NBER WORKING PAPER SERIES

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Working Paper 14845
http://www.nber.org/papers/w14845

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 2009

We are grateful to Andrew Abel, Janice Eberly, Andrea Eisfeldt, Mike Faulkender, Michael Fishman, Albert Kyle, Robert McDonald, Marco Pagano, Gordon Phillips, and Robert Pindyck for their comments. We also thank seminar participants at Columbia Business School, NYU Stern, U.C. Berkeley, University of Maryland, and Northwestern University (Kellogg) for their comments. The views expressed herein are those of the author(s) and do not necessarily reflect the views of the National Bureau of Economic Research.

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April 2009
JEL No. E22,G12,G32,G35

ABSTRACT

This paper proposes a simple homogeneous dynamic model of investment and corporate risk management for a financially constrained firm. Following Froot, Scharfstein, and Stein (1993), we define a corporation's risk management as the coordination of investment and financing decisions. In our model, corporate risk management involves internal liquidity management, financial hedging, and investment. We determine a firm's optimal cash, investment, asset sales, credit line, external equity finance, and payout policies as functions of the following key parameters: 1) the firm's earnings growth and cash-flow risk; 2) the external cost of financing; 3) the firm's liquidation value; 4) the opportunity cost of holding cash; 5) investment adjustment and asset sales costs; and 6) the return and covariance characteristics of hedging assets the firm can invest in. The optimal cash inventory policy takes the form of a double-barrier policy where i) cash is paid out to shareholders only when the cash-capital ratio hits an endogenous upper barrier, and ii) external funds are raised only when the firm has depleted its cash. In between the two barriers, the firm adjusts its capital expenditures, asset sales, and hedging policies. Several new insights emerge from our analysis. For example, we find an inverse relation between marginal Tobin's q and investment when the firm draws on its credit line. We also find that financially constrained firms may have a lower equity beta in equilibrium because these firms tend to hold higher precautionary cash inventories.

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1 Introduction

In the presence of external financing costs, corporate investment, risk management, and financing decisions are closely intertwined. Corporations can create value by managing their cash holdings and by hedging their underlying earnings risk (see e.g. Smith and Stulz (1985) and Graham and Smith (1999)). As Froot, Scharfstein, and Stein (1993) and Kim, Mauer and Sherman (1998) have emphasized, corporate risk management can reduce firms' costs of financing investments by transferring internal funds and structuring external financing so that enough cash is available in the states of nature where investment is most valuable. While this general principle and characterization of the main role of corporate risk management is increasingly well understood, how to translate this prescription into day-to-day risk management policies still remains largely undetermined. Simple questions such as when/how corporations should reduce their cash holdings, or when/how they should replenish their dwindling cash inventory are still not precisely understood. Similarly, the questions of which risks the corporation should hedge and by how much, or whether the firm should undertake an enterprise-wide risk management approach or a piecemeal approach of hedging individual risks, are not well understood.

Our goal is to propose the first elements of a tractable dynamic economic framework, in which optimal corporate investment, asset sales, cash inventory, and risk management policies are easily and precisely characterized. The key building block of our model is the neoclassical q theory of investment, which features a constant returns to scale (AK) production technology, convex adjustment costs (á la Hayashi (1982)), and earnings shocks that are independently and identically distributed ($i.i.d.$).

We add to this model a deadweight external financing cost plus an opportunity cost of hoarding cash, and proceed to derive the firm's optimal cash-inventory, external financing, payout, investment and hedging policies as functions of the firm's underlying risk-return characteristics, investment adjustment technology, and the different financing costs it faces. Although we make the somewhat

strong assumption that productivity shocks are *i.i.d.*, both investment and cash holdings are nevertheless highly persistent due to the firm's constrained optimal investment decisions in response to these shocks.

Importantly, with external financial costs, the firm's investment is no longer determined by equating the marginal cost of investing with marginal q as in the neoclassical model under Modigliani-Miller neutrality (even in the absence of fixed costs of investing).[1] Instead, corporate investment is determined by the following modified investment Euler equation:

$$\text{marginal cost of investing} = \frac{\text{marginal } q}{\text{marginal cost of financing}}.$$

In other words, the investment Euler equation for a financially constrained firm links investment to the *ratio of marginal q to the marginal cost of financing.* When firms are flush with cash, the marginal cost of financing is approximately one, so that this Euler equation is then approximately the same as the classical Euler equation. But when firms have low cash holdings or are close to financial distress, the marginal cost of financing may be much larger than one and can substantially modify the classical investment Euler equation. More interestingly, when credit line is the firm's marginal source of financing, marginal q increases with the firm's leverage, while investment decreases with its leverage. That is, marginal q and investment move in opposite directions.

The logic behind this result is the following. First, an increase in investment helps relax the firm's future borrowing constraint by adding capital that may be pledged as collateral. This explains why marginal q increases with its leverage. Second, the more debt (through credit line) the firm has, the more it wants to move away from the external financing region by engaging in aggressive asset sales. The two effects explain why we may simultaneously observe an increasing marginal q schedule and a decreasing investment schedule as the firm uses more credit (i.e., takes on more

[1] See Abel and Eberly (1994) for a general specification of the q theory of investment under the neoclassic setting with both fixed and variable costs. Their analysis builds on the classical theory of investment of Jorgensen (1963), Lucas and Prescott (1971), and Hayashi (1982). Irreversible fixed costs of investment in particular give rise to 'inaction' regions and generate real options for the firm as in McDonald and Siegel (1986) and Dixit and Pindyck (1994). We do not consider fixed costs of investing in this paper. The standard investment Euler equation may not hold in the presence of fixed costs. See Caballero and Leahy (1996).

leverage).

Another novel economic insight emerging from our analysis concerns the behavior of a financially constrained firm's equity beta in terms of its cash holdings. One would expect equity beta to be higher for a financially constrained firm, as it reflects the firm's exposure to both idiosyncratic and systematic risk, whereas the equity beta of an unconstrained (first-best) firm reflects only the firm's exposure to systematic risk. This intuition is broadly valid in a static setting. However, in a dynamic setting where firms actively manage their cash holdings, a financially constrained firm can have a lower equity beta than an unconstrained firm. The reason is that such a firm is likely to hold a significant proportion of its assets in cash, which has a zero beta, while an unconstrained firm does not hold any cash. In addition, our model shows that returns on real investments depend on the financing constraints. Our model thus provides guidance on how to extend the neoclassical production based asset pricing framework (see Cochrane (1991)) and how to conduct asset pricing tests using investment returns of financially constrained firms.

Much of the empirical literature on firms' cash holdings tries to identify a *target cash-inventory* for a firm by weighing the costs and benefits of holding cash. [2] The implicit idea in this literature is that this target level helps determine when a firm should increase its cash savings and when it should dissave.[3] Our analysis, however, suggests that the notion of a target cash level, or target cash-capital ratio, is too narrow. Instead, a firm's optimal cash inventory policy is better described by a *double-barrier policy* similar to the Baumol-Tobin theory of an individual household's transactions demand for money (Baumol (1952) and Tobin (1956)) and Miller and Orr (1966) theory of firms' demand for money. When the cash-capital ratio hits an endogenous upper barrier, it is optimal for the firm to pay out cash. When the firm runs out of cash, it either closes down or raises outside funds, depending on whether the liquidation value exceeds the continuation value. The firm never

[2] See Almeida, Campello, and Weisbach (2004, 2008), Faulkender and Wang (2006), Khurana, Martin, and Pereira (2006), and Dittmar and Mahrt-Smith (2007).

[3] Recent empirical studies have found that corporations tend to hold more cash when their underlying earnings risk is higher or when they have higher growth opportunities (see e.g. Opler, Pinkowitz, Stulz, and Williamson (1999) and Bates, Kahle, and Stulz (2008)).

issues external equity before depleting its cash reserve. By deferring issuance it postpones incurring external financing costs, yet it can still finance its desired level of investment via internal funds. Thus, our model generates a simple dynamic pecking order of financing where first internal funds are used and external equity issues are a last resort source of funding. In between these two barriers the firm does not sit still but continuously manages its cash reserves by adjusting its investment policy and by dynamically hedging its earnings risk.

When cash holdings are higher, the firm invests more *ceteris paribus*, because the marginal value of cash is smaller. When the firm is approaching the point where its cash reserves are depleted, it optimally scales down its investment and may even engage in asset sales. This way the firm can avoid or postpone raising costly external financing.

In addition to these cash management instruments, the firm can also benefit by hedging its earnings risk and investing in financial assets that are correlated with its underlying earnings risk. The benefit from such hedging is to reduce the volatility of the firm's net earnings and thus to reduce the need for the firm to hold costly cash inventory. Derivatives and cash thus play complementary roles in risk management. Derivatives allow firms to exploit the covariation between the firm's earnings and derivative returns, while cash is a non-state-contingent risk management tool. Derivatives (such as oil or currency futures) help reducing the firm's systematic risk exposure, while cash can also help smooth idiosyncratic risk. Finally, asset sales are also an important tool in managing risk. However, as investment distortions can be very costly, the firm only actively resorts to asset sales in times of distress when replenishing liquidity is very valuable.

Despite the potential technical complications from introducing external financing costs in the neoclassical dynamic model of investment, we are able to characterize the solution via an analytically tractable one-dimensional dynamic optimization problem where the key state variable is the firm's cash-capital ratio. We are also able to give concrete and detailed prescriptions for how a firm should manage its cash reserves and choose its investment and payout policies, given its underlying

production technology, investment opportunities, investment adjustment costs, financing costs, and market interest rates. In particular, we provide comparative statics results for our baseline model and show that when expected profitability is low or when the costs of external financing are high, the firm does not raise new external funds when it runs out of cash, but chooses to liquidate instead. In contrast, for higher expected profitability or lower costs of external financing, the firm prefers raising new costly external funds. For each of these cases we show how the firm's cash-inventory and investment policies vary with earnings volatility and transaction costs.

Through simulations we can also compute the stationary distributions for the firm's cash-capital, investment-capital ratios, firm value-capital ratio as well as the marginal value of financing. Remarkably, we find that under the stationary distribution, firms are most likely to hold sufficient cash to be close to their payout boundary. As a result, average marginal cost of financing is close to unity, even for firms with large external financing costs that result in substantial financing constraints. However, firms respond to these constraints by optimally managing their cash holdings so as to be able to stay away most of the time from financial distress situations where they may need to raise more external funds.

There is only a handful of theoretical analyses of firms' optimal cash, investment and risk management policies. A key first contribution is by Froot, Scharfstein, and Stein (1993), who develop a static model of a firm facing external financing costs and risky investment opportunities. Another more recent contribution by Almeida, Campello, and Weisbach (2008) extends the Hart and Moore (1994) theory of optimal cash holdings by introducing cash-flow and investment uncertainty in a three-period model. The contributions most closely related to ours are:

1. Hennessy and Whited (2005, 2007), who also consider dynamic models of investment for financially constrained firms. The key differences with our setup are that they do not model the firm's cash accumulation process and they explore a model with decreasing returns to scale, which is not as tractable as our constant returns to scale specification. Moreover, they

do not explore the interaction between corporate risk management and investment. Recently, Gamba and Triantis (2008) have extended Hennessy and Whited (2007) to introduce issuance costs of debt and thus explain why firms may simultaneously issue debt and hold cash.

2. Hennessy, Levy, and Whited (2007), who also characterize the investment Euler equation for a financially constrained firm at the payout and equity issuance boundaries. However, they do not integrate the firm's cash and risk management policies with its investment and financing policies.

3. Riddick and Whited (2008), who explore a discrete-time model with decreasing returns to scale, an AR(1) process in logs for earnings, and quadratic investment adjustment costs. They also analyze the firm's optimal cash inventory and investment policy when the firm faces external costs of financing. While their model is more flexible than ours, we are able to exploit the continuous-time and constant returns to scale structure to obtain a more operational characterization of the firm's optimal policy. We also characterize the firm's dynamic hedging policy and its use of credit lines.

4. Decamps, Mariotti, Rochet, and Villeneuve (2006), who also explore a continuous-time model of a firm facing external financing costs. Unlike our set-up, their firm only has a single infinitely-lived project of fixed size, so that they cannot consider the interaction of the firm's real and financial policies. Our model also relates to DeMarzo, Fishman, He, and Wang (2008) which integrate dynamic agency with the q theory of investment (á la Hayashi (1982)) in a continuous-time dynamic optimal contracting framework. Dynamic agency conflicts generate an endogenous financial constraint and induce underinvestment and liquidation in their model.

In contrast to the somewhat thin theoretical literature on corporate risk management, there is a much larger empirical literature exploring the determinants of firms' cash holdings. Much of that

literature focuses on the link between weak corporate governance and firms' excess cash inventories, in particular Pinkowitz, Stulz and Williamson (2006) and Dittmar and Mahrt-Smith (2007) (see Dittmar (2008) for a survey of this literature).

As always with corporate financial decisions, an important determinant of firms' cash inventory policies is taxes. The effect of corporate taxes on firms' payout decisions is explored in Desai, Foley, and Hines (2001) and Foley, Harzell, Titman, and Twite (2007). There is also a large body of empirical research focusing on firms' share repurchase decisions (see again Dittmar (2008) for a survey of the literature on share repurchases that is most relevant to firms' cash inventory policy). Finally, corporate cash policy may also be driven by more strategic considerations, such as building a war-chest to improve the firm's competitive position in product markets (see Haushalter, Klasa, and Maxwell (2007)) or to facilitate corporate acquisitions (see Harford (1999)).

The remainder of the paper proceeds as follows. Section 2 sets up our baseline model. Section 3 proceeds with model solution and qualitative analysis. Section 4 continues with quantitative analysis. Section 5 discusses our model's implication for risks and returns. Section 6 deals with financial hedging and Section 7 extends the baseline model of Section 2 to incorporate credit line financing. Section 8 offer concluding comments.

