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A UNIFIED THEORY OF FIRM SELECTION AND GROWTH

Costas Arkolakis

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ABSTRACT

This paper studies the effects of marketing choice to firm growth. I assume that firm-level growth is the result of idiosyncratic productivity improvements with continuous arrival of new potential producers. A firm enters a market if it is profitable to incur the marginal cost to reach the first consumer and pays an increasing marketing cost to reach additional consumers. The model is calibrated using data on the cross-section of firms and their sales across markets as well as the rate of incumbent firm-exit. The calibrated model quantitatively predicts firm exit, growth, and the resulting firm size distribution in the US manufacturing data. It also predicts a distribution of firm growth rates that deviates from Gibrat's law –i.e. independence of firm size and growth– in a manner consistent with the data.

Costas Arkolakis
Department of Economics
Yale University, 28 Hillhouse Avenue
P.O. Box 208268
New Haven, CT 06520-8268
and NBER
costas.arkolakis@yale.edu

1 Introduction

In the last few decades, economists have focused on explanations for the empirical regularities in the cross-sectional distribution of firm size and the relationship between initial size of surviving firms and their growth rates.[1] A representation of firm technological process that follows Gibrat's law –i.e. independence of the expected growth rate and initial size– has been a key element of theories that attempt to explain such regularities.[2] However, relatively little has been done in understanding the nature of firm demand and how it affects firm dynamics.

This paper attempts to do so by integrating a theory of marketing choice, based on Arkolakis (2010), into a model of firm dynamics in which the growth of firm productivities follows Gibrat's law, as in Luttmer (2007). A firm enters a market if it is profitable to incur the marginal cost to reach the first consumer and can also incur an increasing marginal marketing cost to reach more consumers in the market.

The proposed setup provides a generalization of previous theories of firm growth based on Gibrat's law. In particular, with constant marginal costs to reach additional consumers the model provides a dynamic extension of the Constant Elasticity of Substitution (CES) multi-market (international trade) setup of Melitz (2003)-Chaney (2008) with Gibrat's law embedded in firm behavior. However, with increasing marginal costs the model generates an effective demand for the firms where the demand elasticity declines with firm size and asymptotically tends to the CES demand elasticity. I illustrate that the new model can capture two salient features of the data regarding firm turnover and growth and that it can serve as a reliable framework for a number of quantitative applications.

First, the model offers a simpler setup for the entry and exit of firms that implies that the size of entrants is typically small and roughly equal to the size of exiting firms, as reported for the US manufacturing firm data from Dunne, Roberts, and Samuelson (1988) (henceforth DRS). In a standard sunk cost setup, which assumes a one-time cost of entry to a market, the average size of entrants is larger than that of exiting firms. To model the entry-exit process I follow Kortum (1997) and Eaton and Kortum (2001), and assume that the rate at which new

[1] Unless otherwise noted, I will refer to firm size as firm sales. Sales are typically available for very fine categorizations and also for different markets. See Sutton (1997), Caves (1998) and Axtell (2001) for discussion of the empirical findings on firm size distribution and the inverse relationship between initial size of firms and their exit and growth rates (as well as the variance of their growth rates).

[2] Gibrat's law is combined with endogenous exit of small firms with negative growth rates to generate the inverse exit rate-size relationship. These forces combined lead to higher growth rates of small surviving firms. See for example Klette and Kortum (2004), Lentz and Mortensen (2008), and Luttmer (2007) among others.

ideas arrive is exogenously given. Each idea can be used by a monopolistically competitive firm to produce a differentiated good and potentially generate profits. These ideas become firms only if they are used in production. If not, they enter a "mothball" state until a future shock in their productivity makes production profitable again.

Second, I show that the model with endogenous market penetration costs can reconcile the inverse relationship between firm growth and initial size with the inverse relationship of the variance of firm growth and initial size. In the Luttmer (2007) setup small firms with negative growth rates are faced with an endogenous exit decision. Thus, firms with small initial size that survive, have higher expected growth rates but their variance of growth is lower, compared to firms with larger initial size. This prediction sharply contradicts empirical observations. In the endogenous cost model, the marketing choice for small firms is very volatile, which has as a result a higher variance of growth rate for these firms. In addition, because of the marketing choice, the model implies a strikingly different distribution of growth rates of small firms vis-a-vis a setup where Gibrat's law holds, with some small firms growing at a very fast pace. The higher elasticity of demand implies higher variance of growth but also the potential for higher expected growth for smaller firms.

To quantitatively assess the predictions of the model, I exploit the cross-sectional restrictions that the multi-market structure of the model imposes. Therefore, I use the same parameter values as in the static calibration of the international trade model of Arkolakis (2010) since at each point in time the dynamic model is identical to its static multi-market counterpart. In addition, I calibrate the drift and the variance of the stochastic process that governs firm growth without using information on firm growth. To do so, I once again exploit the multi-market structure of the model and the fact that the elasticity of trade in the model is the shape parameter of the (Pareto) size distribution of firms, as in the static model. This parameter endogenously arises in the model as a function of the drift and the variance of the stochastic process. With this calibration the dynamic model retains contact to its static counterpart and delivers rich dynamic predictions with –effectively– only one additional crucial parameter.

The calibrated model is used to quantitatively explain the exit and growth of sales of US manufacturing firms reported by DRS for a time span of 2 decades. The entry-exit process implied from the model closely matches the exit rates of both incumbent firms and new entrants. Additionally, the benchmark model can generate the small initial size and the sales growth rates of new entrants over time. The fit of the model is notable despite the minimal information that is used for its calibration.

A series of additional evidence regarding the quantitative predictions of the model are also provided. In particular, the model performs well in predicting the size distribution of firms even conditional on their age. This success suggests that the cross-sectional results on firm size are intimately linked to the dynamic ones, hence the desirability of a "unified" theory to analyze them. I also use data for export transactions of Brazilian firms by destination to illustrate that, consistent with the entry-exit process postulated in this model, firms are very likely to return to exporting after a year of no exporting activity. Furthermore, the distribution of growth rates of Brazilian exporters in a destination, conditional on their initial exporting size in that market, is similar to the one predicted by the model.[3]

The theory predicts an inverse relationship of firm size and growth, also discussed by Rossi-Hansberg and Wright (2007) and Luttmer (2011). In the final section of the paper I discuss how empirical tests that were used as evidence against the inverse size-growth relationship (such as measuring growth as suggested by Davis, Haltiwanger, and Schuh (1996)) are not appropriate if the underlying growth of productivities follows a random walk as postulated in this paper and Luttmer (2007).

The demand-based explanation of this paper also implies that all its predictions carry over when considering the sales of a firm in a given (export) market. It is consistent with the growth and exit patterns of Brazilian exporters and the robustness of French exporters size distribution across destinations reported in Eaton, Kortum, and Kramarz (2011). This evidence cannot be explained by the mechanism of financial constraints (see for example Cooley and Quadrini (2001) and Arellano, Bai, and Zhang (2008)) that implies a relationship of firm growth with overall firm size.[4]

The paper follows a large tradition of models of firm dynamics with a continuum of heterogeneous firms. Such models are examined by Jovanovic (1982), Hopenhayn (1992), Klette and Kortum (2004), Luttmer (2007) among others.[5] As in the celebrated work of Yule (1925) and

[3] Impullitti, Irarrazabal, and Opromolla (2006) and Atkeson and Burstein (2010) develop two-country extensions of Luttmer (2007). Irarrazabal and Opromolla (2009) adapt a framework of entry and exit similar to the one in this paper. The authors retain the main assumptions of the fixed cost framework (without assuming sunk costs of exporting) and study the theoretically implied entry-exit patterns into individual destinations.

[4] For a recent theoretical analysis of the effects of firm demand on firm growth see also Foster, Haltiwanger, and Syverson (2010). In models of learning such as Jovanovic (1982), firm growth depends primarily on age rather than on size, which is a key difference from the framework of this paper. In addition, whereas the quantitative predictions of firm-learning models have been relatively unexplored, the relationship of growth to size and age in these models crucially depends on parameterization.

[5] Lentz and Mortensen (2008) and Bernard, Redding, and Schott (2009) develop models of firm dynamics

Simon (1955), I use the two minimal sufficient conditions of random entry-exit and Gibrat's law to generate a cross-sectional distribution with Pareto right tails (see Reed (2001)). Random entry and exit is used in lieu of the assumption of a lower exit or reflective barrier and constitutes a major technical simplification compared to prior related work (see for example Luttmer (2007) or Gabaix (1999)).

The rest of the paper is organized as follows. Section 2 summarizes the quantitative evidence on firm selection, growth, and size distribution. Section 3 and 4 develop the firm-dynamics framework and provide an analytical characterization of its theoretical predictions. Sections 5 and 6 calibrate the model and evaluate its quantitative predictions and Section 7 discusses the implications of the theory for the growth-size debate.

2 Quantitative Facts on Firm Selection, Growth and Sizes

This section summarizes the findings of a set of studies that present empirical regularities regarding the entry and exit of firms, their growth, and their cross-sectional size distribution. In the rest of the analysis, the term "incumbent cohort" refers to firms that were in the market at a certain census year (normalized as year 0).[6] The survivors of that cohort at year t are the firms from the cohort which also sell in the market at year t. The term "entry cohort" refers to firms that enter the market between the current census and the previous one. Thus, by construction, incumbent cohort includes the surviving firms from all past entry cohorts as well as the firms of the current entry cohort. In the next two subsections, and in Figures 2 and 3, I discuss the quantitative facts of DRS on US manufacturing firm selection and growth (also summarized in Table 3). Both figures suggest that firm behavior is roughly independent of the starting year of the cohorts. All the data are based on means across manufacturing 4-digit SIC industries.

extending the theories of Klette and Kortum (2004) and Hopenhayn (1992) respectively. In turn, their models inherit many of the qualitative features of these theories.

[6] In this paper I focus on empirical facts on firms, instead of plants, which can be compared to facts on the behavior of exporters, collected at the firm-level. This choice also justifies the use of a theoretical framework with monopolistically competitive firms. Most of the facts I summarize are also true for plants.

2.1 Facts on Firm Entry and Exit

Figure 2 illustrates the fraction of exiting firms from incumbent and entry cohorts. The fraction of exitors is by construction 0% in year 0 of each cohort and increases as more firms of the cohort exit the market. Two features clearly emerge. First, the exit rates in the census data are very large. After 15 years, only about a quarter of the incumbent cohort firms and around 12% of the entry cohort firms are still active. Second, the exit rates of the entry cohorts are consistently higher than those of the incumbent cohort. Since the incumbent cohorts include firms from the current entry cohorts, exit rates of the entry cohorts account for a large part of the overall incumbent-cohort exit.

An additional robust feature of the data is that the average size of entrants and exitors is very similar. In Table 1, I report the related statistics from DRS illustrating this fact, which will be important for comparison across models:

Fact 1 *The average size of entering and exiting firms is approximately the same.*

2.2 Facts on Firm Growth

I now present evidence from DRS for the growth in the average size of incumbent and entry cohorts in order to illuminate the patterns of firm-growth. In Figure 3, I plot the average sales of surviving firms from incumbent and entry cohorts. The average size of incumbent-cohort firms increases to around 3.2 times the size of all firms in the span of 15 years. Upon entry, the average size of entry-cohort firms is only about 1/3 of the average size of all firms. However, 15 years later the average size of the surviving entry-cohort firms is around 30% larger than the size of all firms. Arguably, much of the growth of the average sales of firms, and especially firms in the entry cohort, is accounted by the fact that the exit rates are high. In relation to Figure 2 notice also that the higher exit rates of entrants are consistent with their smaller average size.

Whereas DRS report statistics aggregated by cohorts, a large literature has been devoted to understanding the relationship of firm growth and size using micro data. Research as early as Mansfield (1962) has demonstrated a robust inverse relationship between size and growth of surviving firms in the data. A similar inverse relationship between the variance of firm growth rates and the size of firms has been identified.[7] These two key features of the relationship of

[7] See for example Sutton (1997) and Caves (1998) for the firm size-growth relationship and Caves (1998) and Sutton (2002) for relationship between firm size and variance of growth.

firm growth rates as a function of firm size can be summarized as follows:

Fact 2 *For surviving firms the growth rate and its variance declines with firm size.*

2.3 Facts on the Firm Size Distribution

Since data for the firm sales distribution are not provided by DRS, I appeal to different data sources. Detailed data on the size distribution of US manufacturing firms are reported by the Small Business Administration. These data report the number of firms in various size bins. The sales distribution is plotted in Figures 4 and 5 where the sales of the firms (divided by mean sales) are in logarithms in both figures while the rank of the firm is not in logarithms in Figure 4. The first graph clearly illustrates that most of the firms in the data are very small. Figure 5 zooms in on the top decile of the firms and indicates that the size distribution of the larger firms is roughly linear in logarithms and thus can be approximated by the Pareto distribution. By comparing these two figures one concludes that the size distribution of *manufacturing* firms appears to exhibit large departures from the Pareto distribution, challenging the view postulated by Axtell (2001). These findings are very similar to the ones reported in Eaton, Kortum, and Kramarz (2011) for French manufacturing firms. In the same figure I overlay the distribution of sales of manufacturing firms in France. The two distributions exhibit remarkable similarity.

