2-15-2010

A Unified Theory of Function Spaces and Hyperspaces: Local Properties

Szymon Dolecki
Burgundy University

Frédéric D. Mynard
Georgia Southern University, fmynard@georgiasouthern.edu

A UNIFIED THEORY OF FUNCTION SPACES AND HYPERSPACES: LOCAL PROPERTIES

SZYMON DOLECKI AND FRÉDÉRIC MYNARD

ABSTRACT. Every convergence (in particular, every topology) τ on the hyperspace $C(X, \$)$ *preimage-wise* determines a convergence $\tau^{\Uparrow}$ on $C(X, Z)$, where X, Z are topological spaces and $\$$ is the *Sierpiński topology*, so that $f \in \lim_{\tau^{\Uparrow}} \mathcal{F}$ if and only if $f^{-1}(U) \in \lim_{\tau} \mathcal{F}^{-1}(U)$ for every open subset U of Z. Classical instances are the *pointwise, compact-open and Isbell* topologies, which are preimage-wise with respect to the topologies, whose open sets are the collections of, respectively, all (openly isotone) *finitely generated, compactly generated* and *compact* families of open subsets of X (compact families are precisely the open sets of the *Scott topology*); the *natural* (that is, *continuous*) convergence is preimage-wise with respect to the *natural hyperspace* convergence.

It is shown that several fundamental local properties hold for a hyperspace convergence τ (at the whole space) if and only if they hold for $\tau^{\Uparrow}$ on $C(X, \mathbb{R})$ at the origin, provided that the underlying topology of X have some $\mathbb{R}$-separation properties. This concerns character, tightness, fan tightness, strong fan tightness, and various Fréchet properties (from the simple through the strong to that for finite sets) and corresponds to various covering properties (like Lindelöf, Rothberger, Hurewicz) of the underlying space X.

This way, many classical results are unified, extended and improved. Among new surprising results: the tightness and the character of the natural convergence coincide and are equal to the Lindelöf number of the underlying space; The Fréchet property coincides with the Fréchet property for finite sets for the hyperspace topologies generated by compact networks.

1. INTRODUCTION

The study of the interplay between properties of a topological space X and those of the associated space $C(X, Z)$ of continuous functions from X to another topological space Z, endowed with convergence structures, is one of the central themes of topology, and an active area interfacing *topology* and *functional analysis*. Most prominent instances for the space Z is the real line $\mathbb{R}$ (with the usual topology) and a two-point set $\{0, 1\}$ with the *Sierpiński topology* $\$:= \{\varnothing, \{1\}, \{0, 1\}\}$ ([1]). By the usual identifications ([2]), $C(X, \$)$ becomes the *hyperspace*, either of open or of closed subsets of X.

The *topology of pointwise convergence* (*pointwise topology, finite-open topology*) is the structure of choice for a sizable share of such investigations, in part because

---

[1]The present paper focuses as well on these cases.

[2]of subsets and maps valued in $\{0, 1\}$. More precisely, the set $C(X, \$)$ is identified with the set of open subsets of X, because the characteristic function of a subset A of X, defined by $\chi_A(x) = 1$ if and only if $x \in A$, is continuous from X to $\$$ if and only if A is open. Of course, $C(X, \$)$ could also be identified with the set of closed subsets of X.

of its obvious relevance to functional analysis (weak topologies). The book [3] of Arhangel'skii gives a thorough account of this still very active area.

The *compact-open topology* is another most frequently studied structure; in functional analysis, compatible locally convex topologies are characterized (via the Arens-Mackey theorem) as those of uniform convergence on some families of compact sets. The book [36] of McCoy and Ntantu treats *pointwise convergence* and *compact-open topology* simultaneously by considering topologies on $C(X, Z)$ with a subbase given by sets of the form

$$(1.1) \qquad [D, U] := \{f \in C(X, Z) : f(D) \subseteq U\},$$

where U ranges over open subsets of Z and D ranges over a network $\mathcal{D}$ of compact subsets of X.

The pointwise topology is the coarsest structure on $C(X, Z)$, for which the natural coupling

$$(1.2) \qquad \langle \cdot, \cdot \rangle : X \times C(X, Z) \to Z$$

is pointwise continuous for each $x \in X$. The *continuous convergence* $[X, Z]$ is the coarsest structure on $C(X, Z)$, for which (1.2) is jointly continuous. Therefore it satisfies the exponential law (^{3}) and, as such, has been called *natural convergence* (e.g., [18]), the terminology that we adopt here. The exceptional role of the natural convergence among all function space structures on $C(X, Z)$ was recognized as early as [1] by Arens and Dugundji, and a compelling case for its systematic use in functional analysis was made by Binz in [7] and more recently and thoroughly by Beattie and Butzmann in [6].

As a consequence, even though this paper is mostly focused on *completely regular topological spaces X,* no a priori assumption is made on the function space convergence structures on $C(X, Z)$. We refer to [13] for basic terminology and notations on convergence spaces.

The *Isbell topology* [28] was conceived by Isbell in a hope to provide the topological modification of the natural convergence on $C(X, Z)$. This is actually the case, when Z is the *Sierpiński topology* (then the Isbell topology becomes the *Scott topology* on the lattice of open sets). It is why the Isbell topology plays a central role when investigating topological spaces from a lattice-theoretic viewpoint [21], [22].

If θ is a convergence structure (in particular a topology) on $C(X, Z)$, we denote by $C_\theta(X, Z)$ the corresponding convergence space.

In [12] and [15], we studied topologies $\alpha(X, Z)$ on $C(X, Z)$ generated by collections α of families of subsets of X (the *Z-dual topology* of α), for which a subbase of open sets consists of

$$(1.3) \qquad [\mathcal{A}, U] := \left\{f \in C(X, Z) : f^-(U) \in \mathcal{A}\right\},$$

where $\mathcal{A} \in \alpha$ and U ranges over the open subsets of Z, and

$$f^-(U) := \{x \in X : f(x) \in U\}$$

is our usual shorthand for $f^{-1}(U)$. If α consists of (openly isotone) *compact families* (of open subsets of X), then $\alpha(X, Z)$ is coarser than the natural convergence $[X, Z]$ (that is, is *splitting* according to a widespread terminology). Pointwise topology,

---

^{3}that is, $[Y, [X, Z]]$ is homeomorphic to $[X \times Y, Z]$ (under $^t f(y)(x) = f(x, y)$).

compact-open topology and the Isbell topology are particular cases of a general scheme (^{4}).

The relationship between convergences and topologies on *functional spaces* $C(X, \mathbb{R})$ and the corresponding convergences and topologies on *hyperspaces* $C(X, \$)$ (of closed sets or of open sets) is a principal theme of this paper. Functional spaces and hyperspaces are intimately related, but also differ considerably for certain aspects. For instance, the $\mathbb{R}$-dual topologies of collections α of compact families on completely regular spaces are completely regular, while \$-dual topologies for the same collections are T_0 but never T_1.

Topologies $\alpha(X, \$)$ and the convergence $[X, \$]$ have a simpler structure than their counterparts $\alpha(X, \mathbb{R})$ and $[X, \mathbb{R}]$. Actually the collection α is (itself) a subbabse of open sets of $\alpha(X, \$)$. Local properties of $\alpha(X, \$)$ are equivalent to some global (covering) properties of X and this equivalence is usually easily decoded. Therefore in the study of the interdependence between X and $C(X, \mathbb{R})$, it is essential to comprehend the relationship between $C(X, \mathbb{R})$ and $C(X, \$)$.

A crucial observation made in [12] was that all the mentioned topologies and convergences on $C(X, \mathbb{R})$ can be characterized preimage-wise with the aid of the corresponding topologies and convergences on $C(X, \$)$.

In the present paper we unify the investigations of local properties of $C(X, \mathbb{R})$ and of $C(X, \$)$ by revealing an abstract connection between them that embraces all the discussed cases.

As a convention $C_\tau(X, \$)$ will always denote the hyperspace of open subsets of X (endowed with τ) and $cC_\tau(X, \$)$ will denote the homeomorphic image of $C_\tau(X, \$)$ under complementation, which is the corresponding hyperspace of closed subsets of X. If F is a set of maps $f : X \to Z$, then $F^-(B) := \{f^-(B) : f \in F\}$ and, for a family $\mathcal{F}$ of sets of such maps, $\mathcal{F}^-(B) := \{F^-(B) : F \in \mathcal{F}\}$. We say that θ is *preimage-wise with respect to τ* if

$$(1.4) \qquad f \in \lim_\theta \mathcal{F} \Longleftrightarrow \forall_{U \in C(Z,\$)} \ f^-(U) \in \lim_\tau \mathcal{F}^-(U).$$

As we shall see, all the topologies $\alpha(X, Z)$, defined via (1.3), in particular the pointwise, compact-open and Isbell topologies, as well as the natural convergence $[X, Z]$ are preimage-wise with respect to their hyperspace cases: $\alpha(X, \$)$ and $[X, \$]$.

This is a special case of the following scheme. Each $h \in C(Z, W)$ defines the *lower conjugate* map $h_* : C(X, Z) \to C(X, W)$ given by $h_*(f) := h \circ f$. Each convergence τ on $C(X, W)$ determines on $C(X, Z)$ the coarsest convergence for which h_* is continuous for every $h \in C(Z, W)$. In the particular case when W is the Sierpiński topology \$, then for each $U \in C(Z, \$)$, the image $U_*(f) \in C(X, \$)$ and

$$(1.5) \qquad U_*(f) = f^-(U).$$

In other words, if an element U of $C(Z, \$)$ is identified with an open set (via the characteristic function), then in the same way $U_*(f)$ is identified with the preimage of U by f. Therefore θ is *preimage-wise with respect to τ*, if the source

$$(1.6) \qquad (U_* : C_\theta(X, Z) \to C_\tau(X, \$))_{U \in C(Z,\$)}$$

---

^{4}(1.1) is a special case, in which $\mathcal{A} = \mathcal{A}_D$ is the family of all the open subsets of X that include D. Then $\alpha = \{\mathcal{A}_D : D \in \mathcal{D}\}$.

is initial, that is, if θ is the coarsest convergence on $C(X, Z)$ making each map $U_* : C(X, Z) \to C_\tau(X, \$)$ continuous (^{5}).

Preimage-wise approach has been implemented in various branches of mathematics (^{6}). In the study of function spaces, Georgiou, Iliadis and Papadopoulos in [11] considered Z-dual topologies of the type (1.3) as well as the topologies on the set $\{f^-(U) : f \in C(X, Z), U \in C(Z, \$)\}$ of the form $\{\mathcal{H}^-(U) : \mathcal{H} \in \theta, U \in C(Z, \$)\}$, where θ is an arbitrary topology on $C(X, Z)$ (^{7}).

The so-called γ-*connection* of Gruenhage, e.g. [26], is a very particular instance of our preimage-wise approach (it describes the neighborhood filter of the whole space X for the pointwise topology on the hyperspace $C(X, \$)$ of open sets). Jordan exploited the γ-connection in [29] establishing a relation between the neighborhood filter of the zero function in $C_p(X, \mathbb{R})$ and the neighborhood filter of the whole space X in $C_p(X, \$)$ with the aid of *composable* and *steady* relations, which enables a transfer of many local properties, like tightness or character, preserved by such relations.

Jordan's paper is a prefiguration of our theory. Actually the first author realized that Jordan's approach can be easily extended to general topologies $\alpha(X, Z)$, encompassing, among others, the topology of pointwise convergence, the compact-open topology and the Isbell topology [12]. On the other hand, the fact that the natural (or continuous) convergence fits (1.6), that is, that $[X, Z]$ is pre-imagewise with respect to $[X, \$]$, was observed before, e.g. [42].

Yet, even though the relationship between hyperspace structures and function space structures has been identified on a case by case basis, and even as an abstract scheme in [11] for topologies, it seems that no systematic use of this situation is to be found in the literature before [12]. In the present paper, we extend the results of [12] to general convergences, simplify some of the arguments and clarify the role of topologicity, and obtain as by-products a wealth of classical results for function space topologies, as well as new results for the natural convergence. In particular, we obtain the surprising result that the character and tightness of the natural convergence on real valued continuous functions coincide, and are equal to the Lindelöf degree of the underlying space.

2. Preimagewise convergences

Let Z be a topological space. If τ is a convergence on $C(X, \$)$, then $\tau^\Uparrow$ is the convergence on $C(X, Z)$ defined by

$$(2.1) \qquad f \in \lim_{\tau^\Uparrow} \mathcal{F} \iff \forall_{U \in C(Z, \$)} f^-(U) \in \lim_\tau \mathcal{F}^-(U).$$

In view of (1.4), $\tau^\Uparrow$ is preimage-wise with respect to τ. If for a convergence θ on $C(X, \$)$ there exists a convergence on $C(X, \$)$ with respect to which θ is preimage-wise, then there is a finest convergence $\theta^\Downarrow$ on $C(X, \$)$ among those τ for which $\theta = \tau^\Uparrow$. Hence, $\tau^{\Uparrow\Downarrow\Uparrow} = \tau^\Uparrow$ for each τ.

---

^{5}As the category of topological spaces and continuous maps is reflective in that of convergence spaces (and continuous maps), the coarsest convergence making the maps U_* continuous is also the coarsest topology with this property, whenever τ is topological.

^{6}Lebesgue says that the idea of preimage-wise study of functions was pivotal for the theory of his integral [33]. Greco characterized minmax properties of real functions in terms of their preimages introducing a counterpart of measurable sets in analysis [23], [24], [25].

^{7}In general, a topology obtained this way is finer than the restriction of a topology τ, for which θ is preimage-wise.

As we shall show Proposition 4.5 below, it is not necessary to test that $f^-(U) \in \lim_\tau \mathcal{F}^-(U)$ for every open subset U of Z in (2.1), but only for the elements of an *ideal* ([8]) *basis* of the topology on Z. In terms of closed sets, (2.1) becomes

$$f \in \lim_{\tau \Uparrow} \mathcal{F} \Longleftrightarrow \forall_{C \in cC(Z,\$)} \, f^-(C) \in \lim_{c\tau} \mathcal{F}^-(C).$$

In the formula above, analogously to (2.1), it is enough test $f^-(C) \in \lim_{c\tau} \mathcal{F}^-(C)$ for the elements of a *filtered* ([9]) *basis (of closed sets)*.

