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A Unified Framework for Hybrid Control: Background, Model, and Theory*

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Abstract

We propose a very general framework for hybrid control problems which encompasses several types of hybrid phenomena considered in the literature. A specific control problem is studied in this framework, leading to an existence result for optimal controls. The "value function" associated with this problem is expected to satisfy a set of "generalized quasi-variational inequalities" which are formally derived.

Key Words: Hybrid control, optimal control, quasi-variational inequalities, autonomous jumps, impulsive jumps.

1 Introduction

Hybrid control systems are control systems that involve both continuous dynamics and controls, as well as discrete phenomena. Some examples include computer disk drives [13], transmissions and stepper motors [9], constrained robotic systems [2], and intelligent vehicle/highway systems [22]. More generally, such systems arise whenever one mixes logical decision-making with the generation of continuous control laws, such as in modern flight control systems.

In this paper, our focus is on the case where the continuous dynamics is modeled by a differential equation

$$\dot{x}(t) = \xi(t), \qquad t \geq 0 \tag{1.1}$$

Here, $x(t)$ is the *continuous component* of the state taking values in some subset of a Euclidean space. $\xi(t)$ is a *controlled vector field* which generally depends on $x(t)$, the *continuous component* $u(t)$ of the control policy, and the aforementioned discrete phenomena. We shall make this more precise later on. The discrete phenomena generally considered are of four types: (1) autonomous switching, (2) autonomous jumps, (3) controlled switching, and (4) controlled jumps.

In this paper, we study a model that subsumes all these phenomena and study an associated control problem. The paper is organized as follows. The next section details the discrete phenomena arising in hybrid systems listed above, including some simple examples. Section 3 reviews models of hybrid

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systems from the control and dynamical systems literature. Section 4 abstracts the phenomena found in hybrid systems to a unified framework, which is related to the other models reviewed. Section 5 defines an optimal control problem in this framework. All assumptions used in obtaining the remaining results are expressly stated there. The necessity of such assumptions is discussed throughout the sequel. The existence of an optimal control for this problem is established in Section 6. Section 7 gives a formal derivation of the associated *generalized quasi-variational inequalities.* Finally, Section 8 concludes with a list of some open issues.

Closely related issues occur in the study of piecewise deterministic processes, an excellent account of which can be found in [12].

Finally, we collect some notation used throughout. First, we make use of the abbreviations ODEs (ordinary differential equations), FA (finite automata/on), and DEDS (discrete event dynamical systems; see [15]). $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{N}$ denote the reals, nonnegative reals, integers, and nonnegative integers, respectively. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and, in an abuse of common notation, $\lceil x \rceil$ denotes the least integer greater than x. $\underline{N}$ denotes the set $\{1, 2, \ldots, N\}$.

Below we deal with continuous time systems, such as ODEs, that are affected by events at discrete instants, such as jumps in state. We will use $[t]$ to denote the time less than or equal to t at which the last "jump" or "event" occurred. If the event is unclear, we will subscript the variable, so that $[t]_p$ denotes the time at which the variable p last jumped. Throughout, $v\lfloor p \rfloor$ and $x[t]$ will be shorthand for $v(\lfloor p \rfloor)$ and $x([t])$, respectively.

Other notation is common. For example, $X \backslash U$ represents the complement of U in X; $\overline{U}$ represents the closure of U, U^o its interior, ∂U its boundary; $f(t^+), f(t^-)$ denote the right-hand and left-hand limits of the function f at t, respectively; a function is right-continuous if $f(t^+) = f(t)$ for all t; $C(X, Y)$ denotes the space of continuous functions with domain X and range Y; v^T denotes the transpose of vector v.

2 Hybrid Phenomena

In this section, we briefly examine the discrete phenomenon that arise in the study of hybrid systems: (1) autonomous switching, (2) autonomous jumps, (3) controlled switching, and (4) controlled jumps.

We also discuss how finite automata may be viewed as evolving in continuous time, which sets the stage for their interacting with ODEs below.

AUTONOMOUS SWITCHING. Here the vector field $\xi(\cdot)$ changes discontinuously when the state $x(\cdot)$ hits certain "boundaries" [19, 21]. The simplest example of this is when it changes depending on a "clock" which may be modeled as a supplementary state variable [9]. An example of autonomous switching is the following:

Example 2.1 *Consider the following model of a system with hysteresis* [21]*:*

$$\begin{aligned} \dot{x}_1 &= x_2 - \phi(x_1) \\ \dot{x}_2 &= H(\psi(x_1, x_2)) - \phi(x_2) \end{aligned}$$

where the multi-valued function H is shown in Figure 1. *The functions ϕ, ψ depend on the exact system under consideration.*

Note that this system is not just a differential equation whose right-hand side is piecewise continuous. There is "memory" in the system, which affects the value of the vector field. Indeed, such a system naturally has a finite automaton associated with the function H, as pictured in Figure 2.

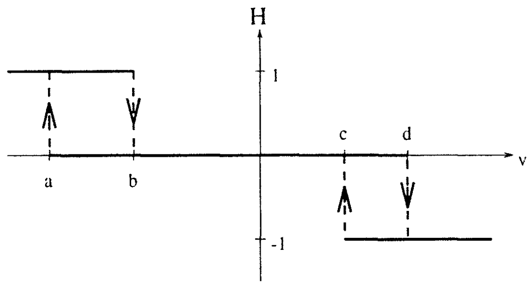

Figure 1: Hysteresis function, H.

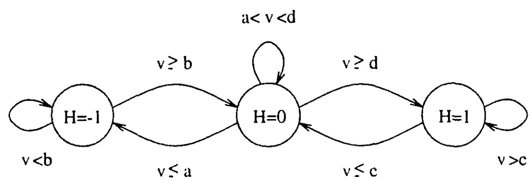

Figure 2: Finite automaton associated with hysteresis function, H.

AUTONOMOUS JUMPS. Here $x(\cdot)$ jumps discontinuously on hitting prescribed regions of the state space [2, 3]. The simplest examples possessing this phenomenon are those involving collisions.

Example 2.2 *Consider the case of the vertical and horizontal motion of a ball of mass m in a room under gravity with constant g (see Figure 3). In this case, the dynamics are given by*

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= 0 \\ \dot{v}_y &= -mg \end{aligned}$$

Further, upon hitting the boundaries $\{(x,y) \mid y = 0 \text{ or } y = C\}$ we instantly set v_y to $-\rho v_y$, where $\rho \in [0,1]$ is the coefficient of restitution. Likewise, upon hitting $\{(x,y) \mid x = 0 \text{ or } x = R\}$ v_x is set to $-\rho v_x$.

CONTROLLED SWITCHING. Here $\xi(\cdot)$ changes abruptly in response to a control command with an associated cost. This can be interpreted as switching between different vector fields [24]. Controlled switching arises, for instance, when one is allowed to pick among a number of vector fields:

$$\dot{x} = f_i(x), \qquad i \in \underline{N}$$

An example of this phenomenon is the following:

Example 2.3 *The following is a simplified model of a manual transmission* [9]*:*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= [-a(x_2) + u]/(1 + v) \end{aligned}$$

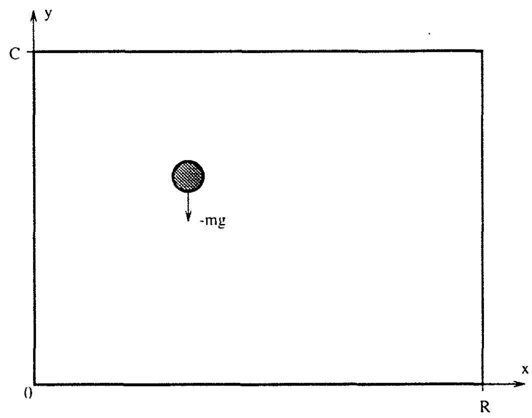

Figure 3: Ball in an enclosed room.

where x_1 is the ground speed, x_2 is the engine RPM, $u \in [0,1]$ is the throttle position, and $v \in \{1,2,3,4\}$ is the gear shift position. The function a is positive for positive argument.

CONTROLLED JUMPS. Here $x(\cdot)$ changes discontinuously in response to a control command, with an associated cost [4]. An simple example is the following:

Example 2.4 *In a simple inventory management model [4], there are a "discrete" set of restocking times $\theta_1 < \theta_2 < \cdots$ and associated order amounts $\alpha_1, \alpha_2, \ldots$. The equations governing the stock at any given moment are*

$$\dot{y}(t) = -\mu(t) + \sum_i \delta(t - \theta_1)\alpha_i$$

where μ represents degradation or utilization dynamics and δ is the Dirac delta function.

If one makes the stocking times and amounts a direct function of y (or t), then these controlled jumps become autonomous jumps.

AUTOMATA. A *digital* or *symbolic automaton* is a quintuple (Q, I, O, ν, η), consisting of the state space, input alphabet, output alphabet, transition function, and output function, respectively. We assume that Q, I, and O are each isomorphic to subsets of $\mathbb{N}$. When these sets are finite, the result is a finite automaton with output. In any case, the functions involved are $\nu : Q \times I \to Q$ and $\eta : Q \times I \to O$. The "dynamics" of the automaton are given by

$$\begin{aligned} q_{k+1} &= \nu(q_k, i_k) \\ o_k &= \eta(q_k, i_k) \end{aligned}$$

Usually, automata are thought of as evolving in "abstract time," where only the ordering of events matters. We can add the notion of time by associating with the kth transition the time at which it occurs:

$$\begin{aligned} q(t_k) &= \nu(q(t_k^-), i(t_k)) \\ o(t_k) &= \eta(q(t_k), i(t_k)) \end{aligned}$$

Finally, this automaton may be thought of as operating in "continuous time" by the convention that the state, input, and output symbols are piecewise right continuous functions. Leading to

$$\begin{aligned} q(t) &= \nu(q(t^-), i(t)) \\ o(t) &= \eta(q(t), i(t)) \end{aligned} \tag{2.1}$$

Here, the state $q(t)$ changes only when the input symbol $i(t)$ changes. Thus, we have the idea of an automaton whose update times are not the abstract members of $\mathbb{N}$, but the event times in $\mathbb{R}$ when the input symbol i changes. Note that in the usual case, a finite automaton can be presented with the same input symbol for two successive time intervals. These situations must be handled differently (e.g., by adding new states and symbols) in the continuous-time version.