2 Model Setup

We begin by describing the firm's physical production and investment technology and its objective function. We then introduce the firm's external financing costs and its opportunity cost of holding cash.

2.1 Production technology

The firm employs only capital as an input for production. The price of capital is normalized to unity. We denote by K and I respectively the level of the capital stock and gross investment. As

is standard in capital accumulation models, the firm's capital stock K evolves according to:

$$dK_t = (I_t - \delta K_t)\, dt, \quad t \geq 0, \tag{1}$$

where $\delta \geq 0$ is the rate of depreciation.

The firm's operating revenue at time t is proportional to its capital stock K_t, and is given by $K_t dA_t$, where dA_t is the firm's revenue or productivity shock over time increment dt. We assume that the firm's cumulative productivity after accounting for systematic risk[4] evolves according to:

$$dA_t = \mu dt + \sigma dZ_t, \quad t \geq 0, \tag{2}$$

where Z is a standard Brownian motion. The parameters $\mu > 0$ and $\sigma > 0$ are the mean and volatility of the productivity shock dA_t. Thus, the revenue shock dA is assumed to be independently and identically distributed (*i.i.d.*). This production specification is often refereed to as the "AK" technology in the macroeconomics literature.[5]

The firm's incremental operating profit dY_t over time increment dt is then given by:

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt, \quad t \geq 0, \tag{3}$$

where I is the cost of the investment and $G(I, K)$ is the additional adjustment cost that the firm incurs in the investment process. Following the neoclassical investment literature (Hayashi (1982)), we assume that the firm's adjustment cost is homogeneous of degree one in I and K. In other words, the adjustment cost takes the homogeneous form $G(I, K) = g(i)K$, where i is the firm's investment capital ratio ($i = I/K$), and $g(i)$ is an increasing and convex function. Our analyses do not depend on the specific functional form of $g(i)$, and to simplify we assume that $g(i)$ is quadratic:

$$g\,(i) = \frac{\theta i^2}{2}, \tag{4}$$

where the parameter θ measures the degree of the adjustment cost.

[4] We leave the details about the risk adjustment to the Appendix.

[5] Cox, Ingersoll, and Ross (1985) develop an equilibrium production economy with the "AK" technology. See Jones and Manuelli (2005) for a recent survey in macro.

Finally, we assume that the firm can liquidate its assets at any time. The liquidation value L_t is proportional to the firm's capital, $L_t = lK_t$, where $l > 0$.

The homogeneity assumption embedded in the adjustment cost and the "AK" production technology allows us to deliver our key results in a parsimonious and analytically tractable way. Adjustment costs may not always be convex and the production technology may exhibit decreasing returns to scale in practice, but these functional forms substantially complicate the formal analysis of dynamic investment models and do not permit a closed-form characterization of investment and financing policies (see Hennessy and Whited (2005, 2007)). As will become clear below, the homogeneity of our model in K allows us to reduce the dynamics to a one-dimensional equation, which is straightforward to solve. See Eberly, Rebelo, and Vincent (2008) for empirical evidence in support of Hayashi homogeneity settings.

2.2 Financing costs

Neoclassical investment models (á la Hayashi (1982)) assume that the firm faces frictionless capital markets and that the Modigliani and Miller (1958) theorem holds. However, in reality, firms face important financing frictions due to incentive, information asymmetry, and transaction cost reasons.[6] Our model incorporates a number of financing costs that firms face in practice and that empirical research has identified, while retaining an analytically tractable setting. The firm may choose to use external financing at any point in time. We assume that the firm incurs a fixed cost of issuing external equity ϕK, which for tractability we take to be proportional to firm size as measured by the capital stock K. This form of fixed costs assumption ensures that the firm does not grow out of the fixed costs. The firm also incurs a proportional issuance cost γ for each unit of external funds it raises. That is, for each incremental dollar the firm raises, it pays $\gamma > 0$. Let H denote the process for the firm's cumulative external financing and hence dH_t the incremental external financing over time dt.

[6] See Jensen and Meckling (1976), Leland and Pyle (1977), and Myers and Majluf (1984), for example.

Let W denote the process for the firm's cash inventory. If the firm runs out of cash ($W_t = 0$), it needs to either raise external funds to continue operating, or liquidate its assets at value lK.[7] If the firm chooses to raise new external funds to continue operating, it must pay the financing costs specified above. The firm may prefer liquidation if the cost of financing is too high relative to the continuation value (when the firm is not productive, e.g., μ is low). Let τ denote the firm's (stochastic) liquidation time, then $\tau = \infty$ means that the firm never chooses to liquidate.

The rate of return that the firm earns on its cash inventory is the risk-free rate r minus a spread $\lambda > 0$ that reflects the fact that retaining cash within the firm is costly. The cost of carrying cash may arise from an agency problem or from tax distortions. Cash retentions are tax disadvantaged because the associated tax rates generally exceed those on interest income (Graham (2000)). Since there is a cost of hoarding cash ($\lambda > 0$), the firm may find it optimal to distribute cash back to shareholders when its cash inventory grows too large.[8]

Let U denote the firm's cumulative non-decreasing payout process to shareholders, and dU_t the incremental payout over time dt. Distributing cash to shareholders may take the form of a special dividend payment or a share repurchase.[9] The benefit of a payout is that shareholders can invest at the risk-free rate r, which is higher than $(r - \lambda)$ the net rate of return on cash within the firm. However, paying out cash also reduces the firm's cash balance, which potentially exposes the firm to current and future under-investment and future external financing costs.

Combining cash flow from operations dY_t given in (3), with the firm's financing policy given by the cumulative payout process U and the cumulative external financing process H, the firm's cash

[7] We extend this specification in Section 7 by allowing the firm to draw on a credit line. In that specification, the firm issues equity or liquidate after it has drawn down its credit line.

[8] If $\lambda = 0$, the firm has no incentives to pay out cash since keeping cash inside the firm does not have any disadvantages, but still has the benefits of relaxing financing constraints. We could also imagine that there are settings under which $\lambda \leq 0$. For example, if we think that the firm may have "better" investment opportunities than investors do, we can think of λ as an excess return. We do not explore this case in this paper as we are interested in a trade-off model for cash holdings.

[9] We cannot distinguish between a special dividend and a share repurchase as we exclude taxes. Note, however, that a commitment to regular dividend payments is suboptimal in our model. We also exclude any fixed or variable payout costs so as not to overburden the model. These can be added to the analysis.

inventory W then evolves according to:

$$dW_t = dY_t + (r - \lambda)\, W_t dt + dH_t - dU_t, \tag{5}$$

where the second term is the interest income (net of the carry cost λ), the third term dH_t is the cash inflow from external financing, and the last term dU_t is the cash outflow to investors, so that $(dH_t - dU_t)$ is the net cash flow from financing. Note that this is a completely general financial accounting equation where dH_t and dU_t are endogenously determined by the firm.

Firm optimality The firm chooses its investment I, its cumulative payout policy U, its cumulative external financing H, and its liquidation time τ to maximize firm value defined below:

$$\mathbb{E}\left[\int_0^\tau e^{-rt}\left(dU_t - dH_t\right) + e^{-r\tau}\left(lK_\tau + W_\tau\right)\right]. \tag{6}$$

The expectation is taken under the risk-adjusted probability. The first term is the discounted value of payouts to shareholders and the second term is the discounted value upon liquidation. Note that optimality may imply that the firm never liquidates. In that case, we simply have $\tau = \infty$. We impose the usual regularity conditions to ensure that the optimization problem is well posed. See the appendix for details.

3 Model Solution

We first describe the firm's optimal investment policy and firm value under the neoclassical benchmark with no capital market frictions. We then characterize the firm's optimal dynamic investment and financing decisions in the presence of external financing and payout costs.

3.1 A neoclassical benchmark

In the absence of costly financing, our model specializes to the neoclassical model of investment with convex adjustment costs. To ensure that the first-best investment policy is well defined, we

assume that the following parameter condition holds:

$$(r+\delta)^2 - 2\left(\mu - (r+\delta)\right)/\theta > 0.$$

Then, under perfect capital markets (when the Modigliani-Miller theorem holds), the firm's first-best investment policy is given by $I^{FB} = i^{FB}K$, where

$$i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2\left(\mu - (r+\delta)\right)/\theta}. \tag{7}$$

The value of the firm's capital stock is $q^{FB}K$, where q^{FB} is Tobin' s q given by:

$$q^{FB} = 1 + \theta i^{FB}. \tag{8}$$

First, the volatility of the productivity shocks σ has no direct impact on the firm's investment decision and firm value (as seen from (7)). However, σ can have an indirect effect on investment since higher systematic volatility increases the cost of capital. More interestingly, even the idiosyncratic component of volatility affects investment and firm value due to its effect via the financing constraint channel, as we will show.

Second, due to the homogeneity property in production (the "AK" production specification and the homogeneity of the adjustment cost function $G(I, K)$ in I and K), marginal q is equal to average (Tobin's) q, as in Hayashi (1982).

Third, gross investment I is positive if and only if the expected productivity μ is higher than $r + \delta$, so we shall assume that $\mu > r + \delta$. Whenever investment is positive, Tobin's q is greater than unity and installed capital earns rents. Intuitively, when $\mu > r + \delta$, the installed capital is more valuable than newly purchased capital. As is standard in the literature, Tobin's q, the ratio between the value of installed capital and that of newly purchased capital, is greater than unity due to the adjustment cost.

We analyze next the firm's optimal investment and financing decisions when it faces costly external financing.

3.2 Firm value $P(K, W)$ and the optimal investment-capital ratio

It is optimal for the firm (facing external financing costs) to hoard some cash to finance its investment and to lower its future financing costs. There are two natural state variables for the firm's optimization problem: the firm's capital stock K and the firm's cash inventory W. Let $P(K, W)$ denote firm value. Then, using the standard principle of optimality, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation for $P(K, W)$ in the interior region for W where $dH_t = 0$ and $dU_t = 0$:

$$rP(K, W) = \max_{I} \; (I - \delta K) P_K + [(r - \lambda)W + \mu K - I - G(I, K)]P_W + \frac{\sigma^2 K^2}{2} P_{WW}. \tag{9}$$

The first term (the P_K term) on the right side of (9) represents the marginal effect of net investment $(I - \delta K)$ on firm value $P(K, W)$. The second term (the P_W term) represents the effect of the firm's expected saving on firm value, and the last term (the P_{WW} term) captures the effects of the volatility of cash holdings W on firm value.

The firm chooses investment I optimally to set the expected return of the firm equal to the risk-free rate r. In the interior region, the firm finances its investment out of its cash inventory only. The convexity of the physical adjustment cost implies that the investment decision in our model admits an interior solution. The investment-capital ratio $i = I/K$ then satisfies the following first-order condition:

$$1 + \theta i = \frac{P_K(K, W)}{P_W(K, W)}. \tag{10}$$

With frictionless capital markets (the Modigliani-Miller world) the marginal value of cash is $P_W = 1$, so that the neoclassical investment formula obtains: $P_K(K, W)$ is the marginal q, which at the optimum is equal to the marginal cost of adjusting the capital stock $1 + \theta i$. With costly external financing, on the other hand, the investment Euler equation (10) captures both real and financial frictions. The marginal cost of adjusting physical capital $(1 + \theta i)$ is now equal to the ratio of marginal q, $P_K(K, W)$, to the marginal cost of financing (or equivalently, the marginal value of

cash), $P_W(K,W)$. Thus, the more costly the external financing (the higher P_W) the less the firm invests, *ceteris paribus*.

A key simplification in our setup is that the firm's two-state optimization problem can be reduced to a one-state problem by exploiting homogeneity. That is, we can write firm value as

$$P(K,W) = K \cdot p(w), \tag{11}$$

where $w = W/K$ is the firm's cash-capital ratio, and reduce the firm's optimization problem to a one-state problem in w. The dynamics of w can be written as:

$$dw_t = ((r-\lambda)w_t + \mu - i(w) - g(i(w)))\,dt + \sigma dZ_t. \tag{12}$$

Instead of solving for firm value $P(K,W)$, we shall instead solve for the firm's value-capital ratio $p(w)$. Note that marginal q is then $P_K(K,W) = p(w) - wp'(w)$, the marginal value of cash is $P_W(K,W) = p'(w)$, and $P_{WW} = p''(w)/K$. Substituting these terms into (9) we then obtain the following ordinary differential equation (ODE) for $p(w)$:

$$rp(w) = (i(w)-\delta)\left(p(w) - wp'(w)\right) + ((r-\lambda)w + \mu - i(w) - g(i(w)))\,p'(w) + \frac{\sigma^2}{2}p''(w). \tag{13}$$

We also obtain an analog of the FOC (10) for the investment-capital ratio $i(w)$ as follows:

$$i(w) = \frac{1}{\theta}\left(\frac{p(w)}{p'(w)} - w - 1\right). \tag{14}$$

To completely characterize the solution for $p(w)$, we must also determine the boundaries $\underline{w}$ at which the firm raises new external funds (or closes down), how much to raise (the target cash-capital ratio after issuance), and $\overline{w}$ at which the firm pays out cash to shareholders.

3.3 The impact of financing costs

In this subsection, we characterize the firm's optimal policy and value. Depending on the parameter values, the firm prefers either liquidation or refinancing by issuing new equity. Case I refers to the setting where liquidation is optimal. In Case II, the cost of equity issuance is small enough that the firm prefers to refinance than to liquidate when it runs out of cash.