The following is an immediate consequence of the definition.

Proposition 2.1. *If* **J** *is a concretely reflective category of convergences,* $C_\tau(X, \$)$ *is an object of* **J**, *then* $C_{\tau \Uparrow}(X, Z)$ *is also an object of* **J**.

In particular, if τ is a topology, a pretopology or a pseudotopology, so is $\tau^{\Uparrow}$.

In the particular important case where $Z = \mathbb{R}$ ([10]), the preimage of a closed set by a continuous function is a *zero set*, because all closed subsets of $\mathbb{R}$ are zero sets. Therefore, a τ-preimage-wise convergence on $C(X, \mathbb{R})$ is determined by the restriction of τ to the cozero sets of X (or the restriction of $c\tau$ to zero sets). More generally, we say that an open subset G of X is *Z-functionally open* if there exist $f \in C(X, Z)$ and $U \in C(Z, \$)$ such that $G = f^-(U)$. Of course, all the elements of $C(X, \$)$ that are not *Z-functionally open* are isolated for $\tau^{\Uparrow \Downarrow}$.

3. Fundamental examples of preimagewise convergences

Recall that the topology of pointwise convergence as well as the compact-open topology on $C(X, Z)$ admit subbases of the form $\{[D, U] : U \in C(Z, \$), D \in \mathcal{D}\}$ where $\mathcal{D}$ is the collection $[X]^{<\infty}$ of finite subsets of X in the former case, and the collection $\mathcal{K}(X)$ of compact subsets of X in the latter, and $[D, U]$ is defined by (1.1). We extend this notation to families of subsets of X by

$$[\mathcal{A}, U] := \bigcup_{A \in \mathcal{A}} [A, U] = \left\{ f : f^-(U) \in \mathcal{A} \right\}.$$

If A is a subset of X then $\mathcal{O}_X(A)$ denotes the collection of open subsets of X that contains A, and if $\mathcal{A}$ is a collection of subsets of X then $\mathcal{O}_X(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{O}_X(A)$. A family $\mathcal{A} \subseteq C(X, \$)$ is called *compact* if $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$ and whenever $\mathcal{B} \subseteq C(X, \$)$ such that $\bigcup_{B \in \mathcal{B}} B \in \mathcal{A}$, there exists a finite subcollection $\mathcal{S}$ of $\mathcal{B}$ such that $\bigcup_{B \in \mathcal{S}} B \in \mathcal{A}$. The collection $\kappa(X)$ of all compact families form a topology on $C(X, \$)$, known as the *Scott topology* (for the lattice of open subsets of X ordered by inclusion) ([11]). The *Isbell topology* on $C(X, Z)$ has a subbase composed of the sets of the form $[\mathcal{A}, U]$ where U ranges over $C(Z, \$)$ and $\mathcal{A}$ ranges over $\kappa(X)$. With the simple observation that

(3.1) $$[\mathcal{O}_X(D), U] = [D, U],$$

whenever U is open, one concludes that the topology of pointwise convergence, the compact-open topology and the Isbell topology are three instances of function space

---

[8]that is, closed under finite unions

[9]that is, closed under finite intersections

[10]More generally, if Z is perfectly normal.

[11]The homeomorphic image of $C_\kappa(X, \$)$ is the hyperspace $cC_\kappa(X, \$)$ of closed subsets of X endowed with the *upper Kuratowski topology*.

topologies determined by some $\alpha \subseteq C(X, \$)$. Indeed, if α is *non-degenerate*, that is, $\alpha \setminus \varnothing \neq \varnothing$, the family

$$(3.2) \qquad \{[\mathcal{A}, U] : \mathcal{A} \in \alpha, U \in C(Z, \$)\}$$

is a subbase for a topology on $C(X, Z)$, denoted $\alpha(X, Z)$. Such topologies have been called *family-open* in [11]. The corresponding topological space is denoted $C_\alpha(X, Z)$.

In view of (3.1) we can, and we will throughout the paper, assume that each $\mathcal{A} \in \alpha$ is *openly isotone*, that is, $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$. The topology of pointwise convergence is obtained when α is the topology $p(X) := \{\bigcup_{F \in \mathcal{F}} \mathcal{O}(F) : \mathcal{F} \subseteq [X]^{<\infty}\}$ on $C(X, \$)$ of *finitely generated families*, while the compact open topology is obtained when α is the topology $k(X) := \{\bigcup_{K \in \mathcal{F}} \mathcal{O}(K) : \mathcal{F} \subseteq \mathcal{K}(X)\}$ on $C(X, \$)$ ([12]) of *compactly generated families*. Of course, the Isbell topology is obtained when α is the topology $\kappa(X)$ of compact families.

Even if $\alpha \subseteq C(X, \$)$ is not a basis for a topology, $\alpha(X, Z) = \alpha^\cap(X, Z)$, where $\alpha^\cap$ is the collection of finite intersections of elements of α, because $\bigcap_{i=1}^n [\mathcal{A}_i, U] = [\bigcap_{i=1}^n \mathcal{A}_i, U]$. Therefore, *we can assume that α is a basis for $\alpha(X, \$)$.*

Proposition 3.1. [12] *If $\alpha \subseteq C(X, \$)$ is non-degenerate then $\alpha(X, Z) = \alpha(X, \$)^\Uparrow$.*

Proof. If $\mathcal{A} \in \alpha$ and $U \in C(Z, \$)$ then

$$U_*^{-1}(\mathcal{A}) = \{f \in C(X, Z) : f^- U \in \mathcal{A}\} = [\mathcal{A}, U]$$

because $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$. Therefore $\alpha(X, Z)$ is indeed the initial topology for the family of maps $(U_* : C(X, Z) \to C_\alpha(X, \$))_{U \in C(Z, \$)}$. $\qquad\square$

By definition, the *natural convergence* $[X, Z]$ on $C(X, Z)$ (also called continuous convergence, e.g., [7], [6]) is the coarsest convergence making the canonical coupling (or evaluation)

$$(3.3) \qquad \langle \cdot, \cdot \rangle : X \times C(X, Z) \to Z$$

continuous ([13]). In other words, $f \in \lim_{[X, Z]} \mathcal{F}$ if and only if for every $x \in X$, the filter $\langle \mathcal{N}(x), \mathcal{F} \rangle$ converges to $f(x)$ in Z, that is, if $U \in \mathcal{O}_Z(f(x))$ there is $V \in \mathcal{O}_X(x)$ and $F \in \mathcal{F}$ such that $\langle V, F \rangle \subseteq U$, equivalently, $F \subseteq [V, U]$. Therefore

Proposition 3.2. $f_0 \in \lim_{[X, Z]} \mathcal{F}$ *if and only if for every open subset U of Z and $x \in X$,*

$$(3.4) \qquad f_0 \in [x, U] \Longrightarrow \exists_{V \in \mathcal{O}_X(x)} [V, U] \in \mathcal{F},$$

if and only if for every open subset U of Z and $x \in X$,

$$(3.5) \qquad x \in f_0^-(U) \Longrightarrow \exists_{F \in \mathcal{F}} \bigcap_{f \in F} f^-(U) \in \mathcal{O}_X(x).$$

In the case where $Z = \$$, the only non-trivial open subset of Z is $\{1\}$ and elements of $C(X, \$)$ are of the form χ_Y for Y open in X. Therefore (3.5) translates into: $Y \in \lim_{[X, \$]} \gamma$ if and only if

$$x \in Y \Longrightarrow \exists_{\mathcal{G} \in \gamma} \bigcap_{G \in \mathcal{G}} G \in \mathcal{O}_X(x),$$

---

[12]Here, $\mathcal{K}(X)$ stands for the set of all compact subsets of X.
[13]$\langle x, f \rangle := f(x)$

In other words, $Y \in \lim_{[X,\$]} \gamma$ if and only if

(3.6)
$$Y \subseteq \bigcup_{\mathcal{G} \in \gamma} \text{int}_X \bigcap_{G \in \mathcal{G}} G.$$

This convergence is often (e.g., [22]) known as the *Scott convergence* (in the lattice of open subsets of X ordered by inclusion). Its homeomorphic image $c[X,\$]$ on the set of closed subsets of X is known as *upper Kuratowski convergence* ([14]).

Proposition 3.3. *(e.g., [42])*

$$[X,Z] = [X,\$]^{\Uparrow}.$$

Proof. In view of (3.5), $f_0 \in \lim_{[X,Z]} \mathcal{F}$ if and only if $f_0^-(U) \subseteq \bigcup_{F \in \mathcal{F}} \text{int}_X \bigcap_{f \in F} f^-(U)$, equivalently,

$$U_*(f_0) \subseteq \bigcup_{\mathcal{G} \in U_*(\mathcal{F})} \text{int}_X \bigcap_{G \in \mathcal{G}} G,$$

for every open subset U of Z. In view of (3.6), we conclude that $f_0 \in \lim_{[X,Z]} \mathcal{F}$ if and only if $U_*(f_0) \in \lim_{[X,\$]} U_*(\mathcal{F})$ for every $U \in C(Z,\$)$, which concludes the proof. $\square$

It follows that if $\tau \leq [X,\$]$ then $\tau^{\Uparrow} \leq [X,Z]$. In other words, the preimage-wise convergence of a splitting convergence is splitting.

That the natural convergence is not in general topological is a classical fact and one of the main motivation to consider convergence spaces. It is well known (see, e.g., [42], [14]) that the topological reflection $T[X,\$]$ of $[X,\$]$ is equal to the *Scott topology* $\kappa(X,\$)$ and we have seen that $\kappa(X,\$) = \kappa(X)$, the collection of all compact openly isotone families on X.

We do not know if for every X there exists a hyperconvergence τ on $C(X,\$)$ such that $T[X,\mathbb{R}] = \tau^{\Uparrow}$.

4. Hyperconvergences

We focus on convergences τ on $C(X,\$)$ that share basic properties with $[X,\$]$ and topologies of the type $\alpha(X,\$)$ ([15]). In particular, we say that τ is *lower* if

$$A \subseteq B \in \lim_{\tau} \gamma \implies A \in \lim_{\tau} \gamma,$$

and *upper regular* if

$$O \in \lim_{\tau} \gamma \implies O \in \lim_{\tau} \mathcal{O}_X^{\natural}(\gamma),$$

where $\mathcal{O}_X^{\natural}(\gamma)$ is generated by $\{\mathcal{O}_X(\mathcal{G}) : \mathcal{G} \in \gamma\}$. Observe that if O_0, O_1 are open subsets of Z and $\mathcal{F}$ is a filter on $C(X,Z)$ then $\mathcal{O}_X^{\natural}(\mathcal{F}^-(O_0)) \leq \mathcal{O}_X^{\natural}(\mathcal{F}^-(O_1))$ whenever $O_0 \subseteq O_1$ ([16]). When considering upper regular convergences, we will often identify a filter γ on $C(X,\$)$ and its upper regularization $\mathcal{O}_X^{\natural}(\gamma)$. With this convention, the previous observation becomes

(4.1)
$$O_0 \subseteq O_1 \implies \mathcal{F}^-(O_0) \leq \mathcal{F}^-(O_1).$$

Proposition 4.1. *Each lower topology on $C(X,\$)$ is upper regular.*

---

[14]Explicitely, if C is a closed subset of X and γ is a filter on $cC(X,\$)$ then $C \in \lim_{c[X,\$]} \gamma$ if and only if $\bigcap_{\mathcal{G} \in \gamma} \text{cl}_X \left(\bigcup_{F \in \mathcal{G}} F \right) \subseteq C$, that is, $\text{adh}_X |\gamma| \subseteq C$ where $|\gamma| := \left\{ \bigcup_{F \in \mathcal{G}} F : \mathcal{G} \in \gamma \right\}$.

[15]We do not treat here *hit-and-miss* convergences, like the *Vietoris topology* or *Fell topology*.

[16]Indeed, if $O_0 \subset O_1$ then $f^-(O_0) \subset f^-(O_1)$, hence $\bigcup_{f \in F} \{P \in C(X,\$) : P \supset f^-(O_0)\} \supset \bigcup_{f \in F} \{P \in C(X,\$) : P \supset f^-(O_1)\}$.

Proof. It is enough to show that if $\mathcal{G} \subseteq C(X, \$)$ is open then $\mathcal{G} = \mathcal{O}_X^\natural(\mathcal{G})$. Let $A \supseteq G \in \mathcal{G}$. Then the principal ultrafilter $A^\bullet$ of A converges to A and therefore to G, because the topology is lower. Because $\mathcal{G}$ is open, $\mathcal{G} \in A^\bullet$ so that $A \in \mathcal{G}$. Hence $\mathcal{G} = \mathcal{O}_X^\natural(\mathcal{G})$. $\square$

Lemma 4.2. *If $X \neq \varnothing$ and $p(X, \$) \leq \tau \leq [X, \$]$, then*

$$\mathrm{cl}_\tau \{A\} = \{O \in C(X, \$) : O \subseteq A\}$$

for each $A \in C(X, \$)$.

Proof. To see that $\mathrm{cl}_\tau \{A\} = \{O \in C(X, \$) : O \subseteq A\}$, note first that

$$\{O \in C(X, \$) : O \subseteq A\} \subseteq \mathrm{cl}_{[X,\$]} \{A\} \subseteq \mathrm{cl}_\tau \{A\} \subseteq \mathrm{cl}_{p(X,\$)} \{A\},$$

where the first inclusion follows from the fact that $[X, \$]$ is lower, and the others from the assumption $p(X, \$) \leq \tau \leq [X, \$]$. Moreover, if $O \in \mathrm{cl}_{p(X,\$)} \{A\}$ then every $p(X, \$)$-open neighborhood of O contains A. In particular, $A \in \mathcal{O}_X(x)$ for each $x \in O$, so that $O \subseteq A$. $\square$

Proposition 4.3. *If $X \neq \varnothing$ and $p(X, \$) \leq \tau \leq [X, \$]$, then τ is T_0 but is not T_1*

Proof. By Lemma 4.2, if $A_1 \neq A_0$, say, there is $x \in A_1 \setminus A_0$, then $\mathrm{cl}_\tau \{A_0\} = \{O \in C(X, \$) : O \subseteq A_0\}$ is τ-closed and contains A_0 but not A_1 and the convergence is therefore T_0. As $X \neq \varnothing$ and $\varnothing \in \mathrm{cl}_\tau \{X\}$, the convergence τ is not T_1. $\square$

We say that a convergence τ on $C(X, \$)$ *respects directed sups* if whenever $\{\gamma_i : i \in I\}$ and $\{B_i : i \in I\}$ are two directed families of filters on $C(X, \$)$ and elements of $C(X, \$)$ respectively, such that $B_i \in \lim_\tau \gamma_i$ for each i in I, we have that $\bigcup_{i \in I} B_i \in \lim_\tau \bigvee_{i \in I} \gamma_i$. A compact, lower, upper regular pseudotopology τ on $C(X, \$)$ that respects directed sups is called a *solid hyperconvergence* ([17]).

