A Unified Framework for Strong Price of Anarchy in Clustering Games

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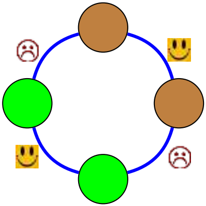
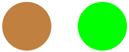
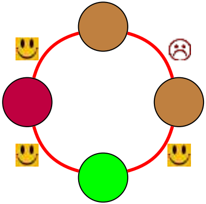
Motivation

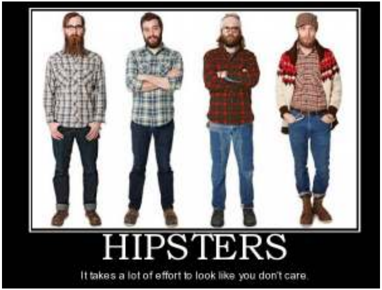


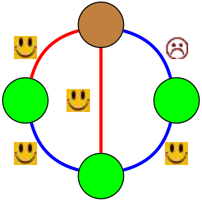
- Mobile phone providers that offer a significant discount for calls between their subscribers.
- Users would benefit the most by subscribing to the provider of the friends with whom they talk most.
-  : Providers.

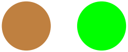
Motivation



- Radio stations broadcast on a limited spectrum of radio frequencies.
- Each station would favor a frequency that is used the least by its nearby stations.
-  : Frequencies.



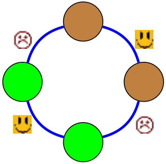


- Agents selecting an identity.
- Each agent aims to have the same identity as similar agents and an identity that is different from dissimilar agents.
-  : Identities.

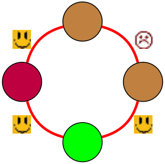
The model

- A graph (V, E) of relationships.

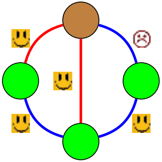
Mobile phones



Frequencies



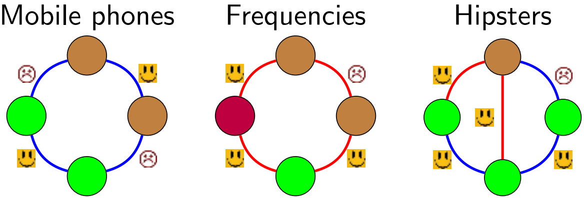
Hipsters



Clustering games

The model

- A graph (V, E) of relationships.
- Each edge e has a type $b_e \in \{-, -\}$, edges may have weights.



Clustering games

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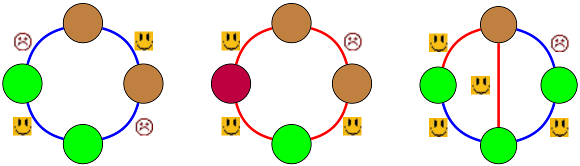
Mobile phones Frequencies Hipsters



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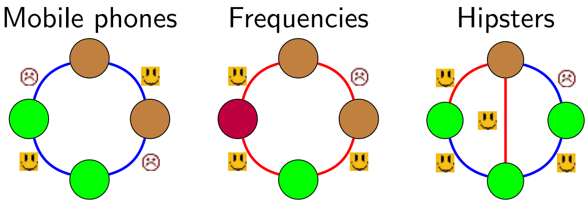
- Each agent (node) i selects σ_i: one of at most k strategies.
  - Denote by $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ the strategy profile (outcome).
  - *Symmetric*: If all agents can select all k strategies.



Clustering games

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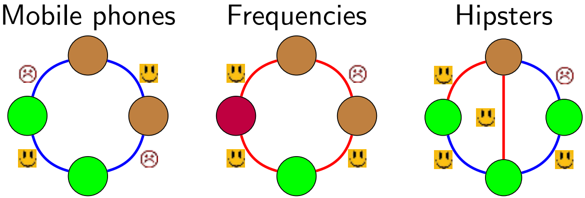
- Each edge e is ☺ or ☹ according to its type b_e and the strategies of the agents.



Clustering games

The model

- A graph (V, E) of relationships.
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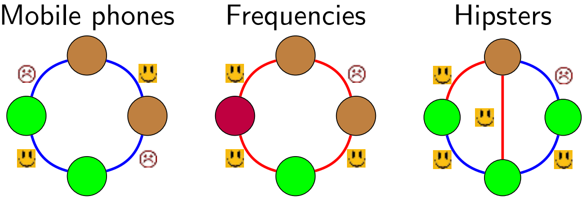
  - *Symmetric*: If all agents can select all k strategies.
- Each edge e is ☺ or ☹ according to its type b_e and the strategies of the agents.
- The utility u_i of agent i is the sum of (weights of) ☺ edges.