3 Review of Models of Hybrid Systems

This section summarizes five models of hybrid systems developed from the dynamical systems and control point of view. Specifically, we review models of Tavernini [21], Back-Guckenheimer-Myers [2], Nerode-Kohn [19], Antsaklis-Stiver-Lemmon [1], and Brockett [9].

For sure, there are many others and no review is attempted here (see [14]). These have been chosen as much for the clarity and rigor of their presentation as for the mechanisms they use to combine discrete and continuous dynamics. Only the models are given here. For further discussion and example systems, the reader is referred to the original papers.

The reader should note that we have liberally changed original notation to place the models in as similar a light as possible.

3.1 Tavernini's Model

Tavernini discusses so-called *differential automata* in [21]. He was motivated to study such systems as a means of modeling hysteretic phenomena such as backlash and friction (cf. Example 2.1).

A *differential automaton*, A, is a triple (S, f, ν) where S is the state-space of A, $S = \mathbb{R}^n \times Q$, $Q \simeq \underline{N}$ is the *discrete state space* of A, and $\mathbb{R}^n$ is the *continuous state space* of A; f is a finite family $f(\cdot, q) : \mathbb{R}^n \to \mathbb{R}^n$, $q \in Q$, of vector fields, the *continuous dynamics* of A; and $\nu : S \to Q$ is the *discrete transition function* of A.

Let $\nu_q \equiv \nu(\cdot, q)$, $q \in Q$. Define $I(q) = \nu_q(\mathbb{R}^n) \backslash \{q\}$, that is, the set of discrete states "reachable in one step" from q. We require that for each $q \in Q$ and each $p \in I(q)$ there exist closed sets

$$M_{q,p} \equiv \nu_q^{-1}(p)$$

The sets $\partial M_{q,p}$ are called the *switching boundaries* of the automaton A. Define $M_q = \bigcup_{p \in I(q)} M_{q,p}$ and define the domain of capture of state q by

$$C(q) \equiv \mathbb{R}^n \backslash M_q = \{x \in \mathbb{R}^n \mid \nu(x, q) = q\}$$

The equations of motion are

$$\begin{aligned} \dot{x}(t) &= f(x(t), q(t)) \\ q(t) &= \nu(x(t), q(t^-)) \end{aligned}$$

with initial condition $[x(0),\ q(0)]^T \in \bigcup_{q \in Q} C(q) \times \{q\}$. The notation t^- indicates that the discrete state is piecewise continuous from the right. Thus, starting at $[x_0, i]$, the continuous state trajectory $x(\cdot)$ evolves according to $\dot{x} = f(x, i)$. If $x(\cdot)$ hits some $\partial M_{i,j}$ at time t_1, then the state becomes $[x(t_1), j]$, from which the process continues. See Figure 4.

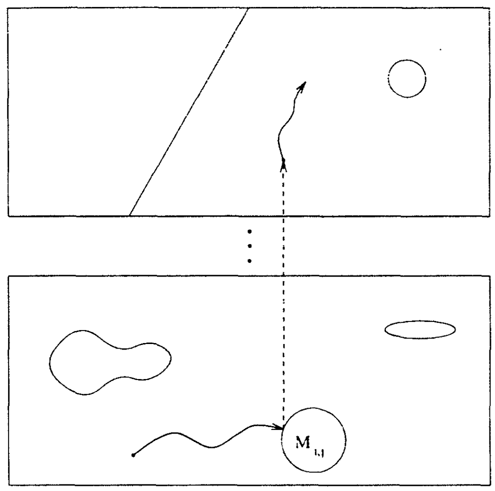

Figure 4: Example dynamics of Tavernini's model.

Tavernini places restrictions on the model above: First each $f(\cdot, q)$, $q \in Q$, is assumed to be globally Lipschitz so that the continuous dynamics are well-behaved. Also, for each $q \in Q$ and $p \in I(q)$, the set $M_{q,p}$ is required to be connected and there must exist a function $g_{q,p} \in C^1(\mathbb{R}^n, \mathbb{R})$ with 0 in its image a regular value such that

$$M_{q,p} = \{x \in \mathbb{R}^n \mid g_{q,p}(x) \geq 0\}$$

Thus, $\nu_q^{-1}(p)$ is an n-submanifold of $\mathbb{R}^n$ with boundary

$$\partial M_{q,p} = \{x \in \mathbb{R}^n \mid g_{q,p}(x) = 0\}$$

which is an $(n-1)$-submanifold of $\mathbb{R}^n$.

Finally, [21] makes the following three key assumptions on differential automata:

- Define $\alpha_q = \min\{\text{dist}(M_{q,p}, M_{q,p'}) \mid p, p' \in I(q), p \neq p'\}$. We require that

  $$\alpha(A) \equiv \min_{q \in Q} \alpha_q > 0$$

  be satisfied. That is, the distance between any two sets with different discrete transitions is bounded away from zero.

- Define $\beta_{q,p} = \min\{\text{dist}(\partial M_{q,p}, \partial M_{p,p'}) \mid p' \in I(p)\}$. We require that the inequality

  $$\beta(A) \equiv \min_{q \in Q} \min_{p \in I(q)} \beta_{q,p} > 0$$

  be satisfied. That is, after a discrete transition, the next set from which another discrete transition takes place is at least a fixed distance away.

- The assumption on $\alpha(A)$ is such that $C(q)$ is an open set with boundary $\partial C(q) = \partial M_q = \bigcup_{p \in I(q)} \partial M_{q,p}$. We require that the inclusions

  $$\partial M_{q,p} \subset C(p), \qquad p \in I(q), q \in Q$$

  be satisfied. That is, after a discrete transition one is found in an open set on which the dynamics are well-defined.

With these assumptions,[1] Tavernini proves that the initial value problem has a unique solution with finitely many switching points. Let $[x(\cdot), q(\cdot)]$ denote the solution corresponding to the initial value $[x_0, q_0]$. Then q defines a sequence of discrete states $q_0, q_1, q_2, \ldots$, with switching points $t_1, t_2, \ldots$, where t_i denotes the time of transition from q_{i-1} to q_i. If $[x'(\cdot), q'(\cdot)]$ denotes the solution when the initial value is $[x'_0, q_0]$ where x'_0 is near x_0, then we should have $[x'(\cdot), q'(\cdot)]$ "near" $[x(\cdot), q(\cdot)]$ in the sense that q' should define the same "discrete trajectory" $q_0, q_1, q_2, \ldots$ with possibly different switching points. However, the corresponding switching points of the two solutions should be close, i.e., $|t'_i - t_i|$ should be "small" whenever $|x'_0 - x_0|$ is "small." Tavernini defines a topology to make this precise. He also shows that the set of points with such a property is an open, dense subset of $C(q_0)$. Finally, Tavernini concentrates on the analysis of numerical approximations of the trajectories of differential automata. See [21] for details.

3.2 Back-Guckenheimer-Myers Model

The framework proposed by Back, Guckenheimer, and Myers in [2] is similar in spirit to the Tavernini model. The model is more general, however, in allowing "jumps" in the continuous state-space and setting of parameters when a switching boundary is hit. This is done through *transition functions* defined on the switching boundaries. Also, the model allows a more general state space.

More specifically, the model consists of a state space

$$S = \bigcup_{q \in Q} S_q, \qquad Q \simeq \{1, \ldots, N\},$$

where each S_q is a connected, open set of $\mathbb{R}^n$. Notice that the sets S_q are not required to be disjoint.

The continuous dynamics are given by vector fields $f_q : S_q \to \mathbb{R}^n$. Also, one has open sets U_q such that $\overline{U}_q \subset S_q$ and ∂U_q is piecewise smooth. For $q \in Q$, the transition functions

$$G_q : S_q \to S \times Q$$

govern the jumps that take place when the state in S_q hits ∂U_q. They must satisfy $\pi_1(G_q(x)) \in \overline{U}_{\pi_2(G_q(x))}$, where π_k is the kth coordinate projection function. Thus, $\pi_1(G_q(x))$ is the "continuous part" and $\pi_2(G_q(x))$ is the "discrete part" of the transition function.

The dynamics are as follows. The state starts at point x_0 in U_i. It evolves according to $\dot{x} = f_i(x)$. If $x(\cdot)$ hits some ∂U_i at time t_1, then the state instantaneously jumps to state ξ in $\overline{U}_j$, where $G(x(t_1)) = (\xi, j)$. From there, the process continues. We will refer to this as the BGM model. See Figure 5, which is taken from [2].

As in [21], it is assumed in [2] that the switching boundaries are fairly regular. In particular, it is assumed that the *switching boundaries* ∂U_q have a concrete representation in terms of the zeros of

$$h_q \equiv \min\{h_{q,1}, \ldots, h_{q,N_q}\}.$$

where the $h_{q,i} : S_q \to \mathbb{R}$ are smooth. The convention then is such that $h_q > 0$ on U_q. Thus, the switching boundaries are $(n-1)$-dimensional Lipschitz continuous manifolds. This does not add much power (over single functions) since Lipschitz functions are strongly approximated by C^1 functions: for every $\epsilon > 0$ a C^1 function can be chosen that coincides with a Lipschitz function except on a set of measure ϵ [17].

The model above is fairly expressive, allowing the modeling of a large variety of phenomena. However, its expressiveness does allow the possibility of some seemingly "anomalous" behavior. For example, since one allows jumping to the boundary of the sets U_i, trajectories may infinitely "cycle"

[1] Actually, the vector fields $f(\cdot, q)$ and switching functions $g_{q,p}$ are assumed to be smooth.

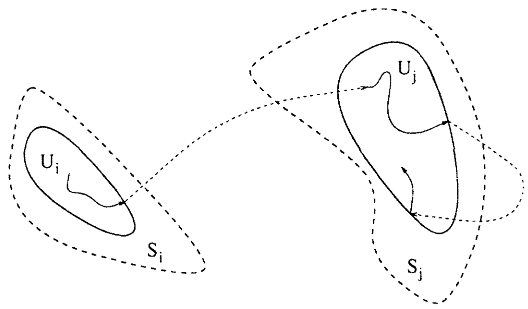

Figure 5: Example dynamics of the BGM model.

if $G(x \in S_j) = (y, i)$ and $G(y \in S_i) = (x, j)$. In a simulation facility, however, such conditions can presumably be detected and reported to the user for interpretation.

