h(t) = \infty, \quad \limsup_{t\to 1-} h(t) = \infty.$$

For $p = -2r$ we obtain $\lim\limits_{t\to 0+} t^p h(t) < \infty$ and hence A2.3 follows. Moreover, we have

$$(t^n\sigma_1'(t))' = \nu t^{n-r/m-2}\Big(-\frac{r}{m}\Big(n-\frac{r}{m}-1\Big) - \Big(n-\frac{r}{m}\Big)\Big(1-\frac{r}{m}\Big)t\Big), \quad t \in (0,1),$$
$$(t^n\sigma_2'(t))' = -\frac{\beta}{m}t^n(1-t^2)^{1/2m-2}\Big((n+1)(1-t^2) + 2t^2\Big(1-\frac{1}{m}\Big)\Big), \quad t \in (0,1).$$

Thus, we can find $\delta_1, \delta_2 \in (0,1)$ which are sufficiently small to guarantee

$$\begin{gathered}(t^n \sigma_1'(t))' \geqslant 0, \quad (t^n \sigma_2'(t))' \leqslant 0, \;\; t \in (0, \delta_1), \\ (t^n \sigma_1'(t))' \geqslant K, \quad (t^n \sigma_2'(t))' \leqslant K, \;\; t \in (1-\delta_2, 1),\end{gathered}$$

where $K = -\nu(n - r/m)(1 - r/m)$. We can see that (2.3) and (2.4) hold and using Theorem 2.2 (ii) we deduce that problem (3.1) has a positive solution $u \in C^1(0,1)$ satisfying (2.2). For $r \in (-1, 0)$, property (3.8) can be proved in the same way as in Case 1.

Finally, we show that if $A = 0$ and $a_1 = 0$, then $u'(1-) = -\infty$. Since $u(1) = 0$, there exists $\xi \in (0,1)$ such that $u^m(t) \leqslant a/2b$ for $t \in [\xi, 1]$. Moreover, we have

$$-\int_\xi^t \frac{\mathrm{d}s}{u^{2m}(s)} \leqslant -\int_\xi^t \frac{\mathrm{d}s}{\sigma_2^{2m}(s)} \leqslant -\frac{1}{2\beta^{2m}} \int_\xi^t \frac{\mathrm{d}s}{1-s} = \frac{1}{2\beta^{2m}} \ln \frac{1-t}{1-\xi}, \quad t \in (\xi, 1).$$

Therefore, by integrating (3.1a), we obtain

$$\begin{aligned} t^n u'(t) &= \xi^n u'(\xi) + \int_\xi^t \frac{s^n}{u^{2m}(s)} (b u^m(s) - a) \,\mathrm{d}s + c \int_\xi^t s^{n+2r} \,\mathrm{d}s \\ &\leqslant \xi^n u'(\xi) + \frac{a}{2} \xi^n \left(-\int_\xi^t \frac{\mathrm{d}s}{u^{2m}(s)} \right) \\ &\leqslant \xi^n u'(\xi) + \frac{a \xi^n}{4\beta^{2m}} \ln \frac{1-t}{1-\xi} + \frac{c}{n+2r+1}, \quad t \in (\xi, 1). \end{aligned}$$

Hence, $\lim\limits_{t \to 1-} t^n u'(t) = u'(1-) = -\infty$. □

From Theorem 3.6 we are now able to derive the following existence result for problem (1.4).

Theorem 3.7. *Problem* (1.4) *has a positive solution* z *such that*

$$\begin{cases} \lim\limits_{s \to \infty} z(s) > 0, \quad \lim\limits_{s \to \infty} s^\gamma z'(s) = -\dfrac{\lambda^2}{8\gamma}, \;\; \gamma \geqslant \dfrac{3}{2}, \\[2ex] \lim\limits_{s \to \infty} z(s) \geqslant 0, \quad \lim\limits_{s \to \infty} \sqrt{s^3}\, z'(s) = -\infty, \;\; \gamma < \dfrac{3}{2}, \end{cases} \tag{3.17}$$

and

$$\begin{cases} z'(1+) \in \mathbb{R}, & A > 0, \\ z'(1+) \in \mathbb{R}, & A = 0, \; b_1 > 0, \\ z'(1+) = \infty, & A = 0, \; b_1 = 0. \end{cases} \tag{3.18}$$