Note that in a solid hyperconvergence, every filter converges. Indeed, every ultrafilter is convergent by compactness, so that every ultrafilter converges to $\varnothing$ because the convergence is lower. As the convergence is pseudotopological, *every filter converges to $\varnothing$ in a solid hyperconvergence.*

Proposition 4.4. *$[X, \$]$ and $\alpha(X, \$)$ are solid hyperconvergences provided that $\alpha \subseteq \kappa(X)$.*

Proof. $[X, \$]$ is well known to be pseudotopological (e.g., [9], [17]). In view of (3.6), it is lower, and compact because every filter converges to $\varnothing$. It is upper regular by Proposition 5.2.

It respects directed sups because if $B_i \in \lim_{[X,\$]} \gamma_i$ for each $i \in I$, where the family $\{\gamma_i : i \in I\}$ is directed, then for each $x \in \bigcup_{i \in I} B_i$ there is i such that $x \in B_i \in \lim_{[X,\$]} \gamma_i$, so that there is $\mathcal{G} \in \gamma_i$ with $x \in \mathrm{int} \left(\bigcap_{G \in \mathcal{G}} G \right)$. As $\mathcal{G} \in \gamma_i \leq \bigvee_{i \in I} \gamma_i$, we have $\bigcup_{i \in I} B_i \subseteq \bigcup_{\mathcal{G} \in \bigvee_{i \in I} \gamma_i} \mathrm{int} \left(\bigcap_{G \in \mathcal{G}} G \right)$.

We have seen that $\alpha(X, \$) \leq [X, \$]$ whenever $\alpha \subseteq \kappa(X)$ because $T[X, \$] = \kappa(X, \$)$, so that $\alpha(X, \$)$ is compact because $[X, \$]$ is. It is lower (and therefore upper regular by Proposition 4.1) because $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$ for each $\mathcal{A} \in \alpha$. To see that it respects directed sups, assume that $B_i \in \lim_{\alpha(X,\$)} \gamma_i$ for each $i \in I$, where the families $\{B_i : i \in I\}$ and $\{\gamma_i : i \in I\}$ are directed, and consider $\mathcal{A} \in \alpha$ containing

---

[17]Notions of *upper convergence, lower regularity* and respecting directed sups for a convergence on $cC(X, \$)$ are defined dually, and a compact lower regular upper pseudotopology on $cC(X, \$)$ that respects directed sups is also called *solid hyperconvergence.*

$\bigcup_{i\in I} B_i$. By compactness of $\mathcal{A}$ there is a finite subset F of I such that $\bigcup_{i\in F} B_i \in \mathcal{A}$. Since $\{B_i : i \in I\}$ is directed, there is $i_F \in I$ such that $\bigcup_{i\in F} B_i \subseteq B_{i_F} \in \mathcal{A}$. Since $B_{i_F} \in \lim_{\alpha(X,\$)} \gamma_{i_F}$, the open set $\mathcal{A}$ belongs to γ_{i_F}, hence to $\bigvee_{i\in I} \gamma_i$. Therefore $\bigcup_{i\in I} B_i \in \lim_{\alpha(X,\$)} \bigvee_{i\in I} \gamma_i$. $\qquad\square$

Proposition 4.5. *If τ is a solid hyperconvergence, $\mathcal{B}$ is an ideal basis for the topology of Z, and $\mathcal{C}$ is a filtered basis of closed sets in Z, then $f \in \lim_{\tau\Uparrow} \mathcal{F}$ if and only if*

$$\forall_{B\in\mathcal{B}} \ f^-(B) \in \lim_\tau \mathcal{F}^-(B),$$

if and only if

$$\forall_{C\in\mathcal{C}} \ f^-(C) \in \lim_{c\tau} \mathcal{F}^-(C).$$

Proof. We only need to show the first equivalence. Assume that $\forall_{B\in\mathcal{B}} \ f^-(B) \in \lim_\tau \mathcal{F}^-(B)$. In view of (2.1), it is enough to show that $f^-(O) \in \lim_\tau \mathcal{F}^-(O)$ whenever $O \in C(Z,\$)$. Consider a family $\{B_i : i \in I\} \subseteq \mathcal{B}$ such that $O = \bigcup_{i\in I} B_i$. Because $\mathcal{B}$ is an *ideal* basis for the topology, we can assume this family to be directed, so that $\{f^-(B_i) : i \in I\}$ is as well. Moreover, $f^-(B_i) \in \lim_\tau \mathcal{F}^-(B_i)$ for each $i \in I$ and in view of (4.1), the family of filters $\{\mathcal{F}^-(B_i) : i \in I\}$ is directed. Since τ respects directed sup,

$$f^-(O) = \bigcup_{i\in I} f^-(B_i) \in \lim_\tau \bigvee_{i\in I} \mathcal{F}^-(B_i).$$

Moreover, $\mathcal{F}^-(O) \geq \bigvee_{i\in I} \mathcal{F}^-(B_i)$ by (4.1) so that $f^-(O) \in \lim_\tau \mathcal{F}^-(O)$, which concludes the proof. $\qquad\square$

5. Interplay between hyperconvergences and the underlying topologies

Recall that for a family $\mathcal{P}$ of subsets of X, we denote $\{\mathcal{O}_X(P) : P \in \mathcal{P}\}$ by $\mathcal{O}_X^\natural(\mathcal{P})$. Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of the same set X *mesh*, in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We write $A\#\mathcal{B}$ for $\{A\}\#\mathcal{B}$.

Proposition 5.1. *The following are equivalent:*
   (1) $\mathcal{R}\#\mathcal{O}_X^\natural(\mathcal{P})$ *in* $C(X,\$)$;
   (2) $\mathcal{P}$ *is a refinement of* $\mathcal{R}$;
   (3) $\mathcal{O}_X^\natural(\mathcal{P}) \leq \mathcal{O}_X^\natural(\mathcal{R})$.

Proof. By definition, $\mathcal{R}\#\mathcal{O}_X^\natural(\mathcal{P})$ if and only if for each $P \in \mathcal{P}$ there is $R \in \mathcal{R}$ with $P \subseteq R$, which means that $\mathcal{P}$ is a refinement of $\mathcal{R}$. Equivalently, for each $P \in \mathcal{P}$ there is $R \in \mathcal{R}$ such that $\mathcal{O}_X(R) \subseteq \mathcal{O}_X(P)$, that is, $\mathcal{O}_X^\natural(\mathcal{P}) \leq \mathcal{O}_X^\natural(\mathcal{R})$. $\qquad\square$

A family $\mathcal{P}$ is said to be an *ideal subbase* if for each finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ there is $P \in \mathcal{P}$ such that $P \supseteq \bigcup \mathcal{P}_0$. Note that $\mathcal{O}_X^\natural(\mathcal{P})$ is a filter base if and only if $\mathcal{P}$ is an ideal subbase ([18]).

If γ is a filter on $C(X,\$)$ then

$$(5.1) \qquad \gamma^\Downarrow := \left\{ \bigcap_{G\in\mathcal{G}} G : \mathcal{G} \in \gamma \right\}$$

is an ideal subbase of the *reduced ideal of* γ.

---

[18]In fact, if $P_0 \cup P_1 \subset P$, then $\mathcal{O}_X(P_0) \cup \mathcal{O}_X(P_1) = \mathcal{O}_X(P_0 \cup P_1) \supset \mathcal{O}_X(P)$.

As usual, we extend, in an obvious way, the convergence of filters to that of their filter bases ([19]). A set of filters $\mathbb{B}$ is a *convergence base* of a convergence τ on Y if for every $y \in Y$ and each $\mathcal{F}$ with $y \in \lim_\tau \mathcal{F}$ there is $\mathcal{B} \in \mathbb{B}$ such that $\mathcal{B} \leq \mathcal{F}$ and with $y \in \lim_\tau \mathcal{B}$.

Proposition 5.2. $[X, \$]$ *admits a convergence base generated by* $\mathcal{O}_X^\natural(\mathcal{P})$*, where the families* $\mathcal{P}$ *are ideal subbases.*

Proof. If $Y \in \lim_{[X,\$]} \gamma$ then, by (3.6), the family

$$\mathcal{P} := \left\{ \text{int}\,(A) : A \in \gamma^\Downarrow \right\}$$

is an open cover of Y. Clearly, $\mathcal{P}$ is an ideal subbase, hence $\mathcal{O}_X^\natural(\mathcal{P})$ is a filter base. As for each $P \in \mathcal{P}$ there is $\mathcal{G} \in \gamma$ such that $P = \text{int}\,\left(\bigcap_{G \in \mathcal{G}} G\right)$ we infer that $\mathcal{G} \subseteq \mathcal{O}_X^\natural(P)$, that is, $\mathcal{O}_X^\natural(\mathcal{P})$ is coarser than γ. Finally $Y \in \lim_{[X,\$]} \mathcal{O}_X^\natural(\mathcal{P})$, because $\bigcap \mathcal{O}_X^\natural(P) = P$ for each $P \in \mathcal{P}$, and thus (3.6) holds. $\square$

Proposition 5.3. *If* $\mathcal{P} \subseteq C(X, \$)$ *is an ideal subbase and* τ *is an upper regular convergence on* $C(X, \$)$ *then*

$$\text{adh}_\tau \mathcal{P} = \lim_\tau \mathcal{O}^\natural(\mathcal{P}).$$

Proof. As $\mathcal{O}^\natural(\mathcal{P}) \# \mathcal{P}$, it is clear that $\lim_\tau \mathcal{O}^\natural(\mathcal{P}) \subseteq \text{adh}_\tau \mathcal{P}$. Conversely, if $U \in \text{adh}_\tau \mathcal{P}$ there is a filter $\eta = \mathcal{O}^\natural(\eta)$ meshing with $\mathcal{P}$ such that $U \in \lim_\tau \eta$. In other words, for each $\mathcal{A} = \mathcal{O}(\mathcal{A}) \in \eta$ there is $P \in \mathcal{P} \cap \mathcal{A}$. Thus $\mathcal{O}(P) \subseteq \mathcal{A}$ and $\mathcal{O}^\natural(\mathcal{P}) \geq \eta$ so that $U \in \lim_\tau \mathcal{O}^\natural(\mathcal{P})$. $\square$

If $\mathcal{P} \subseteq C(X, \$)$, we denote by $\mathcal{P}^\cup$ the ideal base generated by $\mathcal{P}$.

Proposition 5.4. *Let* τ *be a solid hyperconvergence such that* $p(X, \$) \leq \tau \leq [X, \$]$ *and let* $\mathcal{P} \subseteq C(X, \$)$*. Then* $\mathcal{P}$ *is a cover of* U *if and only if* $U \in \text{adh}_\tau \mathcal{P}^\cup$*.*

Proof. If $\mathcal{P}$ is a cover of U so is the ideal base $\mathcal{P}^\cup$, so that $U \in \lim_{[X,\$]} \mathcal{O}^\natural(\mathcal{P}^\cup)$ by Proposition 5.2. Moreover, $\mathcal{O}^\natural(\mathcal{P}^\cup) \# \mathcal{P}^\cup$ so that $U \in \text{adh}_{[X,\$]} \mathcal{P}^\cup \subseteq \text{adh}_\tau \mathcal{P}^\cup$. Conversely, if $U \in \text{adh}_\tau \mathcal{P}^\cup$ then by Proposition 5.3, $U \in \lim_\tau \mathcal{O}^\natural(\mathcal{P}^\cup) \subseteq \lim_{p(X,\$)} \mathcal{O}^\natural(\mathcal{P}^\cup)$. Therefore, by definition of $p(X, \$)$, for each $x \in U$ there is $S \in \mathcal{P}^\cup$ such that $\mathcal{O}(S) \subseteq \mathcal{O}(x)$, that is, $x \in S$. Thus there is $P \in \mathcal{P}$ containing x and $\mathcal{P}$ is a cover of U. $\square$

Corollary 5.5. *If* $\mathcal{P} \subseteq C(X, \$)$ *is an ideal base and* τ *is a solid hyperconvergence such that* $p(X, \$) \leq \tau \leq [X, \$]$ *then*

$$\text{adh}_\tau \mathcal{P} = \lim_\tau \mathcal{O}^\natural(\mathcal{P}) = \lim_{[X,\$]} \mathcal{O}^\natural(\mathcal{P}) = \text{adh}_{[X,\$]} \mathcal{P}$$

consists of those $U \in C(X, \$)$ *for which* $\mathcal{P}$ *is a cover of* U*.*

Corollary 5.5 does not mean that all the pretopological solid hyperconvergences between $p(X, \$)$ and $[X, \$]$ coincide! But their adherences of ideal bases are the same.

---

[19] If $\mathcal{B}$ is a filter base and τ is a convergence, then $y \in \lim_\tau \mathcal{B}$ if $y \in \lim_\tau \mathcal{B}^\uparrow$, where $\mathcal{B}^\uparrow$ is the filter generated by $\mathcal{B}$.

Example 5.6. Let X be an infinite countable set with the discrete topology. In this case $p(X, \$) = [X, \$]$. The hyperset $\mathcal{P} := \{\{x\} : x \in X\}$ is an open cover of X. By definition, $Y \in \mathrm{adh}_{p(X,\$)}\mathcal{P}$ if for each finite subset F of Y there is $A \in \mathcal{O}_X(F) \cap \mathcal{P}$. Hence there is $x \in X$ such that $F \subset \{x\}$, which means that the only finite subsets of Y are singletons, that is, Y is a singleton. On the other hand, $X \in \mathrm{adh}_{p(X,\$)}\mathcal{P}^{\cup}$, because $F \in \mathcal{O}_X(F) \cap \mathcal{P}^{\cup}$ for each finite subset F of Y.