The paper [2] presents computer tools that have been developed by its authors for the simulation of hybrid systems. As an example, Raibert's one-legged hopping robot [20] is looked at under their framework.

3.3 Nerode-Kohn Model

In [19], Nerode and Kohn take an automata-theoretic approach to systems composed of interacting ODEs and FA. The basic philosophy of the models discussed in [19] is given in great generality, with a subsequent specialization to various cases, e.g., deterministic versus non-deterministic. To keep the discussion germane to that so far, we discuss here the so-called "event-driven, autonomous sequential deterministic model" [19, p. 331]. We will refer to it as the NKSD (for sequential deterministic) model. Here, autonomous refers to the fact that the ODEs do not explicitly depend on time, although this is without loss of generality by appending to the state a single equation for t.

The model consists of three basic parts: plant, digital control automaton, and interface. In turn, the interface is comprised of an analog-to-digital (AD) converter and digital-to-analog (DA) converter. See Figure 6.

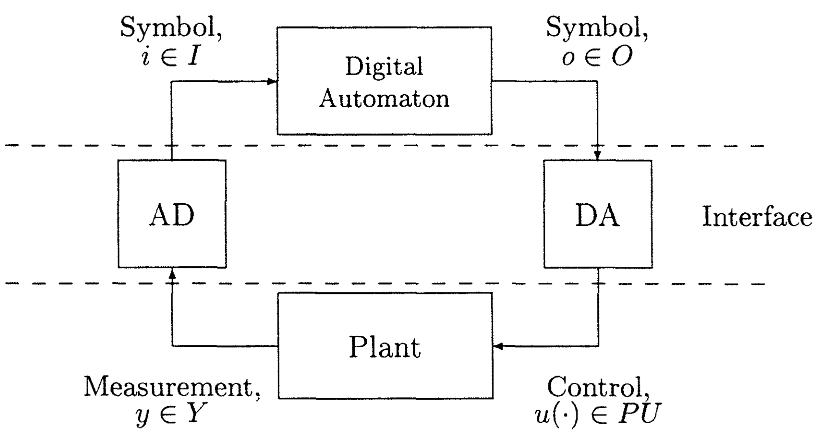

Figure 6: Hybrid system as in Nerode-Kohn model.

The plant is modeled as

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t))\end{aligned}$$

where $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $y \in Y \subset \mathbb{R}^p$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^p$.[2]

The plant is considered to be an input/output automaton in the following sense. The states of the system (in this sequential deterministic case) are merely the usual plant states, members of $\mathbb{R}^n$ [19, p. 333]. The input alphabet is formally taken to be the set of members of $(u(\cdot), \delta_k)$ where δ_k is a positive scalar and $u(\cdot)$ is a member of the set of piecewise right continuous functions in $U^{[0,\infty)}$. Let PU, for piecewise U, denote the latter set. Suppose the plant is in state x_k at time t_k. The "next state" of the transition function from this state with input symbol$(u(\cdot), \delta_k)$ is given by $x_{k+1} \equiv x(t_k + \delta_k)$, where $x(\cdot)$ is the solution on $[t_k, t_k + \delta_k]$ of

$$\dot{x}(t) = f(x(t), u(t - t_k)), \qquad x(t_k) = x_k.$$

Setting $t_{k+1} = t_k + \delta_k$, the process is continued.

The digital control automaton is a symbolic automaton as discussed in Section 2. In general, then, Q, I, and O are each isomorphic to subsets of $\mathbb{N}$. However, the interesting case is where these sets are finite, which is discussed below. As noted before, this automaton may be thought of as operating in "continuous time" by the convention that the state, input, and output symbols are piecewise right continuous functions, leading to 2.1.

It remains to couple these two "automata." This is done through the interface by introducing maps $AD : Y \times Q \to I$ and $DA : O \to PU$. The AD symbols are determined by (FA-state-dependent) partitions of the output space Y. These partitions are not allowed to be arbitrary, but are the "essential parts" of small topologies placed on Y for each $q \in Q$. We explain this later. To each $o \in O$ is associated an open set of PU. The DA signal corresponding to output symbol o is chosen from this open set of plant inputs. The scalar δ_k is a formal construct, denoting the time until the next "event." It is not actually computed or chosen by the digital automaton, nor is it actively used by the plant in computing its update equations.

The dynamics of the above model are then similar to those of the Tavernini model. Two important distinctions arise: input and output for both the ODEs and FA have been included, and the maps AD and DA have been added. Specifically, we have

$$\begin{aligned}\dot{x}(t) &= f[x(t), DA(o(t), t - [t])] \\ y(t) &= h[x(t)] \\ q(t) &= \nu[q(t^-), AD(y(t), q(t^-))] \\ o(t) &= \eta[q(t), AD(y(t), q(t^-))]\end{aligned}$$

Briefly, the combined dynamics is as follows. Assume the continuous state is evolving according to the first equation and that the FA is in state q. Then $AD(\cdot, q)$ assigns to output $y(t)$ a symbol from the input alphabet of the FA. When this symbol changes, the FA makes the associated state transition, causing a corresponding change in its output symbol o. Associated with this symbol is a control input, $DA(o)$, which is applied as input to the differential equation until the input symbol of the FA again changes.

Now, we explain what is meant by the "small topologies" mentioned above, concentrating on the AD map. Nerode and Kohn introduce topologies that make each mapping $AD_q \equiv AD(\cdot, q)$, $q \in Q$, continuous as follows:

[2] We have lumped the control and disturbance signals of [19] into a single signal u.

1. First, take any finite open cover of the output space: $Y = \bigcup_{i=1}^{d} \mathcal{A}_i$, where the $\mathcal{A}_i$ are open in the given topology of Y.

2. Next, find the so-called *small topology*, $\mathcal{T}_Y$, generated by the subbasis $\mathcal{A}_i$. This topology is finite and its open sets can be enumerated, say, as $\mathcal{B}_1, \ldots, \mathcal{B}_K$.

3. Next, find all the non-empty *join irreducibles* in the collection of the $\mathcal{B}_i$ (that is, all non-empty sets $\mathcal{B}_j$ such that if $\mathcal{B}_j = \mathcal{B}_k \cup \mathcal{B}_l$, then either $\mathcal{B}_j = \mathcal{B}_k$ or $\mathcal{B}_j = \mathcal{B}_l$). There are a finite number of such join irreducibles, denoted $\mathcal{C}_1, \ldots, \mathcal{C}_M$.

4. Without loss of generality, let the set of symbols be $I = \{1, \ldots, M\}$ and define the $AD_q(y) = i$ if $\mathcal{C}_i$ is the smallest open set containing y.

5. Create a topology, $\mathcal{T}_I$, on I as follows. For each $i \in I$, declare $\mathcal{D}_i = \{j \mid \mathcal{C}_j \subset \mathcal{C}_i\}$ to be open. Let $\mathcal{T}_I$ be the topology generated by the $\mathcal{D}_i$.

The sets $AD_q^{-1}(i)$, $i \in I$ are the essential parts mentioned above. For a verification that AD_q is continuous, as well as other results on AD and DA maps, see [6].

The starting point of the Nerode-Kohn approach is an assumption that one can only realistically distinguish points up to knowing the open sets in which they are contained. That is what led them to use the small topologies above to encode the plant output symbols. However, the bottom line is that by combining information of inclusion in different open sets, the AD_q functions, $q \in Q$, form partitions of the measurement space. Although the small topologies are meant to provide "reasonable partitions," it is interesting to note that one can still "identify" single points in the model: Consider as a representative example zero in $[-1, 1]$. Then the open sets $[-1, 1]$, $[-1, 0)$, and $(0, 1]$ give information to exactly deduce $x = 0$. Such anomalies lead to a breakdown of the description of the dynamics above in the sense that it is easy to construct examples where the formal input letter to the plant is $(u, 0)$.

The Nerode-Kohn paper develops the underpinning of a theoretical framework for the hybrid continuous/rule-based controllers used by Kohn in applications. Continuity in the small topologies associated with the AD and DA maps above plays a vital role in the theory of those controllers. See [19] and the references therein for details.

3.4 Antsaklis-Stiver-Lemmon Model

In [1], Antsaklis, Stiver, and Lemmon take a DEDS approach to hybrid systems. Conceptually, the model is related to that of Nerode-Kohn, but we quickly review it here. We will refer to it as the ASL model.

Like the NKSD model, the ASL model consists of three basic parts: the plant, the controller, and the interface. Again, see Figure 6. The plant is modeled as a time-invariant, continuous-time system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. Here $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$. The controller is a discrete event system, modeled as a symbolic automaton. We think of it as operating in continuous time as in Section 2:

$$\begin{aligned} q(t) &= \nu(q(t^-), i(t)) \\ o(t) &= \eta(q(t)) \end{aligned}$$

where $q(t) \in Q$, $i(t) \in I$, and $o(t) \in O$, the state space, plant symbols, and controller symbols, respectively. The sets Q, I, and O are unspecified in [1], but we take from context that they are each isomorphic to subsets of $\mathbb{N}$. The maps are $\nu : Q \times I \to Q$ and $\eta : Q \to O$. The subscript k denotes the kth symbol in a sequence. The output map does not depend on the current symbol, which is without loss of generality after adding more states.

The plant and controller communicate through an interface consisting of two memoryless maps, AD and DA. The first map, called the actuating function, $DA : O \to \mathbb{R}^m$, converts a controller symbol to a piecewise constant plant input:

$$u(t) = DA(o(t))$$

The second map, called the plant symbol generating function, $AD : \mathbb{R}^n \to I$, is a function which maps the plant state space to the set of plant symbols as follows

$$i(t) = AD(x(t))$$

The function AD is based upon a partition of the state space, where each element of the partition is associated with one plant symbol. The combined dynamics is similar to that of the NKSD model.

The model is simple but fairly general. The fact that arbitrary partitions are allowed limits what one can prove about the trajectories of this model. Several example systems are given in [1]. Results, mainly from the DEDS point of view, may be found in [1] and the references therein.

3.5 Brockett's Models

Several models of hybrid systems are described in [9]. We only discuss those which combine ODEs and discrete phenomena since that is our focus here. Two models combining difference equations and discrete phenomena are also discussed in [9].