Case I: Optimal Liquidation. Recall that Tobin's q under the frictionless first-best world is higher than the liquidation value per unit of capital l, i.e., $q^{FB} > l$.[10] Under this assumption, it is suboptimal for the firm to liquidate its asset provided that the firm has cash to operate its physical asset. To formally illustrate this point, note that liquidation at any time yields $lK + W = lK + wK$, the sum of the recovery value of the firm's asset and the firm's total cash inventory W that can be distributed to shareholders at no cost. As the firm can always choose to disburse its cash at any time, the value of cash cannot be lower than W, its value when paid out to shareholders. In addition, by deferring liquidation and holding on to its cash the firm retains a valuable option to finance future investment opportunities and to see its earnings potentially grow, which enhances firm value. Therefore, the optimal liquidation boundary is given by $\underline{w} = 0$. Firm value upon liquidation is thus $p(0)K = lK$, implying that

$$p(0) = l. \tag{15}$$

We now turn to the endogenous upper payout boundary $\overline{w}$ for the cash-capital ratio w. Intuitively, when the cash-capital ratio is very high ($w > \overline{w}$), the firm is better off paying out the excess cash ($\overline{w} - w$) to shareholders. That is, it is optimal for the firm to distribute the excess cash as a lump-sum and bring the cash-capital ratio w down to $\overline{w}$. Since firm value must be continuous before and after cash distribution, $p(w)$ is then given by

$$p(w) = p(\overline{w}) + (w - \overline{w}), \quad w > \overline{w}. \tag{16}$$

Since the above equation also holds for w close to $\overline{w}$, we may take the limit and obtain the following condition for the endogenous upper boundary $\overline{w}$:

$$p'(\overline{w}) = 1. \tag{17}$$

At $\overline{w}$ the firm is indifferent between distributing and retaining one dollar, so that the marginal value of cash must equal one, which is the marginal cost of cash to shareholders. Since the payout

[10] Otherwise, the firm should never employ its physical production technology and instead liquidate its capital for its higher value lK.

boundary $\overline{w}$ is optimally chosen, we also have the following "super contact" condition (see, e.g. Dumas (1991)):

$$p''(\overline{w}) = 0. \tag{18}$$

The Hamilton-Jacoby-Bellman equation (13), the investment-capital ratio equation (14), and the associated liquidation boundary (15) and payout boundary conditions (17)-(18) then jointly characterize the firm's value-capital ratio $p(\cdot)$ and optimal dynamic investment and financing decisions.

Case II: Optimal Refinancing. When the firm's expected productivity μ is high and/or its cost of external financing is low, the firm is better off raising costly external financing than liquidating its assets when it runs out of cash. The endogenous upper boundary is determined in the same way as in Case I. The lower boundary, however, is more interesting. First, although the firm can choose to raise external funds at any time, it is optimal for the firm to wait until it runs out of cash, so that $\underline{w} = 0$. The reason is that cash within the firm earns a below-market interest rate $(r - \lambda)$ and there is a time value for the external financing costs. As long as it has cash, the firm can always pay for any level of investment it desires to undertake with its cash. There is thus no role for raising equity locally in our model given that investment is not lumpy. It is always better to defer external financing as long as possible.[11] Our argument highlights the robustness of the pecking order between cash and external financing. That is, the cost of hoarding cash and the costs of raising external funds imply that there is no need to raise external funds unless the firm has to. This is a form of pecking order between internally generated funds and outside financing.

Second, fixed costs in raising equity ($\phi > 0$) induce the firm to raise a lump-sum mK in cash, where $m > 0$ is endogenously chosen. The reason is simply that it is cheaper to raise equity in lumps (i.e. $m > 0$) to economize on the fixed costs.

[11] However, if the cost of financing varies over time particularly when the firm potentially faces stochastic arrival of growth options, the firm may time the market by raising cash in times when financing is cheaper. See Bolton, Chen, and Wang (2009).

Since firm value is continuous before and after equity issuance, $p(w)$ satisfies the following condition when the firm issues equity (at the boundary $\underline{w} = 0$):

$$p(0) = p(m) - \phi - (1 + \gamma)\, m. \tag{19}$$

The right side represents the value-capital ratio $p(m)$ minus both the fixed and the proportional costs of equity issuance, per unit of capital. Since m is optimally chosen, the marginal value of the last dollar raised must equal the marginal cost of external financing, $1+\gamma$. This gives the following smoothing pasting boundary condition at m:

$$p'(m) = 1 + \gamma. \tag{20}$$

Thus, when it is optimal for the firm to refinance rather than liquidate, the HJB equation (13), the investment-capital ratio equation (14), the equity-issuance boundary condition (19), the optimality condition for equity issuance (20), and the endogenous payout boundary conditions (17)-(18) jointly characterize the firm's dynamic investment and financing decisions. This is the global solution for the firm whenever $p(0) > l$.

4 Quantitative Analysis

We now turn to quantitative analysis of our model. For the benchmark case, we set the riskfree rate at $r = 5\%$ and adopt the technological parameter values that Eberly, Rebelo, and Vincent (2008) suggest fit the US data for the neoclassical investment model with constant returns to scale (Hayashi, 1982). The mean and volatility of the productivity shock are $\mu = 21\%$ and $\sigma = 11\%$, respectively; the adjustment cost parameter is $\theta = 4$, and the rate of depreciation is $\delta = 11\%$. When applicable, these numbers are annualized. The implied first-best q in the neoclassical model is then $q^{FB} = 1.54$, and the corresponding first-best investment-capital ratio is $i^{FB} = 13.6\%$.

We then set the annual cash-carrying cost parameter at $\lambda = 1.5\%$, the proportional financing cost at $\gamma = 6\%$ (as suggested in Sufi (2009)) and the fixed cost of financing at $\phi = 5\%$ (consistent

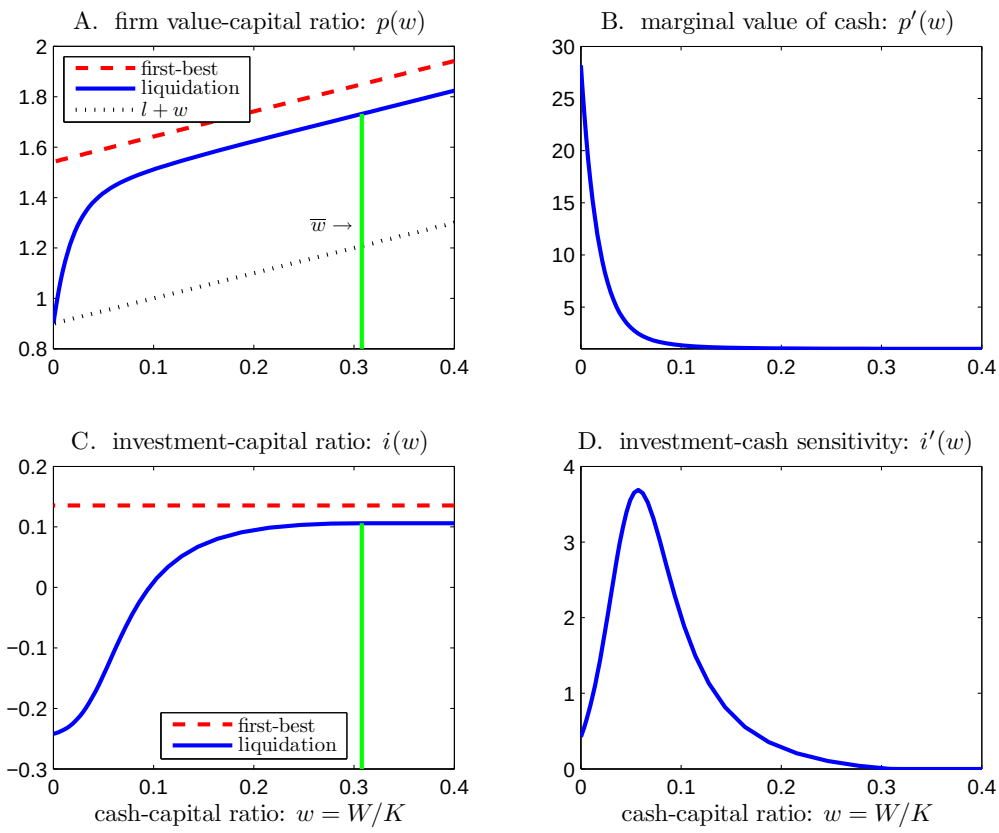

Figure 1: **Case I. Liquidation.** This figure plots the solution in the case when the firm has to liquidate when it runs out of cash ($w = 0$). The parameters are: riskfree rate $r = 5\%$, the mean and volatility of increment in productivity $\mu = 21\%$ and $\sigma = 11\%$, adjustment cost parameter $\theta = 4$, capital depreciation rate $\delta = 11\%$, cash-carrying cost $\lambda = 1.5\%$, and liquidation value-capital ratio $l = 0.9$.

with the evidence of seasonal equity offerings in Altinkilic and Hansen (2000). Finally, for the liquidation value we take $l = 0.9$ (as suggested in Hennessy and Whited (2007)).

Case I: Liquidation. Liquidation is optimal when either the firm's expected productivity is low ($\mu \leq 15\%$) or when external financing costs are very high ($\phi \geq 50\%$). Figure 1 plots the solution under liquidation. In Panel A, the firm's value-capital ratio $p(w)$ starts at l (liquidation value) when cash balances are equal to 0, is concave in the region between 0 and the endogenous payout boundary $\overline{w} = 0.32$, and becomes linear (with slope 1) beyond the payout boundary ($w \geq \overline{w}$). In Section 3, we have argued that the firm will never liquidate before its cash balances hit 0. Panel A

of Figure 1 provides a graphic illustration of this result, where the firm value $p(w)$ lies above the liquidation value $l + w$ (both normalized by capital) for all $w > 0$. In addition, the marginal value of cash increases as the firm becomes more constrained (closer to liquidation), which is confirmed by the concavity of $p(w)$ for $w < \overline{w}$ (i.e. $p''(w) < 0$).

Panel B of Figure 1 plots the marginal value of cash $p'(w) = P_W(K, W)$. It shows that the value-capital ratio $p(w)$ is concave. The external financing constraint makes the firm hoard cash today in order to reduce the likelihood that it will be liquidated in the future. Consider the effect of a mean-preserving spread of cash holdings on the firm's investment policy. Intuitively, the marginal cost from a smaller cash holding is higher than the marginal benefit from a larger cash holding because the increase in the likelihood of liquidation outweighs the benefit from otherwise relaxing the firm's financial constraints. Observe also that the marginal value of cash reaches a staggering value over 25 as w approaches 0. In other words, an extra dollar of cash is worth as high as \$25 to the firm in this region, because it helps keep the firm away from costly liquidation.

Panel C plots the investment-capital ratio $i(w)$ and illustrates under-investment due to the extreme external financing constraints. Optimal investment by a financially constrained firm is always lower than first-best investment $i^{FB} = 0.14$, especially when the firm's cash inventory w is low. Actually, when w is sufficiently low the firm will disinvest by selling assets to raise cash and move away from the liquidation boundary. Note that disinvestment is costly not only because the firm is underinvesting but also because it incurs physical adjustment costs in lowering its capital stock. For the parameter values we use, asset sales (disinvestments) are at the annual rate of over 20% of the capital stock when w is close to zero! The firm tries very hard not to be forced into liquidation, which would permanently eliminate the firm's future growth opportunities. Note also that even at the payout boundary, the investment-capital ratio is only $i(\overline{w}) = 0.1$, about 29% lower than the first best level i^{FB}. Intuitively, the firm is trading off the cash-carrying costs with the cost of underinvestment. It optimally chooses to hoard more cash and to invest more at the payout

boundary when the cash-carrying cost λ is lower.

Next, consider the investment-cash sensitivity measured by $i'(w)$. Observe that $i(w)$ increases with w; the investment-cash sensitivity $i'(w)$ is positive and given by

$$i'(w) = -\frac{1}{\theta}\frac{p(w)p''(w)}{p'(w)^2} > 0. \tag{21}$$

Remarkably, while $i(w)$ is monotonically increasing in w, the investment-cash sensitivity $i'(w)$ is not monotonic in w. Formally, the slope of $i'(w)$ depends on the third derivative of $p(w)$, for which we do not have analytical results. Kaplan and Zingales (1997) have made similar observations on the investment-cash sensitivity in a static setting.

Case II: Refinancing. Consider next the more interesting case of our model when it is optimal for the firm to refinance.[12] As we have argued in Section 3, the firm only uses external financing when necessary, that is when it runs out of cash ($\underline{w} = 0$). Observe that at the financing boundary $\underline{w} = 0$, the firm's value-capital ratio $p(w)$ is strictly higher than l, so that external equity financing is preferred to liquidation in equilibrium.

Figure 2 displays the solutions for both the setting with fixed financing costs ($\phi = 5\%$) and the case without fixed costs ($\phi = 0$). Comparing with Case I, we find that the endogenous payout boundary (marked by the solid vertical line on the right) is $\overline{w} = 0.27$ when $\phi = 5\%$, lower than the payout boundary for the case where the firm is liquidated ($\overline{w} = 0.32$). Not surprisingly, firms are more willing to pay out cash when they can raise new funds in the future. The firm's optimal *return cash-capital ratio* for our parameter values is $m = 0.13$, and is marked by the vertical line on the left in Panel A. Without fixed cost ($\phi = 0$), the payout boundary drops to $\overline{w} = 0.15$, substantially lower than the ones with the fixed costs and the liquidation case. In this case, the firm's return cash-capital ratio is zero. In other words, the firm raises just enough funds to keep w above 0. This is consistent with the intuition that the higher the fixed cost parameter ϕ, the bigger the size of

[12] Throughout the remainder of this paper, we restrict attention to settings with parameters such that external equity financing is preferred to liquidation in equilibrium.