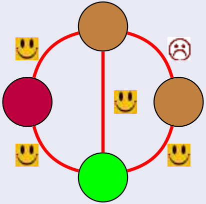
A natural optimization problem

Assign strategies to agents (nodes) in order to maximize the social welfare (SW) – the sum of the agents' utilities.
(σ = outcome)

$SW(\sigma) = \sum_{i \in V} u_i(\sigma) = 2 \cdot \sum_{e \in E} \mathbb{1}_{\{e}$ is  in $_{\sigma\}}$

Example

If all edges are —, we get the Max-k-Cut problem.



Strategic behaviour

In the absence of a central planner, every agent (node) attempts to selfishly maximize utility.

Definition

A Nash equilibrium (NE) is an outcome in which no agent can *strictly* benefit by unilaterally deviating to a different strategy.

However, in many situations agents can coordinate their deviations.

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Definition ([Aum59])

A strong equilibrium (SE) is an outcome for which no coalition of agents can jointly deviate, so that each member *strictly* benefits.

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Definition

A Nash equilibrium (NE) is an outcome in which no agent can *strictly* benefit by unilaterally deviating to a different strategy.

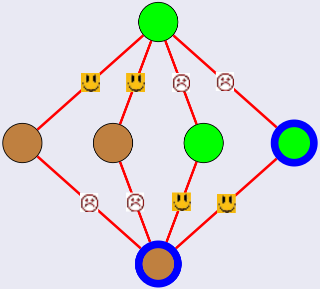
However, in many situations agents can coordinate their deviations.

Definition ([Aum59])

A q-strong equilibrium (SE) is an outcome for which no coalition of agents of size at most q can jointly deviate, so that each member *strictly* benefits.

Coalitional deviations

Nash equilibrium, not 2-strong equilibrium.



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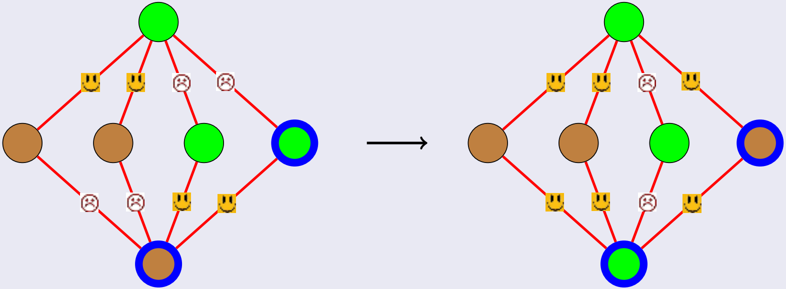
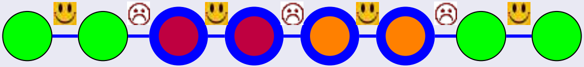


 agents increase utility by 1.

Coalitional deviations

3-strong equilibrium, not 4-strong equilibrium.



Coalitional deviations

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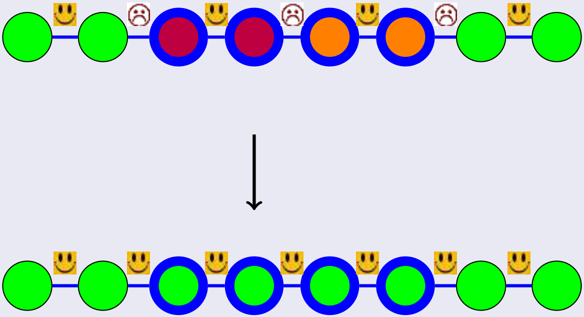
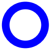


 agents increase utility by 1.

Existence of equilibrium

Theorem

Every clustering game has a Nash equilibrium (since it is a potential game [MS96]).

Theorem

Every clustering game with two strategies has a strong equilibrium.

Extends previous theorems for special cases (Max-Cut and 2-NAE-SAT [GM09], coordination games on graphs [ARSS14]).

Conjecture

Every symmetric clustering game possesses a strong equilibrium.

Extends previous conjecture for Max-k-Cut [GM09].

Quantifying inefficiency

Price of Anarchy (PoA) – the ratio between the social welfare of a worst Nash equilibrium, and that of an unconstrained optimal outcome.

- Quantifies the loss of efficiency due to selfishness.