P r o o f. Problem (3.1) with $n = 3$, $a = \frac{1}{8}$, $b = \mu$, $c = \lambda^2/2$, $r = \gamma - 2$ has the form (1.2). By Lemmas 3.3, 3.4, and 3.5 there exist lower and upper functions σ_1 and σ_2 of problem (1.2) satisfying A2.2. By Theorem 3.6, there is a positive solution u of (1.2) satisfying (2.2), (3.8), and (3.9). Let $r_2 := \max\{|\sigma_2(t)| \colon t \in [0,1]\}$ and let z be defined by

$$z(s) := z\Big(\frac{1}{t^2}\Big) = u(t), \quad t \in (0,1].$$

Then $0 < z(s) < r_2$ for $s \in [1,\infty)$ and z is a solution of problem (1.4). Furthermore, we have

$$-2\sqrt{s^3}\, z'(s) = u'(t).$$

Let $\gamma \geqslant \frac{3}{2}$. Then, by (3.16),

$$\lim_{t\to 0+} u'(t) = \lim_{t\to 0+} \frac{\lambda^2}{4\gamma} t^{2\gamma-3},$$

and

$$\lim_{s\to\infty} s^{\gamma} z'(s) = \lim_{s\to\infty} s^{\gamma-3/2}(s^{3/2} z'(s)) = \lim_{t\to 0+} t^{3-2\gamma}\Big(-\frac{1}{2}u'(t)\Big) = -\frac{\lambda^2}{8\gamma}.$$

Consequently, due to (3.8) and (3.9), z satisfies (3.17) and (3.18). □

4. Numerical approach

Here, we first describe how we approximate solutions of two-point boundary value problems for systems of ordinary differential equations of the form

$$f(t, u'(t), u(t)) = 0, \quad t \in [0,1],$$
$$g(u(0), u(1)) = 0.$$

We assume that the analytical solution u is appropriately smooth and attempt to solve this problem numerically using the collocation method implemented in our Matlab code `bvpsuite`. It is a new version of the general purpose MATLAB code `sbvp`, cf. [4], [5], and [18], which has already been successfully applied to a variety of problems, see for example [9], [10], [11], [19], and [21]. Collocation is a widely used and well-studied standard solution method for two-point boundary value problems, see for example [23] and the references therein. It also proved to be robust in the case of singular boundary value problems.

The code is designed to solve systems of differential equations of arbitrary order. For simplicity of notation we formulate below a problem whose order varies between four and zero, which means that algebraic constraints which do not involve derivatives are also admitted. Moreover, the problem can be given in a fully implicit form

$$F(t, u^{(4)}(t), u^{(3)}(t), u''(t), u'(t), u(t)) = 0, \quad 0 < t \leqslant 1, \tag{4.1a}$$

$$b(u^{(3)}(0), u''(0), u'(0), u(0), u^{(3)}(1), u''(1), u'(1), u(1)) = 0. \tag{4.1b}$$

The program can cope with free parameters, $\lambda_1, \lambda_2, \ldots, \lambda_k$, which will be computed along with the numerical approximation for u,

$$F(t, u^{(4)}(t), u^{(3)}(t), u''(t), u'(t), u(t), \lambda_1, \lambda_2, \ldots, \lambda_k) = 0, \quad 0 < t \leqslant 1, \tag{4.2a}$$

$$b_{\text{aug}}(u^{(3)}(0), u''(0), u'(0), u(0), u^{(3)}(1), u''(1), u'(1), u(1)) = 0, \tag{4.2b}$$

provided that the boundary conditions b_{aug} include k additional requirements to be satisfied by u.

The numerical approximation defined by collocation is computed as follows: On a mesh

$$\Delta := \{\tau_i\colon\ i = 0, \ldots, N\}, \quad 0 = \tau_0 < \tau_1 \ldots < \tau_N = 1,$$

we approximate the analytical solution by a collocating function

$$p(t) := p_i(t), \quad t \in [\tau_i, \tau_{i+1}], \quad i = 0, \ldots, N-1,$$

where we require $p \in C^{q-1}[0,1]$ in the case that the order of the underlying differential equation is q. Here p_i are polynomials of maximal degree $m-1+q$ which satisfy the system (4.1a) at the *collocation points*

$$\{t_{i,j} = \tau_i + \varrho_j(\tau_{i+1} - \tau_i),\ i = 0, \ldots, N-1,\ j = 1, \ldots, m\}, \quad 0 < \varrho_1 < \ldots < \varrho_m < 1,$$

and the associated boundary conditions (4.1b). For $y \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n)^T$, we have

$$|y| := \max_{1 \leqslant k \leqslant n} |y_k|.$$

Let $y \in C[0,1]$, $y\colon\ [0,1] \to \mathbb{R}^n$. For $t \in [0,1]$,

$$|y(t)| := \max_{1 \leqslant k \leqslant n} |y_k(t)|$$

and

$$\|y\|_\infty := \max_{0 \leqslant t \leqslant 1} |y(t)|.$$

Classical theory, cf. [23], predicts that the convergence order for the global error of the method is at least $O(h^m)$, where h is the maximal stepsize, $h := \max_i(\tau_{i+1} - \tau_i)$.