The decomposition of the firm size for firms of different ages has been the topic of a recent contribution by Cabral and Mata (2003), where the authors consider the employment size distribution of Portuguese firms. The authors report non-parametric estimates for the employment size distribution as a function of firm age illustrated in Figure 6. They find that the sizes of younger firms sales are concentrated in the lower ends of the size distribution (entrants are typically of small employment size) whereas the size distribution of the largest firms converges to a log-normal distribution.[8] In the next section I lay out the elements of a simple model that can account for the firm-level facts summarized above.

[8] Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997) study the distribution of sizes of US manufacturing firms in the compustat database, which typically contains larger publicly-traded firms, and they also find evidence for lognormal distribution of firm sizes.

3 The Model

The model introduces market penetration costs, as modeled in Arkolakis (2010), within a firm-dynamics setup similar to Luttmer (2007). I develop a multi-market version of this setup where the decisions of the firms are independent across markets. Thus, the predictions of the model for firm entry-exit and growth apply to the operation of the firm in each market, conditional on its size in that market.

3.1 Model Setup

Time is continuous and indexed by t. The importing market is denoted with an index j and the exporting market with i, where $i, j = 1, ..., N$. At each time t, market j is populated by a continuum of consumers of measure $L_{jt} = L_j e^{g_\eta t}$, where g_η is the growth rate of the population, $g_\eta \geq 0$. I assume that each good ω is produced by a single firm and each firm reaches consumers independently from other firms. Therefore, at a given point in time t, a consumer $l \in [0, L_{jt}]$ has access to a potentially different set of goods Ω^l_{jt}. Firms differ ex-ante only in their productivity, z, and their source market i. I consider a symmetric equilibrium where all firms of type z from market i choose to charge the same price in j, $p_{ijt}(z)$, and reach consumers there with probability, $n_{ijt}(z) \in [0, 1]$. The existence of a large number of firms implies that every consumer from j has access to the same distribution of prices for goods of different types. The existence of a large number of consumers in market j implies that the fraction of consumers reached by a firm of type z from i is $n_{ijt}(z)$ and their total measure is $n_{ijt}(z) L_{jt}$.

Each consumer from market j has preferences over a consumption stream $\{C_{jt}\}_{t\geq 0}$ of a composite good from which she derives utility according to

$$E \int_0^{+\infty} \rho e^{-\rho t} C_{jt}^{\frac{\iota-1}{\iota}} dt ,$$

where $\rho > 0$ is the discount rate and $\iota > 0$ is the intertemporal elasticity of substitution. The composite good is made from a continuum of differentiated commodities

$$C_{jt} = \left(\sum_{i=1}^{N} \int_0^{+\infty} c_{ijt}(z)^{(\sigma-1)/\sigma} dM_{ijt}(z) \right)^{\frac{\sigma}{\sigma-1}}$$

where $c_{ijt}(z)$ is the consumption of a good produced by a firm z in market i and σ is the elasticity of substitution among different varieties of goods where $\sigma > 1$. $dM_{ijt}(z)$ is the density of goods of a given type z from market i that are actually sold to j. Since consumers from market j have access to the same distribution of prices, their level of consumption C_{jt} is the same.

Each household earns labor income w_{jt} from selling its unit labor endowment in the labor market and profits π_{jt} from the ownership of domestic firms. Thus, the demand for good z from i by a consumer from market j is

$$c_{ijt}(z) = \frac{p_{ijt}(z)^{-\sigma}}{P_{jt}^{1-\sigma}} y_{jt} \ ,$$

where $y_{jt} = w_{jt} + \pi_{jt}$ and

$$P_{jt}^{1-\sigma} = \sum_{\upsilon=1}^{N} \int_{0}^{+\infty} p_{\upsilon jt}(z)^{1-\sigma} n_{\upsilon jt}(z) \, dM_{\upsilon jt}(z) \ . \tag{1}$$

Given the definition of the price index, P_{jt}, the budget constraint faced by each consumer is $C_{jt}P_{jt} = y_{jt}$. Thus, the total effective demand in market j for a firm of type z from i is

$$q_{ijt}(z) = n_{ijt}(z) L_{jt} \frac{p_{ijt}(z)^{-\sigma}}{P_{jt}^{1-\sigma}} y_{jt} \ . \tag{2}$$

3.2 Entry and Exit

An 'idea' is a way to produce a good ω. Each idea is exclusively owned and grants a monopoly over the related good. This exclusivity implies a monopolistic competition setup as in Dixit and Stiglitz (1977) and Melitz (2003). However, in my context ideas become firms only if they are used into production. If not, they enter a "mothball" state until a future shock in their productivity makes production profitable. Once ideas are born, they die at an exogenous rate $\delta \geq 0$. To be consistent with balanced growth, I also assume that each market innovates at an exogenous rate $g_B \geq \delta$. This rate will be specified when I construct the balanced growth path and implies that the measure of existing ideas at time t in i is $J_i e^{(g_B - \delta)t}$, where $J_i > 0$ is the initial measure of ideas in i.

New ideas arrive with an initial productivity, $\bar{z}_{it}$, given by

$$\bar{z}_{it} = \bar{z}_i e^{g_E t} \ ,$$

where $\bar{z}_i, g_E > 0$. Productivity evolves over time in a manner specified in the next subsection. The parameter g_E is interpreted as the growth rate of the frontier of new ideas and all new ideas at time t enter with the same productivity. This specification incorporates a form of "creative destruction" since more recent ideas arrive with a higher productivity.[9] In fact, I show that in the balanced growth path there exists a cutoff productivity of profitable operation at each time t, z^*_{ijt}, and this cutoff grows at a rate g_E. z^*_{ijt} is determined by the zero profit condition at each point in time. Since there is no indivisible cost of production or entry, ideas with productivity higher than z^*_{ijt} are used in production and appear as firms in market j.[10]

This setup for entry and exit makes the model substantially more tractable than Luttmer (2007), since there are no forward looking decisions for the firms. Additionally, it captures the possibility that a firm may temporarily shut down as discussed in Lee and Mukoyama (2008). These temporary shutdowns may appear, for example, because of plant retooling or because of classification issues, such as firms going from employment to non-employment status. Mainly due to these classification issues, quantitative information on firm re-entry in the domestic data is scarce. Thus, in order to compute model's statistics for firm entry and exit I replicate as closely as possible the measurement procedures used in the data. However, this phenomenon of temporary exit is prevalent in the exporting data as discussed in section 6.4.

3.3 Firms and Ideas

Ideas can produce in each period for any of their markets using a standard constant returns to scale production function $q(z) = zl$, where l is the amount of labor used in production and z is the labor productivity of the idea at a given point in time. This productivity depends on the date t^b at which it was born and evolves with age a, independently across ideas, according to

$$z_{t^b,a} = \bar{z}_i \exp\left(g_E t^b + g_I a + \sigma_z W_a\right) \ , \tag{3}$$

[9] Extending this simple case to one in which new entrants arrive with different productivities drawn from a non-atomic distribution is straightforward (see, for example, Reed (2002)). In particular, unless entrants are specified to be very large with a high probability the right tails of the distribution will be unaffected. In addition, the process of growth of ideas and firms is not affected by entry.

[10] In the one market model, allowing for free entry of ideas with a fixed amount of labor used for each new idea would imply a setup with identical predictions. The only difference would arise because profits would accrue to labor used for the entry cost as in Arkolakis, Demidova, Klenow, and Rodríguez-Clare (2008). However, such an extension would require further changes in the multi-market framework given that profits from entry arise from the operation into multiple markets.

where $W_a \sim \mathcal{N}(0, a)$ is a Brownian motion with independent increments and the parameter σ_z regulates the volatility of the growth of ideas. Note that the productivity of incumbent ideas is improving on average at a rate g_I. A firm is an idea put to work to produce and market a good.

This specification for the evolution of productivities, borrowed from Luttmer (2007), implies that the expected growth of productivities is independent of firm size. Similar processes have been widely used to represent firm growth since Gibrat (1931). The Brownian motion assumption naturally emerges as the continuous time limit of a firm growth rate that is a discrete-random walk. Notably, the assumption of continuous time is not crucial for the results. What is important for predictions on firm growth is that the growth rates are normally distributed. Continuous time is convenient, however, for analytically characterizing the properties of the model.

In order to sell in a given market, firms pay a market penetration cost which is a function of the number of consumers reached in that market. I model these market penetration costs using the specification of Arkolakis (2010) derived from first principles as a cost of marketing. I assume that these costs are incurred by the firms at each instant of time, analogous to previous models, such as Melitz (2003) and Luttmer (2007), in which a per market fixed cost is required for the firm to operate at each point of time.[11]

The labor required for a firm to reach a fraction of consumers n in a market of population size L is

$$F(n, L) = \begin{cases} \frac{L^\alpha}{\psi} \frac{1-(1-n)^{-\beta+1}}{-\beta+1} & \text{for } \beta \in [0,1) \cup (1, +\infty) \\ -\frac{L^\alpha}{\psi} \log(1-n) & \text{for } \beta = 1 \end{cases} .$$

where $\alpha \in [0, 1]$ and $\psi > 0$. If $\alpha < 1$, the market penetration costs to reach a certain number of consumers decrease with the population size of the market. The parameter β governs the convexity of the marketing cost function: higher β implies more convexity and steeper increases in the marginal cost to reach more consumers. For simplicity, I assume that labor from the destination market is hired for marketing purposes. This specification implies that the total market penetration cost paid by a firm from i that reaches a fraction n of consumers in market j is $w_j F(n, L_j)$.

In addition to the cost to reach consumers, the firm has to pay a variable trade cost modeled

[11] A model that examines state dependence of market penetration costs on previous marketing decisions is left for future research. Drozd and Nosal (2011) and Gourio and Rudanko (2010) develop models where a representative firm's demand is modeled as marketing capital that accumulates over time. My modeling of marketing is static which allows me to analytically characterize the various properties of the model.

using the standard iceberg formulation. This iceberg cost implies that a firm operating in i and selling to market j must ship $\tau_{ij} > 1$ units in order for one unit of the good to arrive at the destination, where I normalize $\tau_{ii} = 1$.

3.4 Firm Optimization

Given the constant returns to scale production technology and the separability of the marketing cost function across markets, the decision of a firm to sell to a given market is independent of the decision to sell to other markets. Total profits of a particular firm are the summation of the profits from exporting activities in all markets $j = 1, ..., N$ (or a subset thereof). Thus, at a given time t, the firm's problem is the same as in Arkolakis (2010), and firm z from i solves the following static maximization problem for each given market j:[12]

$$\pi_{ijt}(z) = \max_{n_{ijt}, p_{ijt}} \left\{ n_{ijt} L_{jt} y_{jt} \frac{p_{ijt}^{1-\sigma}}{P_{jt}^{1-\sigma}} - n_{ijt} L_{jt} y_{jt} \frac{\tau_{ij} p_{ijt}^{-\sigma} w_{it}}{P_{jt}^{1-\sigma} z} - w_{jt} \frac{L_{jt}^{\alpha}}{\psi} \frac{1-[1-n_{ijt}]^{-\beta+1}}{-\beta+1} \right\}$$
$$\text{s.t. } n_{ijt} \in [0,1] \;\; \forall t \; .$$

For any β, the optimal decisions of the firm in the multi-market model are:

$$p_{ijt}(z) = \tilde{\sigma} \frac{\tau_{ij} w_{it}}{z} \tag{4}$$

where

$$\tilde{\sigma} = \sigma / (\sigma - 1) \;\; ,$$

and

$$n_{ijt}(z) = \max \left\{ 1 - \left(\frac{z_{ijt}^*}{z} \right)^{(\sigma-1)/\beta}, 0 \right\} \;\; . \tag{5}$$

z_{ijt}^* is defined by

$$z_{ijt}^* = \sup \{ z : \pi_{ijt}(z) = 0 \} \;\; , \tag{6}$$

and thus

$$z_{ijt}^* = \left[L_{jt}^{1-\alpha} y_{jt} w_{jt}^{-1} (\tilde{\sigma} \tau_{ij} w_{it})^{1-\sigma} \psi P_{jt}^{\sigma-1} / \sigma \right]^{-1/(\sigma-1)} \;\; . \tag{7}$$

Equation (7) reveals that, apart from general equilibrium considerations, the cutoff productivity z_{ijt}^*, and thus the entry-exit decision of the firm, does not depend on the parameter β. Intuitively,

[12] Slighly abusing the notation, I denote the decision of the firm only as a function of its productivity z, supressing time of birth and age information. Given that the optimization decision is static, the current level of productivity is the only state variable. I keep the notation parsimonious throughout the text whenever possible.

β regulates the convexity of marketing costs, but not the level of the cost to reach the very first consumers, and thus it does not affect firm entry and exit.

Substituting (4), (5) and (7) into the expression for sales per firm, (2), and multiplying it by the price, equation (4), the sales of firm z originating from market i in market j can be written as

$$r_{ijt}(z) \equiv p_{ijt}(z)\, q_{ijt}(z) = \begin{cases} L_{jt}^{\alpha} y_{jt} \frac{1}{\tilde{\psi}} \left[e^{\bar{c}_1 \ln\left(z/z_{ijt}^*\right)} - e^{\bar{c}_2 \ln\left(z/z_{ijt}^*\right)} \right] & \text{if } z \geq z_{ijt}^* \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

where

$$\bar{c}_1 = \sigma - 1,\ \bar{c}_2 = (\sigma - 1)\frac{(\beta - 1)}{\beta},\ \tilde{\psi} = \frac{\psi}{\sigma(1 - \bar{\pi})} ,$$

and $\bar{\pi} \equiv \pi_{it}/y_{it}$ is the fraction of profits out of total income. In the balanced growth path equilibrium, this fraction is constant and thus, I suppress its subscripts. Equation (8) reveals that for $\beta = 0$ all firms selling from i to j sell a minimum amount, $L_{jt}^{\alpha} y_{jt}/\tilde{\psi}$, while for $\beta > 0$ this amount is 0. Conditional on entry, more productive firms have higher sales as equation (8) indicates. These firms charge lower prices and thus sell more per consumer (i.e. at the intensive margin). In addition, if $\beta > 0$, they also reach more consumers (i.e. the extensive margin) as implied by equation (5). However, if $\beta = 0$, all entrants in market j optimally choose $n_{ij} = 1$. These differences in β are reflected in firm growth-patterns as I illustrate in section 4.