The first model, which Brockett calls a type B hybrid system, is as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), v\lfloor p \rfloor) \\ \dot{p}(t) &= r(x(t), u(t), v\lfloor p \rfloor) \end{aligned}$$

where $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $p(t) \in \mathbb{R}$, $v\lfloor p \rfloor \in V$, $f : \mathbb{R}^n \times \mathbb{R}^m \times V \to \mathbb{R}^n$, and $r : \mathbb{R}^n \times \mathbb{R}^m \times V \to \mathbb{R}$. Here, X and U are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and V is isomorphic to a subset of $\mathbb{N}$. Also, the rate equation r is required to be nonnegative for all arguments, but need have no upper bound imposed upon it. We will denote such a system as BB, short for Brockett's type B model.

Brockett has mixed continuous and "symbolic" controls by the inclusion of the special "counter" variable p. The control $u(t)$ is the continuous control exercised at time t; the control $v\lfloor p \rfloor$ is the pth symbolic or discrete control, which is exercised at the times when p passes through integer values. In general, one may also introduce (as in [9]) continuous and symbolic output maps:

$$\begin{aligned} y(t) &= c(x(t), v\lfloor p \rfloor) \\ o\lfloor p \rfloor &= \eta(y[t], v\lfloor p \rfloor) \end{aligned}$$

In this case, one may limit f by allowing it to depend only on y instead of the full state x. Note, we have used $[t]$ to denote the value of t at which p most recently became an integer.

Brockett also introduces a type D hybrid system as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), z\lfloor p \rfloor) \\ \dot{p}(t) &= r(x(t), u(t), z\lfloor p \rfloor) \\ z\lceil p \rceil &= \nu(x[t], z\lfloor p \rfloor, v\lfloor p \rfloor) \end{aligned}$$

where $z \in Z$, and Z is isomorphic to a subset of $\mathbb{N}$. Here, $\nu : \mathbb{R}^n \times Z \times V \to Z$, with all other definitions as above except that Z replaces V in those for f and r. Again, u and v are the continuous and discrete

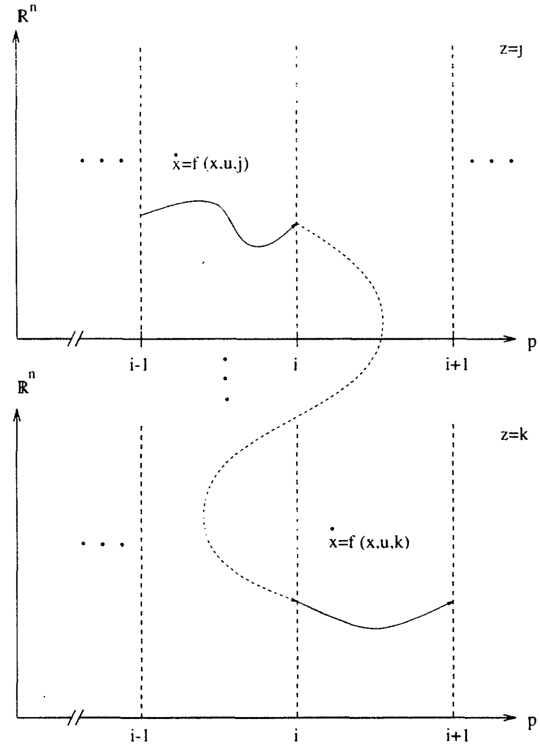

Figure 7: Dynamics of BD model.

controls, respectively. We will denote such a system by BD. We may picture the dynamics as in Figure 7.

The first equation denotes the continuous dynamics and the last equation the "symbolic processing" done by the system. The times when p passes through integer values can be thought of as the discrete event times of the hybrid dynamical system. Thus, we consider BD as a precise, first-order model of interactions of ODEs and DEDS. Once again one may introduce output equations:

$$\begin{aligned} y(t) &= c(x(t), z\lfloor p \rfloor) \\ o\lfloor p \rfloor &= \eta(y[t], z\lfloor p \rfloor) \end{aligned}$$

Finally, Brockett generalizes BD to the case of "hybrid system with vector triggering" (herein, BDV), in which one replaces the single rate and symbolic equations with a finite number of such equations:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), z\lfloor p \rfloor) \\ \dot{p}_i(t) &= r(x(t), u(t), z_i\lfloor p_i \rfloor) \\ z_i\lceil p_i \rceil &= \nu(x[t]_{p_i}, z\lfloor p \rfloor, v_i\lfloor p_i \rfloor) \end{aligned}$$

where $i \in \underline{k}$. Again, outputs may be introduced.

In order for, say, the BB system above to be well-posed, we would like there to exist a unique solution on finite interval $[0, T]$. That is, given $[x(0), p(0)] \in X \times \mathbb{R}$, there should exist unique $x(\cdot)$ and $p(\cdot)$, continuous and differentiable almost everywhere, satisfying the equations. Brockett meets these specifications on any interval in which p does not take on an integral value by requiring f to be

Lipschitz in x, continuous with respect to u, and u to have a finite number of discontinuities. [It is also necessary to assume U bounded]. The result extends if on any finite interval of time p passes through only a finite number of integers, leading to a finite number of discontinuities of the derivatives of x and p in finite time. In general, this requires similar continuity assumptions on r. Consider, for example, the case where $V = \mathbb{N}$, $v\lfloor p \rfloor = (\lfloor p \rfloor + 1)^2$, and $r = v\lfloor p \rfloor$. This leads to $\dot{p} = (\lfloor p \rfloor + 1)^2$, which has finite escape time. Analogous behavior for x results if u is not bounded. In the usual case, however, U, V, and Z are taken compact, avoiding such behavior. Similar discussion holds for models BD and BDV.

In [9], Brockett gives many examples of systems modeled with the above equations, including buffers, stepper motors, and transmissions (cf. Example 2.3).

3.6 Discussion

At the risk of oversimplification, Tavernini, NKSD, and ASL use autonomous switching; BGM uses autonomous switching and autonomous jumps; and BD uses a combination of autonomous and controlled switching.

It is not hard to see that the BGM model contains Tavernini's model. Simply choose $S_i = \mathbb{R}^n$, $U_i = \mathbb{R}^n \backslash M_i$, $i \in \underline{N}$, and $G(x) = (x, j)$ if $x \in M_{i,j}$. One may also show that the NKSD model contains the Tavernini model (see [7]).

From the control perspective, the Tavernini model is an autonomous system and the BGM is essentially so (although one can set parameters on jumps). The NKSD and ASL models focus on the "control automaton," coding the action of the controller in the mappings from continuous states to input symbols through automaton to output symbols and back to controls. Brockett's BD/BDV models allow the possibility of both continuous and discrete controls to be exercised as input to the continuous and symbolic dynamics of the systems, respectively. That is the plant not only responds to the state (or output) of the finite machine, but to continuous commands generated separately as well. One may argue that this is largely a matter of level of modeling. For instance, one can assume (as in NKSD and ASL) that the "low-level" loops have been closed, eliminating the continuous control from the design of the "high-level" ones. Nevertheless, the discussion herein is more in spirit with Brockett's approach.

From the original papers, it is clear that the models above were primarily developed for a variety of purposes: Tavernini and BGM for modeling and simulation, NKSD and ASL for controlling continuous systems with computer programs or "higher level controllers," and Brockett's for modeling the action of (hierarchical) motion control systems. Moreover, there is a direct trade-off between the generality of a model and what one can prove about such a model. Therefore, "containment" of one model in another does not reflect any bias of the more general model's being "superior." Indeed, in the next section we develop a very general, abstract model which captures many hybrid phenomena. Later, however, we place restrictions on this model in order to solve a related control problem.

4 Abstract Model

We first present our over-riding framework in generality. We will refine it later when we set up our control problem. Our state space for $x(\cdot)$ will be $S = \bigcup_{i=1}^{\infty} S_i$ where each S_i is a subset of some Euclidean space $\mathbb{R}^{d_i}$, $d_i \in \mathbb{N}$.[3] Notice that we allow the S_i to overlap and the inclusion of multiple copies of the same space. We also specify *a priori* regions $A_i, C_i, D_i \subset S_i$, $i \in \mathbb{N}$. These are the *autonomous jump sets*, *controlled jump sets*, and *jump destination sets*, respectively. Let A, C, D

[3]The state dimension may change to take into account component failures or change in dynamical description based on discrete events—controlled or autonomous—which change it, *e.g.*, the collision of two inelastic particles.

denote the unions $\bigcup_i A_i$, $\bigcup_i C_i$, $\bigcup_i D_i$, $i \in \mathbb{N}$, respectively. Let U, V be the sets of continuous and discrete controls, respectively. The following maps are assumed to be known:

1. *vector fields* $f_i : S_i \times S_i \times U \to \mathbb{R}^{d_i}$, $i \in \mathbb{N}$.
2. *transition map* $G : A \times V \to D$.
3. *transition delay* $\Delta_1 : A \times V \to \mathbb{R}^+$.
4. *impulse delay* $\Delta_2 : \bigcup_i (C_i \times D_i) \to \mathbb{R}^+$.

The dynamics of the control system can now be described as follows. There is a sequence of *pre-jump times* $\{\tau_i\}$ and another sequence of *post-jump times* $\{\Gamma_i\}$ satisfying $0 = \Gamma_0 \leq \tau_1 < \Gamma_1 < \tau_2 < \Gamma_2 < \cdots \leq \infty$, such that on each interval $[\Gamma_{j-1}, \tau_j)$ with non-empty interior, $x(\cdot)$ evolves according to (1.1) in some S_i, $i \in \mathbb{N}$. At the next pre-jump time (say, τ_j) it jumps to some $D_k \in S_k$ according to one of the following two possibilities:

1. $x(\tau_j) \in A_i$, in which case it must jump to $x(\Gamma_j) = G(x(\tau_j), v_j) \in D$ at time $\Gamma_j = \tau_j + \Delta_1(x(\tau_j), v_j)$, $v_j \in V$ being a control input. We call this phenomenon an autonomous jump.
2. $x(\tau_j) \in C_i$ and the controller chooses to[4] move the trajectory discontinuously to $x(\Gamma_j) \in D$ at time $\Gamma_j = \tau_j + \Delta_2(x(\tau_j), x(\Gamma_j))$. We call this an impulsive jump.