If α is a collection of openly isotone families of subsets of X, we call $\mathcal{P} \subseteq C(X, \$)$ an (open) α-*cover* if $\mathcal{P} \cap \mathcal{A} \neq \varnothing$ for every $\mathcal{A} \in \alpha$. Of course, if $p(X) \subseteq \alpha$ then every open α-cover of X is also an open cover of X. Note that the notion of $p(X)$-cover coincides with the traditional notion ω-cover, and that the notion of $k(X)$-cover coincides with the traditional notion k-cover (see e.g., [36]). It follows immediately from the definitions that

Proposition 5.7. *Let* $\mathcal{P} \subseteq C(X, \$)$ *and let* α *be a topology on* $C(X, \$)$. *Then* $U \in \mathrm{adh}_{\alpha(X,\$)}\mathcal{P}$ *if and only if* $\mathcal{P}$ *is an* α-*cover of* U.

6. Transfer of filters

We shall confer particular attention to the convergence of a filter to the *zero function* for the convergence $\tau^{\Uparrow}$ on $C(X, \mathbb{R})$ that is preimage-wise with respect to a solid hyperconvergence τ on $C(X, \$)$. To that effect, consider a decreasing base of bounded open neighborhoods of 0 in $\mathbb{R}$:

$$(6.1) \qquad\qquad \{W_n : n < \omega\},$$

for instance, let us fix $W_n := \left\{r \in \mathbb{R} : |r| < \frac{1}{n}\right\}$.

Lemma 6.1. $\bar{0} \in \lim_{\tau^{\Uparrow}} \mathcal{F}$ *if and only if* $X \in \lim_\tau \mathcal{F}^-(W_n)$ *for each* $n < \omega$.

Proof. As $\bar{0}^-(O)$ is equal either to X (when $0 \in O$) or to $\varnothing$ (when $0 \notin O$), it follows from (2.1) that the condition is necessary. Conversely, if an open subset O of $\mathbb{R}$ contains 0, then there is $n < \omega$ such that $W_n \subseteq O$, hence $X \in \lim_\tau \mathcal{F}^-(W_n)$ implies that $X \in \lim_\tau \mathcal{F}^-(O)$, because $\mathcal{F}^-(W_n) \leq \mathcal{F}^-(O)$. If now $0 \notin O$ then $\bar{0}^-(O) = \varnothing \in \lim_\tau \mathcal{F}^-(O)$, because τ is a hyperconvergence (hence every filter converges to $\varnothing$). $\qquad\square$

This special case is important, because it is much easier to compare local properties of $\tau^{\Uparrow}$ at $\bar{0}$ with local properties of τ at X than to study analogous properties at an arbitrary $f \in C(X, \mathbb{R})$. Moreover, often a study of the mentioned special case is sufficient for the understanding of this local property at each $f \in C(X, \mathbb{R})$. This is feasible whenever all the translations are continuous for $\tau^{\Uparrow}$, that is, whenever $\tau^{\Uparrow}$ is translation-invariant. It is known that the topology of pointwise convergence, the compact-open topology, the natural convergence and thus the natural topology are translation-invariant. Translations are not always continuous for the Isbell topology (see [16], [30]), but for each topological space X, there exists the finest translation-invariant topology of the form $\alpha(X, \mathbb{R})$ that is coarser than the Isbell topology $\kappa(X, \mathbb{R})$ [15].

Lemma 6.1 suggests that local properties of $\tau^{\Uparrow}$ at $\bar{0}$ "correspond" to local properties of τ at X. The remainder of the paper is devoted to making this statement clear and exploring applications.

If α is a filter on $C(X, \$)$ then, for each (open) subset W of $\mathbb{R}$,

$$(6.2) \qquad\qquad [\alpha, W] := \{[\mathcal{A}, W] : \mathcal{A} \in \alpha\}$$

is a filter base on $C(X, \mathbb{R})$, called the *W-erected filter* of α. Note that

(6.3) $$\alpha \leq \gamma, W \supseteq V \Longrightarrow [\alpha, W] \leq [\gamma, V].$$

The filter on $C(X, \mathbb{R})$ generated by the filter base $\mathcal{V}$ of the neighborhood filter of 0 ([20])

$$\bigcup_{V \in \mathcal{V}} [\alpha, V]$$

does not depend on the choice of a particular neighborhood base of 0 in $\mathbb{R}$ ([21]). We denote it by $[\alpha, \mathcal{N}(0)]$ and call it the *erected filter* of α. In particular, if a base is of the form (6.1), $[\alpha, W_n] \leq [\alpha, W_{n+1}]$ and

(6.4) $$[\alpha, \mathcal{N}(0)] = \bigvee_{n < \omega} [\alpha, W_n].$$

We shall see that if α converges to X in τ then its erected filter converges to the null function in $\tau^{\Uparrow}$. We shall in fact consider a more general case.

Lemma 6.2. *If $\{\alpha_n : n < \omega\}$ is a sequence of filters on $C(X, \$)$ such that $\alpha_n = O_X^\natural(\alpha_n)$, then the sequence of filters $([\alpha_n, W_n])_{n<\omega}$ admits a supremum.*

Proof. If $S_1, \ldots, S_k \in \bigcup_{n<\omega} [\alpha_n, W_n]$, then there are $n_1, \ldots, n_k < \omega$, say, $n_1 \leq \ldots \leq n_k$ and $\mathcal{A}_j \in \alpha_{n_j}$ for $1 \leq j \leq k$ such that $[\mathcal{A}_j, W_{n_j}] \subseteq S_j$, and thus $[\mathcal{A}, W_{n_k}] \subseteq \bigcap_{1 \leq j \leq k} [\mathcal{A}_j, W_{n_j}] \subseteq \bigcap_{1 \leq j \leq k} S_j$, where $\mathcal{A} := \bigcap_{1 \leq j \leq k} \mathcal{A}_j$. As each α_n is based in openly isotone families $\mathcal{A}$, and $[\mathcal{A}, W] \neq \varnothing$ provided that $W \neq \varnothing$, the family $\bigcup_{n<\omega} [\alpha_n, W_n]$ is a filter subbase and generates $\bigvee_{n<\omega} [\alpha_n, W_n]$. $\square$

Theorem 6.3. *If $X \in \lim_\tau \alpha_n$ for each $n < \omega$, then $\bar{0} \in \lim_{\tau^{\Uparrow}} \bigvee_{n<\omega} [\alpha_n, W_n]$.*

Proof. We use Lemma 6.1 to check that $\bar{0} \in \lim_{\tau^{\Uparrow}} \bigvee_{n<\omega} [\alpha_n, W_n]$. Let O be an open subset of $\mathbb{R}$ with 0. Then there is $n < \omega$ such that $O \supseteq W_k$ for $k \geq n$. Then $[\mathcal{A}, W_n]^-(O) = \{f^-(O) : f^-(W_n) \in \mathcal{A}\} \subseteq \mathcal{A}$ for every $\mathcal{A}$. It follows that, $[\alpha_n, W_n]^-(O) \geq \alpha_n$ so that $X \in \lim_\tau [\alpha_n, W_n]^-(O) \subseteq \lim_\tau \left(\bigvee_{n<\omega} [\alpha_n, W_n]\right)^-(O)$. $\square$

Corollary 6.4. *If $X \in \lim_\tau \alpha$ then $\bar{0} \in \lim_{\tau^{\Uparrow}} [\alpha, \mathcal{N}(0)]$.*

In view of (4.1), we have $\bigvee_{n<\omega} [\alpha, W_n] \leq [\alpha, \{0\}]$ ([22]). Thus:

Corollary 6.5. *If $X \in \lim_\tau \alpha$ then $\bar{0} \in \lim_{\tau^{\Uparrow}} [\alpha, \{0\}]$.*

7. Construction of classes of filters

Local properties of a topological space depend on properties of its neighborhood filters. More generally, local properties of a convergence space depend on properties of its convergent filters. To understand how local properties of τ and $\tau^{\Uparrow}$ relate, we first need to understand how the properties of the filter α relate to those of the filter $[\alpha, \mathcal{N}(0)]$ in Corollary 6.4. We will explore this question in details in Section 8. In

---

[20]Indeed, if $B_0, B_1 \in \bigcup_{V \in \mathcal{V}(0)} [\alpha, V]$, then there are $V_0, V_1 \in \mathcal{V}(0)$ and $\mathcal{A}_0, \mathcal{A}_1 \in \alpha$ such that $[\mathcal{A}_0, V_0] \subset B_0$ and $[\mathcal{A}_1, V_1] \subset B_1$, and thus $[\mathcal{A}_0 \cap \mathcal{A}_1, V_0 \cap V_1] \subset [\mathcal{A}_0, V_0] \cap [\mathcal{A}_1, V_1] \subset B_0 \cap B_1$.

[21]In fact, if $\mathcal{V}, \mathcal{W}$ are open bases (of the neighborhood filter of 0) then for each $W \in \mathcal{W}$ there is $V \in \mathcal{V}$ such that $V \subset W$, hence $[\alpha, W] \leq [\alpha, V]$, and conversely.

[22]Here $f^-(0)$ is not open, so that we must use the general definition $[\mathcal{A}, \{0\}] := \{f : \exists_{A \in \mathcal{A}} A \subset f^-(0)\}$.

the present section, we introduce the relevant terminology, as well as examples of local properties to be considered.

Blackboard letters like $\mathbb{D}$ denote classes of filters, and $\mathbb{D}(X)$ denote the set of filters on X of the class $\mathbb{D}$. The class of principal filters is denoted by $\mathbb{F}_0$ and the class of countably based filters is denoted by $\mathbb{F}_1$. More generally, $\mathbb{F}_\kappa$ stands for the class of filters that admit a base of cardinality less than $\aleph_\kappa$.

A convergence (in particular, a topological) space X is called $\mathbb{D}$-*based at* x if whenever $x \in \lim \mathcal{F}$ there is $\mathcal{D} \in \mathbb{D}(X)$, $\mathcal{D} \leq \mathcal{F}$ with $x \in \lim \mathcal{D}$, and $\mathbb{D}$-*based* if it is $\mathbb{D}$-based at each $x \in X$. For example, a convergence (topological) space is first-countable if and only if it is $\mathbb{F}_1$-based.

If $\mathbb{D}$ and $\mathbb{J}$ are two classes of filters, we say that $\mathbb{D}$ is $\mathbb{J}$-*steady* if

$$\mathcal{D} \in \mathbb{D}, \mathcal{J} \in \mathbb{J}, \mathcal{D}\#\mathcal{J} \implies \mathcal{D} \vee \mathcal{J} \in \mathbb{D}.$$

As usual, if $R \subseteq X \times Y$ and $D \subseteq X$ then $RD := \{y \in Y : \exists x \in D, : (x,y) \in R\}$ and $\mathcal{RD} := \{RD : R \in \mathcal{R}, D \in \mathcal{D}\}$.

A class $\mathbb{D}$ is $\mathbb{J}$-*composable* if

$$\mathcal{D} \in \mathbb{D}(X), \mathcal{R} \in \mathbb{J}(X \times Y) \implies \mathcal{RD} \in \mathbb{D}(Y).$$

By convention, we consider that each class $\mathbb{D}$ contains every degenerate filter. In the sequel, classes that are $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady will be of particular interest.

For each set X, we consider the following relations $\Diamond_\kappa$, $\dagger$ and $\triangle$ on $\mathbb{F}(X)$: we write $\mathcal{F}\Diamond_\kappa\mathcal{H}$ if

$$\mathcal{F}\#\mathcal{H} \implies \exists A \in [X]^{\leq \kappa} : A\#\left(\mathcal{F} \vee \mathcal{H}\right);$$

we denote by $\mathcal{F}\triangle_\kappa\mathcal{H}$ the following relation

$$\mathcal{F}\#\mathcal{H} \implies \exists \mathcal{L} \in \mathbb{F}_\kappa : \mathcal{L} \geq \mathcal{F} \vee \mathcal{H}.$$

Finally, we write $\mathcal{F}\dagger_1\mathcal{H}$ if $\mathcal{F} \vee \mathcal{H} \in \mathbb{T}_1$ where $\mathcal{T} \in \mathbb{T}_1$ if

$$(A_n)_{n<\omega}\#\mathcal{T} \implies \exists B_n \in [A_n]^{<\omega} : \left(\bigcup_{n<\omega} B_n\right)\#\mathcal{T},$$

and $\mathcal{F}\dagger_0\mathcal{H}$ if $\mathcal{F} \vee \mathcal{H} \in \mathbb{T}_0$ where $\mathcal{T} \in \mathbb{T}_0$ if

$$(A_n)_{n<\omega}\#\mathcal{T} \implies \exists a_n \in A_n : \{a_n : n \in \omega\}\#\mathcal{T}.$$

If $*$ is a relation on $\mathbb{F}(X)$ and $\mathbb{D} \subseteq \mathbb{F}(X)$, then $\mathbb{D}^* := \{\mathcal{F} \in \mathbb{F}(X) : \forall \mathcal{D} \in \mathbb{D}, \mathcal{F} * \mathcal{D}\}$. Many local topological properties of a space X correspond to the fact that X is $\mathbb{D}^*$-based, for $\mathbb{D} = \mathbb{F}_0$ or $\mathbb{D} = \mathbb{F}_1$.

In particular, a topological space (and by extension, a convergence space) is respectively *Fréchet* ([23]), *strongly Fréchet*, *productively Fréchet*, of κ-*tightness*, *countably fan-tight*, *strongly countably fan-tight*, if it is $\mathbb{F}_0^\triangle$-based, $\mathbb{F}_1^\triangle$-based, $\mathbb{F}_1^{\triangle\triangle}$-based, $\mathbb{F}_1^{\Diamond_\kappa}$-based, $\mathbb{F}_1^{\dagger_1}$-based, $\mathbb{F}_1^{\dagger_0}$-based respectively. Here we gather the just mentioned equivalences:

---

[23]often called Fréchet-Urysohn, but we use the shorter term *Fréchet*.