3.5 Balanced Growth Path Equilibrium

To solve for the cross-sectional distribution of firm sales, I consider the stationary balanced growth path. From expression (8), firm sales are determined by the ratio of firm productivity to the cutoff productivity. Given that, it is convenient to first characterize the stationary distribution of productivities detrended by the rate of growth of the zero profit cutoff,

$$\phi_a = \bar{z}_i \exp\left\{g_E t^b + g_I a + \sigma_z W_a\right\} / \exp\left\{g_E\left(t^b + a\right)\right\} = \bar{z}_i \exp\left\{(g_I - g_E)\, a + \sigma_z W_a\right\} .$$

The logarithm of ϕ_a is a Brownian motion with a drift,

$$s_a = \ln \phi_a = \bar{s}_i + (g_I - g_E)\, a + \sigma_z W_a , \tag{9}$$

where $\bar{s}_i = \ln \bar{z}_i$. s_a will be used as a proxy for the productivity of an idea or the size of a firm after a years given that firms with larger s_a are (weakly) larger in sales, productivity and employment. The term $g_I - g_E$, is the difference between the growth of incumbent ideas and the growth of the frontier of new ideas. Hereafter, I will denote this difference by μ. The probability density of the logarithm of productivities, $s_a = s$, for a given generation of ideas of age $a > 0$ from i is given by the normal density:[13]

$$f_i(s,a) = \frac{1}{\sigma_z \sqrt{a 2\pi}} \exp\left\{ -\left(\frac{s - \bar{s}_i - \mu a}{\sigma_z \sqrt{a}} \right)^2 /2 \right\} . \tag{10}$$

This distribution will allow the model to match the findings of Cabral and Mata (2003) that the size distribution of old firms converges to a log-normal distribution.

The productivity distribution of one age cohort of ideas is not age-stationary. But considering all ideas, across different ages, delivers a stationary cross-sectional distribution of productivities of all ideas from i, $f_i(s)$. In a stationary equilibrium, with entry and exit of ideas, the dynamics of the probability density of each $s \neq \bar{s}_i$ and $\forall i$, are described by a Kolmogorov forward equation,[14]

$$-\mu f_i'(s) + \frac{1}{2}\sigma_z^2 f_i''(s) - g_B f_i(s) = 0 . \tag{11}$$

Intuitively, in a stationary steady state, the net changes at each point s of the distribution (the first two terms) must equal the reduction of the probability density due to entry and exit, at each $s \in (-\infty, \bar{s}_i) \cup (\bar{s}_i, +\infty)$, where

$$\underbrace{g_B}_{\substack{\text{rate of}\\ \text{reduction}}} = \delta + \underbrace{g_B - \delta}_{\substack{\text{entry rate}\\ \text{at point } \bar{s}_i}}.$$

The net changes are due to the stochastic flows of productivities in and out of that point

[13] See for example Harrison (1985) p. 37. $f_i(s,a)$ can be derived as the solution of the differential equation $D_a f_i(s,a) = -\mu f_i'(s,a) + \frac{1}{2}\sigma_z^2 f_i''(s,a)$, with initial condition $f_i(s,a) = \Delta(s - \bar{s}_i)$, where $\Delta(.)$ is the Dirac delta function. Additionally, the realizations of the Brownian motion over different time periods, $s_{a_1}, s_{a_2}, ..., s_{a_n}$, follow a multivariate normal distribution with means $Es_a = s_0 + \mu a$ and covariances $Cov(s_a, s_{a'}) = \sigma_z^2 [\min(a, a')]$. This feature can be used to further scrutinize the model, or to pursue an alternative estimation of its parameters, by looking at the probability distribution of sales and entry exit decisions of individual firms overtime, for researchers that have access to this information.

[14] In an appendix available online, I provide a different proof by explicilty calculating $f(s) = \int_0^{+\infty} e^{-[g_B]a} f(s,a)\, da$. That proof, while more straightforward, provides less intuition on the exact forces that give rise to the shape of the cross sectional distribution of productivities across all ideas. Reed (2001) provides another proof using moment generating functions in which the intuition is also somewhat limited.

described by equation (11).

The density of productivities, $f_i(s)$, has to satisfy a set of conditions. The first requirement is that $-\infty$ is an absorbing barrier which implies

$$\lim_{s\to-\infty} f_i(s) = 0 \ . \tag{12}$$

In addition $f_i(s)$, must be a probability density which implies that

$$f_i(s) \geq 0 \ , \ \ \forall s \in (-\infty, +\infty) \tag{13}$$

and

$$\int_{-\infty}^{\bar{s}_i} f_i(s)\, ds + \int_{\bar{s}_i}^{+\infty} f_i(s)\, ds = 1 \ . \tag{14}$$

Additionally, net inflows into the distribution must equal the net outflows:[15]

$$-\mu\left[f_i(\bar{s}_i-) - f_i(\bar{s}_i+)\right] + \frac{1}{2}\sigma_z^2\left[f_i'(\bar{s}_i-) - f_i'(\bar{s}_i+)\right] = g_B \ . \tag{15}$$

The left-hand side is the net inflows into the distribution from point $\bar{s}_i$. The right-hand side is the outflows from the distribution due to new entry and random exit of ideas. By continuity, the first term in brackets is zero. However, entry of new ideas implies that the distribution is kinked at $\bar{s}_i$. Intuitively, the rate of change of the cdf changes direction at $\bar{s}_i$ because entry happens at that point. The solution of the above system is (see appendix A.2):

$$f_i(s) = \begin{cases} \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{\theta_1(s-\bar{s}_i)} & \text{if } s < \bar{s}_i \\ \frac{\theta_1\theta_2}{\theta_1+\theta_2} e^{-\theta_2(s-\bar{s}_i)} & \text{if } s \geq \bar{s}_i \end{cases} \tag{16}$$

where

$$\theta_1 = \frac{\mu + \sqrt{\mu^2 + 2\sigma_z^2 g_B}}{\sigma_z^2} > 0 \ , \tag{17}$$

$$\theta_2 = -\frac{\mu - \sqrt{\mu^2 + 2\sigma_z^2 g_B}}{\sigma_z^2} > 0 \ . \tag{18}$$

The following assumption guarantees that a time-invariant distribution exists and an ever

[15] This condition results by integrating (11) over all $s \in (-\infty, \bar{s}_i) \cup (\bar{s}_i, +\infty)$, i.e. considering the net inflows from point $\bar{s}_i$ to the rest of the distribution. Similar conditions are used in labor models to characterize the behavior of the distribution at a point of entry to or exit from a particular occupation (see for example Moscarini (2005) and Papageorgiou (2008)).

increasing fraction of ideas is not concentrated in either of the tails of the distribution:[16]

A 1 *: The rate of innovation is positive,* $g_B > 0$*.*

In particular, given that $g_B \geq \delta$, then assuming $\delta = 0$ together with A1 implies that $g_B > \delta$. Using equation (18), A1 also implies that

$$\theta_2 \mu + (\theta_2)^2 \sigma_z^2 / 2 = g_B > 0 \ . \tag{19}$$

The resulting cross-sectional distribution of detrended productivities $\phi \in [0, +\infty)$ is the so-called double Pareto distribution (Reed (2001)) with probability density function:[17]

$$\hat{f}_i(\phi) = \begin{cases} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \frac{\phi^{\theta_1 - 1}}{\bar{z}_i^{\theta_1}} & \text{if } \phi < \bar{z}_i \\ \frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \frac{\phi^{-\theta_2 - 1}}{\bar{z}_i^{-\theta_2}} & \text{if } \phi \geq \bar{z}_i \end{cases} \tag{20}$$

The double Pareto distribution is illustrated in Figure 1. A closer look at the probability density of productivities (equation (20)) reveals that at each moment of time, a constant fraction of ideas $\theta_1 / (\theta_1 + \theta_2)$ is above the threshold $\bar{z}_i$. To keep all the expressions of the model as simple as possible, I assume for the rest of the paper that $1/\psi$ is sufficiently high so that $z_{ijt}^* > \bar{z}_{it}$, $\forall i$, j, t. Thus, the (detrended) cross-sectional distribution of operating ideas (i.e. firms) in each market j is Pareto with shape parameter θ_2. Whereas A1 is necessary for a stationary distribution an additional assumption guarantees that the resulting distributions of firm productivities and sales have a finite mean:

A 2 *: Productivity and sales parameters satisfy*

$$g_B > \max \left\{ \mu + \sigma_z^2 / 2 \ , \ (\sigma - 1) \mu + (\sigma - 1)^2 \sigma_z^2 / 2 \right\} \ .$$

Assumption A2 implies that the entry rate of new ideas is larger than the growth rate of productivities and sales of the most productive incumbent firms. Notice that A2 and A1 (see equation (19)) also imply the common restriction that the Pareto shape coefficient, θ_2, is larger

[16] Under the assumption $\mu > 0$, Pareto distribution emerges in the right-tail of the distribution for the limit case of $\sigma_z \to 0$. However, both $\mu < 0$ and $\sigma_z > 0$ will be essential features of the model in explaining the data as I illustrate in the calibration section.

[17] This distribution can also be thought of as a limit case of the distribution of firms derived by Luttmer (2007) when the exit cutoff goes to $-\infty$. However, in his case, this assumption would imply that firms never exit and that there is no selection in the model.

than both 1 and $\sigma - 1$. To summarize, A1 and A2 imply a set of restrictions, not necessarily independent, between μ, σ_z, σ and g_B.

I will now construct a balanced growth path equilibrium for this economy. To do so I assume that the entry rate of new ideas is

$$g_B = g_\eta (1 - \alpha) + \delta \ , \tag{21}$$

implying that the number of ideas above the entry point will be $\theta_1 / (\theta_1 + \theta_2) J_i e^{g_\eta (1-\alpha) t}$. Aggregate variables, w_{it}, C_{it} grow at a rate g_κ where

$$g_\kappa = g_E + g_\eta (1 - \alpha) / (\sigma - 1) \,. \tag{22}$$

The growth rate of the ideas and thus the varieties adds to the growth rate of the frontier of new productivities, g_E, with a rate that is larger when goods are less substitutable.[18]

Finally, notice that in the balanced growth path the cross-sectional distribution of firm sales and the bilateral trade shares, λ_{ij}, remain unchanged. This means that at each point in time, this model collapses to the endogenous cost model of Arkolakis (2010) when $\beta > 0$ and the fixed cost Chaney (2008) model when $\beta \to 0$.

Proposition 1 *Given A1-A2, and the values of g_κ, g_B given by the equations (22) and (21) respectively there exists a balanced growth path for the economy.*

Proof. *By assumption we have that $L_{it} = L_i e^{g_\eta t}$ and $J_{it} = J_i e^{g_\eta (1-\alpha) t}$, and $\bar{z}_{it} = \bar{z}_i \exp(g_E t)$. Define $z^*_{ijt} = z^*_{ij} e^{g_E t}$, such that $z^*_{ij} > \bar{z}_i$, $w_{it} = w_i e^{g_\kappa t}$, $C_{it} = C_i e^{g_\kappa t}$, $P_{it} = P_i$. Given these assumptions and definitions, the cross-sectional distribution of the productivities of operating firms is Pareto. For each cross section of the model, the share of profits in total income equals $\bar{\pi} = (\sigma - 1) / (\sigma \theta_2)$ (see Arkolakis (2010)) and the market share of i to j equals to*

$$\lambda_{ij} = (\tau_{ij})^{-\theta_2} w_i^{-\theta_2} J_i (\bar{z}_i)^{\theta_2} / \left[\sum_{v=1}^{N} (\tau_{vj})^{-\theta_2} J_v (\bar{z}_v)^{\theta_2} w_v^{-\theta_2} \right] \ . \tag{23}$$

In turn, the equilibrium variables w_{it}, P_{it}, z^*_{ijt} are characterized by the trade balance condition

[18] The equilibrium also requires that the value of the aggregate endowment is finite. In order for this to happen the discount rate must exceed the rate of growth of the economy and thus preference and technology parameters must satisfy $\rho + \frac{1}{\iota} g_\kappa > g_\kappa + g_\eta$. This restriction and in particular the values of the parameters ρ and ι play no essential role in my analysis and will not be discussed altogether in what follows.

$w_i L_i = \sum_v \lambda_{iv} w_v L_v$, $\forall i$, *the price index given by (1),* $\forall i$, *and the productivity cutoff condition given by (7) for* $\forall i, j$. *Simply substituting the guessed values of the variables into these equilibrium equations reveals that the guess is correct since the equations hold for* $\forall t$. *It also allows to solve for the values of* z_{ij}^*, w_i, P_i *using the same equations. Finally,* C_i, *can be solved using the budget constraint completing the construction of the balanced growth path.* ■

Moreover, although it is not necessary for the existence of a balanced growth path, I will, in general, restrict the analysis to parameter values that will allow me to match the facts on firm growth rates as a function of firm size. These parameter values will imply that the productivity growth of firms is not too negative, so that there is positive growth, on average, in the extensive margin of consumers for the smaller firms.