More precisely, for the global error of p, $\|p-u\|_\infty = O(h^m)$ holds uniformly in t. For certain choices of the collocation points the so-called *superconvergence* order can be observed. In the case of Gaussian points this means that the approximation is exceptionally precise at the meshpoints τ_i, $\max_{\tau_i\in\Delta} |p(\tau_i)-u(\tau_i)|_\infty = O(h^{2m})$.

To make the computations more efficient, an adaptive mesh selection strategy based on an a posteriori estimate for the global error of the collocation solution may be utilized. We use a classical error estimate based on mesh halving. In this approach, we compute the collocation solution $p_\Delta(t)$ on a mesh Δ. Subsequently, we choose a second mesh Δ_2 where in every interval $[\tau_i,\tau_{i+1}]$ of Δ we insert two subintervals of equal length. On this new mesh, we compute the numerical solution based on the same collocation scheme to obtain the collocating function $p_{\Delta_2}(t)$. Using these two quantities, we define

$$\mathcal{E}(t) := \frac{2^m}{1-2^m}(p_{\Delta_2}(t)-p_\Delta(t)) \tag{4.3}$$

as an error estimate for the approximation $p_\Delta(t)$. Assume that the global error $\delta(t) := p_\Delta(t)-u(t)$ of the collocation solution can be expressed in terms of the principal error function $e(t)$,

$$\delta(t) = e(t)|\tau_{i+1}-\tau_i|^m + O(|\tau_{i+1}-\tau_i|^{m+1}), \quad t\in[\tau_i,\tau_{i+1}], \tag{4.4}$$

where $e(t)$ is independent of Δ. Then obviously, the quantity $\mathcal{E}(t)$ satisfies $\mathcal{E}(t) - \delta(t) = O(h^{m+1})$ and the error estimate is asymptotically correct. Our mesh adaptation is based on the equidistribution of the global error of the numerical solution. Thus, we define a *monitor function* $\Theta(t) := \sqrt[m]{\mathcal{E}(t)}/h(t)$, where $h(t) := |\tau_{i+1}-\tau_i|$ for $t\in[\tau_i,\tau_{i+1}]$. Now, the mesh selection strategy aims at the equidistribution of

$$\int_{\tilde\tau_i}^{\tilde\tau_{i+1}} \Theta(s)\,\mathrm{d}s$$

on the mesh consisting of the points $\tilde\tau_i$ to be determined accordingly, where at the same time measures are taken to ensure that the variation of the stepsizes is restricted and tolerance requirements are satisfied with small computational effort. Details of the mesh selection algorithm and a proof of the fact that our strategy implies that the global error of the numerical solution is *asymptotically equidistributed* are given in [7].

We now discuss the numerical solution of problem (3.1) whose analytical properties are formulated in Theorem 3.6. For the numerical experiments we specify the following parameter setting:

$$n=3,\ \ m=1,\ \ a=\frac{1}{8},\ \ b=\mu=0,\ \ c=\frac{\lambda^2}{2}=\frac{1}{2},\ \ \lambda=1,\ \ r=\gamma-2,$$

see Theorem 3.6. In order to be able to formulate the first boundary condition in (3.1b), we introduce a new variable $v(t) := t^3 u'(t)$ and transform the scalar boundary value problem (3.1) to an associated boundary value problem for the following system of two implicit differential equations of first order:

$$v'(t) + t^3\Big(\frac{1}{8u^2(t)} - \frac{\mu}{u(t)} - \frac{\lambda^2}{2} t^{2\gamma-4}\Big) = 0, \tag{4.5a}$$

$$v(t) - t^3 u'(t) = 0, \tag{4.5b}$$

$$v(0) = 0, \quad a_0 u(1) + \frac{1}{2} a_1 u'(1) = A, \tag{4.5c}$$

with $t \in [0,1]$. For numerical simulation, problem (4.5) has been rearranged to

$$v'(t)u^2(t) + t^3\Big(\frac{1}{8} - \mu u(t) - \frac{\lambda^2 u^2(t)}{2} t^{2\gamma-4}\Big) = 0, \tag{4.6a}$$

$$v(t) - t^3 u'(t) = 0, \tag{4.6b}$$

$$v(0) = 0, \quad a_0 u(1) + \frac{1}{2} a_1 v(1) = A. \tag{4.6c}$$

4.1. Numerical results

In this section we illustrate the theoretical findings of Theorem 3.6 by appropriate numerical experiments which have been carried out using collocation at 4 Gaussian collocation points. The numerical solution has been calculated on a fixed equidistant mesh with 1000 points. These rather dense grids were necessary for a good visualization of approximations when transforming them from the standard interval $[0,1]$ back to the infinite interval $[1,\infty)$. The error estimate and the residual were also recorded as indicators for the accuracy of the numerical solution. The error estimate was computed from (4.3) by coupling solutions related to meshes with 1000 and 2000 meshpoints. The residual was obtained by substituting the numerical solution p into the system of differential equations (4.6a), (4.6b).