<table>
<thead>
<tr><th>class</th><th>based</th></tr>
</thead>
<tbody>
<tr><td>Fréchet</td><td>$\mathbb{F}_0^{\triangle}$-based</td></tr>
<tr><td>strongly Fréchet</td><td>$\mathbb{F}_1^{\triangle}$-based</td></tr>
<tr><td>productively Fréchet</td><td>$\mathbb{F}_1^{\triangle\triangle}$-based</td></tr>
<tr><td>κ-tight</td><td>$\mathbb{F}_1^{\Diamond_\kappa}$-based</td></tr>
<tr><td>countably fan-tight</td><td>$\mathbb{F}_1^{\dagger_1}$-based</td></tr>
<tr><td>strongly countably fan-tight</td><td>$\mathbb{F}_1^{\dagger_0}$-based</td></tr>
</tbody>
</table>

(7.1)

Examples of $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady classes include the class $\mathbb{F}_n$ of filters with a filter-base of cardinality less than $\aleph_n$ for $n \geq 1$, as well as $\mathbb{F}_1^{\triangle}$, $\mathbb{F}_1^{\triangle\triangle}$, $\mathbb{F}_1^{\Diamond_\kappa}$, $\mathbb{F}_1^{\dagger_1}$ and $\mathbb{F}_1^{\dagger_0}$. The class $\mathbb{F}_0^{\triangle}$ of *Fréchet filters* is $\mathbb{F}_0$-composable but not $\mathbb{F}_1$-steady, and the class $\mathbb{F}_1^{\Diamond\Diamond}$ of *steadily countably tight filters* is $\mathbb{F}_1$-steady but not $\mathbb{F}_0$-composable. See [31] for a systematic study of these concepts and applications to product theorems.

8. Transfer of classes of filters

We notice that the erected filter $[\alpha, \mathcal{N}(0)]$ of α can be reconstructed from α with the aid of compositions of relations as follows. Let $\Delta := \{(f, A, k) : A \subseteq f^-(W_k)\}$ and let Δ_j be the j-th projection of Δ. Let $\mathcal{N}$ stand for the cofinite filter on ω.

Proposition 8.1.
$$[\alpha, \mathcal{N}(0)] = \Delta_1(\Delta_2^- \alpha \vee \Delta_3^- \mathcal{N}).$$

Proof. If $\mathcal{A} \in \alpha$ then $\Delta_2^- \mathcal{A} = \{(f, A, k) : f^-(W_k) \in \mathcal{A}\}$. If $n < \omega$ then $\Delta_3^- \{k : k \geq n\} = \{(f, A, k) : \exists_{k \geq n} A \subseteq f^-(W_k)\}$. Hence

$$\Delta_2^- \mathcal{A} \vee \Delta_3^- \{k : k \geq n\} = \bigcup_{k \geq n} \{(f, A, k) : f^-(W_k) \in \mathcal{A}\},$$

and thus $\Delta_1\left(\Delta_2^- \mathcal{A} \vee \Delta_3^- \{k : k \geq n\}\right) = \bigcup_{k \geq n} \{f : f^-(W_k) \in \mathcal{A}\} = \bigcup_{k \geq n}[\mathcal{A}, W_n]$. Because $W_k \subseteq W_n$ if $k \geq n$, hence $[\mathcal{A}, W_k] \subseteq [\mathcal{A}, W_n]$. Consequently,

$$\Delta_1(\Delta_2^- \alpha \vee \Delta_3^- \mathcal{N}) = \{[\mathcal{A}, W_n] : \mathcal{A} \in \alpha, n < \omega\} = [\alpha, (W_n)_n].$$

$\square$

Corollary 8.2. *If $\mathbb{B}$ is an $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady class of filters and $\alpha \in \mathbb{B}$ then $[\alpha, \mathcal{N}(0)] \in \mathbb{B}$.*

Consider for each n, the relation $[\cdot, W_n] : C(X, \$) \to C(X, \mathbb{R})$. Note that the filter $\bigvee_{n < \omega}[\alpha_n, W_n]$ of Lemma 6.2 is the supremum of the images of the filters α_n under this relation. A class $\mathbb{B}$ of filters is *countably upper closed* if it is closed under countable suprema of increasing sequences. In particular:

Proposition 8.3. *If $\mathbb{B}$ is an $\mathbb{F}_0$-composable and countably upper closed class of filters, and if each $\alpha_n \in \mathbb{B}$, then $\bigvee_{n < \omega}[\alpha_n, W_n] \in \mathbb{B}$.*

Let $\mathcal{W} := \{W_n\}$ be a fixed base of $\mathcal{N}(0)$ in $\mathbb{R}$. Define

(8.1)
$$\mathcal{F}^{\mathcal{N}(0)} := \bigvee_{n < \omega} [\mathcal{F}^-(W_n), W_n],$$

on $C(X, \mathbb{R})$, associated with a filter $\mathcal{F}$ on $C(X, \mathbb{R})$. As $F \subseteq [F^-(W), W]$ for every $F \subset C(X, \mathbb{R})$ and $W \subset \mathbb{R}$,

(8.2)
$$\mathcal{F}^{\mathcal{N}(0)} \leq \mathcal{F}.$$

Proposition 8.4. *For each symmetric open intervals V, W that contain 0, there is a strictly increasing linear map h such that*

$$[\mathcal{F}^-(W), W] = h\left([\mathcal{F}^-(V), V]\right).$$

Proof. A base of $[\mathcal{F}^-(W), W]$ is of the form

$$G_F(W) := \left\{g : g^-(W) \in \left\{f^-(W) : f \in F\right\}\right\} : F \in \mathcal{F}$$

is a base of $[\mathcal{F}^-(W), W]$. Let h be a strictly increasing (linear) map such that $h(V) = W$. Then if $(h \circ g)^-(W) = g^-(V)$. Therefore $g \in G_F(V)$ if and only if $h \circ g \in G_F(W)$, that is, $G_F(W) = h\left(G_F(V)\right)$. □

It follows that, if $W_n = r_n W$, where $W := (-1, 1)$ and $\{r_n\}_n$ is a decreasing sequence tending to 0, then

$$\mathcal{F}^{\mathcal{N}(0)} = \bigvee_{n < \omega} r_n \mathcal{H},$$

where $\mathcal{H} := [\mathcal{F}^-(W), W]$.

Corollary 8.5. *Let $\mathbb{B}$ be a class of filters.*
   (1) *If $\mathbb{B}$ is $\mathbb{F}_0$-composable and countably upper closed, and τ is $\mathbb{B}$-based at X, then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$.*
   (2) *If $\mathbb{B}$ is $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady, and τ is pretopology that is $\mathbb{B}$-based at X, then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$.*

Proof. 1. If $\overline{0} \in \lim_{\tau^{\Uparrow}} \mathcal{F}$ then $X \in \lim_{\tau} \mathcal{F}^-(W_n)$ for each n. Therefore, for each n, there is $\mathcal{B}_n \in \mathbb{B}$ with $X \in \lim_{\tau} \mathcal{B}_n$ and $\mathcal{B}_n \leq \mathcal{F}^-(W_n)$. In view of Theorem 6.3, $\overline{0} \in \lim_{\tau^{\Uparrow}} \bigvee_{n < \omega}[\mathcal{B}_n, W_n]$. By Proposition 8.3, $\bigvee_{n < \omega}[\mathcal{B}_n, W_n] \in \mathbb{B}$. Moreover,

$$\bigvee_{n < \omega}[\mathcal{B}_n, W_n] \leq \mathcal{F}^{\mathcal{N}(0)} \leq \mathcal{F},$$

which concludes the proof.

2. If τ is pretopological, then in the proof above, for each n we can take $\mathcal{B}_n = \mathcal{V}_{\tau}(X)$, so that $\bigvee_{n < \omega}[\mathcal{B}_n, W_n] = [\mathcal{V}_{\tau}(X), \mathcal{N}(0)]$. By Proposition 8.1, $[\mathcal{V}_{\tau}(X), \mathcal{N}(0)] \in \mathbb{B}$. □

A filter α on $C(X, \$)$ valued in openly isotone families, can be reconstructed from its erected filter $[\alpha, \mathcal{N}(0)]$ with the aid of compositions of relations, provided that a separation condition by real-valued continuous functions holds. A family $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$ is *functionally separated* if for every $O \in \mathcal{A}$, there is $A \in \mathcal{A}$ and $h \in C(X, [0, 1])$ such that $h(A) = \{0\}$ and $h(X \setminus O) = \{1\}$. A hyperfilter is called *functionally separated* if it admits a base of functionally separated hypersets. A solid hyperconvergence on $C(X, \$)$ is *functionally separated* if whenever $Y \in \lim \gamma$ then there exists $\alpha \leq \gamma$ such that $Y \in \lim \alpha$ and α is functionally separated.

It follows from [16, Lemma 2.5] that compact families on a completely regular space are functionally separated. Therefore if $\alpha \subseteq \kappa(X)$ then $\alpha(X, \$)$ is functionally separated.

Lemma 8.6. *If X is normal, then $[X, \$]$ is functionally separated. Moreover, for each bounded open neighborhood W of 0 in $\mathbb{R}$, $[X, \$]$ has a base of filters α such that*

$$\alpha = \mathcal{O}^{\sharp}\left(\alpha^{\Downarrow}\right) = \mathcal{O}^{\sharp}\left(\left([\alpha, \mathcal{N}(0)]^-(W)\right)^{\Downarrow}\right)$$

where $\alpha^{\Downarrow}$ is the reduced ideal of α (5.1).

Proof. In view of (3.6), if $O \in \lim_{[X,\$]} \alpha$ then for each $x \in O$ there exists $\mathcal{A}_x \in \alpha$ such that $x \in \text{int}_X (\bigcap_{U \in \mathcal{A}_x} U)$. By regularity, there is a closed neighborhood V_x of x such that $V_x \subseteq \text{int}_X (\bigcap_{U \in \mathcal{A}_x} U)$. As the family $\mathcal{P} := \{\bigcup_{x \in S} V_x : S \in [O]^{<\infty}\}$ is an ideal base, $\mathcal{O}_X^\natural (\mathcal{P})$ is a filter-base on $C(X, \$)$; moreover, $\bigcap_{x \in S} \mathcal{A}_x \subseteq \mathcal{O}_X (\bigcup_{x \in S} V_x)$ for each $S \in [O]^{<\infty}$. Therefore $\alpha \geq \mathcal{O}_X^\natural (\mathcal{P})$ and $O \in \lim_{[X,\$]} \mathcal{O}_X^\natural (\mathcal{P})$. Finally, since $\mathcal{P}$ consists of closed sets and X is normal then $\mathcal{O}_X^\natural (\mathcal{P})$ is functionally separated, which completes the proof.

As shown in the first part of the proof, $[X, \$]$ has a base composed of filters $\alpha = \mathcal{O}_X^\natural (\mathcal{P})$ where $\mathcal{P}$ is an ideal base of closed sets. For each $P \in \mathcal{P}$, each $n \in \mathbb{N}$ consider the corresponding element

$$R := \bigcap_{f \in [P, W_n]} f^- (W)$$

of $([\alpha, \mathcal{N}(0)]^- (W))^\Downarrow$. Then $\mathcal{O}(P) \subseteq \mathcal{O}(R)$ so that $\alpha \geq \mathcal{O} \left({}^\natural ([\alpha, \mathcal{N}(0)]^- (W))^\Downarrow \right)$. Indeed, if $\mathcal{O}(P) \nsubseteq \mathcal{O}(R)$ then $R \nsubseteq P$ and there is $x \in R \setminus P$. By complete regularity, there is a continuous map $h \in C(X, \mathbb{R})$ such that $h(x) = 1 + \sup W$ and $h(P) = \{0\}$. Then $h \in [P, W_n]$ but $h(R) \nsubseteq W$; a contradiction. $\square$

Let us call $\$$-*compatible* a class $\mathbb{B}$ of filters satisfying

$$\beta \in \mathbb{B}(C(X, \$)) \Longrightarrow \mathcal{O}^\natural (\beta^\Downarrow) \in \mathbb{B}(C(X, \$)).$$

Theorem 8.7. *If α is a filter on $C(X, \$)$ and W is an open bounded neighborhood of 0, then $\alpha \leq [\alpha, \mathcal{N}(0)]^- (W)$. If moreover α is functionally separated, then*

$$\alpha = [\alpha, \mathcal{N}(0)]^- (W).$$

Proof. 1. If n is such that $W_n \subseteq W$, then $[\mathcal{A}, W_n]^- (W) \subseteq \mathcal{A}$ for each $\mathcal{A} \in \alpha$. Indeed, if $G \in [\mathcal{A}, W_n]^- (W)$ then there is $A \in \mathcal{A}$ and $f \in C(X, \mathbb{R})$ such that $G = f^- (W)$ and $f(A) \subseteq W_n$. As $W_n \subseteq W$, we infer that $A \subseteq G$, so that $G \in \mathcal{A}$. Consequently $\alpha \leq [\alpha, \mathcal{N}(0)]^- (W) = \bigvee_{n < \omega} [\mathcal{A}, W_n]^- (W)$.

2. If $G \in \mathcal{A}$ then, by the functional separation of $\mathcal{A}$, there is $A \in \mathcal{A}$ and $h \in C(X, \mathbb{R})$ such that $h(A) = \{0\}$ and $h(X \setminus G) = \{\sup W\}$. Therefore, $h \in [\mathcal{A}, W_n]$ for each $n < \omega$, and $h^- (W) \subseteq G$ so that $G \in \mathcal{O}^\natural ([\mathcal{A}, W_n]^- (W))$, hence $[\alpha, W_n]^- (W) \leq \alpha$ for each $n < \omega$. $\square$

In particular, $\alpha \leq \bigwedge_{n < \omega} [\alpha, \mathcal{N}(0)]^- (W_n)$ and if if α is functionally separated, then the equality holds. On the other hand, if α is an ultrafilter then $\alpha = [\alpha, \mathcal{N}(0)]^- (W)$ for any open bounded neighborhood of 0.

Consider the function $W_* : C(X, \mathbb{R}) \to C(X, \$)$ (defined by (1.5)). It follows from Theorem 8.7 that if α is functionally separated, then $\alpha = W_* [\alpha, \mathcal{N}(0)]$, that is, α is the image of $[\alpha, \mathcal{N}(0)]$ by a relation. This observation constitutes a considerable simplification of a construction proposed in [29] for the finite-open topologies and extended to α-topologies (^{24}) in [12].