R 1 *: Productivity and sales parameters satisfy* $\mu(\sigma - 1) + (\sigma - 1)^2 \sigma_z^2 > 0$.

This restriction will hold true in the calibration.

4 Theoretical Predictions of the Model

I will now proceed to describe the theoretical properties of the model and sketch the connection to the empirical findings presented in section 2. Details for the various derivations are in the appendix. To facilitate exposition I will define some additional terms. Aside from the fact that there is exogenous death of ideas, the productivity of an idea can be considered at a given time $\tilde{t}$ as a new process starting from current productivity $z_{\tilde{t}}$. For convenience, I define a proxy of the relative "size" of an idea from a given origin i to a given destination j when a years have elapsed from some reference time $\tilde{t}$ as,

$$s_{ija} \equiv \ln \frac{z_{\tilde{t}+a}}{z^*_{ij\tilde{t}+a}} \text{ , } a \geq 0. \tag{24}$$

s_{ija} follows a Brownian motion with initial condition s_{ij0}, drift μ, and standard deviation σ_z. Notice that given the expression for sales, equation (8), the variable s_{ij0} and the aggregate variables summarize current firm behavior in market j. In particular, if $s_{ij0} < 0$ the firm does not currently sell in market j.

4.1 Firm Entry and Exit

I first illustrate analytical relationships for firm and cohort survival rates that provide a more intuitive interpretation of the workings of firm-selection in the model. In particular, the survival function for a firm of initial size $s_{ij0} = s_0 > 0$, $S_{ij}(a|s_{ij0} = s_0)$, is defined as the probability of selling in market j after a years conditional on initial size in the market, and is given by

$$S_{ij}(a|s_{ij0} = s_0) = e^{-\delta a}\Phi\left(\frac{s_0 + \mu a}{\sigma_z\sqrt{a}}\right) . \tag{25}$$

where $\Phi(\cdot)$ denotes the cdf of the normal distribution. This expression implies that firms with larger initial size in a market, $s_{ij0} = s_0 > 0$, have higher probability of selling in this market next period. Notice that the survival function only depends on the size of the firm in the market, s_0, implying that the probability of exit of a firm in a market depends on its relative size there.

Integrating the firm survival rates across different initial sizes the model delivers an analytical characterization of the survival rates of a given "incumbent" cohort of firms from i that sell to j,

$$S_{ij}(a) = e^{-\delta a}\left[\Phi\left(\frac{\mu}{\sigma_z}\sqrt{a}\right) + e^{a\left(\frac{\theta_2^2\sigma_z^2}{2}+\theta_2\mu\right)}\Phi\left(-\frac{\theta_2\sigma_z^2 + \mu}{\sigma_z}\sqrt{a}\right)\right] . \tag{26}$$

This expression is strictly decreasing in a if $\mu < 0$.[19] The expression depends on the productivity distribution parameter θ_2 that regulates the relative density of firms around the exit productivity cutoff, z^*_{ijt}, and thus the number of firms that are likely to exit in near the future.

Overall, the model is qualitatively consistent with the evidence on firm exit illustrated in section 2. In particular, consistent with Figure 2 the model generates high attrition for the new entrants, since these firms enter with small sizes, and also for the incumbent cohorts, if the distribution has enough density around z^*_{ijt}. These results are driven by the productivity process adapted by Luttmer (2007) and the entry-exit process postulated in this paper, by dispensing of his assumption of a sunk cost of entry. This feature of the model implies that the average size of entrants is small and by construction equal to the size of exitors, consistent with Fact 1. Notice that given the process for individual productivities, the parameters β and σ do not play any role in entry and exit. These two parameters are crucial for firm growth, which I discuss next.

[19] Since the empirically relevant case will turn out to be $\mu < 0$ I will mainly discuss the prediction of the model under this restriction in the main text.

4.2 Firm Growth

Given the assumption of Gibrat's law in productivity growth and the CES preferences specification, two distinct forces act so that Gibrat's law does not hold for the growth of firms in each market: the selection effects and the market penetration technology. I analyze each of these forces separately.

4.2.1 Firm Selection and Firm Growth

I first examine the changes in the mean and the variance of the natural logarithm of sales for the case of $\beta \to 0$. The moments of the logarithm of sales function can be obtained using the moment generating function. I define the growth over the period of a years as $\hat{G}_{ija} = \log r_{ij\tilde{t}+a}(s_{ija}) - \log r_{ij\tilde{t}}(s_{ij0})$. The expected firm growth given initial size is[20]

$$E\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) = (\alpha g_\eta + g_\kappa)\, a + (\sigma - 1)\, \mu a + (\sigma - 1)\, \sigma_z \sqrt{a} m \left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) . \quad (27)$$

where $m(x) = \varphi(x)/\Phi(-x)$ is the inverse Mills ratio, with $\varphi(x)$ the pdf of the standard normal distribution. The third term of this expression appears because of selection and is decreasing in size, s_0, and converging to 0 for large s_0 (see appendix A.1, property P1). Thus, the force of selection by itself implies that growth rates are declining in initial size. Gibrat's law is approximately true for the largest firms, which are unaffected by the selection forces.

The variance of firm growth given initial size is

$$\mathcal{V}\left(\hat{G}_{ija}|s_{ija} > 0, s_{ij0} = s_0\right) = (\sigma - 1)^2 \sigma_z^2 a \left\{1 - m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right)\left[m\left(-\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) + \frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right]\right\} .$$

The term in the brackets incorporates the effects of selection and can be shown that it is increasing in its argument, $\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}$.[21] In turn, $\mathcal{V}$ is increasing in s_0 and in fact for large s_0 it converges

[20] The correction for the selection bias is different from the specification of Heckman (1979) in that entry and sales decision are perfectly correlated in my case (both driven by productivity shocks). Partial correlation can be generated, for example, if there exists randomness in a term that would influence entry but is not perfectly correlated to sales. The obvious candidate term in this model is the parameter $1/\psi$. The econometric techiques developed to adjust for selection bias by Heckman (1979) could be appropriate for this case. Such an approach has been used by Evans (1987b).

[21] The proof can be found in Sampford (1953). More generally, the result that the left truncated variance is decreasing in the truncation point (and thus is increasing in the size of the firm) holds for all distributions with logconcave pdf (see An (1998)). This set of distributions includes the normal. Similar monotonicity results cannot be obtained for the variance of log sales because the lognormal distribution is neither logconcave nor logconvex in its entire domain.

to $(\sigma-1)^2 \sigma_z^2 a$. Straightforward intuition implies that given that the normal distribution of growth rates is unimodal, censoring of the negative growth rates will reduce the variance of firm growth rates.

A number of instructive conclusions can be derived from the above derivations. First, the selection mechanism alone implies that surviving small firms grow faster than larger firms. Second, the same mechanism also implies that the variance of firm growth would increases with firm size, an implication in sharp contrast to Fact 2.[22]

4.2.2 Market Penetration Technology and Firm Growth

I now turn to study the effects of different specifications in the market penetration technology on the growth of firms, independently from the selection effect. To do so, I characterize the instantaneous growth rate of the firm in each market, which is not affected by entry and exit. This analysis can be performed by applying Ito's lemma to expression (8) for firms with initial size in the market of $s_{ij0} = s_0 > 0$,[23]

$$\frac{dr_{ij\tilde{t}}(s_0)}{r_{ij\tilde{t}}(s_0)} = \left[\alpha g_\eta + g_\kappa + \mu \frac{h'(s_0)}{h(s_0)} + \frac{1}{2}\sigma_z^2 \frac{h''(s_0)}{h(s_0)}\right] da + \left[\sigma_z \frac{h'(s_0)}{h(s_0)}\right] dW \ , \tag{28}$$

where

$$h(s_0) = e^{\bar{c}_1 s_0} - e^{\bar{c}_2 s_0} \ .$$

In equation (28) the first and second parenthetical terms represent the (instantaneous) growth, $E(dr/r)$, and the standard deviation of growth of a firm of size s_0 respectively. Proposition 2 characterizes the relationship between the instantaneous growth rates of firms of size $s_{ij0} = s_0$ in a given market for different values of β:

Proposition 2 *Given A1-A2 and R1,*

a) If $\beta \to 0$ the instantaneous growth rate of all firms is the same.

[22] In the Klette and Kortum (2004) model, the variance unconditional on survival is inversely proportional to firm size. The decrease in the variance with firm size happens since the sales of the firm are proportional to the number of goods that the firm has. Since each good has the same variance, the total variance of firm sales is inversely proportional to firm size in that model.

[23] See Dixit and Pindyck (1994) chapter 3 for the details of Ito's lemma and related derivations. Since Brownian motion paths exhibit infinite variation for any given time interval standard calculus does not apply. The application of Ito's Lemma requires the sales function to have a continuous second derivative. The function $h(s)$ does so for $s > 0$ but it does does not attain continuous derivatives at $s = 0$.

b) There exist a $\beta' \in (0, +\infty)$, such that $\forall\ \beta > \beta'$, then $\partial (E(dr/r))/\partial s_0 < 0$, and $\forall \beta < \beta'$, then $\partial (E(dr/r))/\partial s_0 > 0$, for all firms with $s_0 > 0$.

Proof. *To prove part (a) of proposition 2 I use De l' Hospital rule to compute the terms in expression (28) for $s_{ij0} = s_0$,*

$$h'(s_0)/h(s_0) \overset{\beta \to 0}{\longrightarrow} (\sigma - 1) \ , \ \frac{h''(s_0)}{h(s_0)} \overset{\beta \to 0}{\longrightarrow} (\sigma - 1)^2 \ . \tag{29}$$

To prove part (b) I look at the derivative of the first parenthetical term in expression (28) with respect to s_0. In appendix A.4, I show that the sign of this derivative is negative if and only if

$$\beta \geq \frac{(\sigma - 1)^2 \sigma_z^2}{2\left[\mu(\sigma - 1) + (\sigma - 1)^2 \sigma_z^2\right]} > 0 \ . \tag{30}$$

If R1 is not satisfied there does not exists a value of β for which the growth rates are decreasing in size. The growth rate for very large firms, $s_0 \to \infty$, is the same as the growth rate of all firms for $\beta \to 0$. ■

For $\beta > 0$, the model with endogenous market penetration costs also predicts an inverse relationship between the sales of firms in a market and the instantaneous variance of their growth rates for that market as illustrated in the next proposition:

Proposition 3 *Given A1-A2*

a) If $\beta \to 0$, the instantaneous variance of the growth rate is independent of firm initial size.

b) If $\beta > 0$, the instantaneous variance of the growth rate is higher the smaller the firm initial size.

Proof. *See appendix A.4.* ■

Overall, the endogenous cost model, with a high enough β, reconciles the inverse size-growth relationship with the inverse size-variance of growth relationship, as indicated in Fact 2. The intuition for both results is simple: for a given percentage change in firm productivity changes in the marketing margin are large for firms with few consumers. Thus, for firm of small relative size the effective demand elasticity is very large, and their growth rate and volatility can be large as well. The largest firms make only small adjustments to their marketing margin and the demand elasticity for these firm is asymptotically constant, as discussed in the proof of Proposition 2.

5 Calibration

The goal of this section is to determine the parameters of the model without using information on the growth of firm sales. The procedure that I propose allows to evaluate the ability of the model to *predict* the relationship between firm size and firm growth without using micro-data on individual firm-growth rates. In order to do so, I exploit estimates from the trade literature by mapping the multi-market structure of the model to information on exports to different countries.

As a rule, I choose the parameters that affect the cross-section of country trade flows and firm sales, θ_2, β, and $\sigma - 1$, using the results of the estimation of Eaton and Kortum (2002), and information from the French exporting dataset of Eaton, Kortum, and Kramarz (2011) as exploited for the static version of this model in Arkolakis (2010). The moments in the French dataset allow me, given θ_2, to identify β and $\sigma - 1$ independently from other parameters of the model. In particular, these two parameters affect the predictions of the model for firm-size distribution –and firm growth– but do not impact the prediction of the model for firm selection (since they do not affect the evolution and steady state distribution of firm *productivities*).

To calibrate the stochastic process of firm productivities and the parameters determining the balanced growth path, information on firm exit rates and US macroeconomic aggregates is used. US data are easily accessible for all these statistics. Table 2 provides a summary of the model calibration that I discuss in detail below and the sources used.

One potential concern is the use of moments from two different countries for the calibration of the model. While I have access to different moments from US and France both these countries are developed economies. In addition, the distribution of sales of manufacturing firms in France and the USA displays remarkable similarity as argued in Section 2. Having pointed out the similarities across the two datasets I proceed using them to calibrate the model's parameters.

5.1 Parameters from the static model

For the calibration of the parameters that determine the cross-section of sales I follow Arkolakis (2010) since each cross-section of the dynamic model is identical to that setup. This paragraph briefly describes his procedure. The parameter α governs firm entry as a function of the population of the market and is set to $\alpha = .44$ to match the entry of French exporting firms into markets with different population size. The parameter β and the ratio $\tilde{\theta} = \theta_2 / (\sigma - 1)$, jointly determine the cross-sectional sales heterogeneity. The choice of $\beta = .915$ and $\tilde{\theta} = 1.645$ implies that the model matches the size advantage in the domestic market (France) of prolific exporters compared to firms that export little or not at all.