First, we set $a_0 = 1$, $a_1 = 0$ and $A = 1$. According to Theorem 3.6 this means that $u'(1-) \in \mathbb{R}$. The corresponding numerical results for two different values of γ, both covering the case $r > -\frac{1}{2}$, can be found in Figs. 1 and 2.

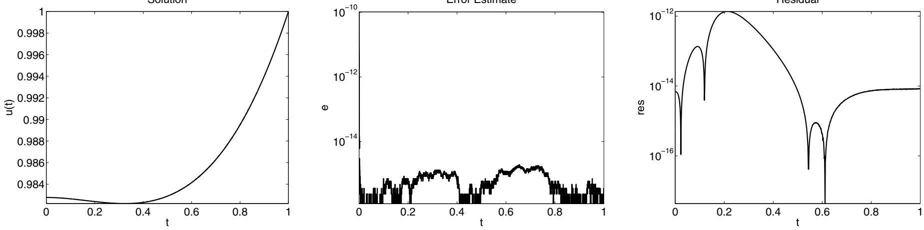

Figure 1. Problem (4.6), $\gamma = 2.5$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

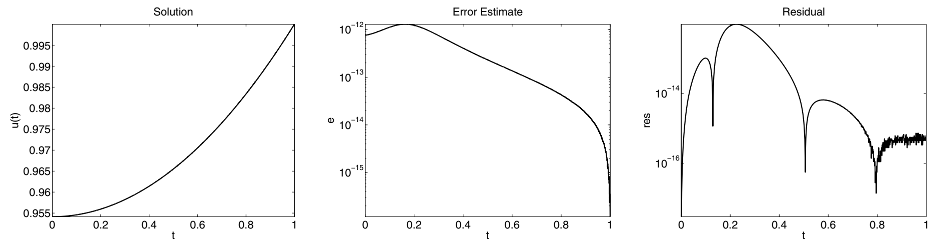

Figure 2. Problem (4.6), $\gamma = 2$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

In both the figures $u(0) > 0$ and $u'(0+) = 0$, as was predicted by Theorem 3.6. Moreover, both the error estimate and the residual are very small thus indicating an excellent accuracy of the approximation. In Fig. 3 the results for $r = -\frac{1}{2}$ are depicted. Again, $u(0) > 0$ is clearly visible. Here we have $n = 3$, $c = \frac{1}{2}$ and therefore, $u'(0+) = c/n \approx 0.167$ which is in a good agreement with Theorem 3.6.

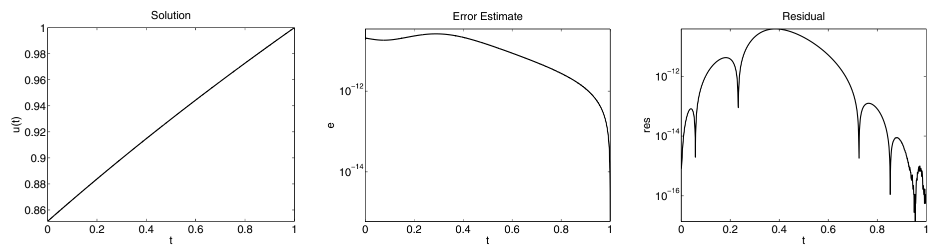

Figure 3. Problem (4.6), $\gamma = 1.5$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

Finally, Figs. 4 and 5 show the last case $r < -\frac{1}{2}$.

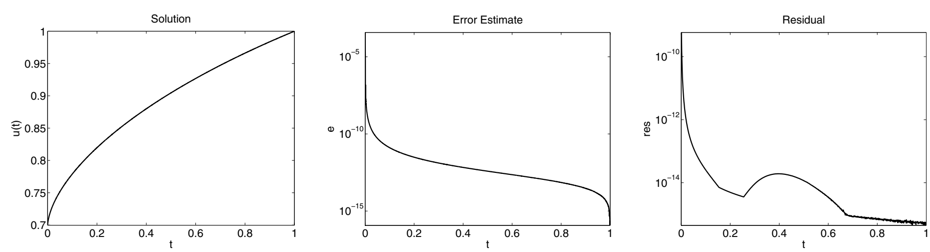

Figure 4. Problem (4.6), $\gamma = 1.3$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

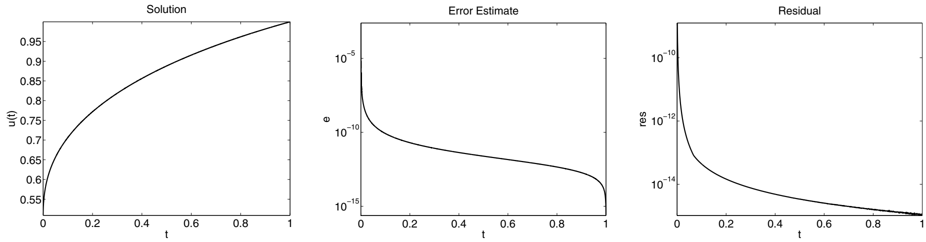

Figure 5. Problem (4.6), $\gamma = 1.2$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

For both settings, $u(0) \geqslant 0$ and $u'(0+) = \infty$.