If $\mathbb{B}$ is a class of filters, let $\mathbb{B}^\wedge$ denote the class of filters than can be represented as an infimum of filters of the class $\mathbb{B}$.

Corollary 8.8. *Let $\mathbb{B}$ be an $\mathbb{F}_0$-composable class of filters.*

---

^{24}where α is a collection of compact families including all the finitely generated ones

(1) *Let τ be a functionally separated solid hyperconvergence. If $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ then τ is $\mathbb{B}$-based at X.*
(2) *If $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ then $P\tau$ is $\mathbb{B}^{\wedge}$-based at X.*
(3) *If $\mathbb{B}$ is $\$$-compatible and $[X, \mathbb{R}]$ is $\mathbb{B}$-based at $\overline{0}$, then $[X, \$]$ is $\mathbb{B}$-based at X.*

Proof. (1). Let α be a functionally separated filter on $C(X, \$)$ such that $X \in \lim_{\tau} \alpha$. By Corollary 6.4, $\overline{0} \in \lim_{\tau^{\Uparrow}} [\alpha, \mathcal{N}(0)]$. Therefore, there is $\mathcal{G} \in \mathbb{B}$ such that $\overline{0} \in \lim_{\tau^{\Uparrow}} \mathcal{G}$ and $\mathcal{G} \leq [\alpha, \mathcal{N}(0)]$, hence $X \in \lim_{\tau} \mathcal{G}^{-}(W)$. In view of Theorem 8.7,

$$\alpha = [\alpha, \mathcal{N}(0)]^{-}(W) \geq \mathcal{G}^{-}(W),$$

and $\mathcal{G}^{-}(W) \in \mathbb{B}$ by $\mathbb{F}_0$-composability.

(2). If, in the proof above, α is an ultrafilter, then the assumption of functional separation is not needed. Now the vicinity filter of X for $P\tau$ is

$$\begin{aligned} \mathcal{V}_{\tau}(X) \quad &= \quad \bigwedge \{\alpha : \alpha \in \mathbb{U}(C(X, \$)), X \in \lim_{\tau} \alpha\} \\ &= \quad \bigwedge \{\mathcal{G}^{-}(W_1) : \alpha \in \mathbb{U}(C(X, \$)), X \in \lim_{\tau} \alpha\}. \end{aligned}$$

Therefore $\mathcal{V}_{\tau}(X) \in \mathbb{B}^{\wedge}$.

(3). In the proof of (1) above, if $\tau = [X, \$]$ then by Lemma 8.6, we can assume $\alpha = \mathcal{O}^{\sharp}([\alpha, \mathcal{N}(0)]^{-}(W))^{\Downarrow} \geq \mathcal{O}^{\sharp}(\mathcal{G}^{-}(W))^{\Downarrow}$. By $\$$-compatibility and $\mathbb{F}_0$-composability, $\mathcal{O}^{\sharp}(\mathcal{G}^{-}(W))^{\Downarrow} \in \mathbb{B}$ and $X \in \lim_{[X, \$]} \mathcal{O}^{\sharp}(\mathcal{G}^{-}(W))^{\Downarrow}$ by Proposition 5.2. $\qquad \square$

Combining Corollaries 8.5 and 8.8, we obtain:

Corollary 8.9. *Let $\mathbb{B}$ be an $\mathbb{F}_0$-composable class of filters.*

(1) *Let τ be a functionally separated solid hyperconvergence.*
  (a) *If $\mathbb{B}$ is countably upper closed then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ if and only if τ is $\mathbb{B}$-based at X.*
  (b) *If $\mathbb{B}$ is $\mathbb{F}_1$-steady and if τ is pretopological, then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ if and only if τ is $\mathbb{B}$-based at X.*
(2) *If $\mathbb{B}$ is countably upper closed and $\$$-compatible, then $[X, \mathbb{R}]$ is $\mathbb{B}$-based if and only if $[X, \$]$ is $\mathbb{B}$-based at X.*

In view of Lemma 8.6, we have in particular:

Corollary 8.10. *Let $\mathbb{B}$ be an $\mathbb{F}_0$-composable class of filters.*

(1) *If $\mathbb{B}$ is $\mathbb{F}_1$-steady and $\alpha \subseteq \kappa(X)$, then $C_{\alpha}(X, \mathbb{R})$ is $\mathbb{B}$-based at $\overline{0}$ if and only if $C_{\alpha}(X, \$)$ is $\mathbb{B}$-based at X.*
(2) *If $\mathbb{B}$ is countably upper closed and either X is normal or $\mathbb{B}$ is $\$$-compatible, then $[X, \mathbb{R}]$ is $\mathbb{B}$-based if and only if $[X, \$]$ is $\mathbb{B}$-based at X.*

In particular, if $\mathcal{D}$ is a compact network on a completely regular space X, we consider $\alpha_{\mathcal{D}} := \mathcal{O}_X^{\natural}(\mathcal{D})$. Then $C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})$ is a topological group and if γ is a cardinal function corresponding to a $\mathbb{F}_1$-steady and $\mathbb{F}_0$-composable class of filters, like *character* χ, *tightness* t, *fan-tightness* vet, and *strong fan-tightness* vet*, then

$$(8.3) \qquad \qquad \gamma(C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})) = \gamma(C_{\alpha_{\mathcal{D}}}(X, \$), X).$$

As mentioned before, translations need not be continuous for the Isbell topology on $C(X, \mathbb{R})$. However, the fine Isbell topology $\overline{\kappa}(X, \mathbb{R})$ is always translation-invariant and the neighborhood filter of f for the fine Isbell topology is $f + \mathcal{N}_{\kappa}(\overline{0})$

[16, Theorem 4.1], which implies that the translations are continuous for the Isbell topology if and only if this Isbell topology coincides with the fine Isbell topology.

On the other hand, for every X there exists the finest translation-invariant topology $\Sigma(X, \mathbb{R})$ that is an $\mathbb{R}$-dual topology of $\Sigma(X) \subseteq \kappa(X)$, hence coarser than the Isbell topology $\kappa(X, \mathbb{R})$ [15]. Therefore

$$\gamma(C_{\overline{\kappa}}(X, \mathbb{R})) = \gamma(C_\kappa(X, \$), X) \text{ and } \gamma(C_\Sigma(X, \mathbb{R})) = \gamma(C_\Sigma(X, \$), X).$$

We will see in the next section that calculating invariants for $C_{\alpha_\mathcal{D}}(X, \$), C_\kappa(X, \$)$ and $[X, \$]$ in terms of X is often easy. This way, we will recover a large number of known results, as well as obtain new ones.

9. Character and tightness

Theorem 9.1. *(e.g., [38]) The tightness and the character of $[X, \$]$ coincide.*

Proof. As the tightness is not greater than character, we need only prove that $\chi([X, \$], Y) \leq t([X, \$], Y)$. Assume that $t([X, \$], Y) = \lambda$ and let $Y \in \lim_{[X, \$]} \gamma$. By Proposition 5.2, there exists an ideal subbase $\mathcal{P}$ that is an open cover of Y such that $Y \in \lim_{[X, \$]} \mathcal{O}_X^\natural(\mathcal{P})$ and $\mathcal{O}_X^\natural(\mathcal{P}) \leq \gamma$. It is clear that $\mathcal{P} \# \mathcal{O}_X^\natural(\mathcal{P})$, hence there is a family $\mathcal{S}_0 \subseteq \mathcal{P}$ such that $\text{card} \mathcal{S}_0 \leq \lambda$ and $\mathcal{S}_0 \# \mathcal{O}_X^\natural(\mathcal{P})$. The family $\mathcal{S} := \mathcal{S}_0^\cup$ is a subfamily of $\mathcal{P}$, because $\mathcal{P}$ is an ideal, $\text{card} \mathcal{S} \leq \lambda$ and, a fortiori $\mathcal{S} \# \mathcal{O}_X^\natural(\mathcal{P})$. In view of Proposition 5.1, $\mathcal{O}_X^\natural(\mathcal{P}) \leq \mathcal{O}_X^\natural(\mathcal{S})$ and $\mathcal{O}_X^\natural(\mathcal{S})$ is a filter-base, so that $Y \in \lim_{[X, \$]} \mathcal{O}_X^\natural(\mathcal{S})$. Moreover $\mathcal{O}_X^\natural(\mathcal{P}) \geq \mathcal{O}_X^\natural(\mathcal{S})$ because $\mathcal{S} \subseteq \mathcal{P}$, so that $\mathcal{O}_X^\natural(\mathcal{P}) = \mathcal{O}_X^\natural(\mathcal{S})$ has a filter base of cardinality not greater than λ. $\square$

An immediate consequence of Corollary 5.5 and Theorem 9.1 is (the known fact [38]) that

$$(9.1) \qquad\qquad t([X, \$], U) = \chi([X, \$], U) = L(U)$$

at each $U \in C(X, \$)$, where $L(U)$ is the Lindelöf degree of U.

The α-*Lindelöf number* $\alpha L(U)$ a subset U of X is the smallest cardinal λ such that every open α-cover of U has an α-subcover of U of cardinality not greater than λ. In view of Corollary 5.5, we have if $p(X) \subseteq \alpha \subseteq \kappa(X)$, then an ideal base $\mathcal{P} \subseteq C(X, \$)$ is an open cover of $U \in C(X, \$)$ if and only if it is an α-cover of U. Therefore

$$(9.2) \qquad\qquad L(U) \leq \alpha L(U)$$

for each open subset U of X.

It follows immediately from Proposition 5.7 that

$$(9.3) \qquad\qquad \alpha L(U) = t(\alpha(X, \$), U).$$

In view of Corollary 8.10 (1) and of the fact that the class $\mathbb{F}_1^\diamond$ is $\mathbb{F}_1$-steady and $\mathbb{F}_0$-composable, we obtain:

Theorem 9.2. *Let α be a topology on $C(X, \$)$ such that $p(X) \subseteq \alpha \subseteq \kappa(X)$. Then*

$$\alpha L(X) = t(\alpha(X, \$), X) = t(C_\alpha(X, \mathbb{R}), \overline{0}).$$

A similar result was announced in [12, Corollary 3.3], but the provided proof was not correct. In particular, if $\alpha = \alpha_\mathcal{D}$ where $\mathcal{D}$ is a network of compact subsets of X, then $C_{\alpha_\mathcal{D}}(X, \mathbb{R})$ is a topological group and

$$(9.4) \qquad\qquad \alpha_\mathcal{D} L(X) = t\left(C_{\alpha_\mathcal{D}}(X, \mathbb{R})\right).$$

This is exactly [36, Theorem 4.7.1]. Indeed, in [36], a $\mathcal{D}$-cover of X, where $\mathcal{D}$ is a network of closed subsets of X, that is, a family of subsets of X such that every member of $\mathcal{D}$ is contained in some member of this family. McCoy and Ntantu define the $\mathcal{D}$-Lindelöf degree of X as the least cardinality λ such that every open $\mathcal{D}$-cover has a $\mathcal{D}$-subcover of cardinality not greater than λ, and establish that $t\left(C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})\right)$ is equal to the $\mathcal{D}$-Lindelöf degree of X [36, Theorem 4.7.1]. It is immediate that the $\mathcal{D}$-Lindelöf degree of X is equal to $\alpha_{\mathcal{D}} L(X)$. Instances include:

Corollary 9.3. *(e.g., [36, Corollary 4.7.2]) $C_k(X, \mathbb{R})$ is countably tight if and only if every open k-cover has a countable k-subcover.*

Corollary 9.4. *(e.g., [3]) The following are equivalent:*
  (1) *$C_p(X, \mathbb{R})$ is countably tight;*
  (2) *every open ω-cover has a countable ω-subcover;*
  (3) *X^n is Lindelöf for every $n \in \omega$.*

Note that (2) $\Longleftrightarrow$ (3) in the corollary above uses the observation that

(9.5) $$pL(X) = \sup\{L(X^n) : n \in \omega\},$$

a proof of which can be found for instance in [36, Corollary 4.7.3.].

Proposition 9.5. $\kappa L(U) = t(\kappa(X, \$), U) = t([X, \$], U) = L(U).$

Proof. In view of $T[X, \$] = \kappa(X, \$)$ and of (9.1),

$$t(\kappa(X, \$), U) = t(T[X, \$], U) \le t([X, \$], U) = \chi([X, \$], U) = L(U),$$

because $t(X) \ge t(PX) \ge t(TX)$ for any convergence space X [38, Proposition 2.1]. In view of Theorem 9.2 and (9.2)

$$L(U) \le \kappa L(U) = t(\kappa(X, \$), U).$$

$\square$

In particular $L(X) = \kappa L(X)$, hence for the Isbell topology $\kappa(X, \mathbb{R})$ and fine Isbell topology $\overline{\kappa}(X, \mathbb{R})$, we conclude that

Corollary 9.6.
$$L(X) = t(C_\kappa(X, \mathbb{R}), \overline{0}) = t(C_{\overline{\kappa}}(X, \mathbb{R})).$$

It was shown in [5] that if X is Čech-complete then $t(C_k(X, \mathbb{R})) = L(X)$. We can refine this result as follows ([25]):

Corollary 9.7. *If X is a (completely regular) consonant topological space then*
$$t(C_k(X, \mathbb{R})) = L(X).$$

Proof. X is consonant if and only if $T[X, \$] = C_k(X, \$)$. In view of (8.3), we have $t(C_k(X, \mathbb{R})) = t\left(C_k(X, \$), X\right)$. But $t\left(C_k(X, \$), X\right) = t\left(T[X, \$], X\right) = L(X)$, which concludes the proof. $\square$

The natural convergence $[X, \mathbb{R}]$ is a convergence group, in particular translation-invariant. Therefore, in view of Corollary 8.10 (2),

(9.6) $$\chi([X, \mathbb{R}]) = \chi([X, \$], X),$$

because the class $\mathbb{F}_\lambda$ is $\$$-compatible, $\mathbb{F}_0$-composable, and countably upper closed for every cardinal λ. Although the class of countably tight filters is not countably

---

[25]Every Čech-complete space is consonant [14, Theorem 4.1], but not conversely.

upper closed, we are in a position to see that $t([X, \mathbb{R}]) = t([X, \$], X)$. Indeed, $t([X, \mathbb{R}]) \leq \chi([X, \mathbb{R}])$ and, in view of Corollary 8.8 (2), $t(P[X, \$], X) \leq t([X, \mathbb{R}])$, because $(\mathbb{F}_1^{\Diamond})^{\wedge} = \mathbb{F}_1^{\Diamond}$. Therefore

$$L(X) = t(T[X, \$], X) \leq t(P[X, \$], X) \leq t([X, \mathbb{R}]) \leq \chi([X, \mathbb{R}]) = \chi([X, \$], X) = L(X).$$

Corollary 9.8.

$$\begin{aligned}
L(X) &= \chi([X, \$], X) = t([X, \$], X) = t(T[X, \$], X) \\
&= \chi([X, \mathbb{R}]) = t([X, \mathbb{R}]).
\end{aligned}$$

Note that $L(X) = \chi([X, \mathbb{R}])$ is a corollary of [19, Theorem 1] of Feldman. However, the surprising fact that $\chi([X, \mathbb{R}]) = t([X, \mathbb{R}])$ seems to be entirely new.