Moreover, I exploit the mapping to the static model in order to calibrate the values of θ_2 and σ, given the value of $\tilde{\theta}$. The parameter θ_2 is key in determining the aggregate elasticity of bilateral trade flows and the welfare properties of a wide class of multi-market models as argued by Arkolakis, Costinot, and Rodríguez-Clare (2010). Thus, I use the estimate for the trade elasticity of Eaton and Kortum (2002), $\theta_2 = 8.28$, which falls in the middle of the range of estimates reviewed by Anderson and Van Wincoop (2004). I retain this parameter fixed when I look at the predictions of the fixed cost model. Given $\theta_2 = 8.28$ the value of σ that is consistent with $\tilde{\theta} = 1.645$ is $\sigma = 6.02$. This value of σ is in the ballpark of the estimates of Broda and Weinstein (2006) and also implies a markup of around 20% which is consistent with values reported in the literature (see Martins, Scarpetta, and Pilat (1996)).[24] For the fixed cost model, $\beta = 0$, I set $\tilde{\theta} = 1.49$, which is the benchmark calibration of Arkolakis (2010) for that model, which given $\theta_2 = 8.28$ requires $\sigma = 6.55$ for that model. The effect of changing $\tilde{\theta}$ in the fixed cost model is discussed in the specific exercises.

5.2 Parameters governing dynamics

To determine the value of the parameters that govern the aggregate dynamics of the model, I use macroeconomic data for the US economy and US manufacturing census data from DRS. The parameters g_η, g_κ, g_E govern primarily aggregate dynamics. The growth of the population from 1960 onwards in the US is around 1.22% and the growth rate of real GDP per capita is around 2%. Thus, I set $g_\eta = .0122$ and $g_\kappa = .02$. Given the definition of g_κ, the growth of the technological frontier of new ideas is

$$g_E = g_\kappa - g_\eta (1 - \alpha) / (\sigma - 1) = 0.0187 .$$

The parameters δ, g_I and σ_z, which govern firm dynamics, must also be specified. In the

[24] Whereas I use the mapping of my model to the Eaton and Kortum (2002) setup to use their estimate of 8.28 Luttmer picks the value of the coefficient to be 9.56 to match the estimate of the upper tail of the size distribution by Axtell (2001), $\tilde{\theta} = 1.06$, with a choice of $\sigma = 10$. This difference makes for most of the (small) differences in the calibrated parameters, g_I, g_E and σ_z that I obtain below versus the ones used by Luttmer (2007).

model, δ regulates the exogenous death rate. Given that the probability of endogenous exit for firms with large size is (practically) 0, I calibrate δ by looking at the death rate of these firms. This information is obtained by the US Manufacturing Census during the period 1996-2004, where the tabulation of the largest manufacturing firms is the one for 500 or more employees. The data indicate an average exit rate of 0.89% per year for these firms, in turn, δ is set at 0.0089.

The parameters g_I and σ_z govern productivity (and thus firm-) dynamics. Note that, θ_2, which is an explicit function of these two parameters (equation (18)), was calibrated to the value of 8.28. Thus, to jointly calibrate g_I and σ_z, requires one more moment from the data which is a function of these two parameters in the model. I obtain this information from the data by looking at the cohort exit rates of US manufacturing firms as reported by DRS. I use the exit rate of 42% in the first 4 years for the first cohort analyzed by DRS. Using equations (18) and (26) together with the empirical values for the elasticity of trade and the cohort exit rates a simple method of moments implies the values $g_I = 0.24\%$ and $\sigma_z = 6.64\%$. These parameter values imply that $\mu = g_I - g_E = -1.63\%$ for the incumbent firms.

Notice that μ, σ_z, and δ are present in equations (18) and (26) while σ and β do not affect these relationships. Additionally, for the *firm* statistics generated by the model the difference $\mu = g_I - g_E$ is important and not the value of the parameters separately.[25] The crucial parameters in this calibration are β, σ, μ and σ_z. Given that θ_2 is a function of the last two the dynamic model has only one extra key parameter than its static counterpart. Thus, the model maintains contact with previous static heterogeneous firm theories but nevertheless generates a series of reliable quantitative predictions on firm dynamics studied below.

Discussion of the calibrated parameters With the calibrated parameters, the expected growth rates of the largest firms are very close to zero, consistent with the related numbers reported by Davis, Haltiwanger, Jarmin, and Miranda (2007) and Davis, Haltiwanger, Jarmin, Krizan, Miranda, Nucci, and Sandusky (2009). The standard deviation of the sales of the largest firms converges to $(\sigma - 1)\,\sigma_z = 33\%$, around 2 to 3 times what is reported in the data by Davis, Haltiwanger, Jarmin, and Miranda (2007) and Comin and Mulani (2007) and also larger than other estimations of models of firm dynamics as in Hopenhayn and Rogerson (1993) and Lee and Mukoyama (2008).[26]

[25] Thus, there is an additional degree of freedom in the calibration. I use this extra degree and chose g_E so that g_κ is the same in all model specifications even if σ's might differ (equation 22).

[26] This shortcoming of the model is the topic of Luttmer (2011).

An obvious question that arises from this calibration is whether the assumption of a random walk in logarithms is the right representation for firm productivities. Ultimately, this question is one of empirical nature, so that it falls beyond the scope of this paper. However, while the predictions of the model with random walk will be tested in various dimensions, it worth pointing out two things. First, despite the fact that this model is specified with a random walk in productivities it still generates an autocorrelation of firm sales of less than one. Second, even for models that assume that firm productivities follows an AR(1) process, very often the estimates of the autoregressive coefficient of the productivity process that are more consistent with the data are close to one (see for example Hopenhayn and Rogerson (1993), Lee and Mukoyama (2008), and Alessandria and Choi (2007)).

6 Quantitative Results

I now turn to look at the predictions of the calibrated model for firm exit, growth and the firm-size distribution. Effectively, all of the predictions of the model below are out of sample predictions.

6.1 Firm Exit

In the next two subsections, I study the implications of the model for the patterns of exit and sales growth for incumbent and entry cohorts. To compare the outcomes of the model with the DRS statistics I look at the predictions of the model every five years, effectively conducting a five-year census in the model generated data. To measure entrants and exitors I resemble as closely as possible the DRS methodology and classify a firm that exits in a census-year and re-enters in a later census as an exitor when it leaves and as an entrant when it re-enters.[27]

The predictions of the calibrated model for the cumulative exit rates of incumbent and new cohorts are illustrated in Table 3. The model closely matches the exit rate of incumbent cohorts for a period of two decades as well as the entry cohort exit rates for a period of 15 years. The high exit rates for the first cohort years are the implication of the high concentration of firms close to the cutoff productivity. The fraction of firms concentrated close to this cutoff is naturally higher among entry cohorts firms, which is the feature of the model that accounts for the differences in exit rates among entry and incumbent cohorts.

[27] See footnote 13 in DRS.

Discussion of the results: sunk costs of entry The success of the model in generating the exit patterns in the US census data challenges the view that sunk costs are *necessary* to explain the entry and exit behavior of firms. Whereas the existence of a sunk cost would not refrain the model from generating the exit rates observed in the US data this existence would nevertheless imply that the average size of entrants is larger than the average size of exitors. Different calibrations of the Luttmer (2007) model (analyzed in appendix A.5) suggest that the model requires the average size of entrants to be about 15%-25% larger than the average size of exitors in order to match the exit rates in the data, which comes in sharp contrasts to the evidence presented in Table 1.[28] For the above reasons, notwithstanding its tractability, the marketing theory that I introduce appears to be a valuable tool for the analysis of the entry and exit patterns of firms. Below, I also discuss its implications for firm growth and sizes that crucially depend on β and σ.

6.2 Firm Growth

To illuminate the prediction of the two models in terms of firm growth Figure 7 presents a scatterplot of the ratio of firm final to initial size on firm initial size (its initial sales percentile). The graph effectively illustrates the distribution of firm growth rates conditional on firm initial size.

Starting from the simpler case of the calibrated model with $\beta = 0$, in the right panel of the Figure, firms with negative productivity growth may select out of the market, the more likely so the lower their initial size. This selection mechanism implies that the expected growth rate is inversely related to size (expression (27)) but it also implies that the variance of the distribution of firm growth rates increases with firm size, in sharp contradiction with Fact 2. Introducing the marketing choice of firms implies that the distribution of growth rates is fundamentally different, with the case of $\beta = .915$ featuring some small firms with phenomenally high growth rates. Thus, the relationship between firm growth and initial size is still inverse. In addition, due to the large volatility of the marketing margin, the endogenous cost model reconciles this relationship with the inverse relationship between the variance of firm growth and initial firm size.

[28] A model with large sunk costs generates hazard rates for entry cohorts that are initially increasing, as argued by Ruhl and Willis (2008). This implication is counterfactual both for the DRS data and other datasets as noted by Ericson and Pakes (1998).

I also compare the predictions of the model to the average size of incumbent and new cohorts in DRS in Table 3. The benchmark model can match the small contribution of the entering cohort but eventually overpredicts the size of the survivors in this entry cohort (and underpredicts the average size of firms in the incumbent cohort). Thus, the model probably implies more growth for the new entrants/small firms than what is seen in the DRS data. The main tension between the fixed cost model and the data is that the model overpredicts the size of the entrants. Decreasing $\tilde{\theta}$ (increasing σ) increases the dispersion of sizes between small and large firms for both incumbent and entry cohorts. As illustrated in the online theory appendix, for $\tilde{\theta} = 1.25$ the fixed model achieves satisfactory predictions in this dimension but for higher and lower $\tilde{\theta}$ the model substantially deviates from the data. As it will be obvious from the above discussion, different values of $\tilde{\theta}$ do not improve other key shortcomings of the fixed cost model.

In Figure 8 I illustrate the quantitative predictions of the model regarding the variance of growth rates using available moments for publicly traded US manufacturing firms analyzed by Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997) and tabulated by employment bins (1-10 employees, 10-100 etc). In general, the endogenous cost model overpredicts the variance of growth rates observed in the data (except for the size bin of 10-100 employees), but captures the inverse relationship of variance of firm growth and firm size. The fixed cost model predicts a (slightly) increasing relationship.

Discussion of the results: declining sales elasticity and learning I have argued that carefully modeling marketing costs within a setup where productivities follow Gibrat's law can bring the predictions of the model related to firm growth very close to what is observed to the data. This modeling of marketing generates a demand elasticity that declines with firm size, so that the theory lends itself to a variety of different interpretations, as discussed in Arkolakis (2010) (e.g. heterogeneous consumer tastes, multiple products etc).

A different explanation from the one I propose is the idea of learning as suggested e.g. by Jovanovic (1982) or Eaton, Eslava, Krizan, Kugler, and Tybout (2009). Learning could serve as an explanation for the relationship between firm exit and growth on size or age. The calibration proposed for the model in this paper can be considered to have an advantage versus that approach. On the one hand, the predictions of a learning model depend on the parameterization of the demand function, the distribution of prior productivities, and the current shock to productivities. This challenging task has received only little attention in the literature so far.[29] On

[29] See the discussion of Jovanovic (1982) Dunne, Roberts, and Samuelson (1989) and Ericson and Pakes (1998).

the other hand, the calibration proposed for this paper generates plausible predictions whereas it is not at all clear that the learning explanation by itself can has as much predictive power.

6.3 Firm Size Distribution

The predictions of the model for the overall firm size distribution are the ones analyzed by Arkolakis (2010) and are illustrated, for completeness, in Figure 9. Both models can match the Pareto right tail of the distribution but only the model with $\beta > 0$ can explain the existence of many small firms in the data (deviations from Pareto). In the model with $\beta > 0$ small firms endogenously reach very few consumers leading, at the same time, to a failure of Gibrat's law and to a size distribution more curved than the Pareto for that model.

Figure 10 illustrates the predictions of the model for the size distribution of firms with different ages.[30] Due to selection and the stochastic evolution of productivities, the size distribution shifts to the right with age. In addition, the endogenous cost model *quantitatively* generates the main features in the Brazilian data of Cabral and Mata (2003): it implies enough dispersion for the distribution of young firms whereas the distribution of older firms eventually approximates a log-normal distribution. Two differences arise with the fixed cost model, which falls short of predicting the employment size distribution observed in the Portuguese data. First, given that entrants are concentrated around the entry point, the absence of the marketing decision implies a small dispersion in the size distribution, and thus almost all young firms are concentrated around the minimum employment size. Second, because of the fact that selection forces act only on the left tail of the distribution, absent of the marketing choice the shape of the distribution of old firms is not symmetric. This choice primarily affects the left tail of the distribution and thus it remedies the shortcomings of the fixed cost model by generating a roughly symmetric distribution for the older firms.

6.4 Additional Evidence: International Trade

To subject the theory into further scrutiny and evaluate its predictions for turnover and growth into different markets I use the Brazilian manufacturing exporting transactions data to the

For a related structural approach see Abbring and Campbell (2003).

[30] Cabral and Mata (2003) use the largest worker tenure in the firm to approximate firm age. To generate statistics from the model as comparable as possible to theirs I consider a firm as an exitor the first year it has zero sales. Thus, a 15 year old firm is one that operates continuously for 15 years. I also calibrate ψ to match the average size of manufacturing Portuguese firms.

different Brazilian exporting destinations. These data only contain information on export sales but all the theoretical predictions of the model apply to the entry-exit and sales behavior of the firm in a given destination market.[31] The data cover the universe of Brazilian merchandise exporting transactions from 1990-2001 and are described in Molina and Muendler (2008) and Arkolakis and Muendler (2010). I aggregate these data at the firm-destination level and consider the top 50 exporting destinations.