We now set $A = 0$ and leave all the other parameters unchanged. According to Theorem 3.6 this results in $u'(1-) = -\infty$. Figs. 6 to 10 show the corresponding numerical runs for $\gamma = 2.5,\ 2,\ 1.5,\ 1.3,\ 1.2$, respectively.

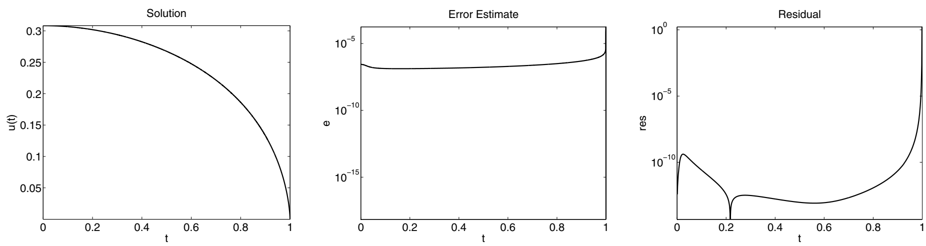

Figure 6. Problem (4.6), $A = 0$, $\gamma = 2.5$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

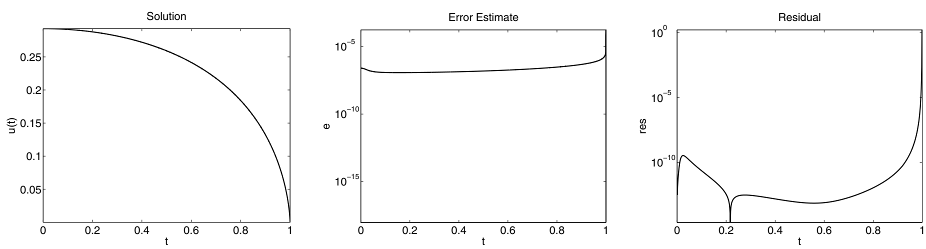

Figure 7. Problem (4.6), $A = 0$, $\gamma = 2$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

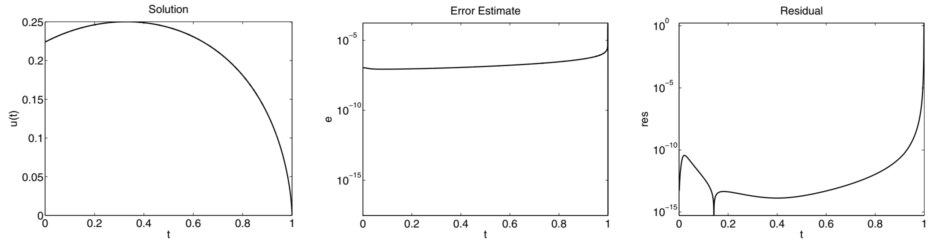

Figure 8. Problem (4.6), $A = 0$, $\gamma = 1.5$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

Again, $u'(0+) \approx 0.167$.

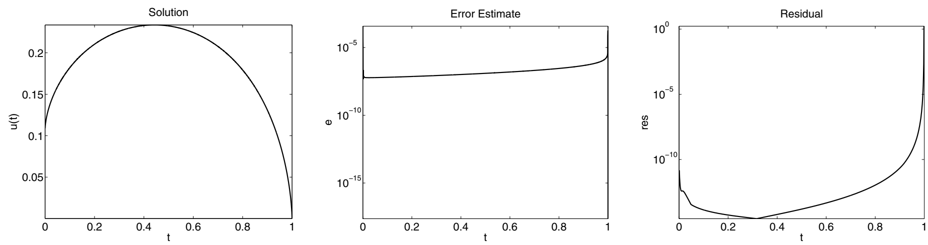

Figure 9. Problem (4.6), $A = 0$, $\gamma = 1.3$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

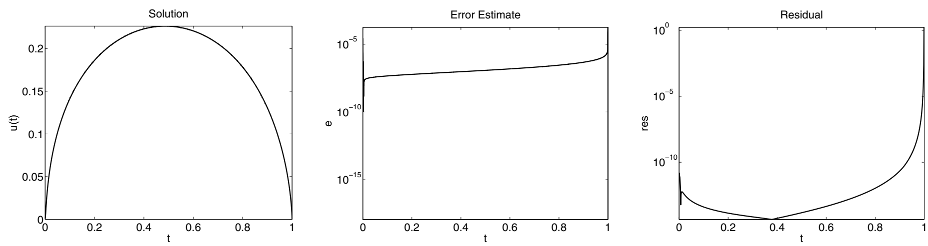

Figure 10. Problem (4.6), $A = 0$, $\gamma = 1.2$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

The last setting discussed in Theorem 3.6 is $A = 0$ and $a_1 > 0$. We use $a_1 = 2$, all the other parameters remain unchanged, see Figs. 11 to 15 for the numerical simulations corresponding to the above values of γ.