As we have seen, character and tightness coincide for $[X, \$]$ as well as for $[X, \mathbb{R}]$, but they do not for $\alpha(X, \$)$ (and therefore not for $\alpha(X, \mathbb{R})$). By definition the character of $C_\alpha(X, \$)$ at U does not exceed λ if there is $\{\mathcal{A}_\beta : \beta \leq \lambda\} \subseteq \alpha$ such that $U \in \mathcal{A}_\beta$ for each β and for each $\mathcal{A} \in \alpha$ such that $U \in \mathcal{A}$, there is $\beta \leq \lambda$ such that $\mathcal{A}_\beta \subseteq \mathcal{A}$. In particular $\chi(C_\alpha(X, \$), X) \leq \lambda$ if there is a subset γ of α of cardinality at most λ such that each element of α contains an element of γ. In the particular case where $\alpha = \alpha_{\mathcal{D}}$ for a network $\mathcal{D}$ of closed subsets of X, the condition above translates to: $\chi(C_{\alpha_{\mathcal{D}}}(X, \$), X) \leq \lambda$ if there is $\mathcal{S} \subseteq \mathcal{D}$ with $|\mathcal{S}| \leq \lambda$ such that every element of $\mathcal{D}$ is contained in an element of $\mathcal{S}$, that is, if $\mathcal{D}$ contains a $\mathcal{D}$-cover (in the sense of [36]) of cardinality at most λ. In other words,

$$\chi(C_{\alpha_{\mathcal{D}}}(X, \$), X) = \mathcal{D}a(X),$$

where $\mathcal{D}a(X)$ is the $\mathcal{D}$-Arens number of X, as defined in [36]. In view of Corollary 8.10 (1), we recover [36, Theorem 4.4.1]:

Corollary 9.9. *If $\mathcal{D}$ is a network of compact subsets of X then:*

$$\chi(C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})) = \chi(C_{\alpha_{\mathcal{D}}}(X, \$), X) = \mathcal{D}a(X).$$

Since $C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})$ is a topological group it is metrizable whenever it is first-countable. Therefore, instances of this result include that $C_p(X, \mathbb{R})$ is metrizable if and only if X is countable, and that $C_k(X, \mathbb{R})$ is metrizable if and only if X is hemicompact.

We can more generally define, for $\alpha \subseteq \kappa(X)$, the α-Arens number $\alpha a(X)$ of X as the least cardinal λ such that there is a subset γ of α of cardinality at most λ such that each element of α contains an element of γ, and we have

$$\chi(C_\alpha(X, \mathbb{R}), \overline{0}) = \chi(C_\alpha(X, \$), X) = \alpha a(X).$$

The α-Arens number seems however somewhat intractable unless $\alpha = \alpha_{\mathcal{D}}$ for a network $\mathcal{D}$ of closed subsets of X.

10. Fan-tightness and strong fan-tightness

As the classes of countably fan-tight and strongly countable fan-tight filters (7.1) are $\mathbb{F}_1$-steady and $\mathbb{F}_0$-composable, Corollary 8.10 (1) applies to the effect that

$$\begin{aligned}
(10.1) \qquad \text{vet}(C_\alpha(X, \$), X) &= \text{vet}(C_\alpha(X, \mathbb{R}), \overline{0}); \\
\text{vet}^*(C_\alpha(X, \$), X) &= \text{vet}^*(C_\alpha(X, \mathbb{R}), \overline{0}).
\end{aligned}$$

It is straightforward from the definitions and Proposition 5.7 that $\text{vet}(C_\alpha(X, \$), U)$ (resp. $\text{vet}^*(C_\alpha(X, \$), X)$) is equal to the minimal cardinality λ such that if for each family $\{\mathcal{P}_\gamma : \gamma < \lambda\}$ of open α-covers of U there are subsets $\mathcal{V}_\gamma \subseteq \mathcal{P}_\gamma$ of cardinality

less than λ (resp. $P_\gamma \in \mathcal{P}_\gamma$) for each $\gamma < \lambda$, such that $\bigcup_{\gamma < \lambda} \mathcal{V}_\gamma$ (resp. $\{P_\gamma : \gamma < \lambda\}$) is an α-cover of U. Let us call the cardinal numbers defined above the α-*Hurewicz* $\alpha H(X)$ and α-*Rothberger* $\alpha R(X)$ *numbers of* X, respectively. In this terminology, we have:

$$(10.2) \qquad \qquad \mathrm{vet}(C_\alpha(X,\$), U) \quad = \quad \alpha H(U),$$

$$(10.3) \qquad \qquad \mathrm{vet}^*(C_\alpha(X,\$), U) \quad = \quad \alpha R(U),$$

for each open subset U of X. In particular, [35, Theorem 1] and [35, Theorem 2] stating that $cC_p(X,\$)$ and $cC_k(X,\$)$ have countable strong fan-tightness if and only if $pR(U) = \omega$ and $kR(U) = \omega$ for each open subset U of X, respectively, are instances of (10.3) for $\alpha = p(X)$ and $\alpha = k(X)$. Similarly, [35, Theorem 9] and [35, Theorem 10] characterizing countable fan-tightness of $cC_p(X,\$)$ and $cC_k(X,\$)$ respectively, are instance of (10.2) for $\alpha = p(X)$ and $\alpha = k(X)$ respectively.

Combining (10.1) and (10.2), we have:

$$(10.4) \qquad \mathrm{vet}(C_\alpha(X,\$), X) \quad = \quad \mathrm{vet}(C_\alpha(X,\mathbb{R}), \overline{0}) = \alpha H(X);$$

$$(10.5) \qquad \mathrm{vet}^*(C_\alpha(X,\$), X) \quad = \quad \mathrm{vet}^*(C_\alpha(X,\mathbb{R}), \overline{0}) = \alpha R(X).$$

Let $s = \{\mathcal{O}(x) : x \in X\}$. Note that $C_s(X,\$) = C_p(X,\$)$. An infinite topological space X has the Hurewicz property [4] (also often called Menger Property, e.g. [35]) if and only if $sH(X) := H(X) = \omega$ and X has the Rothberger property (e.g., [37], [41]) if and only if $sR(X) := R(X) = \omega$. An argument similar to [36, Corollary 4.7.3.] was used to show (9.5) and can be adapted to show that

$$(10.6) \qquad \qquad \begin{aligned} pH(X) &= \sup\{H(X^n) : n \in \omega\}; \\ pR(X) &= \sup\{R(X^n) : n \in \omega\}. \end{aligned}$$

Note that (10.4) particularizes to [34, Theorem 1] when $\alpha = \alpha_{\mathcal{D}}$ where $\mathcal{D}$ is a network of compact subsets of X. Combined with (10.6), we obtain:

Corollary 10.1. (1) [4],[34, Theorem 2]

$$\mathrm{vet}(C_p(X,\mathbb{R})) = \sup\{H(X^n) : n \in \omega\},$$

so that $C_p(X,\mathbb{R})$ is countably fan-tight if and only if X^n has the Hurewicz property for each $n < \omega$.

(2)

$$\mathrm{vet}^*(C_p(X,\mathbb{R})) = \sup\{R(X^n) : n \in \omega\},$$

so that $C_p(X,\mathbb{R})$ is countably strongly fan-tight if and only if X^n has the Rothberger property for each $n < \omega$.

On the other hand, for $\alpha = k(X)$, we obtain in particular:

Corollary 10.2. [32]

(1) $C_k(X,\mathbb{R})$ is countably fan-tight if and only if for every sequence $(\mathcal{P}_n)_{n<\omega}$ of k-covers, there are finite subsets $\mathcal{V}_n \subseteq \mathcal{P}_n$ for each n such that $\bigcup_{n<\omega} \mathcal{V}_n$ is a k-cover.

(2) $C_k(X,\mathbb{R})$ is countably strongly fan-tight if and only if for every sequence $(\mathcal{P}_n)_{n<\omega}$ of k-covers, there are $P_n \in \mathcal{P}_n$ for each n such that $\{P_n : n < \omega\}$ is a k-cover.

11. Fréchet properties

An obstacle to applying the results of Section 8 to the Fréchet property is that the class of Fréchet filters, while $\mathbb{F}_0$-composable, fails to be $\mathbb{F}_1$-steady. The results apply to the strong Fréchet property though, whose associated class of filters is both $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady. We have seen that tightness and character coincide for $[X, \$]$ and $[X, \mathbb{R}]$. Therefore these spaces are Fréchet if and only if they are strongly Fréchet if and only if they are countably tight if and only if they are first-countable. On the other hand,

Theorem 11.1. *Let $p(X) \subseteq \alpha \subseteq \kappa(X)$. The following are equivalent:*

(1) $C_\alpha(X, \mathbb{R})$ *is strongly Fréchet at* $\overline{0}$;
(2) $C_\alpha(X, \$)$ *is strongly Fréchet at* X;
(3) *For every decreasing sequence* $(\mathcal{P}_n)_{n \in \omega}$ *of open α-covers, for each $n < \omega$ there exists $P_n \in \mathcal{P}_n$ so that each $\mathcal{A} \in \alpha$ contains all but finitely many of the elements of* $(P_n)_{n \in \omega}$.

Proof. The equivalence between (1) and (2) follows from Corollary 8.10 (1), and the equivalence between (2) and (3) follows immediately from the definition of strongly Fréchet and Proposition 5.7. □

The Fréchet property for function spaces can nevertheless be characterized with our results in the special case of $\alpha = \alpha_\mathcal{D}$ for a network $\mathcal{D}$.

Following [27], we call a topological space X *Fréchet-Urysohn for finite sets at* $x \in X$, or FU_{fin} at x, if for any $\mathcal{P} \subseteq [X]^{<\infty}$ such that each $U \in \mathcal{O}_X(x)$ contains an element of $\mathcal{P}$, there is a sequence $(P_n)_{n \in \omega} \subseteq \mathcal{P}$ such that each $U \in \mathcal{O}_X(x)$ contains all but finitely many elements of $(P_n)_{n \in \omega}$. We call a filter $\mathcal{F}$ an *FU_{fin}-filter* if for any $\mathcal{P} \subseteq [X]^{<\infty}$ such that $\mathcal{P} \geq \mathcal{F}$, there is a sequence $(P_n)_{n \in \omega} \subseteq \mathcal{P}$ such that $(P_n)_{n \in \omega} \geq \mathcal{F}$. Let $\mathbb{FU}_{fin}$ denote the corresponding class of filters. Clearly, a space is FU_{fin} at x if it is $\mathbb{FU}_{fin}$-based at x.

Theorem 11.2. *Let $\mathcal{D}$ be a network of compact subsets of X and $Y \in C(X, \$)$. If $C_{\alpha_\mathcal{D}}(X, \$)$ is Fréchet at Y then $C_{\alpha_\mathcal{D}}(X, \$)$ is Fréchet-Urysohn for finite sets at Y.*

Proof. Let β be a family of finite subsets of $C(X, \$)$ such that for each $D \in \mathcal{D}$ containing Y, there is $\mathcal{P} \in \beta$ with $\mathcal{P} \subseteq \mathcal{O}_X(D)$. In other words, $D \subseteq \bigcap_{P \in \mathcal{P}} P$. Since the intersection is finite, $\bigcap_{P \in \mathcal{P}} P \in \mathcal{O}_X(D)$. Therefore, $Y \in \mathrm{cl}_{\alpha_\mathcal{D}} \left\{ \bigcap_{P \in \mathcal{P}} P : \mathcal{P} \in \beta \right\}$. As $C_{\alpha_\mathcal{D}}(X, \$)$ is Fréchet at Y, there is a sequence $(\mathcal{P}_n)_{n \in \omega}$ of elements of β such that $Y \in \lim_{\alpha_\mathcal{D}} \left(\bigcap_{P \in \mathcal{P}_n} P \right)_{n \in \omega}$. In other words, for each $Y \subseteq D \in \mathcal{D}$, there is n_D such that $\bigcap_{P \in \mathcal{P}_n} P \in \mathcal{O}_X(D)$ for each $n \geq n_D$, so that $\mathcal{P}_n \subseteq \mathcal{O}_X(D)$ for each $n \geq n_D$, which proves that $C_{\alpha_\mathcal{D}}(X, \$)$ is FU_{fin} at Y. □

The method of the proof does not work for general topologies $\alpha(X, \$)$ with $\alpha \subseteq \kappa(X)$, because compact families do not need to be filters. In particular, there remains the following problem (of course, for dissonant X):

Problem 11.3. *Does the Fréchet property and the FU_{fin} property coincide for the Scott topology $C_\kappa(X, \$)$?*

It is known (e.g., [40]) that a FU_{fin} topological space is α_2 (in the sense of [2][26]). Therefore Theorem 11.2 implies that in $C_{\alpha_{\mathcal{D}}}(X, \$)$ the Fréchet property implies α_2, and *a fortiori* α_3 and α_4, in particular implies the strong Fréchet property.