In the online data appendix I replicate the Figures 2 and 3 and argue that qualitatively the patterns of exit and growth of the Brazilian exporters are very similar to the ones illustrated in the US manufacturing data by DRS. The main difference that stands out in the export data is the higher exit rate of exporters in their early years. For example the exit rate of entry cohorts is around 15 percentage points higher in the export data for both incumbent and entry cohorts. The resulting stronger effects of selection are also reflected in a higher growth rate for the surviving exporters. Notice that the robust feature of the US manufacturing census data that the average size of entrants and exitors is approximately the same repeats itself in the Brazilian data, a finding already pointed out by Eaton, Eslava, Kugler, and Tybout (2008) for Colombian manufacturing exporters.[32]

Exporting transactions data are well-suited to test the predictions of the model regarding the i) entry-exit process ii) and the distribution of firm growth rates conditional on initial size. In particular, exporting transactions data allow to monitor firms that continuously enter and exit from certain markets even if their size in these markets is very small. Regarding the first prediction, the calibrated model predicts that a firm that exits a market has a high probability of exporting again there in the near future. For example, in the model, 36% of the times that a firm exits a market it will return to the same destination in the next 3 years. For the Brazilian data the mean of this probability across the sample years is strictly positive, but slightly lower, around 28%. To test the second prediction I look at Brazilian exporters and their growth at the top Brazilian destination, the United States. Figure 11 creates a scatterplot of conditional

[31] In the online data appendix, I test the prediction that the growth of the firm in each market depends on its size in the market and not its overall size (captured by a firm fixed effect) and find strong support for this hypothesis. Correlation on the sales of the firms across different destinations can be added in the model, without affecting this result, by assuming that the productivity (or equivalently some random demand shock) for each destination is the weighted sum of independent Brownian motions. The specification of the weights can determine the correlation across destinations.

[32] For the robustness of the exporter-size distribution see Eaton, Kortum, and Kramarz (2011) and Arkolakis and Muendler (2010). See also Eaton, Eslava, Kugler, and Tybout (2008) and Albornoz, Corcos, Ornelas, and Pardo (2009) for evidence on the growth-size relationship of exporters.

growth rates of exporters comparable to Figure 7 for the model. The distribution of growth rates is very much in line with the predictions of the model with $\beta > 0$: some small firms exhibit extremely high growth rates and the distribution of small firm growth rates has much higher variance (more details available in the online data appendix). Despite its simplicity the endogenous cost model captures the main salient features of the growth rates of exporters.

7 Application: The Size-Growth Debate

Do small firms grow faster? This question has been at the center of an academic and policy debate over the past three decades. The evidence for violations of the independence of firm growth rate and size (i.e. violations of Gibrat's law) reviewed in section 2.2 are cited as justification for differential treatment for small businesses (see for example Birch (1981, 2010)).

The comprehensive econometric analysis by Evans (1987a,b) shows that the negative growth-size relationship is robust to controlling for sample truncation caused by the exit of smaller firms. Methodologically, Davis, Haltiwanger, and Schuh (1996) and Haltiwanger, Jarmin, and Schuh (2010) challenge the inverse size-growth relationship in the basis of the interaction of size classification and possible regression to the mean –the tendency of firms that experience a growth shock in one period to experience an opposite shock in the next one–. They instead propose an alternative measure of firm growth by considering firm initial size as its mean size in the two periods. Using this metric they find that the size growth relationship disappears in many of their specifications.

Whereas I have argued that small firms grow faster, many of them at very high growth rates, when I use the metric of growth proposed by Davis, Haltiwanger, and Schuh (1996) to measure growth in the model the inverse size growth relationship disappears, as discussed in more detail in the online theoretical appendix. This happens since the productivity growth in the model is not a mean reverting process and thus there is no miss-classification bias for smaller firms and of course no measurement error. Thus, small firms are being misclassified in high initial sizes, *exactly* when they grow fast, a weakness of their metric also pointed out by Neumark, Wall, and Zhang (2011). Of course, whether the productivity process of firms is better represented by a random walk or a mean reverting process is an unresolved question where perhaps future empirical work will be of most value. Instead this paper illustrates how new theories can generate predictions that test the size growth relationship in a variety of dimensions.

8 Conclusion

This paper develops a simple unified framework to analyze firm selection and growth. The framework is based on the modeling of marketing choice –and thus demand– at the firm-level. The model allows for a parsimonious calibration that nevertheless implies a good fit to a number of statistics in the data. The success of this framework suggests that carefully modeling marketing costs could be a promising avenue for a deeper understanding of firm dynamics.

A key modeling simplification used in this paper is that firms incur the marketing cost to reach consumers in each period, without being able to build continuing customer relationships. This simplification made the analysis highly tractable by keeping firm efficiency as the only current decision state of the firm. Other approaches, such as the customer capital models of Drozd and Nosal (2011) and Gourio and Rudanko (2010), or the buyer-seller matching model of Eaton, Eslava, Krizan, Kugler, and Tybout (2009) explore various ways of formalizing this accumulation process. This research may lead to models that account for the differences in the short-run versus long-run behavior of firms and also the importance of age in their decisions.

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Statistics/Census Year	1963	1967	1972	1977	1982	mean
Entrants Relative Size	-	.352	.396	.308	.346	.350
Exitors Relative Size	.353	.399	.338	.351	-	.360

Table 1: Size of Firms Entering and Exiting Relative to All Firms Source: Numbers calculated using market shares (Table 2) and exit rates (Table 8) from DRS

Benchmark Param.	Value	Source/Target
Cross Section		**Cross-sectional exporting data**
α	0.44	Arkolakis (2010)
θ_2	8.28	Eaton & Kortum (2002)
σ	6.02	Sales advantage of prolific exporters in France:
β	.915	Arkolakis (2010), Eaton, Kortum & Kramarz (2011)
Balanced Growth		**US macroeconomic aggregates**
g_η	0.0122	US population growth
g_κ	0.02	US GDP growth
g_E	0.0187	US GDP growth
Idiosyncratic Product.		**US manufacturing Census data**
δ	0.0089	Death rate of firms with 500+ employees
g_I	0.0024	Exit rates of 1963 cohort from Dunne, Roberts
σ_z	0.0664	& Samuelson (1988) (& the value of θ_2)

Table 2: Model Parameterization

Statistic / Cohort year	0	5	10	15
Entry cohort exit rate				
Data (mean '67-'77 cohorts)		0.62	0.79	0.88
model		0.63	0.80	0.88
Inc. cohort exit rate				
Data (mean '67-'77 cohorts)		0.48	0.65	0.76
model		0.47	0.67	0.79
Entry cohort mean sales				
Data (mean '67-'77 cohorts)	0.35	0.61	0.99	1.32
model ($\beta > 0$)	0.38	0.89	1.37	1.85
model ($\beta = 0$)	0.61	0.92	1.23	1.55
Inc. cohort mean sales				
Data (mean '67-'77 cohorts)	1.02*	1.65	2.37	3.07
model ($\beta > 0$)	1.00	1.63	2.22	2.82
model ($\beta = 0$)	1.00	1.41	1.83	2.27

Table 3: Cohort Exit Rates and Average Sizes in the Data and the Model Source: DRS and model simulations. Mean sales are constructed from DRS for the 1967, 1972, and 1977 censuses that were conducted every 5 years using exit rates and market shares. Approximation error appears due to this construction and the fact that numbers are representing means across 4-digit industries

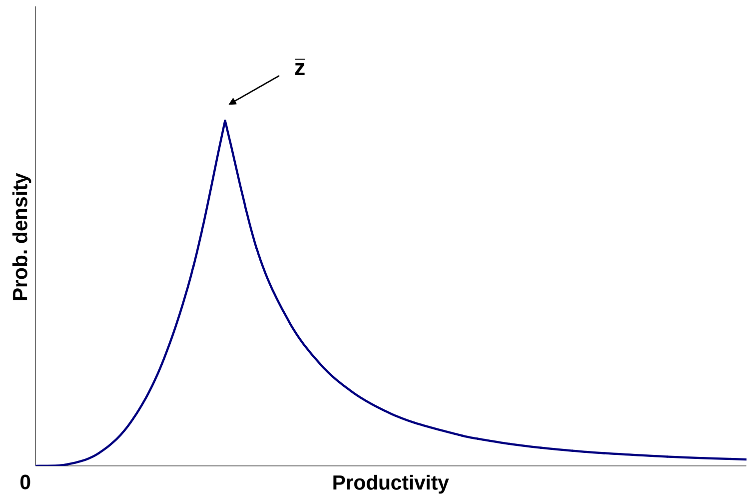

Figure 1: Double Pareto distribution

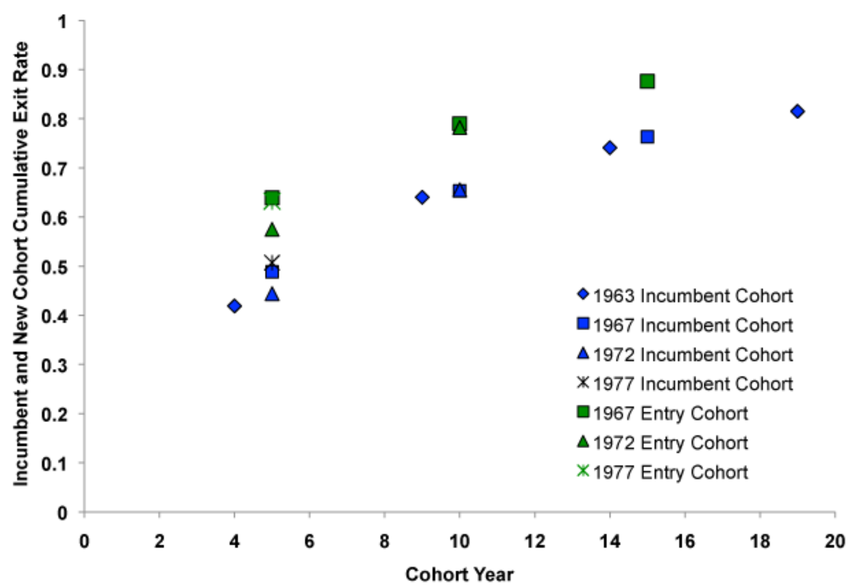

Figure 2: Incumbent and Entry Cohorts Exit Rate in the US Manufacturing Census

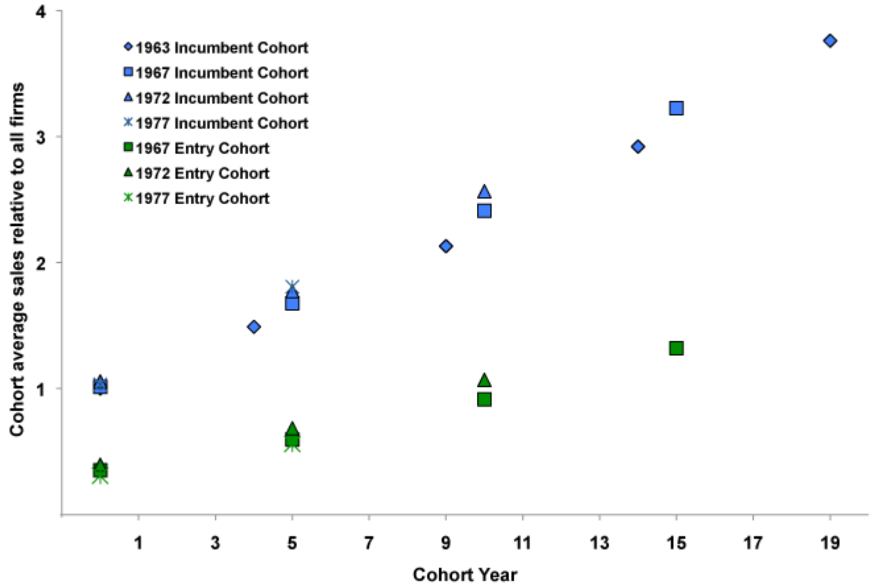

Figure 3: Incumbent and Entry Cohort Average Sales (of surviving firms) in the US Manufacturing Data

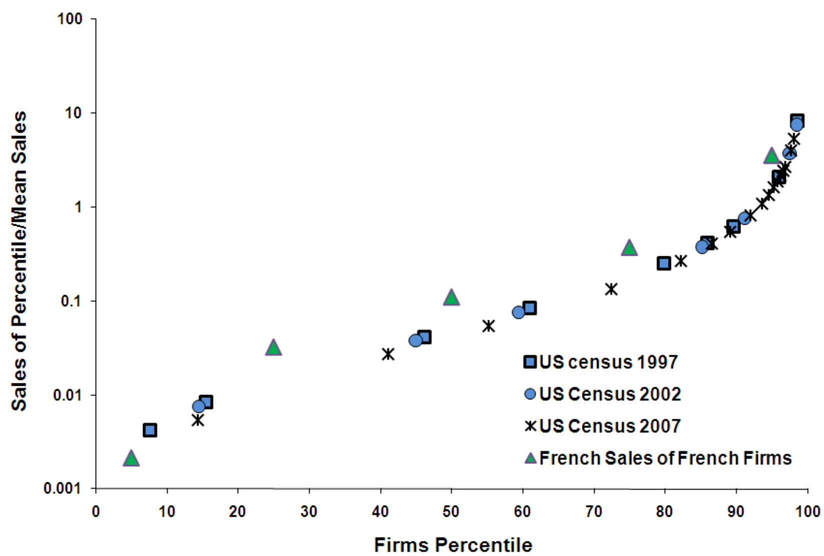

Figure 4: Distribution of total sales of US manufacturing census firms and of French sales of French manufacturing.
Source: US data are obtained from the Small Business Administration and French data from Eaton, Kortum, and Kramarz (2011). Note: For the US census firms the maximum point of each bin reported and the number of firms included in the bin is used to plot the sales size of the firms and the corresponding percentiles. Mean sales are referring to mean sales of all US manufacturing census firms and mean sales in France of French manufacturing firms.