See Figure 8.

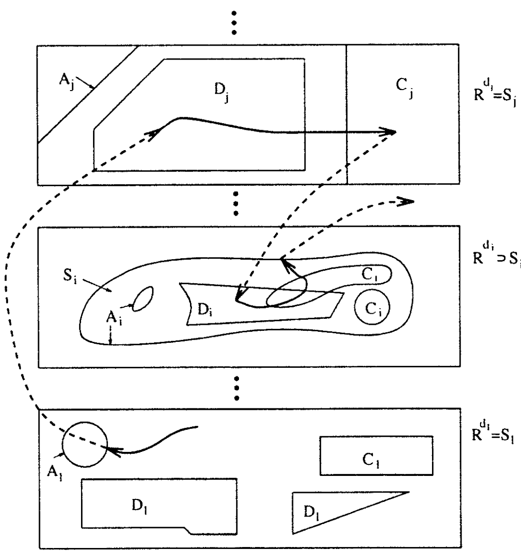

Figure 8: Example dynamics of our model.

For $t \in [0, \infty)$, let $[t] = \max_j \{\Gamma_j \mid \Gamma_j \leq t\}$. The vector field $\xi(t)$ of (1.1) is given by

$$\xi(t) = f_i(x(t), x[t], u(t)) \tag{4.1}$$

where i is such that $x(t), x[t] \in S_i$ and $u(\cdot)$ is a U-valued control process.

[4] It does not have to

To avoid confusion, the shorthand

$$G(x, v; i) = (x'; j)$$

will sometimes be used to explicitly denote the transition from $x \in A_i \subset S_i$ (with discrete control v) to $x' \in D_j \subset S_j$.

We will now show how this framework encompasses the discrete phenomenon of Section 2 and how it compares to the models of Section 3.

If a set of parameters or controls is countable and discrete, such as a set of strings, we may take it to be isomorphic with a subset of $\mathbb{N}$. On the other hand, consider a set of parameters or controls, U, where U is a compact, connected, locally connected metric space U. By the Hahn-Mazurkiewicz theorem [16], U is the continuous image of $[0, 1]$ under some map and thus we may set $U = [0, 1]$ without any loss of generality. Thus, we may assume below without any loss of generality that parameters and controls take values in a subset $P \subset \mathbb{R}^m$.

AUTONOMOUS SWITCHING. We show that autonomous switching can be viewed as a special case of autonomous jumps, which are taken care of next. Consider the differential equation with parameters

$$\dot{x} = f(x, p)$$

$x \in \mathbb{R}^n$, $p \in P \subset \mathbb{R}^m$ closed, $f : \mathbb{R}^n \times P \to \mathbb{R}^n$ continuous. Let, $\nu : \mathbb{R}^n \times P \to P$ be the function governing autonomous switching. For example, in the Tavernini model, ν is the "discrete dynamics."

Then, since $\mathbb{R}^n$ has the universal extension property [18], we can extend f to a continuous function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Now, consider the ODE on $\mathbb{R}^{n+m}$:

$$\begin{aligned} \dot{x} &= F(x, \xi) \\ \dot{\xi} &= 0 \end{aligned}$$

$x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous. Let, the transition function be $G : \mathbb{R}^n \times P \to \mathbb{R}^n \times P$ with $G(x, p) = (x, \nu(x, p))$.

AUTONOMOUS JUMPS. This is clearly taken care of with the sets A_i.

CONTROLLED SWITCHING. A system with controlled switching is described by

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \in \mathbb{R}^d$$

where $u(\cdot)$ is a piecewise constant function taking values in $U \subset \mathbb{R}^m$ and $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is a map with sufficient regularity. There is a strictly positive cost associated with the switchings of $u(\cdot)$. In our framework, let $x'(\cdot) = [x(\cdot),\ u(\cdot)]^T$ be the new state process with dynamics

$$\dot{x}'(t) = f'(x'(t)), \qquad f'(\cdot) = [f(\cdot),\ 0]^T$$

taking values in $S = \bigcup_{i=0}^{\infty} S_i$ where each S_i is a copy of $\mathbb{R}^d \times U$. Set $C_i = D_i = S_i$, $A_i = \emptyset$ for $i \in \mathbb{N}$. Switchings of $u(\cdot)$ now correspond to impulsive jumps with the associated costs.

CONTROLLED JUMPS. This is clearly taken care of with the sets C_i.

AUTOMATA. A variety of automata are automatically subsumed by inclusion of the Tavernini, BGM, NKSD, ASL, and Brockett models, which is demonstrated next.

TAVERNINI AND BGM MODELS. Let primed symbols denote those in our model with the same notation as those for BGM. It is obvious that our model includes the BGM model by choosing $S_i' = \overline{S_i}$, $A_i = \partial S_i \cup \partial U_i$, $D_i = \overline{U_i}$, $C_i = \emptyset$, and

$$G'(x;j) = (\pi_1(G(x));\ \pi_2(G(x)))$$

for $x \in U_j$. G' need not be defined on $\partial S_j \backslash \partial U_j$, but for completeness we may define $G'(x;j) = (x;j)$ for $x \in \partial S_j \backslash \partial U_j$. Since BGM contains Tavernini's model, our model does as well.

NKSD AND ASL MODELS. Our model includes the ASL model. First, choose $S_i = \mathbb{R}^n \times \mathbb{R}^3$, $i \in I$. Then, note that the sets $AD^{-1}(i)$, $i \in I$, form a partition of Y. Define the sets $M_j = h^{-1}(AD^{-1}(j))$ and define

$$A_i \equiv \bigcup_{j \neq i, j \in I} M_j$$

The define

$$\tilde{f}_i = [f,\ 0,\ 0,\ 0]$$

with dimensions representing x, q, i, and o. The model is complete by specifying

$$G(x;i) = (x, \nu(q,j), j, \eta(q); j)$$

if $x \in M_j \subset A_i$.

Inclusion of NKSD is similar. However, since the resulting partitions depend on q one must use multiple copies of S_i and $\tilde{f}_i$ as above, one for each AD_q, $q \in Q$. We must also append the state t to the state vector S (and use $[t]$), with the obvious differential equations/transitions. Finally, η depends on both q and j in this case.

BROCKETT'S MODELS. Our model includes Brockett's BD model by choosing $S = \mathbb{R}^n \times \mathbb{R}^4$ and defining

$$\tilde{f} = [f,\ r,\ 0,\ 0,\ 0]$$

with dimensions representing x, $q = p - \lfloor p \rfloor$, $i = \lfloor p \rfloor$, v, and z. Also, set $A = \mathbb{R}^n \times \{1\} \times \mathbb{R}^3$, $D = \mathbb{R}^n \times \{0\} \times \mathbb{N}^3$, and $G((x,1,i,v,z),v') = (x,0,i+1,v',\nu(x,z,v))$. BB is seen to be included in the same manner, but removing the state dimension for z. It is clear that this can be extended to include BDV.

SETTING PARAMETERS AND TIMERS. A system which, upon hitting boundaries, sets parameters from an arbitrary compact set $P \subset \mathbb{R}^p$ can be modeled in our framework by redefining $S_i' = S_i \times \mathbb{R}^p$, $V' = V \times P$ and defining $f_i' : S_i \times S_i \times U \to \mathbb{R}^{d_i} \times \mathbb{R}^p$ as

$$f_i'(x,p,y,q,u) = [f_i(x,y,u),\ 0]^T$$

and $G' : A \times P \times V \times P \to D \times P$ as

$$G'(x,p,v,p') = [G(x,v),\ p']^T$$

each for all possible arguments. Likewise, one can redefine the switching cost and delay appropriately.

A system which sets timers upon hitting boundaries can be modeled by a vector of the rate equations in Brockett's BDV model of hybrid systems, which in turn can be modeled in our framework as previously discussed.

5 The Control Problem

In this section, we define a control problem and elucidate all assumptions used in deriving the results of the sequel.

5.1 Problem

Let $a > 0$ be a *discount factor.* We add to our previous model the following known maps:

1. *running cost* $k : S_i \times S_i \times U \to \mathbb{R}^+$.
2. *transition cost* $c_1 : A \times V \to \mathbb{R}^+$.
3. *impulse cost* $c_2 : \bigcup_i (C_i \times D) \to \mathbb{R}^+$, satisfying for all $i, j \in \mathbb{N}$ the conditions

$$c_2(x, y) \geq c_0 > 0, \qquad \forall x \in C_i,\ y \in D \tag{5.1}$$

$$c_2(x, y) < c_2(x, z) + e^{-a\Delta_2(x,z)} c_2(z, y), \quad \forall x \in C_i,\ z \in D \cap C_j,\ y \in D \tag{5.2}$$

Thus, autonomous jumps are done at a cost of $c_1(x(\tau_j), v_j)$ paid at time τ_j; impulsive jumps at a cost of $c_2(x(\tau_j), x(\Gamma_j))$ paid at time τ_j.

In addition to the costs associated with the jumps as above, the controller also incurs a *running cost* of $k(x(t), x[t], u(t))$ per unit time during the intervals $[\Gamma_{j-1}, \tau_j)$, $j \in \mathbb{N}$. The total discounted cost is defined as

$$\int_{\mathbb{T}} e^{-at} k(x(t), x[t], u(t))\, dt + \sum_i e^{-a\sigma_i} c_1(x(\sigma_i), v_i) + \sum_i e^{-a\zeta_i} c_2(x(\zeta_i), x(\zeta_i')) \tag{5.3}$$

where $\mathbb{T} = \mathbb{R}^+ \backslash (\bigcup_i [\tau_i, \Gamma_i))$, $\{\sigma_i\}$ (respectively $\{\zeta_i\}$) are the successive pre-jump times for autonomous (respectively impulsive) jumps and ζ_j' is the post-jump time for the jth impulsive jump. The *decision* or *control* variables over which (5.3) is to be minimized are the continuous control $u(\cdot)$, the discrete control $\{v_i\}$ exercised at the pre-jump times of autonomous jumps, the pre-jump times $\{\zeta_i\}$ of impulsive jumps, and the associated *destinations* $\{x(\zeta_i')\}$. As for the periods $[\tau_j, \Gamma_j)$, we shall follow the convention that the system remains frozen during these intervals. Note that (5.1) rules out from consideration infinitely many impulsive jumps in a finite interval and (5.2) rules out the merging of post-jump time of an impulsive jump with the pre-jump time of the next impulsive jump.