Definition

$$\text{PoA} = \frac{\text{SW of worst NE}}{\text{SW of an optimal outcome}} \leq 1$$

Remark

A lower bound is a positive result, and an upper bound is a negative result.

Quantifying inefficiency

Strong Price of Anarchy (SPoA) – the ratio between the social welfare of a worst Strong equilibrium, and that of an unconstrained optimal outcome.

- Quantifies the loss of efficiency due to selfishness, and assuming coordination capabilities.

Definition

$$\text{SPoA} = \frac{\text{SW of worst SE}}{\text{SW of an optimal outcome}} \leq 1$$

Remark

A lower bound is a positive result, and an upper bound is a negative result.

Quantifying inefficiency

q-Strong Price of Anarchy (q-SPoA) – the ratio between the social welfare of a worst q-Strong equilibrium, and that of an unconstrained optimal outcome.

- Quantifies the loss of efficiency due to selfishness, and assuming limited coordination capabilities.

Definition

$$\text{q-SPoA} = \frac{\text{SW of worst q-SE}}{\text{SW of an optimal outcome}} \leq 1$$

Remark

A lower bound is a positive result, and an upper bound is a negative result.

Previous work

- $z(q) = \frac{q-1}{n-1}$

Class Name	Case Description			Result	
	— / —	# of Str.	Sym	PoA	SPoA
Max-Cut	—	2	$\surd$	1/2 "folklore"	2/3 [GM09]
2-NAE-SAT	— / —	2	$\surd$	1/2 [GM09]	2/3 [GM09]
Max-k-Cut	—	k	$\surd$	$\frac{k-1}{k}$ [Hoe07]	$\left[\frac{k-1}{k-\frac{1}{2(k-1)}}, \frac{k-1}{k-\frac{1}{2}}\right]$ [GM10]
				q-SPoA	
Coordination games on graphs	—	k	$\times$	$\left[\frac{z(q)}{2}, \frac{z(q)}{2} + \frac{z(q)^2}{4-2\cdot z(q)}\right]$ [ARSS14]	

Clustering games are $(1/2, 0)$-coalitionally smooth games [BSTV14], therefore SPoA $\geq \frac{1}{2}$.

Goal

Construct a unified recipe for quantifying the degradation of social welfare (i.e., q-SPoA) in various settings that fall into the class of clustering games.

Our contribution

1. We provide a unified framework for computing the q-SPoA in clustering games.
2. We use our framework to recover previous results on special cases.
3. We use our framework to establish new q-SPoA bounds on previously studied games.
4. We identify new settings that fall into the class of clustering games and establish q-SPoA bounds for them.

The framework - High level

Proving a lower bound on the q-SPoA

1. For each coalition K of size at most q obtain an expression for the lower bound on the welfare of K in equilibrium.
2. Infer a generic expression for a lower bound for coalition of any size.
3. Use combinatorial reasoning for each special case to substitute terms in the generic expression to derive a meaningful lower bound.

Price of Anarchy results (positive results)

$z(q) = \frac{q-1}{n-1}$ (so NE $\Rightarrow z(q) = 0$ and SE $\Rightarrow z(q) = 1$).

Symmetric games

- New special case:
  Only — edges: q-SPoA $\geq \frac{2+(k-2)\cdot z(q)}{2k-z(q)}$
  - PoA $\geq \frac{1}{k}$, SPoA $\geq \frac{k}{2k-1}$
- Only — edges: q-SPoA $\geq \frac{k-1}{k-\frac{1}{2(k-1)}\cdot z(q)}$
- Both — and — edges: q-SPoA $\geq \frac{2+(k-2)\cdot z(q)}{2k-\frac{1}{k-1}\cdot z(q)}$

Asymmetric games (clustering games in general)

- q-SPoA $\geq \frac{z(q)}{2}$

Upper bounds (negative results)

Proposition (a tight bound on SPoA)

The symmetric case with a line graph of — edges with $2k$ nodes and k strategies for each player has a SPoA of $\frac{k}{2k-1}$

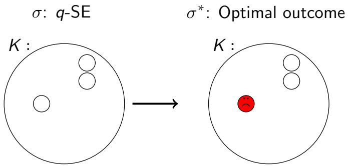
Proposition (a tight bound on PoA)

There exists a symmetric coordination games on a graph with k strategies, with PoA $= 1/k$.

Theorem (Upper bound in Max-Cut, for $q << n$)

For any $\epsilon > 0$ and $q = O(n^{1-\epsilon})$, the q-SPoA of Max-Cut is $1/2$.