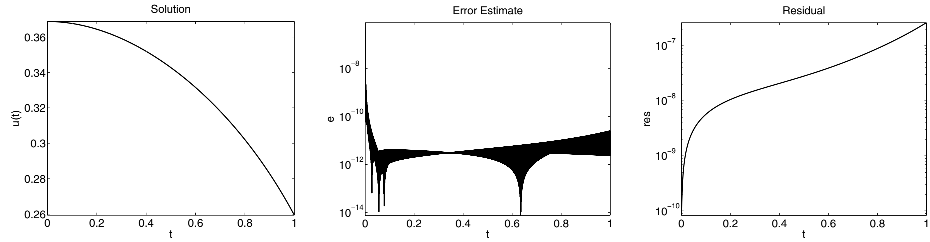

Figure 11. Problem (4.6), $A = 0$, $a_1 > 0$, $\gamma = 2.5$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

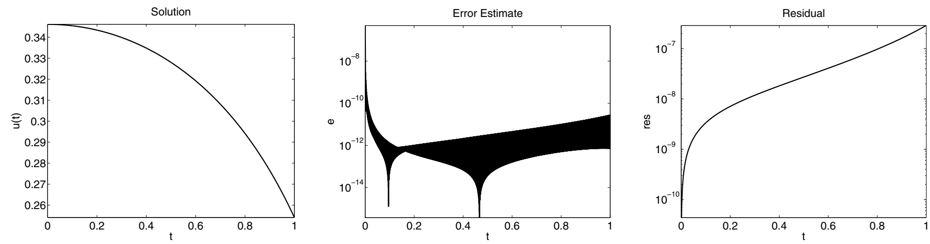

Figure 12. Problem (4.6), $A = 0$, $a_1 > 0$, $\gamma = 2$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

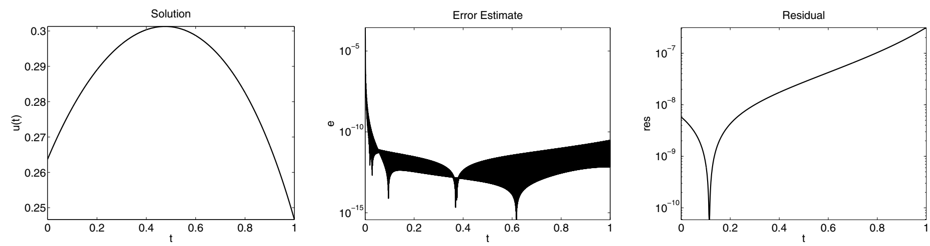

Figure 13. Problem (4.6), $A = 0$, $a_1 > 0$, $\gamma = 1.5$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

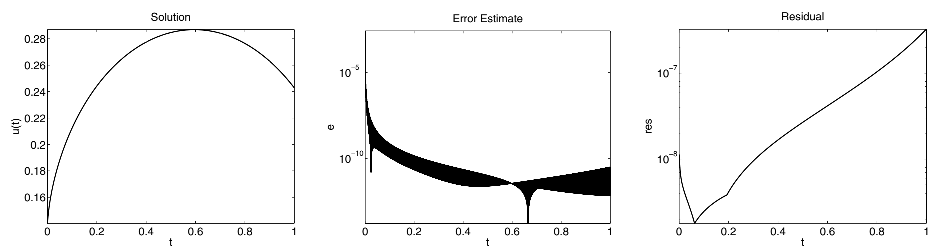

Figure 14. Problem (4.6), $A = 0$, $a_1 > 0$, $\gamma = 1.3$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

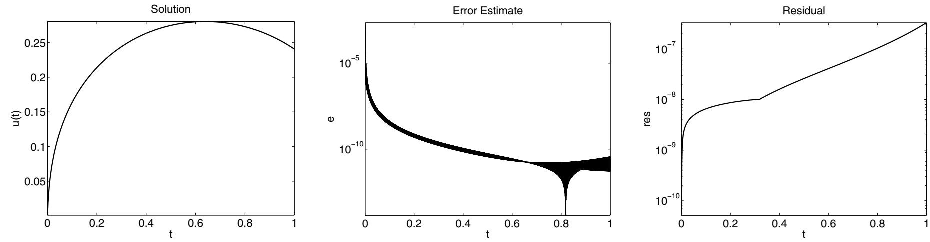

Figure 15. Problem (4.6), $A = 0$, $a_1 > 0$, $\gamma = 1.2$: The numerical approximation for the solution component $u(t)$, the error estimate and the residual.