Our framework clearly includes conventional impulse control [4].

5.2 Assumptions

Throughout the sequel, we make use of the following further assumptions on our abstract model, which are collected here for clarity and convenience.

For each $i \in \mathbb{N}$, the following hold: S_i is the closure of a connected open subset of Euclidean space $\mathbb{R}^{d_i}$, $d_i \in \mathbb{N}$, with Lipschitz boundary ∂S_i. $A_i, C_i, D_i \subset S_i$ are closed. In addition, ∂A_i is Lipschitz and contains ∂S_i.

The maps G, Δ_1, Δ_2, c_1, c_2, and k are bounded uniformly continuous. The vector fields f_i, $i \in \mathbb{N}$, are bounded (uniformly in i), uniformly Lipschitz continuous in the first argument, uniformly equicontinuous with respect to the rest. U, V are compact metric spaces. Below, $u(\cdot)$ is a U-valued control process, assumed to be measurable.

All the above are fairly mild assumptions. The following are more technical assumptions. They may be traded for others as discussed in Section 8. However, in the sequel we construct examples pointing out the necessity of such assumptions or ones like them.

Assumption 1 *$d(A_i, C_i) > 0$ and $\inf_{i \in \mathbb{N}} d(A_i, D_i) > 0$, d being the appropriate Euclidean distance.*

Assumption 2 *Each D_i is bounded and for each i, there exists an integer $N(i) < \infty$ such that for $x \in C_i$, $y \in D_j$, $j > N(i)$, $c_2(x, y) > \sup_z J(z)$.*

Assumption 3 *For each i, ∂A_i is an oriented C^1-manifold without boundary and at each point x on ∂A_i, $f_i(x, z, u)$ is "transversal" to ∂A_i for all choices of z, u. By this we require that (1) the flow lines be transversal in the usual sense[5] and (2) the vector field does not vanish on ∂A_i.*

Assumption 4 *Same as Assumption 3 but with C_i replacing A_i.*

6 Existence of Optimal Controls

Let $J(x)$ denote the infimum of (5.3) over all choices of $u(\cdot), \{v_i\}, \{\zeta_i\}, \{x(\zeta_i')\}$ when $x(0) = x$. We have

Theorem 6.1 *A finite optimal cost exists for any initial condition.*

Proof Let F, K, Q be bounds of the f_i, k, and c_1, respectively. Then, choosing to make no controlled jumps and using arbitrary u, v we have that

$$J(x) \leq K \int_0^\infty e^{-at}\, dt + \sum_i e^{-a\sigma_i} Q \leq K/a + Q \sum_i e^{-a\sigma_i}$$

Let $\beta = \inf_{i \in \mathbb{N}} d(A_i, D_i)$. Then $\sigma_{i+1} - \sigma_i \geq \beta/F$, so the second term is bounded by $Q \sum_{i=1}^\infty (e^{-a\beta/F})^i$, which converges. $\square$

The following corollary is immediate from the argument above:

Corollary 6.2 *There are only finitely many autonomous jumps in finite time.*

To see why an assumption like Assumption 1 is necessary for the above results, one need only consider the following one-dimensional example:

Example 6.3 *Let $S_i = [0, 2]$, $A_i = \{0, 2\}$, and $f_i(\cdot, \cdot, \cdot) \equiv -1$ for each $i \in \mathbb{N}$. Also for each i, define $C_i = \emptyset$, $D_i = 1/i^2$ and $G(A_i, \cdot) \equiv 1/(i+1)^2$. Finally, let $\Delta_1(\cdot, \cdot) \equiv 0$ and $c_1(\cdot, \cdot) \equiv 1$. Starting in S_1 at $x(0) = 1$, we see that*

$$x\left(\sum_{i=1}^N \frac{1}{i^2}\right) = \frac{1}{(N+1)^2}$$

Since the sum of inverse squares converges, we will accumulate an infinite number of jumps and infinite cost by time $t = \pi^2/6$.

Next, we show that $J(x)$ will be attained for all x if we extend the class of admissible $u(\cdot)$ to "relaxed" controls. The "relaxed" control framework [23] is as follows: We suppose that $U = \mathcal{P}(U')$, defined as the space of probability measures on a compact space U' with the topology of weak convergence [5]. Also

$$\begin{aligned} f_i(x, z, u) &= \int f_i'(x, z, u) u(dy), \qquad i \in \mathbb{N} \\ k(x, z, u) &= \int k'(x, z, u) u(dy) \end{aligned}$$

for suitable $\{f_i'\}$, k' satisfying the appropriate continuity/Lipschitz continuity requirements. The relaxed control framework and its implications in control theory are well known and the reader is referred to [23] for details.

[5] Transversality implies that ∂A_i is $(d_i - 1)$-dimensional.

Theorem 6.4 *An optimal trajectory exists for any initial condition.*

Proof Fix $x(0) = x_0 \in S_{i_0}$, $i_0 \in \mathbb{N}$. Consider a sequence

$$(x^n(\cdot), u^n(\cdot), \{v_i^n\}, \{\sigma_i^n\}, \{\zeta_i^n\}, \{\tau_i^n\}, \{\Gamma_i^n\}), \qquad n \in \mathbb{N},$$

associated with our control system, with the obvious interpretation, such that $x^n(0) = x_0$ for all n and the corresponding costs decrease to $J(x_0)$. Let $y^n(\cdot)$ denote the solution of

$$\dot{y}^n(t) = f_{i_0}(y^n(t), x_0, u^n(t)), \qquad y^n(0) = x_0, n \in \mathbb{N} \tag{6.1}$$

Then $x^n(\cdot)$, $y^n(\cdot)$ agree on $[0, \tau_1^n)$. Since $\{f_i\}$ are bounded, $\{y^n(\cdot)\}$ are equicontinuous bounded in $C(\mathbb{R}^+; \mathbb{R}^{d_{i_0}})$, hence relatively sequentially compact by the Arzela-Ascoli theorem. The finite nonnegative measures $\eta^n(dt, dy) = dt\ u^n(t, dy)$ on $[0,T] \times U'$ are relatively sequentially compact in the topology of weak convergence by Prohorov's theorem [5]. $\{\tau_1^n\}$ are trivially relatively compact in $[0, \infty]$. Thus dropping to a subsequence if necessary, we may suppose that $y^n(\cdot) \to y^\infty(\cdot)$, $\eta^n(dt, dy) \to \eta^\infty(dt, dy)$, $\tau_1^n \to \tau_1^\infty$ in the respective spaces. Clearly η^∞ disintegrates as $\eta^\infty(dt, dy) = dt\ u^\infty(t, dy)$. Rewrite (6.1) as

$$y^n(t) = x_0 + \left(\int_0^t f_{i_0}(y^n(s), x_0, u^n(s)) - f_{i_0}(y^\infty(s), x_0, u^n(s))\, ds\right) + \int_0^t f_{i_0}(y^\infty(s), x_0, u^n(s))\, ds$$

for $t \geq 0$. By the uniform Lipschitz continuity of f_{i_0}, the term in parentheses tends to zero as $n \to \infty$. Since $\eta^n \to \eta^\infty$, the last term, in view of the relaxed control framework, converges to

$$\int_0^t f_{i_0}(y^\infty(s), x_0, u^\infty(s))\, ds$$

for $t \in [0, T]$. Since T was arbitrary a standard argument allows us to extend this claim to $t \in [0, \infty)$. (We use [5, Theorem 2.1(v), p. 12] and the fact that $\eta^\infty(\{t\} \times U') = 0$.) Hence $y^\infty(\cdot)$, $u^\infty(\cdot)$ satisfy (6.1) with $n = \infty$. Since $d(C_{i_0}, A_{i_0}) > 0$, either $\tau_i^n = \sigma_i^n$ for sufficiently large n, or $\tau_i^n = \zeta_i^n$ for sufficiently large n. Suppose the first possibility holds. Then $y^\infty(\tau_1^\infty) = \lim x^n(\tau_1^n) \in A_{i_0}$. Let $v_i^n \to v_i^\infty$ in V along a subsequence. Then $c_1(x^n(\tau_1^n), v_1^n) \to c_1(y^\infty(\tau_1^\infty), v_1^\infty)$, $\Delta_1(x^n(\tau_1^n), v_1^n) \to \Delta_1(y^\infty(\tau_1^\infty), v_1^\infty)$, $\Gamma_1^n \to \Gamma_1^\infty \equiv \tau_1^\infty + \Delta_1(y^\infty(\tau_1^\infty), v_1^\infty)$. Set $x^\infty(\cdot) = y^\infty(\cdot)$ on $[0, \tau_1^\infty]$ and $x^\infty(\Gamma_1^\infty) = G(x^\infty(\tau_1^\infty), v_1^\infty)$. Then

$$\int_0^{\tau_1^n} e^{-at} k(x^n(t), x_0, u^n(t))\, dt \to \int_0^{\tau_1^\infty} e^{-at} k(x^\infty(t), x_0, u^\infty(t))\, dt \tag{6.2}$$

If the second possibility holds instead, one similarly has $y_1(\tau_1^\infty) \in C_{i_0}$. Then Assumption 2 ensures that $\{x^n(\Gamma_1^n)\}$ is a bounded sequence in D and hence converges along a subsequence to some $y' \in D$. Then, on dropping to a further subsequence if necessary, $c_2(x^n(\tau_1^n), x^n(\Gamma_1^n) \to c_2(y^\infty(\tau_1^n), y')$, $\Delta_2(x^n(\tau_1^n), x^n(\Gamma_1^n)) \to \Delta_2(y^\infty(\tau_1^n), y')$. Set $x^\infty(\cdot) = y^\infty(\cdot)$ on $[0, \tau_1^\infty]$, $\Gamma_1^\infty = \tau_1^\infty + \Delta_2(y^\infty(\tau_1^\infty), y')) = \lim \Gamma_1^n$ and $x^\infty(\Gamma_1^\infty) = y'$. Again (6.2) holds. Note that in both cases, $x^\infty(\cdot)$ defined on $[0, \tau_1^\infty]$ is an admissible segment of a controlled trajectory for our system. The only way it would fail to be so is if it hit A_{i_0} in $[0, \tau_1^\infty)$. If so, $x^n(\cdot)$ would have to hit A_{i_0} in $[0, \tau_1^n)$ for sufficiently large n by virtue of Assumption 3, a contradiction.