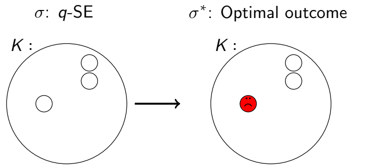
The renaming process



$$\boxed{|K| \leq q}$$

Since σ is a q-SE, one agent doesn't benefit from deviating, rename this agent to 1.

The renaming process



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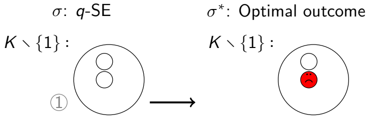
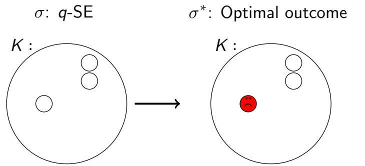


Since σ is a q-SE, one agent doesn't benefit from deviating, rename this agent to 2.

The renaming process



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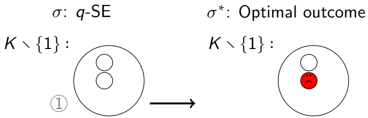


Since σ is a q-SE, one agent doesn't benefit from deviating, rename this agent to 2.

Result

For $|K| \leq q$:

$$\underbrace{\sum_{i \in K} u_i(\sigma)}_{\text{total welfare of K}} \geq \sum_{i \in K} u_i(\text{agents } i \ldots |K| \text{ deviate to } \sigma^*)$$

The renaming process

Result of the renaming process

For $|K| \leq q$:

$$\sum_{i \in K} u_i(\sigma) \geq \sum_{i \in K} u_i(\text{agents } i \ldots |K| \text{ deviate to } \sigma^*)$$

The renaming process

Result of the renaming process

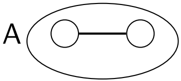
For $|K| \leq q$:

$$\begin{aligned}\sum_{i \in K} u_i(\sigma) \geq \sum_{i \in K} u_i(\text{agents } i \ldots |K| \text{ deviate to } \sigma^*) \\ = & u_1(\sigma_1, \sigma^*_{-1}) + \\ & u_2(\sigma_1, \sigma_2, \sigma^*_{-\{1,2\}}) + \\ & \ldots \\ & u_{|K|}(\sigma_1, \ldots, \sigma_{|K|}, \sigma^*_{-K})\end{aligned}$$

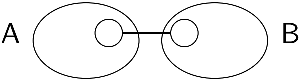
- The utilities of different agents are taken at different outcomes.
- An entangled outcome is not necessarily a stable point nor an optimal outcome.
- Therefore, decomposition is needed.

Decomposition I

1. $\mathcal{B}$: Edges that are both in σ and σ^*.
2. $\mathcal{O}$: Edges that are only in σ^*.
3. $\mathcal{E}$: Edges that are only in σ.
4. $\mathcal{I}^A$: Edges that are in the *interior* of A:



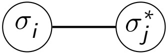
5. $\delta^{A,B}$: Edges that are in the *cut* of A and B:



Decomposition II

6. $1^{(\sigma_K^*, \sigma_{-K})}$: Edges that are by the outcome $(\sigma_K^*, \sigma_{-K})$.

7. $[K]^{\sigma, \sigma^*}$: Edges from the interior of K, where each edge between two agents that are renamed to $i < j$, is when colored



Lemma

For every q-strong equilibrium σ, optimal outcome σ^*, and a set of players K of size at most q:

$$SW_K(\sigma) \geq \mathcal{I}^K \cap (\mathcal{B} + \mathcal{O}) + [K]^{\sigma,\sigma^*} + \delta^{K,K^c} \cap 1^{(\sigma^*_K,\sigma_{-K})}$$

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- For a larger coalition A, sum over all $K \subseteq A$, $|K| = q$ and normalize.
- For $D = \{i : \sigma_i \neq \sigma_i^*\}$, split δ^{K,K^c} to $\delta^{K,D^c} \cup \delta^{K,D \setminus K}$

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- For a larger coalition A, sum over all $K \subseteq A$, $|K| = q$ and normalize.
- For $D = \{i : \sigma_i \neq \sigma^*_i\}$, split δ^{K,K^c} to $\delta^{K,D^c} \cup \delta^{K,D\setminus K}$
- And then you get something like this:

$$SW_D(\sigma) \geq \frac{q-1}{|D|-1} \cdot \left(\mathcal{I}^D + \delta^{D,D^c}\right) \cdot (\mathcal{B} + \mathcal{O})$$
$$+ \binom{|D|-1}{q-1}^{-1} \sum_{\substack{K \subseteq D \\ |K|=q}} \cdot \left([K]^{\sigma,\sigma^*} + \delta^{K,D\setminus K} \cdot 1^{(\sigma^*_K,\sigma_{-K})}\right) \cdot (\mathcal{B} + \mathcal{O} + \mathcal{E})$$

Symmetry

When all players have the same strategy space:

- π: a permutation over the strategy space.
- σ_π: The outcome where each player i plays $\pi(\sigma_i)$.