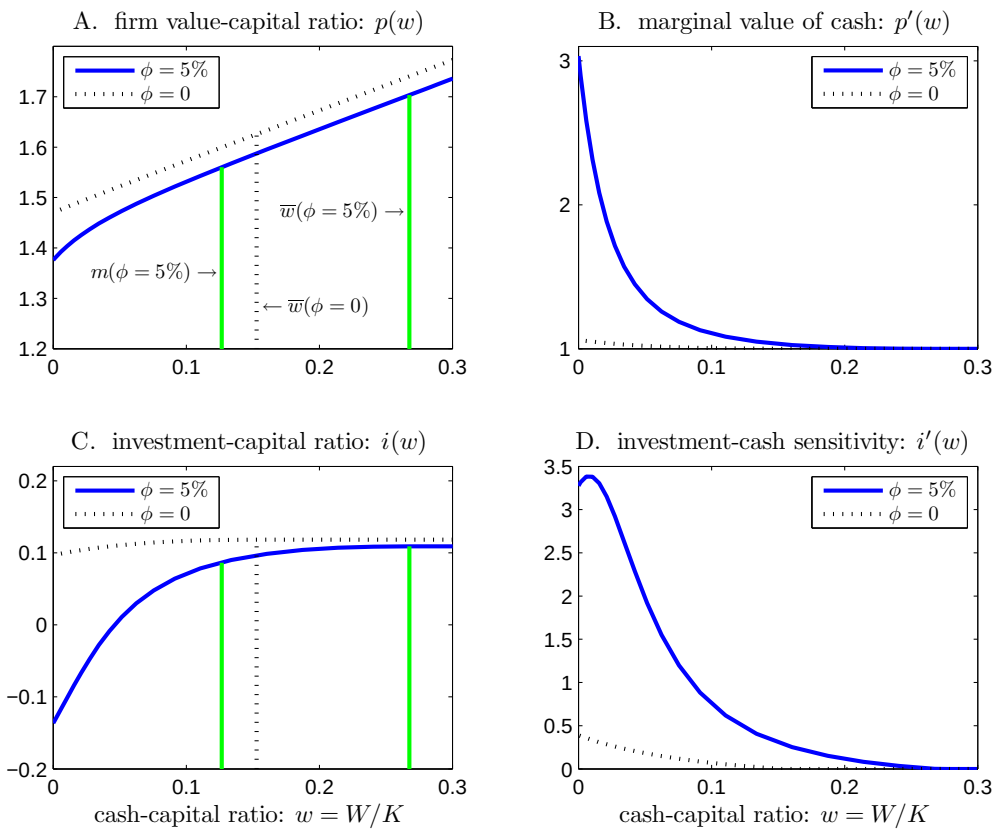

Figure 2: **Case II. Optimal refinancing at $w = 0$.** This figure plots the solution in the case of refinancing. The parameters are: riskfree rate $r = 5\%$, the mean and volatility of increment in productivity $\mu = 21\%$ and $\sigma = 11\%$, adjustment cost parameter $\theta = 4$, capital depreciation rate $\delta = 11\%$, cash-carrying cost $\lambda = 1.5\%$, proportional and fixed financing costs $\gamma = 6\%$, $\phi = 5\%$.

refinancing (higher *return cash-capital ratio* m) each time the firm raises cash.

Panel B plots the marginal value of cash $p'(w)$, which is positive and decreasing, confirming that $p(w)$ is strictly concave for $w \leq \overline{w}$. Conditional on issuing equity and having paid the fixed financing cost, the firm optimally chooses the return cash-capital ratio m such that the marginal value of cash $p'(m)$ is equal to the marginal cost of issuance $1 + \gamma$. To the left of the return cash-capital ratio m, the marginal value of cash $p'(w)$ lies above $1 + \gamma$, reflecting the fact that the fixed cost component in raising equity increases the marginal value of cash. When the firm runs out of cash, the marginal value of cash is around 3, much higher than $1 + \gamma = 1.06$.

As in the previous case, the investment-capital ratio $i(w)$ is increasing in w. Higher fixed

cost component effectively increases the severity of financing constraints, and therefore leads to more underinvestment. This is particularly true in the region to the left of the return cash-capital ratio m, where the investment-capital ratio $i(w)$ drops rapidly and even involves asset sales (about 13% of total capital when w approaches 0). Note that asset sales go down quickly ($i'(w) > 3$) when w is close to zero. This is because both asset sales and equity issuance are very costly. In contrast, removing the fixed financing costs greatly alleviates the under-investment problem, and the investment-capital ratio $i(w)$ becomes essentially flat except for very low w.

Average q, marginal q, and investment. We now turn to the model's predictions on average and marginal q. We take the firm's *enterprise value* – the value of all the firm's marketable claims minus cash, $P(K, W) - W$ – as our measure of the value of the firm's capital stock. Average q, denoted by $q_a(w)$, is then the firm's enterprise value divided by its capital stock:

$$q_a(w) = \frac{P(K, W) - W}{K} = p(w) - w. \tag{22}$$

In our model where financing is costly, marginal q, denoted by $q_m(w)$, is given by

$$q_m(w) = \frac{d\left(P(K, W) - W\right)}{dK} = p(w) - wp'(w) = (p(w) - w) - \big(p'(w) - 1\big)\, w. \tag{23}$$

Recall that in the neoclassical setting (Hayashi (1982)), average q equals marginal q. In our model, average q differs from marginal q due to the external financing costs. An increase in the capital stock K has two effects on the firm's enterprise value. The first is captured by the term $(p(w) - w)$ and reflects the direct effect of an increase in capital on firm value, holding w fixed. This term is equal to average q. The second term $(p'(w) - 1)\, w$ reflects the effect of external financing costs on firm value through w. Increasing the capital stock mechanically lowers the cash-capital ratio $w = W/K$ for a given cash inventory W. As a result, the firm's financing constraint becomes tighter and firm value drops, *ceteris paribus*.

Figure 3 plots the average and marginal q for the liquidation case, the refinancing case with no fixed costs ($\phi = 0$) and the refinancing cost with fixed costs ($\phi = 5\%$). The average and marginal

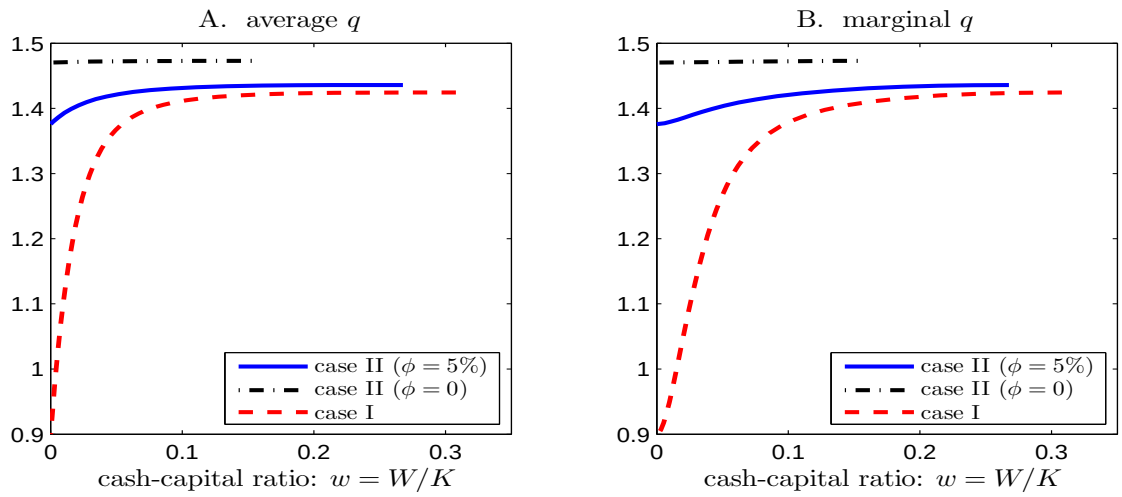

Figure 3: **Average q and marginal q.** This figure plots the average q and marginal q from the three special cases of the model. The parameters, when applicable, are: riskfree rate $r = 5\%$, the mean and volatility of increment in productivity $\mu = 21\%$ and $\sigma = 11\%$, adjustment cost parameter $\theta = 4$, capital depreciation rate $\delta = 11\%$, cash-carrying cost $\lambda = 1.5\%$, proportional and fixed financing costs $\gamma = 6\%$, $\phi = 5\%$.

q are below the first best level, $q^{FB} = 1.54$ in all three cases, and they become lower as external financing becomes more costly. The marginal value of cash $p'(w)$ is always larger than one due to costly external financing. As a result, average q increases with w. Also, the concavity of $p(w)$ implies that marginal q increases with w. From (22) and (23), we see that $p'(w) > 1$ and $w > 0$ imply that $q_m(w) > q_a(w)$, as displayed in Figure 3.

Stationary distributions of w, $p(w)$, $p'(w)$, $i(w)$, average q, and marginal q. We next investigate the stationary distributions for the key variables tied to optimal firm policies. To make the distributions empirically relevant, we compute them under the physical probability measure.[13] Figure 4 shows the distributions for the cash-capital ratio w, the value-capital ratio $p(w)$, the marginal value of cash $p'(w)$, and the investment-capital ratio $i(w)$. Since $p(w), p'(w), i(w)$ are all monotonic in Case II, the densities for their stationary distributions are connected with that of w through (the inverse of) their derivatives.

[13]The link between the physical and risk-adjusted measure is explained in the Appendix.

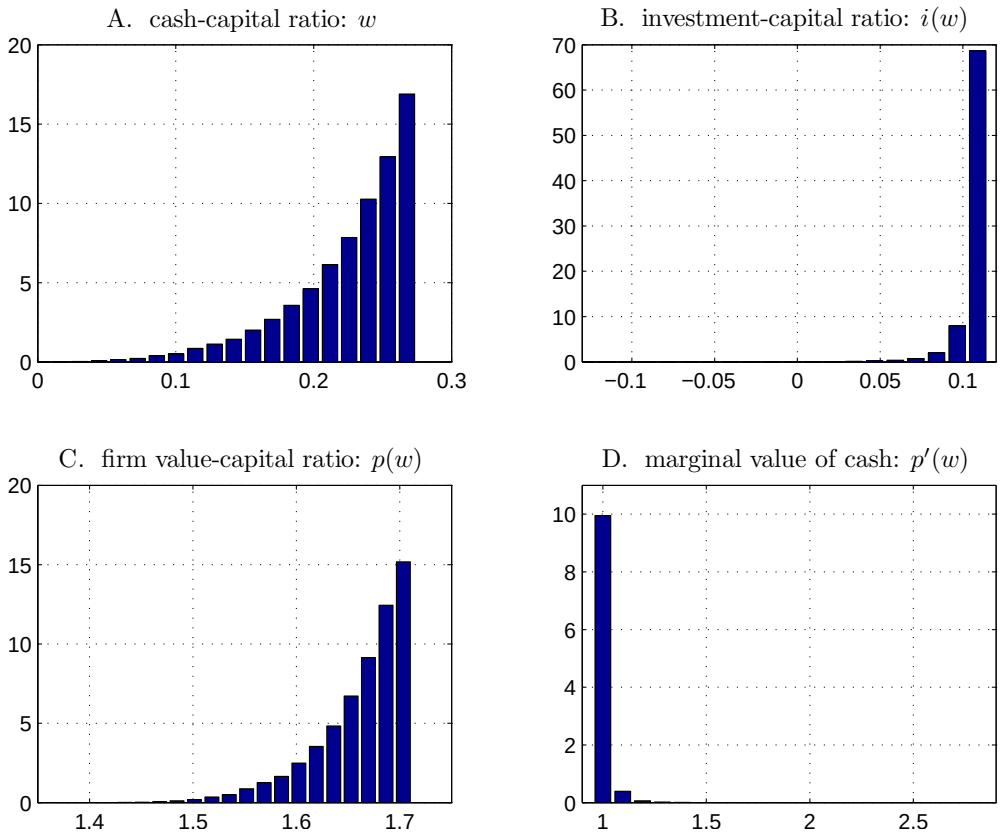

Figure 4: **Stationary distributions in the case of refinancing.** This figure plots the stationary distributions of 4 variables in Case II. The parameters are: risk free rate $r = 5\%$, the mean and volatility of increment in productivity $\mu = 21\%$ and $\sigma = 11\%$, adjustment cost parameter $\theta = 4$, capital depreciation rate $\delta = 11\%$, cash-carrying cost $\lambda = 1.5\%$, proportional and fixed financing costs $\gamma = 6\%$, and $\phi = 5\%$.

Strikingly, the cash holdings of a firm are relatively high most of the time, and hence the probability mass for $i(w)$ and $p(w)$ is concentrated at the highest values in the relevant support of w. The marginal value of cash $p'(w)$ is therefore also mostly around unity. Thus, the firm's optimal cash management policies appear to be effective at shielding it from severe financing constraints and underinvestment most of the time.

Table 1 reports the mean, median, standard deviation, skewness, and kurtosis for w, $i(w)$, $p(w)$, $p'(w)$, average q ($q_a(w)$) and marginal q ($q_m(w)$). Not surprisingly, all these variables have skewness. Other than the marginal value of cash $p'(w)$, all remaining five variables have negative skewness with medians larger than the respective means. The positive skewness for $p'(w)$ is consistent with

Table 1: Moments from the stationary distribution of the refinancing case

This table reports the population moments for cash-capital ratio (w), investment-capital ratio ($i(w)$), marginal value of cash ($p'(w)$), average q ($q_a(w)$), and marginal q ($q_m(w)$) from the stationary distribution in Case II.

	w	$i(w)$	$p'(w)$	$q_a(w)$	$q_m(w)$
mean	0.226	0.105	1.010	1.435	1.434
median	0.239	0.108	1.001	1.436	1.435
std	0.043	0.010	0.037	0.001	0.004
skewness	-1.375	-7.289	15.129	-12.296	-5.011
kurtosis	4.778	86.061	416.580	253.866	40.499

the negative skewness of all the other five variables, as $p'(w)$ is highest for low values of w due to the concavity of $p(w)$. Note also that all these variables have *fat tails*. Interestingly, the kurtosis values for $p'(w)$ and $q_a(w)$ are large, despite their small standard deviations and the small difference between the mean and median values of both $p'(w)$ and $q_a(w)$.