Now repeat this argument for $\{x^n(\Gamma_1^\infty + \cdot)\}$ in place of $\{x^n(\cdot)\}$. The only difference is a varying but convergent initial condition instead of a fixed one, which causes only minor alterations in the proof. Iterating, one obtains an admissible trajectory $x^\infty(\cdot)$ with cost $J(x_0)$. □

It is easy to see why Theorem 6.4 may fail in absence of Assumption 2:

Example 6.5 *Suppose, for example,* $k(x, z, u) \equiv \alpha_i$ *and* $c_1(x, v) \equiv \beta_i$ *when* $x \in S_i$, $c_2(x, y) \equiv \gamma_{i,j}$ *when* $x \in S_i$, $y \in S_j$, *with* α_i, β_i, $\gamma_{i,j}$ *strictly decreasing with* i, j. *It is easy to conceive of a situation where the optimal choice would be to "jump to infinity" as fast as you can.*

The theorem may also fail in the absence of Assumption 3 as the following two-dimensional system shows:

Example 6.6

$$\begin{aligned} \dot{x}_1(t) &= 1, \qquad x_1(0) = 0 \\ \dot{x}_2(t) &= u, \qquad x_2(0) = 0 \end{aligned}$$

with $u \in [0, 1]$ *and cost*

$$\int_0^\infty e^{-t} \min\{|x_1(t) + x_2(t)|, 10^{20}\}\, dt$$

with the provision that the trajectory jumps to $[10^{10}, 10^{10}]$ *on hitting a certain curve* A. *For* A, *consider two possibilities:*

1. *the line segment* $\{x_1 = 1, -1 \leq x_2 \leq 0\}$, *a* C^1*-manifold with boundary;*
2. *the circle* $\{(x_1, x_2) \mid (x_1 - 1)^2 + (x_2 + 1)^2 = 1\}$, *a* C^1*-manifold without boundary, but the vector field* $(1, u)$ *with* $u = 0$ *is not transversal to it at* $(1, 0)$.

It is easy to see that the optimal cost is not attained in either case.

Also, it is not enough that the flow lines for each control be transversal in the usual sense as the following one-dimensional example shows:

Example 6.7 *Let* $S_1 = S_2 = \mathbb{R}^+$.

$$f_1(x, y, u) = -x + u, \qquad f_2(x, y, u) = 0, \qquad u \in [-1, 0]$$

with running cost $\min\{K, |x|\}$ *and* $G(0, \cdot; 1) \equiv (K; 2)$. *Choosing, for example,* $K > 1$ *one sees that the optimal cost cannot be attained for any* $1 \geq x(0) > 0$.

Coming back to the relaxed control framework, say that $u(\cdot)$ is a *precise* control if $u(\cdot) = \delta_{q(\cdot)}(dy)$ for a measurable $q : [0, \infty) \to U'$ where δ_z denotes the Dirac measure at $z \in U'$. Let M denote the set of measures on $[0, T] \times U'$ of the form $dt\, u(t, dy)$ where $u(\cdot)$ is a relaxed control, and M_0 its subset corresponding to precise controls. It is known that M_0 is dense in M with respect to the topology of weak convergence [23]. In conjunction with the additional assumption Assumption 4 below, this allows us to deduce the existence of ϵ-optimal control policies using precise $u(\cdot)$, for every $\epsilon > 0$.

Theorem 6.8 *Under Assumptions* 2–4, *for every* $\epsilon > 0$ *an* ϵ*-optimal control policy exists wherein* $u(\cdot)$ *is precise.*

Proof Recall the setup of Theorem 6.4. Consider the time interval $[0, \tau_1^\infty]$. Let $\overline{u}^n(\cdot)$, $n \in \mathbb{N}$, be precise controls such that $dt\, \overline{u}^n(t, dy) \to \eta^\infty(dt, dy) = dt\, u^\infty(t, dy)$ in the topology of weak convergence. Let $\overline{y}^n(\cdot)$, $n \in \mathbb{N}$, denote the corresponding solutions to (6.1). Now τ_1^∞ equals either σ_1^∞ or ζ_1^∞. Suppose the former holds. As in the proof of Theorem 6.4, we have $\overline{y}^n \to y^\infty(\cdot)$ in $C([0, \infty), S_{i_0})$. Using Assumption 3 as in the proof of Theorem 6.4, one verifies that

$$\overline{\sigma}_1^n \equiv \inf\{t \geq 0 \mid \overline{y}^n(t) \in A_{i_0}\} \to \sigma_1^\infty$$

Thus for any $\delta > 0$, we can take n large enough such that

$$\begin{aligned} |\sigma_1^\infty - \overline{\sigma}_1^n| &< \delta \\ \sup\{\|\overline{y}^n(t) - y^\infty(t)\| \mid 0 \le t < \sigma_1^\infty \vee \overline{\sigma}_1^n\} &< \delta \\ |\overline{\sigma}_1^n + \Delta_1(\overline{y}^n(\overline{\sigma}_1^n), v_1^\infty) - \Gamma_1^\infty| &< \delta \end{aligned}$$

Set $\overline{x}^n(\cdot) = \overline{y}^n(\cdot)$ on $[0, \overline{\sigma}_1^n]$ and $\overline{x}^n(\overline{\sigma}^n + \Delta_1(\overline{x}^n(\overline{\sigma}_1^n), v_1^\infty)) = G(\overline{x}^n(\overline{\sigma}_1^n), v_1^\infty)$ (corresponding to control action v_1^∞). The latter may be taken to lie in the open δ-neighborhood of $x^\infty(\Gamma_1^\infty)$ by further increasing n if necessary. In case $\tau_1^\infty = \zeta_1^\infty$, one uses Assumption 4 instead to conclude that $\overline{y}^n(t'_n) \in C_{i_0}$ for some t'_n in the δ-neighborhood of τ_1^∞ for n sufficiently large. Set $\overline{\zeta}_1^n = t'_n$, $\overline{x}^n(\cdot) = \overline{y}^n(\cdot)$ on $[0, \overline{\zeta}^n)$. By further increasing n if necessary, we may also ensure that

$$\begin{aligned} \{\overline{x}^n(t) \mid t \in [0, \overline{\zeta}_1^n)\} \cap A_{i_0} &= \emptyset \\ \sup\{\|\overline{y}^n(t) - x^\infty(t)\| \mid 0 \le t \le \overline{\zeta}_1^n \wedge \overline{\zeta}_1^\infty\} &< \delta \\ |\overline{\zeta}_1^n + \Delta_2(\overline{x}^n(\overline{\zeta}_1^n), x^\infty(\Gamma_1^\infty)) - \Gamma_1^\infty| &< \delta \end{aligned}$$

Set

$$\overline{x}^n(\overline{\zeta}_1^n + \Delta_2(\overline{x}^n(\overline{\zeta}_1^n), x^\infty(\Gamma_1^\infty))) = x^\infty(\Gamma_1^\infty)$$

It is clear how to repeat the above procedure on each interval between successive jump times to construct an admissible trajectory $\overline{x}^n(\cdot)$ with cost within ϵ of $J(x_0)$ for a given $\epsilon > 0$. □

Remarks. If $\{\overline{f}_i(x, z, y) \mid y \in U'\}$ are convex for each x, z, a standard selection theorem [23] allows us to replace $u^\infty(\cdot)$ by a precise control which will then be optimal. Even otherwise, using Caratheodory's theorem (which states that each point in a compact subset of $\mathbb{R}^n$ is expressible as a convex combination of at most $n+1$ of its extreme points) and the aforementioned selection theorem, one may suppose that for $t \ge 0$, the support of $u^\infty(t)$ consists of at most $d_i + 1$ points when $x(t) \in S_i$.

7 The Value Function

In the foregoing, we had set $[0] = 0$ and thus $x[0] = x(0) = x_0$. More generally, for $x(0) = x_0 \in S_{i_0}$, we may consider $x[0] = y$ for some $y \in S_{i_0}$, making negligible difference in the foregoing analysis. Let $V(x, y)$ denote the optimal cost corresponding to this initial data. Then in dynamic programming parlance, $(x, y) \mapsto V(x, y)$ defines the "value function" for our control problem.

In view of Assumption A3, we can speak of the *right side* of ∂A_i as the side on which $f_i(\cdot,\cdot,\cdot)$ is directed towards ∂A_i, $i \in \mathbb{N}$. A similar definition is possible for the right side of ∂C_i (in light of Assumption A4).

Definition 7.1 (From the right) *Say that* $(x_n, y_n) \to (x_\infty, y_\infty)$ *from the right in* $\bigcup_i (S_i \times S_i)$ *if* $y_n \to y_\infty$ *and either* $x_n \to x_\infty \notin \bigcup_i(\partial A_i \cup \partial C_i)$ *or* $x_n \to x_\infty \in \bigcup_i(\partial A_i \cup \partial C_i)$ *from the right side.*

V is said to be *continuous from the right* if $(x_n, y_n) \to (x_\infty, y_\infty)$ from the right implies $V(x_n, y_n) \to V(x_\infty, y_\infty)$.

Theorem 7.2 *V is continuous from the right.*

Proof Let $(x_n, y_n) \to (x_\infty, y_\infty)$ from the right in $\bigcup_i(S_i \times S_i)$ and let $\overline{x}^n(\cdot)$, $n \in \mathbb{N} \cup \{\infty\}$, denote optimal trajectories for initial data (x_n, y_n) respectively. By dropping to a subsequence of $n \in \mathbb{N}$ if necessary, obtain as in Theorem 6.4 a limiting admissible trajectory $x'(\cdot)$ for initial data (x_∞, y_∞) with cost (say) α such that $V(x_n, y_n) \to \alpha \ge V(x_\infty, y_\infty)$. Suppose $\alpha > V(x_\infty, y_\infty) + 3\epsilon$ for some $\epsilon > 0$. Starting from $x^\infty(\cdot)$, argue as in Theorem 6.8 to construct a trajectory $\tilde{x}^n(\cdot)$ with initial data (x_n, y_n)

for n sufficiently large, so that the corresponding cost does not exceed $V(x_\infty, y_\infty) + \epsilon$. At the same time, $V(x_n, y_n) \geq \alpha - \epsilon > V(x_\infty, y_\infty) + 2\epsilon$ for n sufficiently large, which contradicts the fact that $V(x_n, y_n)$ is the optimal cost for initial data (x_n, y_n). The claim follows. □

Again, Example 6.7 shows the necessity of the vector field's not vanishing on ∂A_i.