Lemma (Permutation invariance)

For every outcome σ and permutation π, the  *edges are identical in σ and σ_π.*

Corollary

The sets of edges $\mathcal{B}, \mathcal{O}, \mathcal{E}$ are invariant when replacing σ^* with σ^*_π

Symmetry

Left-hand side

- $D_\pi = \{i : \sigma_i \neq \pi(\sigma_i^*)\}$
- From previous lemma:

$$SW_{D_\pi}(\sigma) \geq \frac{q-1}{|D_\pi|-1} \cdot \left(\mathcal{I}^{D_\pi} + \delta^{D_\pi, {D_\pi}^c}\right) \cdot (\mathcal{B} + \mathcal{O}) + \binom{|D_\pi|-1}{q-1}^{-1} \sum_{\substack{K \subseteq D_\pi \\ |K|=q}} \cdot \left([K]^{\sigma,\sigma^*} + \delta^{K, D_\pi \setminus K} \cdot 1^{(\sigma_K^*, \sigma_{-K})}\right) \cdot (\mathcal{B} + \mathcal{O} + \mathcal{E})$$

- Sum over all permutations.

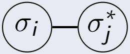
$$\sum_\pi SW_{D_\pi}(\sigma) = \underbrace{(k-1)(k-1)!}_{k = \#\text{ of strategies}} \cdot SW(\sigma)$$

Symmetry

Right-hand side

The following properties are used to quantify the right-hand side:

- Permutation invariance.
- Both in $[K]^{\sigma,\sigma^*}$ and $\delta^{K, D \setminus K} \cap 1^{(\sigma^*_K, \sigma_{-K})}$, edges look like:



  And for the set $D = \{i : \sigma_i \neq \sigma^*_i\}$, j **changes** color.
- The type of edge (— / —) implies how many times it is when summing over all π.

Combining RHS and LHS

$$(k-1)(k-1)! \cdot SW(\sigma) \geq \text{some factor} \cdot SW(\sigma^*)$$

Future work

- Solve the conjecture for existence of strong equilibrium
- Close gaps (SPoA in Max-k-Cut, etc.)
- More meaningful upper bounds for q-SPoA.
- Extend analysis to handle other solution concepts (mixed, correlated, coarse correlated equilibria).
- Try to use our analysis to shed light on coalitional dynamics.

Thank you!

Example - Symmetric coordination games on graphs

Theorem

The SPoA of symmetric coordination games on graphs with k strategies is at least $\frac{k}{2k-1}$.

Proof.

1. Recall that $D = \{i : \sigma_i \neq \sigma_i^*\}$

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2. Therefore: $(\sigma_D^*, \sigma_{-D}) = \sigma^* = \mathcal{B} + \mathcal{O}$ *notation abused

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3. Plug to lemma:

$$SW_D(\sigma) \geq \left(\mathcal{I}^D + \delta^{D,D^c}\right) \cdot (\mathcal{B} + \mathcal{O}) + [D]^{\sigma,\sigma^*}$$

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3. Plug to lemma:

$$SW_D(\sigma) \geq \left(\mathcal{I}^D + \delta^{D,D^c}\right) \cdot (\mathcal{B} + \mathcal{O}) + [D]^{\sigma,\sigma^*}$$

4. All edges are —, therefore, changing strategy to *one* node of an edge which is ☺ in σ or σ_π^*, surely makes it ☹.
   $\Rightarrow [D_\pi]^{\sigma,\sigma_\pi^*} = 0$.

Example - Symmetric coordination games on graphs

Proof (Cont.)

5. Sum over all π

$$\sum_{\pi} SW_{D_\pi}(\sigma) \geq \sum_{\pi} \left(\mathcal{I}^{D_\pi} + \delta^{D_\pi, D_\pi^c}\right) \cdot (\mathcal{B} + \mathcal{O})$$

Example - Symmetric coordination games on graphs

Proof (Cont.)