Existing empirical research on corporate cash inventory has mostly focused on firms' average holdings (the first entry in the first row of Table 1) and highlighted that average holdings have increased in recent years. Our model gives a more complete picture of the dynamics of firm capital expenditures and cash holdings. It provides predictions on the time series behavior of firm's investment and financing policies, their valuation, as well as the cross-sectional distribution of cash holdings, and the joint distribution of cash holdings, investment, Tobin' s q, and the frequency of external financing.

As is apparent from Table 1, average cash holdings provide an incomplete and even misleading picture of firms' cash management, investment and valuation. Indeed, one observes that even though the median and the mean of the firm's marginal value of cash $p'(w)$ is close to unity, with only a standard deviation of 0.037, there is a huge kurtosis (416.5) indicating that firms are exposed to potentially large financing costs even if their marginal value of cash is close to unity on average.[14]

[14] These findings may help explain why Gomes (2001) finds no cash-flow effect in his investment regressions based on simulated data. If under the stationary distribution most firms' cash holdings are bunched close to the payout

Also, despite the tight distributions for average q and $i(w)$, the mean and median of $q_a(w)$ are 1.44, which is about 6.5% lower than $q^{FB} = 1.54$, the average q for a firm without external financing costs. Similarly, the mean and median of $i(w)$ is 0.11, which is about 23% lower than $i^{FB} = 0.14$, the investment-capital ratio for a firm without external financing costs. Therefore, simply looking at the difference between the mean and the median or even the standard deviation for these variables, one can end up with a misleading description of firms' financing constraints. The observation that the ratio of the median to mean marginal value of cash $p'(w)$ is close to unity, in particular, does not imply that firm financing constraints are small. The endogeneity of firms' cash holdings mitigates the time-varying impact of financing costs on investment, but the effects remain large on average.

Comparative Statics We close this section with a comparative statics analysis of firm cash holdings and investment for the following six parameters: $\mu, \theta, r, \sigma, \phi, \lambda$. We divide these parameters into two categories. The first three (μ, θ, r) are parameters on the physical side and have direct effects on investment (see i^{FB} in equation(7)); the rest (σ, ϕ, λ) only affect investment and firm value through financing constraints. We examine the effects of these parameters through their impact on the distributions of cash holdings and investment in Figure 5 and 6.

In Figure 5, the left panels (A, C, and E) plot the cumulative stationary distributions (CDF) of the cash holdings w, and the right panels (B, D, and F) plot the cumulative distributions of firm investments i. As panel A highlights, when mean productivity increases (from $\mu = 16\%$ to $\mu = 21\%$) firms tend to hold more cash. That is, the cumulative distributions of firms for higher values of μ first-order stochastically dominate the distributions for lower values of μ. This is intuitive, since the return on investment increases with μ so that the shadow value of cash increases. Still, one might expect firms to spend their cash more quickly for higher μ as the value of investment opportunities

boundary then indeed one should not find a *cash-flow* effect on investment on average. Note also that the key variable for investment of financially constrained firms is the firm's cash-capital ratio and not the firm's cash-flow.

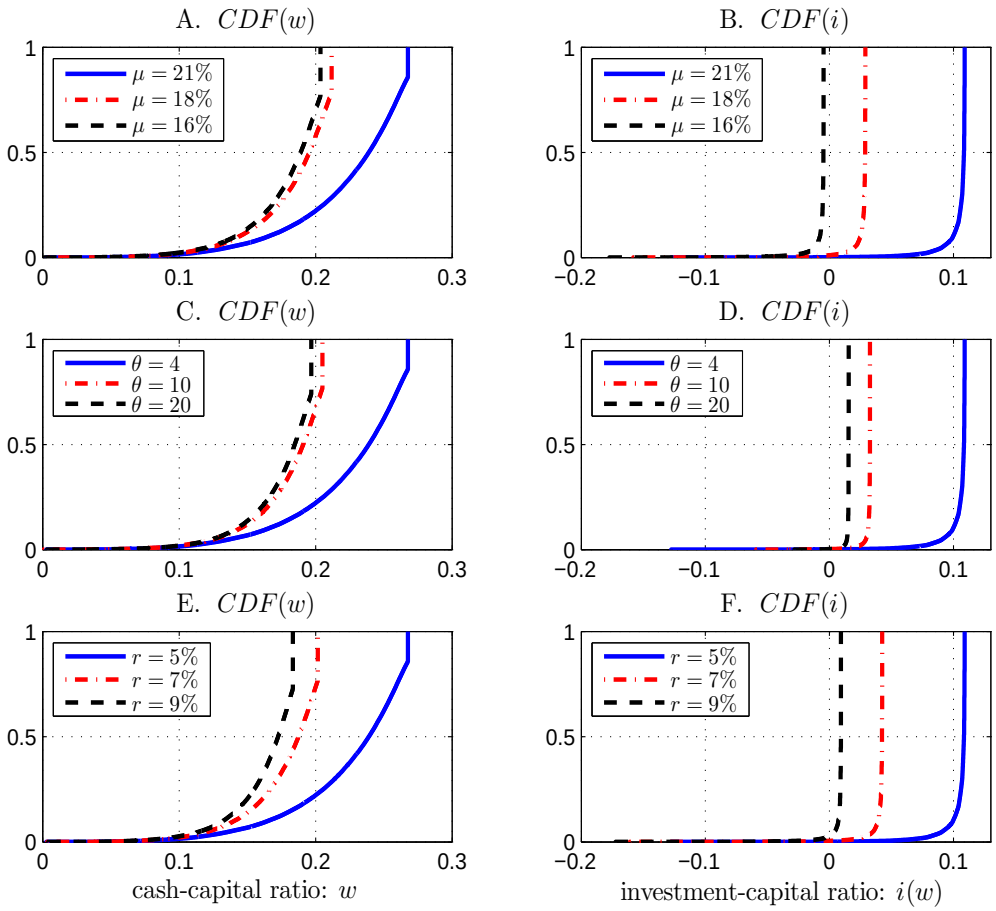

Figure 5: **Comparative statics I: μ, θ, and r.** This figure plots the cumulative distribution function for the stationary distribution of cash-capital ratio (w) and investment-capital ratio ($i(w)$) for different values of the mean of productivity shocks μ, investment adjustment cost θ, and interest rate r.

rises, so that the net effect on firm cash holdings is ambiguous a priori. In our baseline model, the net effect on w of a higher μ is positive, because investment adjustment costs induce firms to only gradually increase their investment outlays in response to an increase in μ.

The effect of an increase in μ on investment is highlighted in Panel B. Again, firms respond to an increase in μ by increasing investment. At $\mu = 16\%$ firms are disinvesting as $i(w)$ is negative for all firms. At $\mu = 18\%$ nearly all firms are making positive investments, with most firms bunched at an investment level of roughly $i(w) = 0.03$. Finally, for $\mu = 21\%$ most firms are investing $i(w) = 0.11$.

The effects of an increase in investment adjustment cost θ and interest rate r on cash holdings

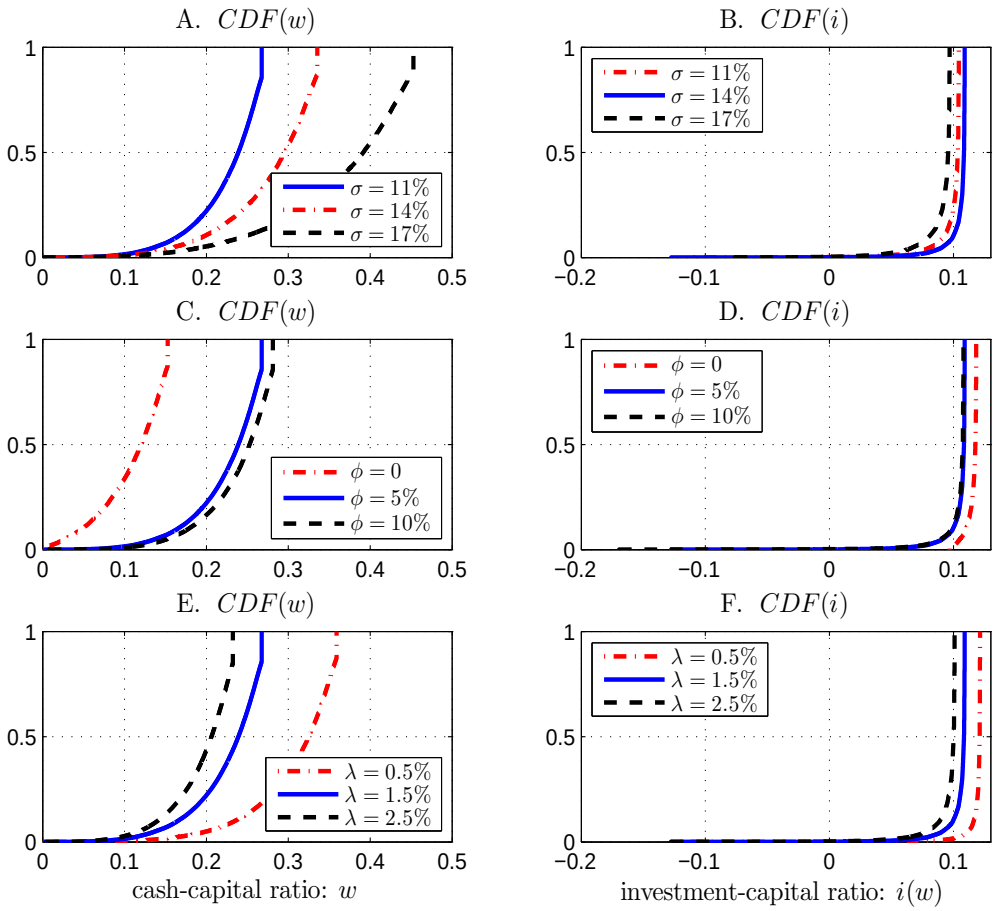

Figure 6: **Comparative statics II: σ, ϕ, and λ.** This figure plots the cumulative distribution function for the stationary distribution of cash-capital ratio (w) and investment-capital ratio ($i(w)$) for different values of the volatility of productivity shocks σ, fixed costs of external financing ϕ, and carry cost of cash λ.

and investment are also quite intuitive. As panel D shows, an increase in θ has a negative effect on investment. If firms invest less, one should expect their cash holdings to increase almost mechanically. However, this turns out not to be the case. Firms have a lower shadow value of cash if they anticipate lower future investment outlays. Therefore they end up holding less cash, as is illustrated in panel C. Similar comparative statics hold for increases in the risk-free rate r: with higher interest rates firms invest less and therefore hold less cash. This is indeed the case, as is illustrated in panels E and F.

The effects of an increase in the idiosyncratic volatility of productivity shocks are shown in

panels A and B of Figure 6, where the stationary distribution is plotted for values of $\sigma = 11\%$, $\sigma = 14\%$ and $\sigma = 17\%$. We change σ while holding systematic volatility fixed, so that the risk-adjusted growth rate μ is unaffected. Again, it is intuitive that firms respond to greater underlying volatility of productivity shocks by holding more cash. Higher cash reserves, in turn, tend to raise the average cost of investment, so that one might expect a higher σ to induce firms to scale back investment. Similarly, an increase in external costs of financing ϕ ought to induce firms to increase their precautionary cash holdings and to scale back their capital expenditures. This is exactly what our model predicts, as shown in panels C and D. The effect of an increase in the carry cost λ ought to be to induce firms to spend their cash more readily, by disbursing it more frequently to shareholders or investing more aggressively. Interestingly, although cash holdings decrease with λ, as seen in panel E, the net effect on investment is negative, as panel F shows. A higher λ makes it more expensive for firms to maintain its buffer-stock cash holdings and indirectly raises the cost of investment.

Finally, one clear difference between Figure 5 and 6 is that, unlike the physical parameters, the parameters σ, ϕ, λ have rather limited effects on investment. This result implies that firms can effectively adjust their cash/payout/financing policies in response to changes in financing or cash management costs, leaving little impact on the real side (investment).

5 Risk and Return

In this section, we investigate how the firm's investment, financing, and cash management policies affect the risk and return on the equity of the firm. In order to highlight the impact of financing constraints on the firm's risk and returns, we adopt the benchmark asset pricing model (CAPM), which measures the riskiness of an asset with its market beta. Let r_m and σ_m denote the expected return and volatility of the market portfolio, and let ρ be the correlation coefficient between the firm's productivity process A and the returns of the market portfolio.

Without financial frictions (the Modigliani-Miller world), the firm implements the first-best investment policy. Its expected return is constant and given by the classical CAPM formula

$$\mu^{FB} = r + \beta^{FB} (r_m - r), \tag{24}$$

where the firm's constant equity beta reflects its exposure to systematic risk

$$\beta^{FB} = \frac{\rho\sigma}{\sigma_m} \frac{1}{q^{FB}}. \tag{25}$$

We can derive an analogous *conditional* CAPM expression for the instantaneous risk-adjusted return $\mu^r(w)$ of a *financially constrained firm* (that is, a firm facing external financing costs) by applying Ito's lemma (see e.g. Duffie (2001)):

$$\mu^r(w) = r + \beta(w) (r_m - r), \tag{26}$$

where

$$\beta(w) = \frac{\rho\sigma}{\sigma_m} \frac{p'(w)}{p(w)} \tag{27}$$

is the financially constrained firm's conditional beta, which reflects the firm's exposure to *both systematic and idiosyncratic risk.*

Indeed, equation (27) highlights the fact that the equity β for a financially constrained firm is monotonically decreasing with its cash-capital ratio w. The cash-capital ratio w has two effects on the conditional beta of a financially constrained firm: first, an increase in w relaxes the firm's financing constraint and reduces underinvestment. As a result, the risk of holding the firm's equity is lower. Second, the firm's asset risk is also reduced as a result of the firm holding a greater share of its assets in cash (whose beta is zero). Both channels imply that the conditional beta $\beta(w)$ and the required rate of return $\mu^r(w)$ decreases with w.