All numerical results show a good agreement with Theorem 3.6. Both the error estimates[2] and the residuals show that the solutions accuracy is excellent. To visualize solutions of problem (1.4) posed on the semi-infinite interval, we have to transform the numerical solution obtained on $[0, 1]$ back to the original interval $[1, \infty)$. To this end we use

$$z(s) := z\Big(\frac{1}{t^2}\Big) = u(t), \quad s \in [1, \infty), \quad t \in (0, 1],$$

to obtain the values for $z(s)$.

We again discuss three different settings, where for all experiments $b_0 = a_0 = 1$. For $A = 1$ and $b_1 = \frac{1}{2}a_1 = 0$, Fig. 16 shows the numerical solution of (1.4) displayed on a short and a long interval.

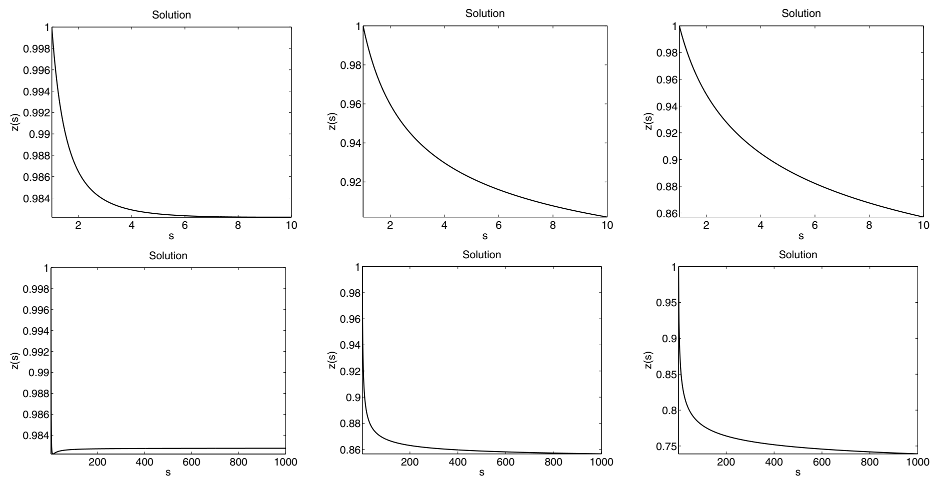

Figure 16. Problem (1.4), $A = 1$, $b_1 = 0$: Solution $z(s)$ on the interval $[1, 10]$ (above) and interval $[1, 1000]$ (below) for values of $\gamma = 2.5$, $\gamma = 1.5$ and $\gamma = 1.3$ (from left to right).

[2] Often within the level of the machine accuracy of MATLAB.

For a better illustration of the solution behavior for $\gamma = 2.5$ displayed on the long interval in Fig. 16, we depict this solution in Fig. 17 on three further intervals of smaller length, see also Fig. 1.

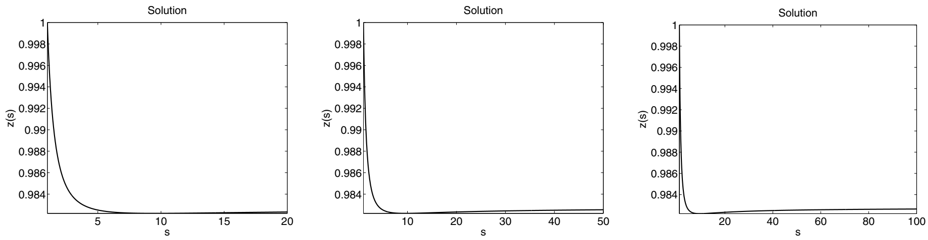

Figure 17. Problem (1.4), $A = 1$, $b_1 = 0$, $\gamma = 2.5$: Solution $z(s)$ on the intervals $[1, 20]$, $[1, 50]$, and $[1, 100]$ (from left to right).

For $\gamma \geqslant \frac{3}{2}$, $z'(1+) \in \mathbb{R}$ holds and we know that the solution of (4.6) is positive with $\lim_{s\to\infty} z(s) > 0$. Also, for $\lambda = 1$, $\lim_{s\to\infty} s^\gamma z'(s) = -\frac{1}{8}\gamma^{-1}$. In principle, we should be able to verify this latter limit using the values of the numerical solution at the meshpoints approaching zero and the relation

$$s^\gamma z'(s) = -v(t)/(2t^{2\gamma}) =: w(t), \tag{4.7}$$

cf. (4.6b). For $\gamma = 1.5$ (and $\gamma = 1.6$) we have plotted $w(t)$ using its values at the meshpoints and found out that $w(0) - (-\frac{1}{8}\gamma^{-1}) \approx 10^{-5}$.