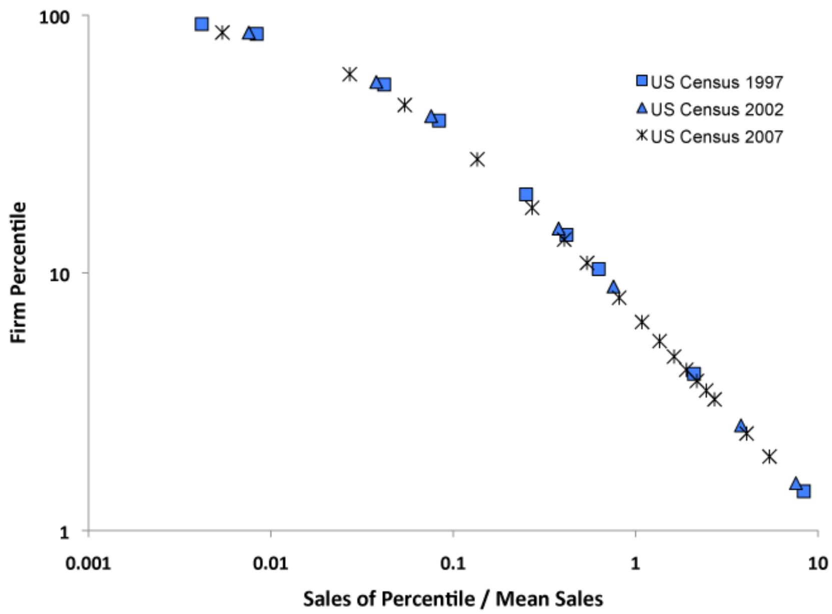

Figure 5: Distribution of total sales of US manufacturing census firms.
Source: Small Business Administration. Note: I use the maximum point of each reported bin and the number of firms included in the bin to plot the sales size of the firms and the corresponding percentiles.

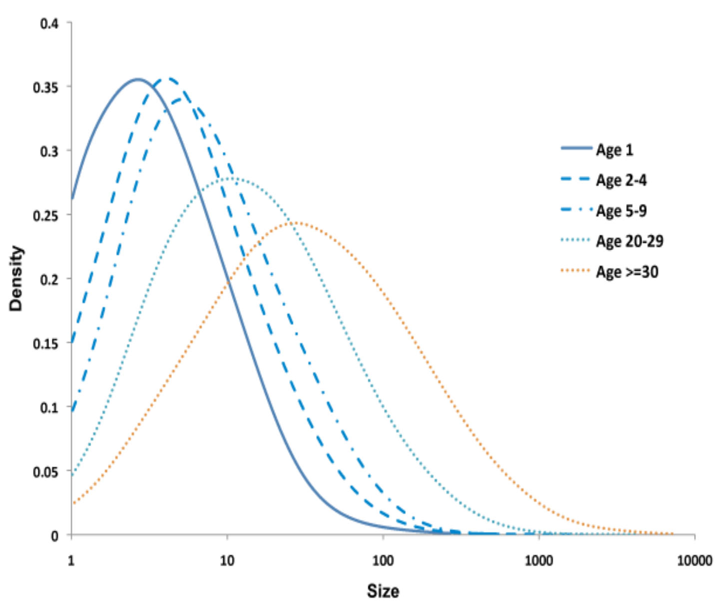

Figure 6: Distribution of Sales by Age Cohort. Source Cabral and Mata (2003).

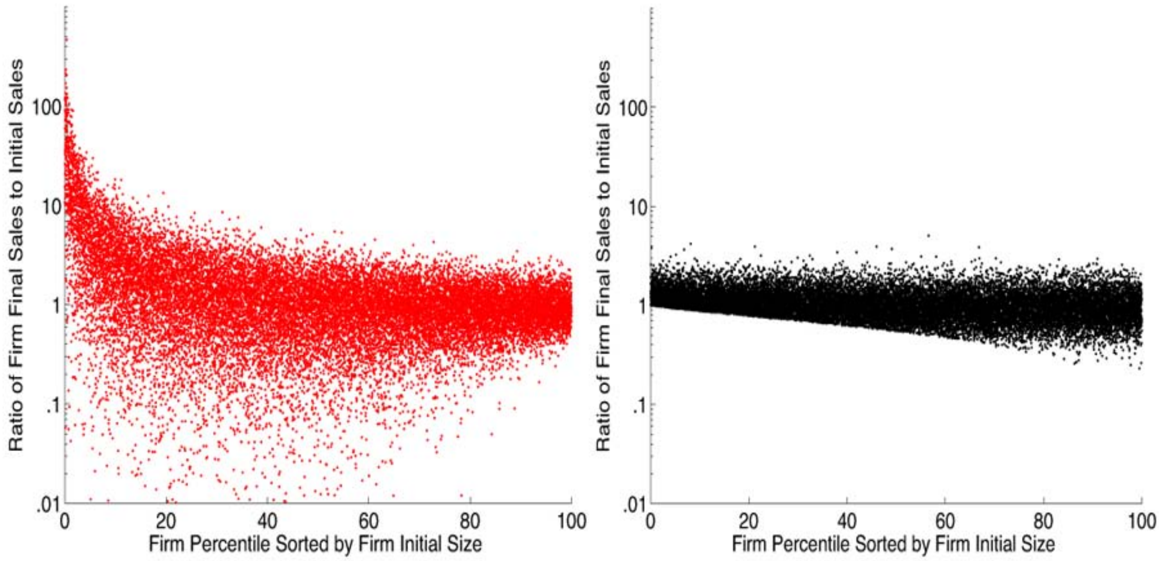

Figure 7: Firm Percentiles and Firm Growth Rates in the Calibrated Model. Endogenous Cost (left panel) and Exogenous Cost (right panel).

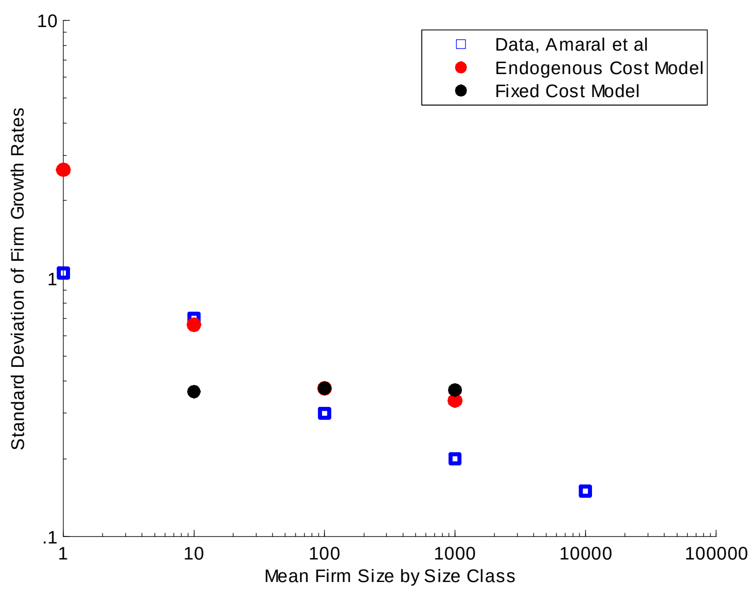

Figure 8: Standard Deviation of Firm Growth Rates and Firm Initial Employment size. Source: Data for US Manufacturing firms in compustat tabulated by Amaral, Buldyrev, Havlin, Leschhorn, Mass, Salinger, Stanley, and Stanley (1997).

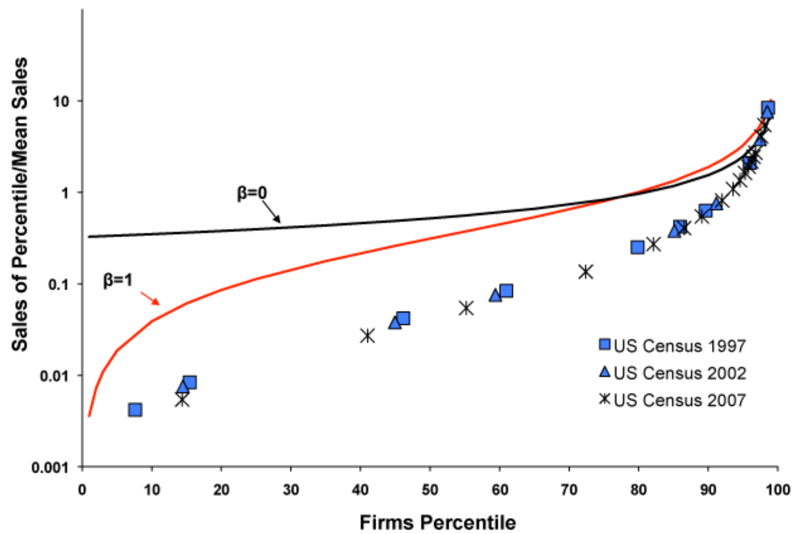

Figure 9: Size Distribution of US Manufacturing Firms and the Model.

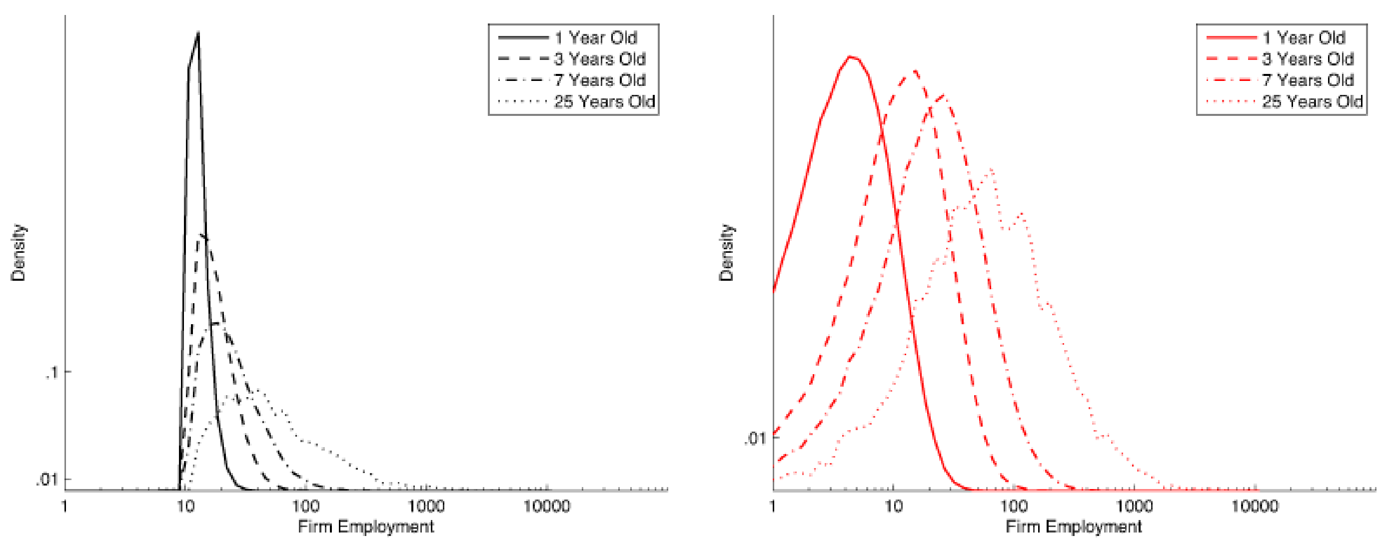

Figure 10: Employment Distribution for Firm with Different Ages in the Endogenous Cost (left panel) and Fixed Cost (right panel) Models.

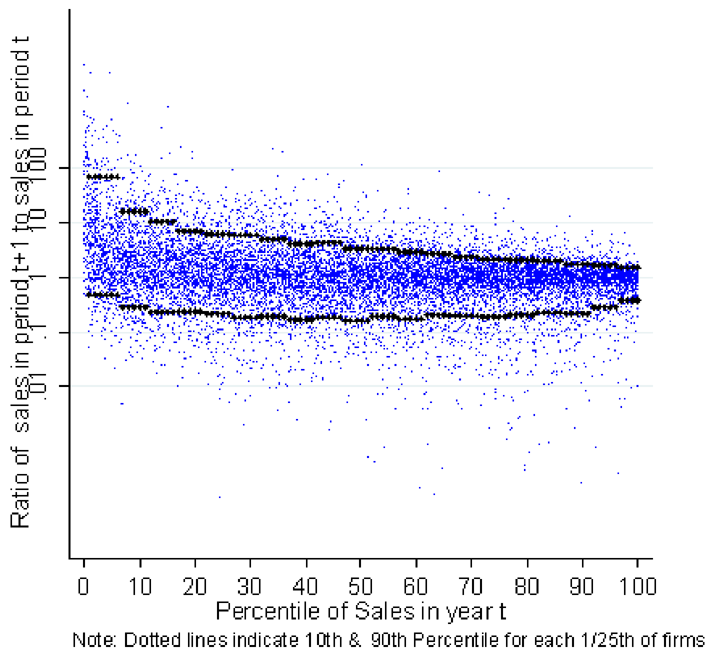

Figure 11: Brazilian Exporter Percentiles and Exporter Growth Rates in the United States

Source: Brazilian exporting transactions data 1990-2001.