This is in contrast to the *constant* equity beta for an unconstrained firm (given in (25)). Importantly, our analysis highlights how idiosyncratic risk is priced for a financially constrained firm. Idiosyncratic risk, as much as systematic risk, exposes the firm to external financing costs. When

the firm faces external financing costs, it effectively behaves like a risk-averse agent that requires compensation for exposure to idiosyncratic risk.

Interestingly, when w is sufficiently high, the beta for a firm facing external financing costs can be even lower than the beta for the neoclassical firm (facing no financing costs). This can be seen in Figure 7. We may also illustrate this point by rewriting the conditional beta as follows:

$$\beta(w) = \frac{\rho\sigma}{\sigma_m}\frac{p'(w)}{(p(w) - w) + w} = \frac{\rho\sigma}{\sigma_m}\frac{p'(w)}{q_a(w) + w}, \tag{28}$$

where $q_a(w) = p(w) - w$ is the firm's *average* q (the ratio of the firm's *enterprise value* and its capital stock). Although $q_a(w) < q^{FB}$ and $p'(w) > 1$, the second term, w, in the denominator of $\beta(w)$ can be so large that $\beta(w) < \beta^{FB}$. Thus, as firms facing external financing costs engage in optimal risk management by hoarding cash, the buffer stock of cash holdings can make them even safer than neoclassical firms facing no financing costs and holding no cash.

Panel A of Figure 7 plots the firm's value-capital ratio $p(w)$ for three different levels of idiosyncratic volatility ($\xi = 5\%$, $\xi = 15\%$, and $\xi = 30\%$). The other parameter values for this calculation are $r_m - r_f = 6\%$, $\sigma_m = 20\%$, and the systematic volatility is fixed at $\rho\sigma = 8.8\%$ (assuming $\rho = 0.8$ when $\sigma = 11\%$). As expected, it shows that firm value is higher and the payout boundary $\overline{w}$ is lower for lower levels of idiosyncratic volatility.

Panel B of Figure 7 plots the *marginal value of cash* ($p'(w)$) for the same three levels of idiosyncratic volatility. It shows, as expected, that $p'(w)$ is decreasing in w for each level of idiosyncratic volatility. The figure also reveals that for high values of w, the marginal value of cash ($p'(w)$) is higher for higher levels of idiosyncratic volatility. But, more surprisingly, for low values of w the marginal value of cash is actually decreasing in idiosyncratic volatility. The reason is simply that when the firm is close to financial distress, a dollar is more valuable for a firm with lower idiosyncratic volatility.

Panel C plots the *investment-capital ratio* for the three different levels of idiosyncratic volatility. We see again that for sufficiently high w, investment is decreasing in idiosyncratic volatility, whereas

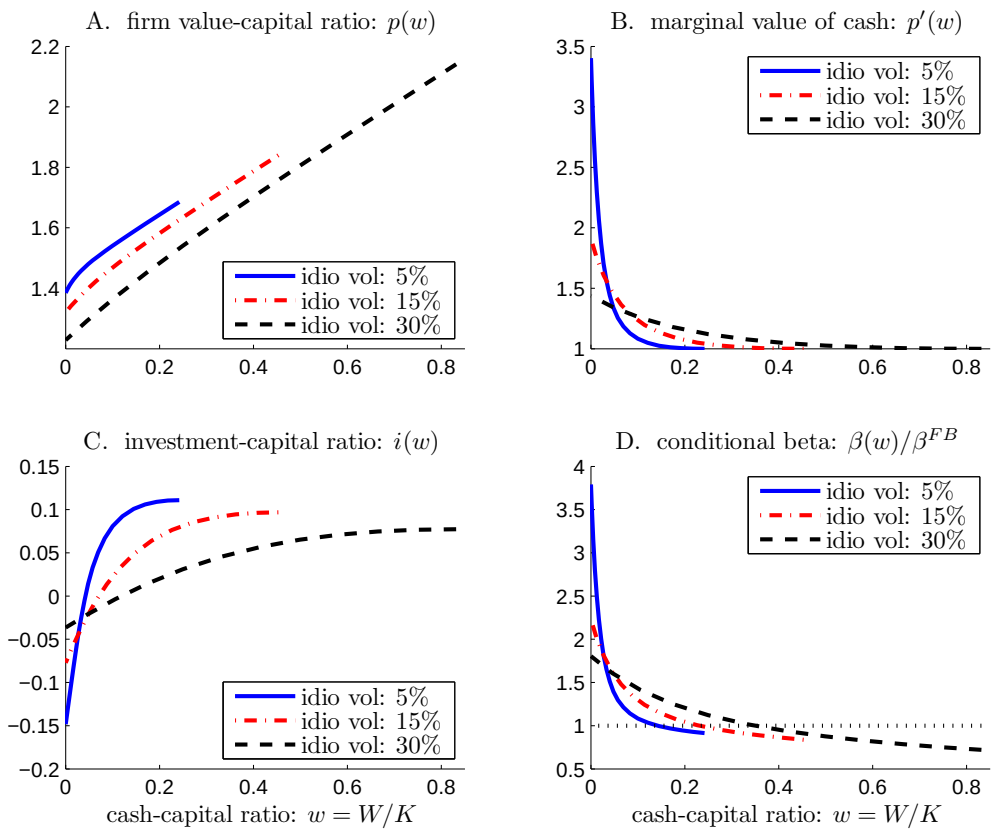

Figure 7: **Idiosyncratic volatility, firm value, investment, and beta.** Fixing all other parameters, for three different levels of idiosyncratic volatility (5%, 15%, 30%), this figure plots the firm value-capital ratio, marginal value of cash, investment-capital ratio, and the ratio of the conditional beta of a constrained firm to that of an unconstrained firm (first best).

for low w, it is increasing. That is, when w is low, firms with low idiosyncratic volatility engage in more asset sales. Again, this latter result is driven by the fact that a marginal dollar has a higher value for a firm with lower idiosyncratic volatility. Therefore, such a firm will sell more assets to replenish its cash holdings.

Panel D plots *conditional betas* normalized by the first-best equity beta: $\beta(w)/\beta_{FB}$. At low levels of w, the firm's normalized beta $\beta(w)/\beta_{FB}$ can approach a value as high as 4 for idiosyncratic volatility $\xi = 5\%$. On the other hand, $\beta(w)$ is actually lower than β^{FB} for high w. As we have explained above, this is due to the fact that a financially constrained firm hoards significant amounts of cash, a perfectly safe asset, so that the mix of a constrained firm's assets may actually be safer

than the asset mix of an unconstrained firm, which does not hoard any cash. The important empirical implication that follows from these observations is that a constrained firm's equity beta cannot be unambiguously ranked relative to the equity beta of an unconstrained firm. It all depends on the constrained firm's cash holdings. The inverse relation between equity returns and corporate cash holdings has been documented by Dittmar and Mahrt-Smith (2007) among others. Importantly, our analysis points out that this inverse relation may not just be due to agency problems, as they emphasize, but may also be driven by the changing asset risk composition of the firm.

Panel D also reveals important information about equity beta in the cross section. A constrained firm's equity beta is not monotonic in its underlying idiosyncratic volatility. For large cash-capital ratio w, the equity beta is increasing in the idiosyncratic volatility. However, when the level of w is low, firms with low idiosyncratic volatility actually have higher equity beta. The rankings of beta are driven by the ratio $p'(w)/p(w)$, which can be inferred from the top two panels. Finally, for a cross-section of firms with heterogeneous production technologies and external financing costs, it is crucial to take account of the endogeneity of cash holdings to understand the firm's cash holding choices. Indeed, a firm with high external financing costs is more likely to hold a lot of cash, but its conditional beta (and expected return) may still be higher than for a firm with low financing costs. Thus, a positive relation between equity returns and corporate cash holdings in the cross section, although inconsistent with the within-firm predictions above, may still be consistent in a richer formulation of our model with cross-sectional firm heterogeneity.

We close this section by briefly considering the implications of costly external financing for the *internal rate of return* (IRR) of an investor who seeks to purchase shares in an all-equity firm for a fixed buy-and-hold horizon T. We use the same parameter values as the beta calculation above, except that we fix the total earnings volatility to $\sigma = 11\%$ (the benchmark value). Starting with a given initial value w_0 and fixing a holding period T, we simulate sample paths of productivity shocks. On each path, we use the optimal decision rules to determine the dynamics of cash holdings,

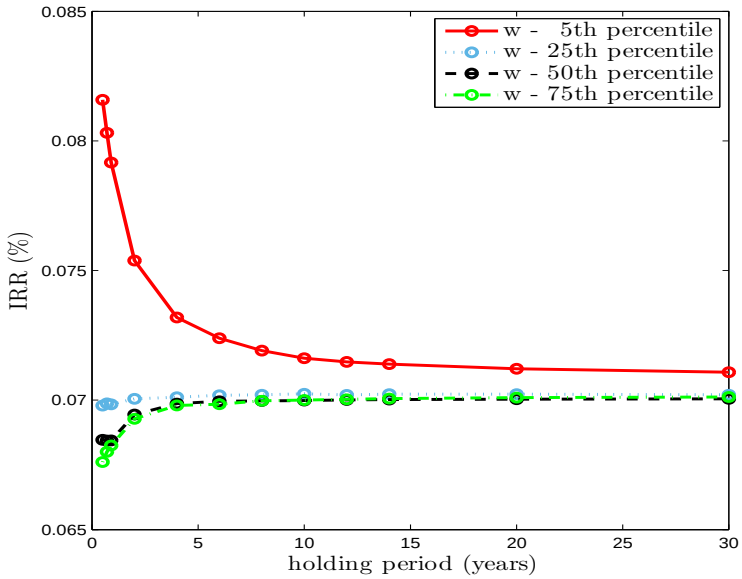

Figure 8: **Conditional IRR.** This figure plots the conditional internal rate of return from investing in the firm in Case II over different horizons at different levels of cash-capital ratios w. These values of w correspond to $5, 25, 50, 75$th percentile of the stationary distribution.

investment, financing, and payout to shareholders and to compute the value of the firm at time T. We then compute the IRR for the simulated cash flows from the investment. We report the IRR solutions in Figure 8 for investment horizon (holding period) T ranging from 0 to 30 years and for firms with initial w ranging from the 5% lowest to the 75% highest cash-holding firms in the population (note that a firm among the 5% lowest cash inventory holders would have a cash-capital ratio smaller than $w = 0.14$). For an investor with a very short investment horizon (say, less than a year), buying shares in an all-equity firm among the 5% lowest holders of cash may require a return as high as 8.4%, compared to 6.7% for an otherwise identical firm among the 25% highest holders of cash.

6 Dynamic Hedging

In addition to cash inventory management, the firm can also reduce its cash-flow risk by investing in financial assets (an aggregate market index, options, or futures contracts) which are correlated

with its own business risk. Consider, for example, the firm's hedging policy using market index futures. Let F denote the futures price. Under the risk-adjusted probability, the futures price evolves according to:

$$dF_t = \sigma_m F_t dB_t, \tag{29}$$

where σ_m is the volatility of the market portfolio, and B is a standard Brownian motion that is partially correlated with Z_t (the Brownian motion in (2)), $E\left[dB_t dZ_t\right] = \rho dt$.

Let ψ_t denote the fraction of total cash W_t that the firm invests in the futures contract. Futures contracts often require that the investor hold cash in a margin account, and there is typically a cost for holding cash in this account. Let κ_t denote the fraction of the firm's cash W_t held in the margin account ($0 \leq \kappa_t \leq 1$), and let ϵ denote the unit cost on cash held in the account. We assume that the firm's futures position (in absolute value) cannot exceed a constant multiple π of the amount of cash $\kappa_t W_t$ in the margin account.[15] That is, we require

$$|\psi_t W_t| \leq \pi \kappa_t W_t. \tag{30}$$

As the firm can costlessly reallocate cash between the margin account and its regular interest-bearing account at any time, the firm will optimally hold the minimum amount of cash necessary in the margin account to minimize the incremental interest cost ϵ. Therefore, optimality implies that the inequality (30) holds as an equality. When the firm takes a hedging position in a futures index, its cash-capital ratio then evolves as follows:

$$dW_t = K_t\left(\mu dA_t + \sigma dZ_t\right) - \left(I_t + G_t\right) dt + dH_t - dU_t + (r - \lambda) W_t dt - \epsilon \kappa_t W_t dt + \psi_t \sigma_m W_t dB_t. \tag{31}$$

Before analyzing optimal firm hedging constrained by costly margin requirements, we first investigate the case where there are no margin requirements for hedging.

[15] For simplicity, we abstract from any variation of margin requirement, so that π is constant.