In Fig. 18 the numerical solution of (1.4) for $A = 0$ and $b_1 = 0$ is shown.

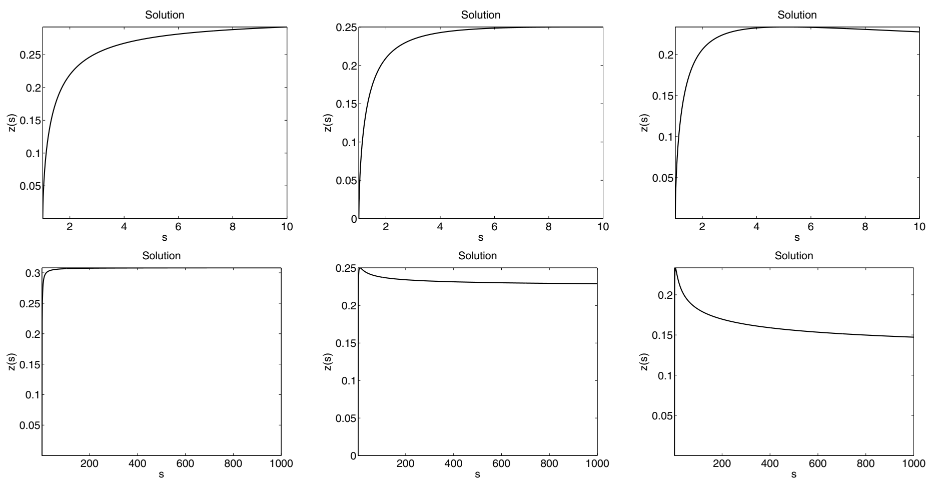

Figure 18. Problem (1.4), $A = 0$, $b_1 = 0$: Solution $z(s)$ on the interval $[1, 10]$ (above) and interval $[1, 1000]$ (below) for values of $\gamma = 2.5$, $\gamma = 1.5$ and $\gamma = 1.3$ (from left to right).

Here, as expected, $z'(1+) = \infty$ holds for all values of γ. Also, $z(s) \geqslant 0$.

Finally, we consider $A = 0$ and $b_1 = 1$. The numerical results for this setting and the above five values of γ are given in Fig. 19. With $z'(1+) \in \mathbb{R}$, $\lim_{s\to\infty} s^\gamma z'(s) \approx -\frac{1}{8}\gamma^{-1}$ for $\gamma = 1.5$, and $\lim_{s\to\infty} z(s) \geqslant 0$ the numerical solution again very well reflects the properties of the analytical solution.

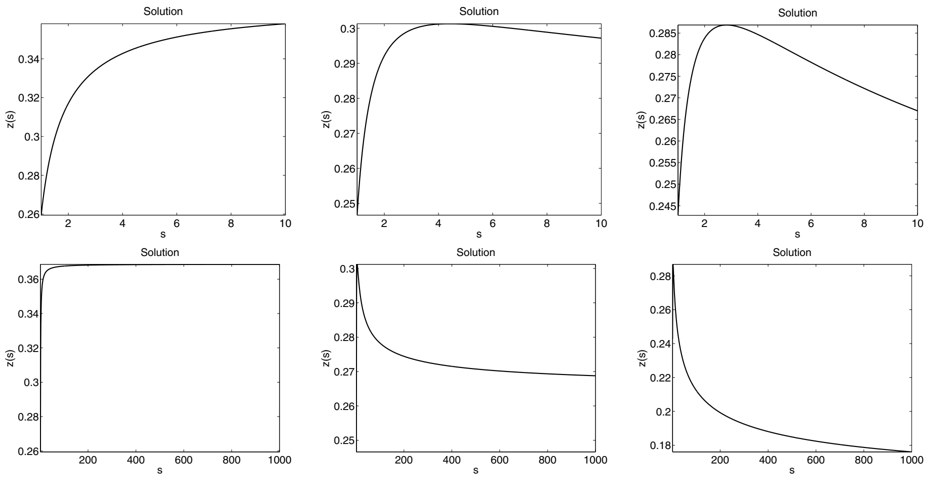

Figure 19. Problem (1.4), $A = 0$, $b_1 = 1$: Solution $z(s)$ on the interval $[1, 10]$ (above) and interval $[1, 1000]$ (below) for values of $\gamma = 2.5$, $\gamma = 1.5$ and $\gamma = 1.3$ (from left to right).

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