A Appendix

A.1 Preliminary definitions and facts

In the various proofs and derivations of this appendix I am going to use the following definitions and well known facts for the Normal distribution quoted as **properties P**. Notice that the pdf of the simple normal distribution with mean 0 and variance 1 is given by $\varphi(x) = \frac{e^{-x^2}/2}{\sqrt{2\pi}}$ and the cdf by $\Phi\left(\frac{x-\mu}{\sigma_z}\right) = \frac{1}{\sigma_z\sqrt{2\pi}}\int_{-\infty}^{x} e^{-(\tilde{x}-\mu)^2/2\sigma_z^2} d\tilde{x}$.

P 1 *The inverse mill's ratio of the Normal, $\varphi(x)/\Phi(-x)$, is increasing in x, $\forall x \in (-\infty, +\infty)$.*

P 2 *The error function is defined by:* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-(\tilde{x})^2} d\tilde{x}$.

P 3 $\Phi(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$, *where $\Phi(x)$ is the cdf of the standard normal cdf*

P 4 $\int e^{-\tilde{c}_1 x^2 + \tilde{c}_2 x} dx = e^{(\tilde{c}_2)^2/4(\tilde{c}_1)}\sqrt{\pi}\operatorname{erf}\left(\frac{2\tilde{c}_1 x - \tilde{c}_2}{2\sqrt{\tilde{c}_1}}\right) / \left(2\sqrt{\tilde{c}_1}\right)$, *for some constants $\tilde{c}_1$, $\tilde{c}_2 > 0$*

A.2 Deriving the Stationary Distribution of Productivities

A simple guess for the solution of the Kolmogorov equation (11) is $f(s) = A_1 e^{\theta_1 s} + A_2 e^{-\theta_2 s}$ where θ_1 and $-\theta_2$ are given by the two solutions of the quadratic equation $\frac{1}{2}\sigma_z^2\theta_i^2 - (g_I - g_E)\theta_i - g_\eta(1-\alpha) = 0$, where $i = 1, 2$. Using condition (12) set $A_2 = 0$ for $s < \bar{s}_i$ and using the requirement that $f(s)$ is a probability density set $A_1 = 0$ for $s \geq \bar{s}_i$.

Finally, from the characterization of the flows at the entry point (15), I pick A_1, A_2 such that

$$\frac{1}{2}\sigma_z^2\left(A_1\theta_1 e^{\theta_1\bar{s}_i} + A_2\theta_2 e^{-\theta_2\bar{s}_i}\right) = g_\eta(1-\alpha) \ ,$$

which in combination with (14) that gives

$$\int_{-\infty}^{\bar{z}_i'} A_1 e^{\theta_1 s} ds + \int_{\bar{z}_i'}^{+\infty} A_2 e^{-\theta_2 s} ds = 1 \ ,$$

imply that $A_1 = \theta_1\theta_2/(\theta_1+\theta_2)\, e^{-\theta_1\bar{s}_i}$, $A_2 = \theta_1\theta_2/(\theta_1+\theta_2)\, e^{\theta_2\bar{s}_i}$. Notice that the solutions also satisfy the first term in the LHS of (15) since they imply that $f(\bar{s}_i-) = f(\bar{s}_i+)$. In other words the distribution is continuous, but the derivative has a kink at $\bar{s}_i$.

A.3 Exit Rates and Firm Sales

Firm Survival in a market The objective is to compute the probability that a firm will be selling in a market after a years (so that $s_{ija} \geq 0$), conditional on the initial productivity of the firm today, $s_{ij0} = s_0$. I denote this probability by $S(a|s_0)$ and thus, using expression (10),

$$S(a|s_0) = e^{-\delta a}\int_0^{+\infty} \frac{e^{-\left(\frac{s_a - s_0 - \mu a}{\sigma_z\sqrt{a}}\right)^2/2}}{\sigma_z\sqrt{a2\pi}} ds_a$$

which using change of variables yields equation (25) in the main text.

Cohort Survival Rates The expression to be derived is expression (26): the probability that a firm in an incumbent cohort, among all the currently operating firms, $s_{ij0} \geq 0$, also operates after time a has elapsed, $s_{ija} \geq 0$. If I denote this probability as $\Pr(s_{ija} \geq 0 | s_{ij0} \geq 0)$, then taking in account random death the cohort survival rate is $S_{ij}(a) = e^{-\delta a} \Pr(s_{ija} \geq 0 | s_{ij0} \geq 0)$. I first derive the probability,

$$\begin{aligned} \Pr(s_{ija} \geq 0 | s_{ij0} \geq 0) &= \int_0^{+\infty} \int_0^{+\infty} \Pr(s_{ija} = s_a | s_{ij0} = s_0) \frac{\Pr(s_{ij0} = s_0)}{\Pr(s_{ij0} \geq 0)} ds_a ds_0 \\ &= \int_0^{+\infty} \frac{\Pr(s_{ij0} = s_0)}{\Pr(s_{ij0} \geq 0)} \int_0^{+\infty} \Pr(s_{ija} = s_a | s_{ij0} = s_0) ds_a ds_0 \end{aligned} \tag{31}$$

Using equation (16) the conditional density of productivities is given by

$$\frac{\Pr(s_{ij0} = s_0)}{\Pr(s_{ij0} \geq 0)} = \theta_2 e^{-\theta_2 (s_0 - 0)} \ . \tag{32}$$

The inner integral of expression (31) is given by equation (25). Thus, by replacing expressions (25), (32) in (31) and using integration by parts,

$$\Pr(s_{ija} \geq 0 | s_{ij0} \geq 0) = \Phi\left(\frac{\mu\sqrt{a}}{\sigma_z}\right) + \int_0^{+\infty} e^{-\theta_2 s_0} \frac{1}{\sigma_z \sqrt{a}} \varphi\left(\frac{s_0 + \mu a}{\sigma_z \sqrt{a}}\right) ds_0 \ .$$

Using the definition of the error function P2 and property P4 the integral of the last expression equals to

$$\frac{e^{-\frac{1}{2}\frac{\mu^2}{\sigma_z^2}a}}{\sigma_z\sqrt{a2\pi}} \left| \frac{e^{\frac{\left(\frac{\mu}{\sigma_z^2}+\theta_2\right)^2}{4\frac{1}{2\sigma_z^2 a}}}\sqrt{\pi}}{\sqrt{2\frac{1}{2\sigma_z^2 a}}} \operatorname{erf}\left(\frac{2\frac{1}{2\sigma_z^2 a}x + \frac{\mu}{\sigma_z^2} + \theta_2}{2\sqrt{\frac{1}{2\sigma_z^2 a}}}\right) \right|_{x=0}^{x=+\infty} = e^{\frac{\sigma_z^2 a}{2}(\theta_2)^2 + \theta_2 \mu a} \Phi\left(-\left(\frac{\mu}{\sigma_z} + \theta_2 \sigma_z\right)\sqrt{a}\right).$$

where I used property P3 for the last equality. Combining the expressions with the random death term gives the survival function, $S_{ij}(a)$, expression (26).

In the online appendix I show that $S_{ij}(a)$ is increasing in μ, and if $\mu < 0$, $S_{ij}(a)$ is decreasing in a, $DS_{ij}(a) < 0$. The results are applications of the properties of the normal distribution.

Expected Log Sales Here I sketch the derivations for the expected growth and variance of growth of the log sales of firms described in section 4.2.1. To derive that I have to derive moments of the natural logarithm of sales of the firm after time a has elapsed for $\beta \to 0$,

$$\ln L_{j\tilde{t}+a}^{\alpha} y_{j\tilde{t}+a} \frac{1}{\tilde{\psi}} + (\sigma - 1) s_{ij\tilde{t}+a} \ . \tag{33}$$

The first term is deterministic so that derivations are easy. To compute the moments of the second term I can compute the moment generating function (MGF) of this term. I start by computing the moment generating function of some variable $\tilde{s}_a$ that is normally distributed as $(\sigma - 1) s_{ij\tilde{t}+a}$ but with different parameters. Let the mean be $\tilde{\mu}$, the variance $\tilde{\sigma}^2$ and the lower threshold $\tilde{x}$ the values of which I will specify below. The MGF is (for some $\tilde{c} \in R$)

$$E\left(e^{\tilde{c}s_a}|s_0, s_a \geq 0\right) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \frac{\int_{\tilde{x}}^{\infty} e^{\tilde{c}x} e^{-\frac{1}{2}\left(\frac{x-\tilde{\mu}}{\tilde{\sigma}}\right)^2} dx}{1 - \Phi\left(\frac{\tilde{x}-\tilde{\mu}}{\tilde{\sigma}}\right)}$$

$$= e^{-\frac{(\tilde{\mu})^2 - \left[(\tilde{\sigma})^2\tilde{c}+\tilde{\mu}\right]^2}{2(\tilde{\sigma})^2}} \frac{\int_{\tilde{x}}^{\infty} \frac{1}{\tilde{\sigma}\sqrt{2\pi}} e^{-\frac{\left[x-(\tilde{\sigma})^2\tilde{c}-\tilde{\mu}\right]^2}{2(\tilde{\sigma})^2}} dx}{1 - \Phi\left(\frac{\tilde{x}-\tilde{\mu}}{\tilde{\sigma}}\right)} = e^{\tilde{\mu}\tilde{c}+\frac{\tilde{\sigma}^2\tilde{c}^2}{2}} \frac{1 - \Phi\left(\frac{\tilde{x}-\tilde{\mu}-\tilde{\sigma}^2\tilde{c}}{\tilde{\sigma}}\right)}{1 - \Phi\left(\frac{\tilde{x}-\tilde{\mu}}{\tilde{\sigma}}\right)}$$

where in the last equality I used the definition of the cdf of the normal distribution. I can now adjust the parameters of the distribution so that they correspond to the current firm sales size and the underlying stochastic process: $\tilde{\mu} = (\sigma - 1) s_0 + (\sigma - 1) \mu a$, $\tilde{\sigma} = (\sigma - 1) \sigma_z \sqrt{a}$, $\tilde{x} = 0$. Finally, I can compute the moments of the second term of equation (33) by computing the successive derivatives of the MGF wrt to $\tilde{c}$.

A.4 Proofs of Proposition 2 and 3

Proof of Proposition 2 The proof of the proposition requires that $\partial\left(\mu\frac{h'(s)}{h(s)} + \frac{\sigma_z^2}{2}\frac{h''(s)}{h(s)}\right)\Big/\partial s \leq 0$. Extended derivations for this proposition given in an online appendix imply that it is equivalent to show that

$$\mu(\sigma - 1)\left[\left(1 - \tilde{\beta}\right)/\tilde{\beta}\right] e^{-s\frac{(\sigma-1)}{\beta}} + \frac{\sigma_z^2}{2}(\sigma - 1)^2\left[\left(1 - \tilde{\beta}^2\right)/\tilde{\beta}^2\right] e^{-s\frac{(\sigma-1)}{\beta}} \leq 0$$

so that for $\tilde{\beta} = \beta/(\beta - 1)$ I need to show (notice that $e^{s\frac{(\sigma-1)}{\beta}} \geq 1$, for $s \geq 0$)

$$-\left(\tilde{\beta}\right)^2\left[\mu(\sigma - 1) + (\sigma - 1)^2\frac{\sigma_z^2}{2}\right] + \mu(\sigma - 1)\tilde{\beta} + (\sigma - 1)^2\frac{\sigma_z^2}{2} \leq 0\ .$$

This expression after some manipulations gives the condition in equation (30). Notice that if $\mu(\sigma - 1) + (\sigma - 1)^2\sigma_z^2 < 0$ there does not exist a $\beta \in [0, +\infty)$ that satisfies the inequality.

Proof of Proposition 3 The proof of the proposition uses Ito's Lemma. In particular, the variance of the instantaneous growth rate of firms is given by the square of the second bracketed term in expression (28). Given (29) this term equals to $\sigma_z^2(\sigma - 1)^2$ for $\beta \to 0$. For the second part of the proposition, given $\beta > 0$, the derivative of the term is always negative. Thus, the instantaneous variance of growth rates of firms selling to a destination is inversely related to their size there. In the limit for $s_{ij0} \to +\infty$ the term tends to $\sigma_z^2(\sigma - 1)^2$ completing the proof of the proposition.

A.5 Sunk Costs

I consider different calibrations of the sunk cost model and the implied difference for the entrants and exitors that they imply. The key equation that I use from the model of Luttmer (2007) is his equation (19). Using this equation, the exercise that I perform is to consider the average difference of exitors to entrants that will imply a 60% 5-year exit rate for the entry cohort (as in the US manufacturing data). For the calibrated parameters that I consider the model requires an average size of entrants to exitors that is around 22% higher whereas with the calibration of Luttmer this number is around 25%. If given the calibration of the rest of the parameters in Luttmer I choose a lower $\sigma = 6.5$ (instead of his choice of $\sigma = 10$) the average size difference is at around 15%, still much larger than zero albeit lower.