6.1 Optimal hedging with no frictions

With no margin requirement ($\pi = \infty$), the firm carries all its cash in the regular interest-bearing account and is not constrained in the size of the futures positions ψ it can take. Firm value $P(K, W)$ then solves the following HJB equation:

$$\begin{aligned} rP(K,W) = \max_{I,\psi} \; & (I - \delta K)\, P_K(K,W) + ((r-\lambda)W + \mu K - I - G(I,K))\, P_W(K,W) \\ & + \frac{1}{2}\left(\sigma^2 K^2 + \psi^2 \sigma_m^2 W^2 + 2\rho\sigma_m \sigma \psi W K\right) P_{WW}(K,W) \end{aligned} \tag{32}$$

The only difference between (32) and the HJB equation (9) with no hedging is the coefficient of the volatility term (the last term on the second line), which is now affected by hedging. Since firm value $P(K, W)$ is concave in W, so that $P_{WW} < 0$, the optimal hedging position ψ is determined simply by minimizing that coefficient with respect to ψ. The FOC for ψ is:

$$\left(\psi \sigma_m^2 W^2 + \rho\sigma_m \sigma W K\right) P_{WW} = 0.$$

Solving for ψ, we obtain the firm's optimal hedging demand:

$$\psi^*(w) = -\frac{\rho\sigma}{w\sigma_m}. \tag{33}$$

Thus, controlling for size (capital K), the firm hedges more when its cash-capital ratio w is low. Intuitively, the benefit of hedging is greater when the marginal value of cash $p'(w)$ is high. Substituting $\psi^*(w)$ into the HJB equation (32) and exploiting homogeneity, we obtain the following ODE for the firm's value-capital ratio under hedging:

$$rp(w) = (i(w) - \delta)\left(p(w) - wp'(w)\right) + ((r-\lambda)w + \mu - i(w) - g(i(w)))\, p'(w) + \frac{\sigma^2\left(1-\rho^2\right)}{2} p''(w). \tag{34}$$

Note that the ODE above is the same as (13) in the case without hedging except for the variance reduction from σ^2 to $\sigma^2(1-\rho^2)$.

In sum, the benefit of frictionless hedging is to reduce the volatility of the firm's earnings. Reducing the volatility lowers the likelihood that the firm will need to engage in costly external

financing and thereby increases firm value. Through optimal hedging the firm effectively removes its systematic shocks and thus lowers the volatility of its earnings. The remaining idiosyncratic volatility is not hedgeable and hence continues to affect firm value via costly external financing.

6.2 Optimal hedging with margin requirements

Next, we consider the more realistic setting with a margin requirement given by (30). The firm then faces both a cost of hedging and a constraint on the size of its hedging position. As a result, the firm's HJB equation now takes the following form:

$$\begin{aligned} rP(K,W) = \max_{I,\psi,\kappa} & \left(I-\delta K\right) P_K(K,W) + \left((r-\lambda)W + \mu K - I - G(I,K) - \epsilon\kappa W\right) P_W(K,W) \\ & + \frac{1}{2}\left(\sigma^2 K^2 + \psi^2\sigma_m^2 W^2 + 2\rho\sigma_m\sigma\psi W K\right) P_{WW}(K,W) \end{aligned} \tag{35}$$

subject to:

$$\kappa = \min\left\{\frac{|\psi|}{\pi}, 1\right\}. \tag{36}$$

Equation (36) indicates that there are two candidate solutions for κ (the fraction of cash in the margin account): one interior and one corner. If the firm has sufficient cash, so that its hedging choice ψ is not constrained by its cash holding, the firm sets $\kappa = |\psi|/\pi$. This choice of κ minimizes the cost of the hedging position subject to meeting the margin requirement. Otherwise, when the firm is short of cash, it sets $\kappa = 1$, thus putting all its cash in the margin account to take the maximum feasible hedging position: $|\psi| = \pi$.

The direction of hedging (long ($\psi > 0$) or short ($\psi < 0$)) is determined by the correlation between the firm's business risk and futures return. With $\rho > 0$, the firm will only consider taking a short position in futures as we have shown. If $\rho < 0$, the firm will only consider taking a long position. Without loss of generality, we focus on the case where $\rho > 0$, so that $\psi < 0$.

First, consider the cash region with an interior solution for ψ (where the fraction of cash allocated to the margin account is given by $\kappa = -\psi/\pi < 1$). The FOC with respect to ψ is:

$$\frac{\epsilon}{\pi} W P_W + \left(\sigma_m^2 \psi W^2 + \rho\sigma_m\sigma W K\right) P_{WW} = 0.$$

Using homogeneity, we may simplify the above equation and obtain:

$$\psi^*(w) = \frac{1}{w}\left(\frac{-\rho\sigma}{\sigma_s} - \frac{\epsilon}{\pi}\frac{p'(w)}{p''(w)}\frac{1}{\sigma_s^2}\right). \tag{37}$$

Consider next the cash region where w is small. The benefit of hedging is high in this region ($p'(w)$ is high when w is small). The constraint $\kappa \leq 1$ is then binding, hence $\psi^*(w) = -\pi$ for $w \leq w_-$, where the endogenous cutoff point w_- is the unique value satisfying $\psi^*(w_-) = -\pi$ in (37).

Finally, when w is sufficiently high, the firm chooses not to hedge, as the net benefit of hedging approaches zero while the cost of hedging remains bounded away from zero. More precisely, we have $\psi^*(w) = 0$ for $w \geq w_+$, where the endogenous cutoff point w_+ is the unique solution of $\psi^*(w_+) = 0$ using (37).

In summary, there are three endogenously determined regions for optimal hedging. For sufficiently low cash ($w \leq w_-$), the firm engages in maximum feasible hedging ($\psi(w) = -\pi$). All the firm's cash is in the margin account. In the (second) interior region $w_- \leq w \leq w_+$, the firm chooses its hedge ratio $\psi(w)$ according to equation (37) and puts up just enough cash in the margin account to meet the requirements. For high cash holdings ($w \geq w_+$), the firm does not engage in any hedging to avoid the hedging costs.

We now provide quantitative analysis of the impact of hedging on the firm's decision rules and firm value. We choose the following parameter values: $\rho = 0.8$, $\sigma_m = 20\%$ (the same as in Section 5); $\pi = 5$, corresponding to 20% margin requirement; $\epsilon = 0.5\%$; the remaining parameters are those for the baseline case in Section 4.

In Figure 9, several striking observations emerge from the comparisons of the *frictionless hedging*, the *hedging with costly margin requirements*, and the *no hedging* solutions.

First, Panel A makes apparent the extent to which hedging may be constrained by the margin requirements. On the one hand, when $w > w_+ = 0.16$, the firm chooses not to hedge at all because the benefits of hedging are smaller than the costs due to margin requirements. On the other hand,

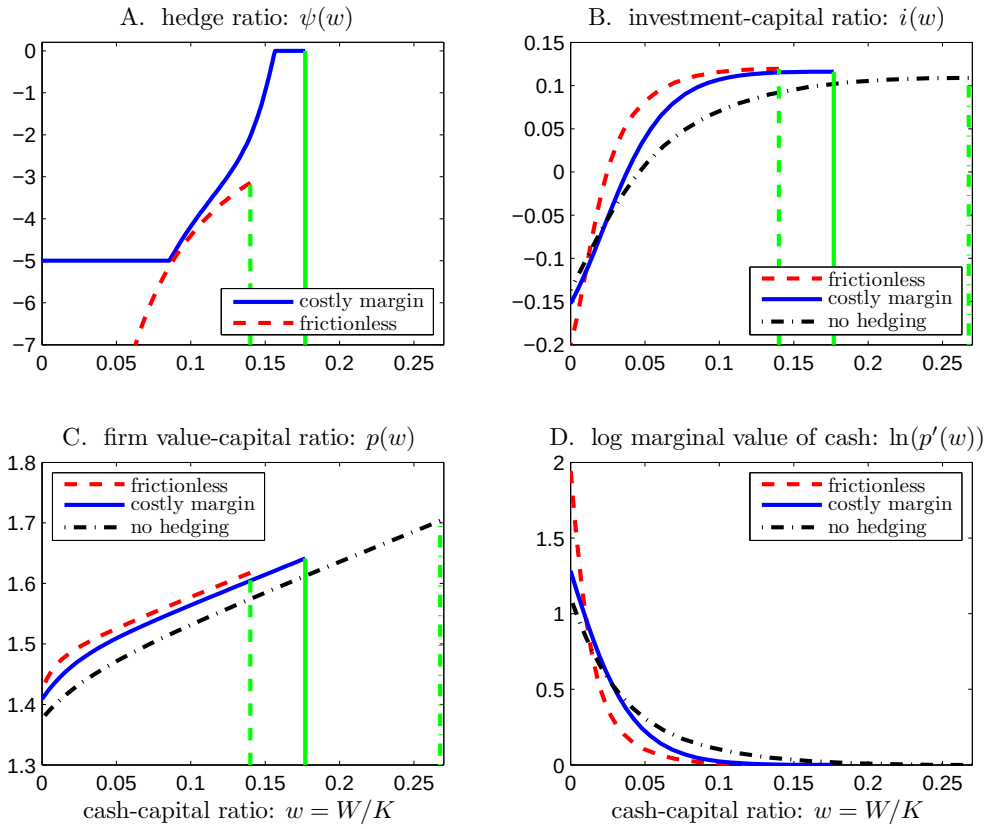

Figure 9: **Optimal hedging.** This figure plots the optimal hedging and investment policies, the firm value-capital ratio, and the marginal value of cash for Case II with hedging (with or without margin requirements). In Panel A, the hedge ratio for the frictionless case is cut off at -7 for display.

it hits the maximum hedge ratio for $w < w_- = 0.08$. Thus, just when hedging is most valuable, the firm will be significantly constrained in its hedging capacity. As a result, the firm effectively faces higher uncertainty under costly hedging than under frictionless hedging. It follows that the firm chooses to postpone payouts to shareholders (the endogenous upper boundary $\overline{w}$ shifts from 0.14 to 0.18). The firm also optimally scales back its hedging position in the middle region due to the costs of hedging.

Second, Panel B reveals the surprising result that for low cash-capital ratios, the firm may *underinvest* even more when it is able to optimally hedge (whether with or without costly margin requirements) than when it cannot hedge at all. This is surprising, as one would expect the firm's

underinvestment problem to be mitigated by hedging. After all, hedging reduces the firm's earnings volatility and thus should reduce the need for precautionary cash balances. This rough intuition is partially correct as, indeed, the firm does invest more for sufficiently high values of w, when it engages in hedging.

But why should the firm invest less or disinvest more for low values of w? The reason can be found in Panels C and D. Panel C plots $p(w)$ under the three settings and confirms the intuition that hedging increases firm value. As expected, $p(w)$ is highest under frictionless hedging and lowest without hedging. However, remarkably, not only is $p(w)$ higher with hedging, but the marginal value of cash $p'(w)$ is also higher, when w is low. Panel D plots the log marginal values of cash under the three solutions. Observe that the marginal value of cash is actually higher for low values of w, when the firm engages in hedging. With a higher marginal value of cash, it is then not surprising that the firm sells its assets more aggressively and hedges its operation risk in order to lower the likelihood of using costly external financing.

How much value does hedging add to the firm? We answer this question by computing the net present value (NPV) of optimal hedging to the firm for the case with costly margin requirements. The NPV of hedging is defined as follows. First, we compute the *cost of external financing* as the difference in Tobin's q under the first-best case and q under case II without hedging. Second, we compute the *loss in adjusted present value* (APV), which is the difference in the Tobin's q under the first-best case and q under hedging with a costly margin. Then, the difference between the costs of external financing and the loss in APV is simply the value created through hedging. On average, when measured relative to Tobin's q under hedging with a costly margin, the costs of external financing is about 7%, the loss in APV is about 5%, so that the NPV of costly hedging is of the order of 2%, a significant creation of value to say the least for a purely financial operation.

7 Credit line

Our baseline model of Section 2 can be extended to allow the firm to draw down a credit line. This is an important extension to consider, as many firms in practice are able to secure such lines, and for these firms, access to a credit line is an important alternative source of *liquidity* than cash.[16]

We model the credit line as a source of funding the firm can draw on at any time it chooses up to a limit. We set the credit limit to a maximum fraction of the firm's capital stock, so that the firm can borrow up to cK, where $c > 0$ is a constant. The logic behind this assumption is that the firm must be able to post collateral to secure a credit line and the highest quality collateral does not exceed the fraction c of the firm's capital stock. We may thus interpret cK to be the firm's short-term debt capacity. We also assume that the firm pays a constant spread α over the risk-free rate on the amount of credit it uses. That is, the firm pays interest on its credit at the rate $r + \alpha$. Sufi (2009) shows that a firm on average pays 150 basis points over LIBOR on its credit lines. This essentially completes the description of a credit line in our model. We leave other common clauses of credit lines-such as commitment fees and covenants-as well as the endogenous determination of the limit cK to future research.

Since the firm pays a spread α over the risk-free rate to access credit, it will optimally avoid using its credit line or other costly external financing before exhausting its internal funds (cash) to finance investment. The firm does not pay fixed costs in accessing the credit line, so it also prefers to first draw on the line before tapping equity markets as long as the interest rate spread α is not too high.[17] Our model thus generates a pecking order among internal funds, credit lines and external equity financing.

As in the baseline model of Section 3, in the cash region, the firm value-capital ratio $p(w)$

[16] In future work, we plan to add to the model the availability of long-term debt financing and study the interactions between debt and cash.

[17] When α is high and equity financing costs (ϕ, γ) are low, the firm may not exhaust its credit line before accessing external equity markets. However, for our parameter values, we find that the pecking order results apply between the credit line and external equity.