We shall now formally derive the *generalized quasi-variational inequalities* $V(\cdot,\cdot)$ is expected to satisfy. Let $\mathcal{C} = \bigcup_i (C_i \times S_i)$ and $E \subset \mathcal{C}$ the set on which

$$V(x,y) = \min_{z \in D} \left\{ c_2(x,z) + e^{-a\Delta_2(x,z)} V(z,z) \right\} \tag{7.1}$$

where $i \in \mathbb{N}$ is such that $x, y \in S_i$. For $(x,y) \in E$, if $x(t) = x$ and $x[t] = y$, an optimal decision (not necessarily the only one) would be to jump to a z where the minimum on the right hand side of (7.1) is obtained. On the other hand, for $(x,y) \in \mathcal{C} \backslash E$,

$$V(x,y) < \min_{z \in D} \left\{ c_2(x,z) + e^{-a\Delta_2(x,z)} V(z,z) \right\}$$

with i as above and it is not optimal to execute an impulsive jump. For $x \in A_i$, however, an autonomous jump is mandatory and thus

$$V(x,y) = \min_{v} \left\{ c_1(x,v) + e^{-a\Delta_1(x,v)} V(G(x,v), G(x,v)) \right\}$$

Suppose E is a closed subset of $\bigcup_i (S_i \times S_i)$. Let $H = E \cup (\bigcup_i (A_i \times S_i))$, with $M = (\bigcup_i (S_i \times S_i)) \backslash H$. Let $(x,y) \in M^o$, with $x, y \in S_{i_0}$ (say). Let O be a bounded open neighborhood of (x,y) in M^o with a smooth boundary ∂O and $\nu = \inf\{t \geq 0 \mid (x(t), y) \notin O\}$, where $x(\cdot)$ satisfies

$$\dot{x}(t) = f_{i_0}(x(t), y, u(t)), \qquad x(0) = x, t \in [0, \nu] \tag{7.2}$$

Note that y is a fixed parameter here. By standard dynamic programming arguments, $V(x,y)$, $x \in \overline{O}$, y as above, is also the value function for the "classical" control problem of controlling (7.2) on $[0, \nu]$ with cost

$$\int_0^\nu e^{-at} k(x(t), y, u(t))\, dt + e^{-a\nu} h(x(\nu), y)$$

where $h(\cdot,\cdot) \equiv V(\cdot,\cdot)$ on ∂O. It follows that $V(x,y)$, $(x,y) \in O$ is the viscosity solution of the Hamilton-Jacobi equation for this problem [11], *i.e.*, it must satisfy (in the sense of viscosity solutions) the p.d.e.

$$\min_u \{ \langle \nabla_x V(x,y), f_{i_0}(x,y,u) \rangle - aV(x,y) + k(x,y,u) \} = 0 \tag{7.3}$$

in O and hence on M^o. (Here ∇_x denotes the gradient in the x variable.) Elsewhere, standard dynamic programming heuristics suggest that (7.3) holds with '=' replaced by '≤'.

Based on the foregoing discussion, we propose the following system of *generalized quasi-variational inequalities* for $V(\cdot,\cdot)$: For $(x,y) \in S_i \times S_i$,

$$V(x,y) \leq \min_{z \in D} \left\{ c_2(x,z) + e^{-a\Delta_2(x,z)} V(z,z) \right\} \text{ on } \mathcal{C} \tag{7.4}$$

$$V(x,y) \leq \min_{v} \left\{ c_1(x,v) + e^{-a\Delta_1(x,v)} V(G(x,v), G(x,v)) \right\} \text{ on } \bigcup_i (A_i \times S_i) \tag{7.5}$$

$$\min_u \{ \langle \nabla_x V(x,y), f_i(x,y,u) \rangle - aV(x,y) + k(x,y,u) \} \leq 0 \tag{7.6}$$

and

$$\left(V(x,y) - \min_{z \in D} \left\{ c_2(x,z) + e^{-a\Delta_2(x,z)} V(z,z) \right\} \right) \cdot \left(\min_u \{ \langle \nabla_x V(x,y), f_i(x,y,u) \rangle - aV(x,y) + k(x,y,u) \} \right) = 0 \text{ on } \mathcal{C} \tag{7.7}$$

((7.7) states that at least one of (7.4), (7.6) must be an equality on $\mathcal{C}$.) (7.4)–(7.7) generalize the traditional *quasi-variational inequalities* encountered in impulse control [4]. We do not address the issue of well-posedness of (7.4)–(7.7). The following "verification theorem," however, can be proved by routine arguments.

Theorem 7.3 *Suppose* (7.4)–(7.7) *has a "classical" solution* V *which is continuously differentiable "from the right" in the first argument and continuous in the second. Suppose* $x(\cdot)$ *is an admissible trajectory of our control system with initial data* (x_0, y_0) *and* $u(\cdot)$, $\{v_i\}$, $\{\sigma_i\}$, $\{\zeta_i\}$, $\{\tau_i\}$, $\{\Gamma_i\}$ *the associated controls and jump times, such that the following hold:*

1. *For a.e.* $t \in \mathbb{T}$, i *such that* $x(t) \in S_i$,

$$\langle \nabla_x V(x(t), x[t]), f_i(x(t), x[t], u(t))\rangle + k(x(t), x[t], u(t)) = \min_u \{\langle \nabla_x V(x(t), x[t]), f_i(x(t), x[t], u)\rangle + k(x(t), x[t], u)\}$$

2. *For all* i,

$$V(x(\sigma_i), x[\sigma_i]) = c_1(x(\sigma_i), v_i) + \exp\{-a\Delta_1(x(\sigma_i), v_i)\}V(G(x(\sigma_i), v_i), G(x(\sigma_i), v_i))$$

3. *For all* i,

$$V(x(\zeta_i), x[\zeta_i]) = c_2(x(\zeta_i), x(\zeta_i')) + \exp\{-a\Delta_2(x, x(\zeta_i'))\}V(x(\zeta_i'), x(\zeta_i'))$$

Then $x(\cdot)$ *is an optimal trajectory.*

Going back to Example 6.7 with this theory, we see that

Example 7.4 *Consider Example 6.7 except with the controls restricted in* $[-1, -\epsilon]$, $0 < \epsilon < 1$. *Then the flows are transversal and do not vanish on* $A_1 = \{0\}$ *for any* u. *In this case, one can solve for the optimal control. For example, if* $K > 1/\epsilon$, *one can show that* $u(\cdot) \equiv -\epsilon$ *is optimal.*

8 Conclusions

We examined the phenomena that arise in hybrid systems and reviewed several models of hybrid systems from the literature. We then proposed a very general framework for hybrid control problems which encompasses these hybrid phenomena. A specific control problem was then studied in this framework, leading to an existence result for optimal controls. The "value function" associated with this problem is expected to satisfy a set of "generalized quasi-variational inequalities" which were formally derived.

The foregoing presents some initial steps towards developing a unified "state space" paradigm for hybrid control. Several open issues suggest themselves. We conclude with a brief list of some of the more striking ones.

1. A daunting problem is to characterize the value function as the unique viscosity solution of the generalized quasi-variational inequalities (7.4)–(7.7).

2. Many of our assumptions can possibly be relaxed at the expense of additional technicalities or traded off for alternative sets of assumptions that have the same effect. For example, the condition $d(C_i, A_i) > 0$ could be dropped by having c_2 penalize highly the impulsive jumps that take place too close to A_i. (In this case, Assumption 4 has to be appropriately reformulated.)

3. Example 6.6 show that Assumption 3 cannot be dropped. In the autonomous case, however, the set of initial conditions that hit a C^∞ manifold are of measure zero [21]. Thus, one might hope that an optimal control would exist for almost all initial conditions in the absence of Assumption 3. The system of Example 6.7 showed this to be false. Likewise, in the systems of Example 6.6 we have, respectively, no optimal control for the sets

$$\{(x_1, x_2) \mid x_2 \leq 0, x_1 < 1, x_2 + 1 > x_1\}$$

and
$$\{(x_1, x_2) \mid x_2 \leq 0, x_1 < a, x_2 + a > x_1\} \cup \left([0,1] \times [-a/2, 0] - \overline{B([1,\ -1]^T, 1)}\right)$$

where $a = 2 - \sqrt{2}$ and $B(x, r)$ denotes the ball of radius r about the point x.

It remains open how to relax the conditions Assumptions 3 and 4. This might be accomplished through additional continuity assumptions on G, Δ_1, and c_1.

4. An important issue here is to develop good computational schemes to compute near-optimal controls, which is currently a topic of further research. See [10] for some related work.

   This is a daunting problem in general as the results of [7] show that the hybrid systems models discussed in Section 3 can simulate arbitrary Turing machines (TMs), with state dimension as small as three. It is not hard to conceive of (low-dimensional) control problems where the cost is less than 1 if the corresponding TM does not halt, but is greater than 3 if it does. Allowing the possibility of an impulsive jump at the initial condition that would result in a cost of 2, one sees that finding the optimal control is equivalent to solving the halting problem.

5. Another possible extension is in the direction of replacing S_{i_0} by smooth manifolds with boundary embedded in a Euclidean space. See [8] for some related work.

6. In light of Definition 7.1, all the proofs seem to hold if Assumption 1 is relaxed to only consider distances "from the right," that is if $\inf_i d_+(A_i, D_i) > 0$, with

$$d_+(A_i, D_i) \equiv \inf_{t>0, u(\cdot), x \in D_i} E_i^t(x, u(\cdot)) \in A_i$$

   where $E_i^t(x, u(\cdot))$ denotes the solutions under f_i with initial condition x and control $u(\cdot)$ in $U^{[0,t)}$. Here, time can be used as a "distance" in light of the uniform bound on the f_i; we consider $t > 0$ by adding that caveat that if we jump directly onto A_i, we do not make another jump until we hit it again. Presumably one must also make some transversality or continuity assumptions for well-posedness. This would allow the results to extend to many more phenomena, including those examples in [8].

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