Lemma 11.4. *The class $\mathbb{FU}_{fin}$ is $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady.*

Proof. Let $\mathcal{F} \in \mathbb{FU}_{fin}(X)$, $A \subseteq X \times Y$ and let $\mathcal{P} \subseteq [Y]^{<\infty}$ such that $\mathcal{P} \geq A\mathcal{F}$. In other words, for each $F \in \mathcal{F}$ there is $P_F \in \mathcal{P}$ such that $P_F \subseteq AF$. Hence for each $y \in P_F$ there is $x_y \in F$ such that $(x_y, y) \in A$. Let $Q_F := \{x_y : y \in P_F\}$ and let $\mathcal{Q} := \{Q_F : F \in \mathcal{F}\}$. Then $\mathcal{Q} \subseteq [X]^{<\infty}$ such that $\mathcal{Q} \geq \mathcal{F}$. Therefore there is a sequence $(F_n)_{n \in \omega} \subseteq \mathcal{F}$ such that $(Q_{F_n})_{n \in \omega} \geq \mathcal{F}$. It is easy to see that $(P_{F_n})_{n \in \omega} \geq A\mathcal{F}$, which shows that $\mathbb{FU}_{fin}$ is $\mathbb{F}_0$-composable.

The class $\mathbb{FU}_{fin}$ is $\mathbb{F}_0$-steady because if $\mathcal{P} \geq A \vee \mathcal{F}$ there is $\mathcal{P}_0 \subseteq \mathcal{P}$ such that $\mathcal{P}_0 \geq A \vee \mathcal{F}$ and $\mathcal{P}_0 \subseteq [A]^{<\infty}$. Moreover, by [40] or [27, Theorem 20], $\mathbb{FU}_{fin} \times \mathbb{F}_1 \subseteq \mathbb{FU}_{fin}$ (in terms of of [31]), hence [31, Theorem 20(1)], $\mathbb{FU}_{fin}$ is therefore also $\mathbb{F}_1$-steady. $\qquad\square$

Theorem 11.5. *Let $\mathcal{D}$ be a network of compact subsets of X. The following are equivalent:*

(1) *$C_{\alpha_{\mathcal{D}}}(X, \$)$ is Fréchet at X;*
(2) *$C_{\alpha_{\mathcal{D}}}(X, \$)$ is FU_{fin} at X;*
(3) *$C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})$ is FU_{fin};*
(4) *$C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})$ is Fréchet;*
(5) *For every open $\mathcal{D}$-cover $\mathcal{C}$ of X, there exists a countable subfamily $\mathcal{S}$ of $\mathcal{C}$ such that every $D \in \mathcal{D}$ is contained in all but finitely many elements of $\mathcal{S}$.*

Proof. (1) $\Longleftrightarrow$ (2) follows from Theorem 11.2. (1) $\Longleftrightarrow$ (5) follows immediately from the definitions. (2) $\Longleftrightarrow$ (3) follows from Corollary 8.10 (1), because the class of FU_{fin} filters is $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady and $C_{\alpha_{\mathcal{D}}}(X, \mathbb{R})$ is a topological group. (3) $\Longrightarrow$ (4) and (2) $\Longrightarrow$ (1) are obvious, and (4) $\Longrightarrow$ (1) follows from Corollary 8.8 (2), because $\mathbb{F}_0^{\triangle} = (\mathbb{F}_0^{\triangle})^{\wedge}$. $\qquad\square$

Note that the equivalence (4) $\Longleftrightarrow$ (5) is [36, Theorem 4.7.4]. In the case $\alpha_{\mathcal{D}} = p(X)$, the equivalences (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) are due to [20].

The case $\alpha_{\mathcal{D}} = p(X)$ generalizes [10, Proposition 5 (1)] stating that $cC_p(X, \$)$ is α_2 whenever it is Fréchet. On the other hand, when $\alpha_{\mathcal{D}} = k(X)$, [10, Proposition 5 (2)] is generalized in two ways: we only need to assume that $cC_k(X, \$)$ is Fréchet (rather than the more stringent condition of strict Fréchetness) and we obtain that $cC_k(X, \$)$ is FU_{fin} rather than α_2.

Note however that while the Fréchet property is equivalent to sequentiality and even to being a k-space for $C_p(X, \mathbb{R})$ and $C_k(X, \mathbb{R})$ (e.g., [39]), these properties are not equivalent for the corresponding hyperspaces. For instance, an example of a space X for which $C_k(X, \$)$ is sequential but not Fréchet is given in [8, p. 275]. Therefore, the results of Section 8 in general do not apply to sequentiality.

---

[26]A topological space X has property α_2 (at x) if for each sequence $(\sigma_n)_{n \in \omega}$ of sequences converging to x, there is a sequence σ convergent to x such that for each $n \in \omega$, the set $\sigma_n \cap \sigma$ is infinite.

12. Appendix: dual convergences

We have seen that each non-degenerate $\alpha \subseteq C(X, \$)$ composed of openly isotone families defines a *Z-dual topology* $\alpha(X, Z)$ on $C(X, Z)$ via (3.2). Note that $f \in \lim_{\alpha(X,Z)} \mathcal{F}$ if and only if

$$(12.1) \qquad \forall_{O \in \mathcal{O}_Z} \forall_{\mathcal{A} \in \alpha} \; f \in [\mathcal{A}, O] \Longrightarrow [\mathcal{A}, O] \in \mathcal{F}.$$

In view of the characterization (3.4) of the natural convergence, it is natural to consider for each collection α of (openly isotone) families on X the *Z-dual convergence* $[\alpha, Z]$ defined by: $f \in \lim_{[\alpha,Z]} \mathcal{F}$ if and only if

$$(12.2) \qquad \forall_{O \in \mathcal{O}_Z} \forall_{\mathcal{A} \in \alpha} \; f \in [\mathcal{A}, O] \Longrightarrow \exists_{A \in \mathcal{A}} \; [A, O] \in \mathcal{F}.$$

Distinct collections α of families of open sets generate distinct topologies on $C(X, Z)$ provided that the elements of $C(X, Z)$ separate these families in X. Such a separation is assured for example by the Z-regularity of X and the compactness of the elements of α (see [15, Proposition 2.1]). In contrast, all the collections α including $p(X)$ and included in $\kappa(X)$ give rise the same convergence, which turns out to be the *natural convergence*.

Theorem 12.1. *The dual convergence $[\alpha, Z]$ is equal to the natural convergence $[X, Z]$ for each collection α such that $p(X) \leq \alpha \leq \kappa(X)$.*

Proof. We first show that $[X, Z] \geq [\kappa(X), Z]$. To this end, assume that $f_0 \in \lim_{[X,Z]} \mathcal{F}$ and let $f_0 \in [\mathcal{A}, O]$ where O is Z-open and $\mathcal{A} \in \kappa(X)$. It follows that $f_0^-(O) \in \mathcal{A}$. If $x \in f_0^-(O)$ then there is $V_x \in \mathcal{O}(x)$ such that $V_x \subseteq f_0^-(O)$ and $[V_x, O] \in \mathcal{F}$. By the compactness of $\mathcal{A}$, there is a finite subset B of $f_0^-(O)$ such that $V := \bigcup_{x \in B} V_x \in \mathcal{A}$. On the other hand, $[V, O] = \bigcap_{x \in B} [V_x, O] \in \mathcal{F}$ showing that $f_0 \in \lim_{[\kappa(X),Z]} \mathcal{F}$.

As $[\kappa(X), Z] \geq [p(X), Z]$, it is now enough to show that $[p(X), Z] \geq [X, Z]$. Suppose that $f_0 \in \lim_{[p(X),Z]} \mathcal{F}$ and let $x \in X, O \in \mathcal{O}_Z$ be such that $f_0 \in [x, O]$, equivalently $f_0^-(O) \in \mathcal{O}_X(x)$, or else, $f_0 \in [\mathcal{O}_X(x), O]$. By the assumption, there is $V \in \mathcal{O}_X(x)$ such that $[V, O] \in \mathcal{F}$, that is, $f_0 \in \lim_{[X,Z]} \mathcal{F}$. $\qquad\square$

Note that, since $[A, O] \subseteq [\mathcal{A}, O]$ for each $A \in \mathcal{A}$,

$$(12.3) \qquad [\alpha, Z] \geq T[\alpha, Z] \geq \alpha(X, Z).$$

References

1. R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific J. Math. **1** (1951), 5–31.
2. A. V. Arhangel'skii, *The frequency spectrum of a topological space and the classification of spaces*, Math. Dokl. **13** (1972), 1185–1189.
3. ———, *Topological function spaces*, Kluwer Academic, Dordrecht, 1992.
4. A.V. Arhangel'skii, *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Soviet Math. Dokl. **33** (1986), 396–399.
5. R.N. Ball and A.W. Hager, *Network character and tightness of the compact-open topology*, Comment. Math. Univ. Carolin. **47** (2006), no. 3, 473–482.
6. R. Beattie and H. P. Butzmann, *Convergence Structures and Applications to Functional Analysis*, Kluwer Academic, 2002.
7. E. Binz, *Continuous convergence in $C(X)$*, Springer-Verlag, 1975, Lect. Notes Math. 469.
8. L. Holà C. Costantini and P. Vitolo, *Tightness, character and related properties of hyperspace topologies*, Top. Appl. **142** (2004), 245–292, to appear.
9. G. Choquet, *Convergences*, Ann. Univ. Grenoble **23** (1947-48), 55–112.
10. Giuseppe Di Maio, Lj. D. R. Kočinac, and Tsugunori Nogura, *Convergence properties of hyperspaces*, J. Korean Math. Soc. **44** (2007), no. 4, 845–854. MR MR2334529 (2008k:54008)

11. Georgiou D.N., Illiadis S.D., and Papadopoulos B.K., *On dual topologies*, Top. Appl. **140** (2004), no. 1, 57–68.
12. S. Dolecki, *Properties transfer between topologies on function spaces, hyperspaces and underlying spaces*, Mathematica Pannonica **19** (2008), no. 2, 243–262.
13. ———, *An initiation into convergence theory*, Contemporary Mathematics 486, vol. Beyond Topology, pp. 115–161, A.M.S., 2009.
14. S. Dolecki, G. H. Greco, and A. Lechicki, *When do the upper Kuratowski topology (homeomorphically, Scott topology) and the cocompact topology coincide?*, Trans. Amer. Math. Soc. **347** (1995), 2869–2884.
15. S. Dolecki, F. Jordan, and F. Mynard, *Group topologies coarser than the Isbell topology*, to appear.
16. S. Dolecki and F. Mynard, *When is the Isbell topology a group topology?*, to appear in Top. Appl.
17. ———, *Hyperconvergences.*, Appl. Gen. Top. **4** (2003), no. 2, 391–419.
18. Martín Escardó, Jimmie Lawson, and Alex Simpson, *Comparing Cartesian closed categories of (core) compactly generated spaces*, Topology Appl. **143** (2004), no. 1-3, 105–145. MR MR2080286 (2005f:54047)
19. W. A. Feldman, *Axioms of countability and the algebra $C(X)$*, Pacific J. Math. **47** (1973), 81–89.
20. J. Gerlits and Z. Nagy, *Some properties of $C(X)$,*, Top. Appl. **14** (1982), 151–161.
21. G. Gierz, K. H. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott, *A compedium of continuous lattices*, Springer-Verlag, Berlin, 1980.
22. G. Gierz, K.H. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott, *Continuous lattices and domains*, Encyclopedia of Mathematics, vol. 93, Cambridge University Press, 2003.
23. G. Greco, *Sur la mesurabilité d'une fonction numérique par rapport à une famille d'ensembles*, Rend. Semin. Mat. Univ. Podova **65** (1981), 163–176.
24. ———, *Decomposizioni di semifiltri e γ-limiti sequenziali in reticoli completamente distributivi*, Ann. Mat. pura e appl. **137** (1984), 61–82.
25. ———, *Minimax theorems and saddling transformations*, J. Math. An. Appl. **147** (1990), 180–197.
26. G. Gruenhage, *Products of Fréchet spaces*, Top. Proc. **30** (2006), no. 2, 475–499.
27. G. Gruenhage and P. Szeptycki, *Fréchet-Urysohn for finite sets*, Topology Appl. **151** (2005), no. 1-3, 238–259. MR MR2139755 (2006i:54004)
28. J. Isbell, *Function spaces and adjoints*, Math. Scand. **36** (1975), 317–339.
29. F. Jordan, *Productive local properties of function spaces*, Top. Appl. **154** (2007), no. 4, 870–883.
30. ———, *Coincidence of function space topologies*, Top. Appl. **157** (2010), no. 2, 336–351.
31. F. Jordan and F. Mynard, *Compatible relations of filters and stability of local topological properties under supremum and product*, Topology Appl. **153** (2006), 2386–2412.
32. Lj.D.R. Kočinac, *Closure properties of function spaces*, Appl. Gen. Top. **4** (2003), no. 2, 255–261.
33. H. Lebesgue, *Sur le développement de la notion d'intégrale*, Mat. Tidsskrift **B** (1926), 54–74.
34. S. Lin, *Tightness of function spaces*, Appl. Gen. Top. **7** (2006), no. 1, 103–107.
35. G. Di Maio, Lj.D.R. Kočinac, and E. Meccariello, *Selection principles and hyperspace topologies*, Top. Appl. **153** (2005), 912–923.
36. R. A. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions*, Springer-Verlag, 1988.
37. A.W. Miller and D.H. Fremlin, *On some properties of Hurewicz, Menger and Rothberger*, Fund. Math. **129** (1988), 17–33.
38. F. Mynard, *First-countability, sequentiality and tightness of the upper Kuratowski convergence*, Rocky Mountain J. of Math. **33** (2003), no. 3, 1011–1038.
39. E. G. Pytkeev, *On the sequentiality of spaces of continuous functions*, Communications Moscow Math. Soc. (1982), 190–191.
40. E. A. Reznichenko and O. V. Sipacheva, *Properties of Fréchet-Uryson type in topological spaces, groups and locally convex spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1999), no. 3, 32–38, 72. MR MR1711871 (2000h:54006)
41. M. Scheepers, *Combinatorics of open covers I: Ramsey Theory*, Top. Appl. **69** (1996), 31–62.

42. F. Schwarz, *Powers and exponential objects in initially structured categories and application to categories of limits spaces*, Quaest. Math. **6** (1983), 227–254.

Mathematical Institute of Burgundy, Burgundy University, B.P. 47 870, 21078 Dijon, France

Department of Mathematical Sciences, Georgia Southern University, PB 8093, Statesboro GA 30460, U.S.A.
  E-mail address: dolecki@u-bourgogne.fr
  E-mail address: fmynard@georgiasouthern.edu