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$$\sum_{\pi} SW_{D_\pi}(\sigma) \geq \sum_{\pi} \left(\mathcal{I}^{D_\pi} + \delta^{D_\pi, D_\pi^c}\right) \cdot (\mathcal{B} + \mathcal{O})$$

6. Left-hand side $= (k-1)(k-1)!SW(\sigma)$

Example - Symmetric coordination games on graphs

Proof (Cont.)

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$$\sum_{\pi} SW_{D_\pi}(\sigma) \geq \sum_{\pi} \left(\mathcal{I}^{D_\pi} + \delta^{D_\pi, D_\pi^c}\right) \cdot (\mathcal{B} + \mathcal{O})$$

6. Left-hand side $= (k-1)(k-1)! SW(\sigma)$
7. Right-hand side, using permutation invariance:

$$\left(\sum_{\pi} \mathcal{I}^{D_\pi}\right) \mathcal{B} + \left(\sum_{\pi} \delta^{D_\pi, D_\pi^c}\right) \mathcal{B} + \left(\sum_{\pi} \mathcal{I}^{D_\pi}\right) \mathcal{O} + \left(\sum_{\pi} \delta^{D_\pi, D_\pi^c}\right) \mathcal{O}$$

Example - Symmetric coordination games on graphs

Proof (Cont.)

All edges are —. Therefore:

$$\left(\sum_{\pi} \mathcal{I}^{D_\pi}\right) \cdot \mathcal{B} = (k-1)(k-1)! \cdot \mathcal{B}$$

For every $e \in \mathcal{B}$:

$$e \in \mathcal{I}^{D_\pi} \Leftrightarrow \pi(\sigma_i^*) \neq \sigma_i$$

- $k-1$ options to fix $\pi(\sigma_i^*)$
- $(k-1)!$ options to set the other $(k-1)$ values of π.

Example - Symmetric coordination games on graphs

Proof (Cont.)

All edges are —. Therefore:

$$\left(\sum_{\pi} \delta^{D_\pi, D_\pi^c}\right) \cdot \mathcal{B} = 0$$

If $e \in \mathcal{B}$, then e can never be in the cut.

Example - Symmetric coordination games on graphs

Proof (Cont.)

All edges are —. Therefore:

$$\left(\sum_{\pi} \mathcal{I}^{D_\pi}\right) \cdot \mathcal{O} = (k-2)(k-1)! \cdot \mathcal{O}$$

For every $e \in \mathcal{O}$

$$e \in \mathcal{I}^{D_\pi} \Leftrightarrow \{\pi(\sigma_i^*)\} \cap \{\sigma_i, \sigma_j\} = \phi$$

- $(k-2)$ options to fix $\pi(\sigma_i^*)$
- $(k-1)!$ options to set the other $(k-1)$ values of π.

Proof (Cont.)

All edges are —. Therefore:

$$\left(\sum_{\pi} \delta^{D_\pi, D_\pi^c}\right) \cdot \mathcal{O} = 2(k-1)! \cdot \mathcal{O}$$

For every $e \in \mathcal{O}$:

- $e \in \delta^{D_\pi, D_\pi^c}$ in exactly two disjoint events:

$$\pi(\sigma_i^*) = \sigma_i \quad \text{or} \quad \pi(\sigma_j^*) = \sigma_j$$

Example - Symmetric coordination games on graphs

Proof (Cont.)

8. In total:

$$(k-1)(k-1)!SW(\sigma) \geq (k-1)(k-1)! \cdot \mathcal{B} + k! \cdot \mathcal{O}$$

Proof (Cont.)

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$$(k-1)(k-1)!SW(\sigma) \geq (k-1)(k-1)! \cdot \mathcal{B} + k! \cdot \mathcal{O}$$

Which equals:

$$(k-1)SW(\sigma) \geq (k-1) \cdot \mathcal{B} + k \cdot \mathcal{O} = k \cdot (\mathcal{B} + \mathcal{O}) - \mathcal{B}$$

Example - Symmetric coordination games on graphs

Proof (Cont.)

Which equals:

$$(k-1)SW(\sigma) \geq (k-1) \cdot \mathcal{B} + k \cdot \mathcal{O} = k \cdot (\mathcal{B} + \mathcal{O}) - \mathcal{B}$$

9. Since $SW(\sigma^*) = 2\,(\mathcal{B} + \mathcal{O})$:

$$2(k-1)SW(\sigma) \geq k \cdot SW(\sigma^*) - SW(\sigma)$$

□

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