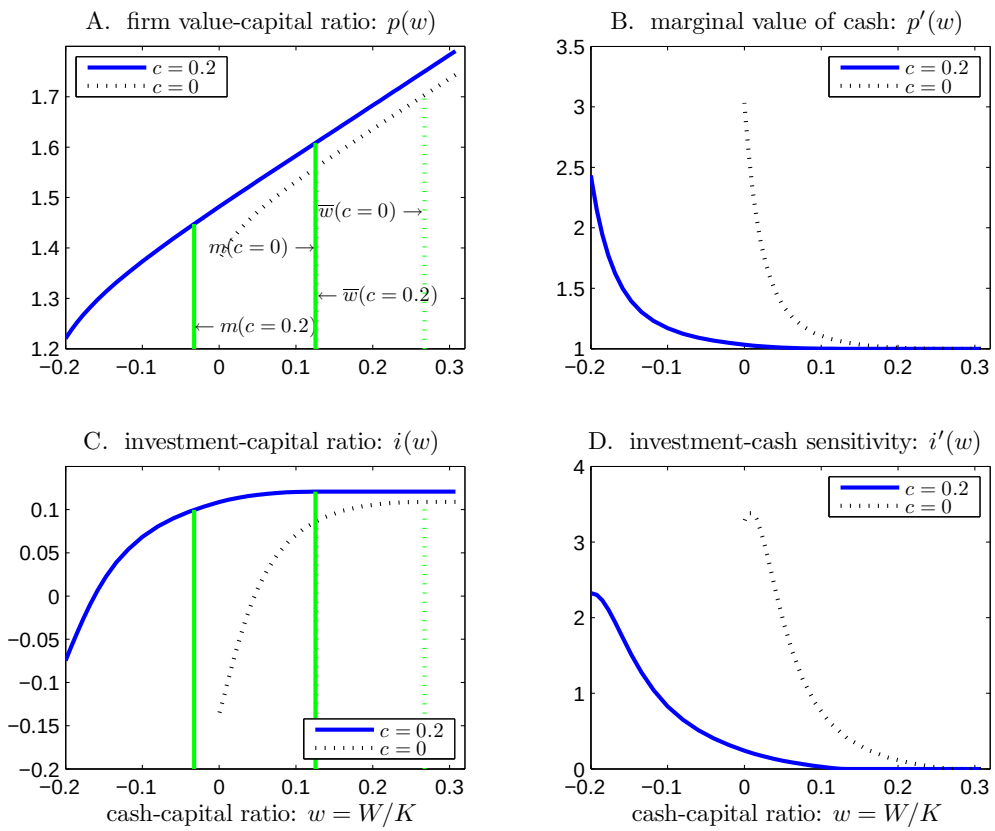

Figure 10: **Credit line.** This figure plots the model solution with credit line and external equity. Each panel plots for two scenarios: one without credit line ($c = 0$) and the other with credit line ($c = 20\%$). The spread on the credit line is $\alpha = 1.5\%$ over the risk-free rate r.

satisfies the ODE in (13), and has the same boundary conditions for payout (17-18). When credit is the marginal source of financing (credit region), $p(w)$ solves the following ODE:

$$rp(w) = (i(w) - \delta)\left(p(w) - wp'(w)\right) + ((r + \alpha)w + \mu - i(w) - g(i(w)))\, p'(w) + \frac{\sigma^2}{2} p''(w), \quad w < 0 \tag{38}$$

When the firm exhausts its credit line before issuing equity, the boundary conditions for the timing and the amount of equity issuance are similar to the ones given in Section 3. That is, we have $p(-c) = p(m) - \phi - (1 + \gamma)(m + c)$, and $p'(m) = 1 + \gamma$. Finally, $p(w)$ is continuous and smooth everywhere, including at $w = 0$, which gives two additional boundary conditions.

Figure 10 plots $p(w)$, the marginal value of liquidity $p'(w)$, the investment-capital ratio $i(w)$,

and the investment-cash sensitivity $i'(w)$, when the firm has access to a credit line. As can be seen from the figure, having access to a credit line increases the firm's value-capital ratio $p(w)$. This is to be expected, as access to a credit line provides a cheaper source of external financing than equity under our chosen parameter value for the spread on the credit line: $\alpha = 0.015$. Second, observe that with the credit line option the firm hoards significantly less cash, and the payout boundary $\overline{w}$ drops from 0.27 to 0.13 when the credit line increases from $c = 0$ to $c = 20\%$ of the firm's capital stock. Third, without access to a credit line ($c = 0$), the firm raises lumpy amounts of equity mK (with $m = 0.13$ for $\phi = 5\%$) when it runs out of cash. In contrast, when $c = 20\%$, the firm raises $0.17K$ in a new equity offering when it has exhausted its credit line, so as to pay off most of the debt it has accumulated on its credit line. But, note that for our baseline parameter choices, the firm still remains in debt after the equity issuance, as $m = -0.03$. Fourth, the credit line substantially lowers the marginal value of liquidity. Without the credit line, the marginal value of cash at $w = 0$ is $p'(0) = 3.03$, while with the credit line ($c = 20\%$), the marginal value of cash at $w = 0$ is $p'(0) = 1.03$, and the marginal value of cash at the point when the firm raises external equity is $p'(-c) = 2.43$.

It follows that a credit line substantially mitigates the firm's underinvestment problem as can be seen in Panel C in Figure 10. Without a credit line ($c = 0$), the firm engages in significant asset sales ($i = -13.7\%$) when it is about to run out of cash. With a credit line, however ($c = 20\%$), the firm's investment-capital ratio is $i(0) = 10.9\%$ when it runs out of cash ($w = 0$). Even when the firm has exhausted its credit line (at $w = -20\%$), it engages in much less costly asset sales ($i(-c) = -7.5\%$). Finally, observe that the investment-cash sensitivity is substantially lower when the firm has access to a credit line. For example, when the firm runs out of cash, the investment-cash sensitivity $i'(0) = 0.2$, much smaller than $i'(0) = 3.3$ when the firm has no credit line and has to issue external equity to finance investment.

Next, we turn to the effect of *liquidity* (cash and credit) on marginal q and investment.

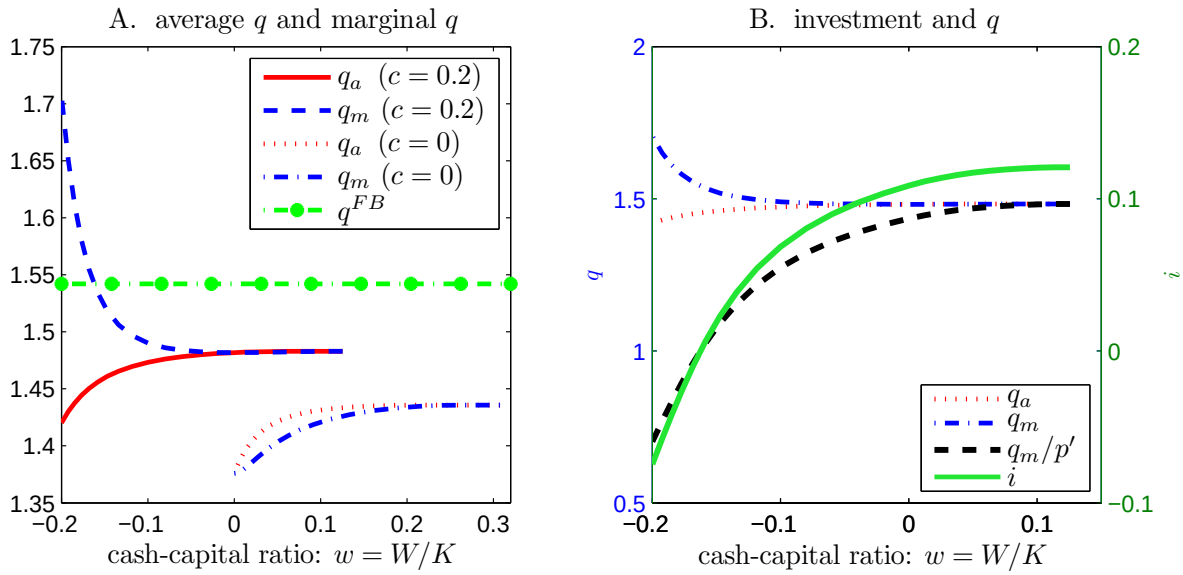

Figure 11: **Investment and q with credit line.** The left panel plots the average q (q_a) and marginal q (q_m) from the case with credit line ($c = 0.2$) and without credit line ($c = 0$). The right panel plots the average q, marginal q, the ratio of marginal q to marginal value of liquidity (q_m/p') on the left axis, and investment-capital ratio (i) on the right axis. These results are for the case with credit line ($c = 0.2$).

The left panel of Figure 11 plots the firm's marginal and average q for two identical firms: one with a credit line ($c = 20\%$), and the other without a credit line ($c = 0$). Recall that:

$$q_m(w) = q_a(w) - (p'(w) - 1)w. \tag{39}$$

For a firm without a credit line, marginal q lies below average q, because $p'(w) > 1$ for $w \in (0, \overline{w})$. The intuition is that a unit increase in capital K lowers the firm's cash-capital ratio $w = W/K$, which makes the firm more financially constrained, thus lowering marginal q.

In contrast, when the firm has access to credit lines and is in the credit region ($W < 0$), marginal q lies above average q. In this region, increasing K raises the firm's debt capacity (credit line limit cK) and lowers its book leverage, which relaxes the firm's borrowing constraint, thus making the value of marginal q higher. Formally, when $w < 0$, the second term in (39) is positive so that $q_m > q_a$. Moreover, note that $q_m'(w) = -p''(w)w < 0$ in the credit region. Thus, marginal q is decreasing in w when $w \leq 0$, opposite to the case when $w > 0$.

Remarkably, although average q is always below the first-best q, marginal q may exceed the first-best marginal q when the firm is in the credit region, as can be seen in Figure 11. Moreover, observe that the quantitative differences between average and marginal q are much larger in the credit region than in the cash region.

A widely held belief in the empirical literature on corporate investment is that marginal q is a more accurate measure than average q of the firm's investment opportunities. This is indeed true in the Modigliani-Miller world (without fixed costs of investment), but it is not generally valid in a world where firms face financial constraints. The right panel of Figure 11 shows that although the investment-capital ratio $i(w)$ increases with w in the credit region, marginal q actually decreases with w. As a result, marginal q and investment move in opposite directions in the credit region. To understand this seemingly counterintuitive result, we must look at the investment Euler equation for a firm facing financial constraints. It is clear from this equation that investment is driven by the ratio of marginal q to the marginal value of liquidity $p'(w)$. Now in the credit region, both the marginal value of liquidity $p'(w)$ and marginal q are high when the firm is close to its credit limit. Indeed, the marginal value of liquidity $p'(w)$ increases at a higher rate than marginal q when the firm uses up more of its credit line (i.e. when we move to the left in the credit region), and as a result, the investment-capital ratio falls when the firm uses more credit.

Finally, we turn to the analysis of the stationary distribution of firms, when firms have access to a credit line. To understand the different behavior in the cash and the credit regions, we report the first four moments of the distribution plus the medians of the variables of interests (w, $i(w)$, $p'(w)$, $q_a(w)$, and $q_m(w)$) for each region. The most significant observation is that the availability of credit makes the firm's stationary distribution for these variables much less skewed and fat-tailed in the cash region. Because liquidity is more abundant with a credit line, the firm's marginal value of cash is effectively unity throughout the cash region. However, the skewness and fat-tails of the distribution now appear in the credit region (note, for example, the high kurtosis (117) for marginal

q in the credit region.) Although the firm has a credit line of up to 20% of its capital stock, it only uses about 4% of its line on average. The reason is that the firm does not spend much time around the credit line limit. The risk of facing a large fixed cost of equity induces the firm to immediately move away from its credit limit.

The cash-capital ratio w, $i(w)$, $q_a(w)$, and $q_m(w)$ are all skewed to the left in the cash region, as in our baseline model without a credit line. The intuition is similar to the one provided in the baseline model. Moreover, because of the firm's optimal buffer-stock cash holding, there is effectively no variation in the cash region for the firm's investment and value. Note also that the mean and median of marginal q and average q are all equal to 1.483, up to the third decimal point. Even for the investment-capital ratio $i(w)$, the difference between its median and mean values only appear at the third decimal point.

Unlike in the cash region, not only is the marginal value of credit $p'(w)$ skewed to the left, but so is marginal q in the credit region. The left skewness of marginal q and $p'(w)$ are both driven by the fact that every so often the firm hits the credit limit and incurs large financing costs. In other words, there is much more variation in the credit region than in the cash region for marginal q and the marginal value of liquidity $p'(w)$. As marginal q and the marginal value of liquidity move in the same direction in the credit region, there is, however, much less variation in $i(w)$, which is monotonically related to the ratio $q_m(w)/p'(w)$.

8 Conclusion

We have shown how in the presence of external financing and payout costs the firm's optimal investment, financing and risk management policies are all interconnected. The firm's optimal cash inventory policy takes the form of a double-barrier policy where the firm raises new funds only when its cash inventory is entirely depleted, and pays out cash to shareholders only when its cash-capital ratio hits an endogenous upper barrier. In between these two barriers, the firm continuously adjusts

Table 2: Conditional moments from the stationary distribution of the credit line model

This table reports the population moments for cash-capital ratio (w), investment-capital ratio ($i(w)$), marginal value of cash ($p'(w)$), average q ($q_a(w)$), and marginal q ($q_m(w)$) from the stationary distribution in the case with credit line.

	w	$i(w)$	$p'(w)$	$q_a(w)$	$q_m(w)$
			A. credit line region		
mean	-0.042	0.094	1.085	1.479	1.485
median	-0.032	0.100	1.060	1.480	1.482
std	0.035	0.019	0.082	0.005	0.009
skewness	-1.194	-2.912	5.049	-3.926	8.754
kurtosis	4.177	14.908	43.820	25.977	116.770
			B. cash region		
mean	0.087	0.119	1.005	1.483	1.483
median	0.095	0.120	1.001	1.483	1.483
std	0.034	0.003	0.007	0.000	0.000
skewness	-0.724	-1.930	1.994	-2.540	-1.122
kurtosis	2.489	6.157	6.463	9.222	3.007

its investment to take account of positive or negative cash-flow shocks. In addition the firm also hedges its earnings risk by investing in financial assets that are not perfectly correlated with its underlying business risk. The benefit of such hedges is to reduce the volatility of the firm's cash inventory.

Our model of the firm's optimal risk management policy is both tractable and operational, to the extent that the firm's policy can be tied down entirely to observable parameter values, such as the volatility and growth of earnings, the costs of external financing and the opportunity cost of hoarding cash. In a number of respects, though, our model is simple. It does not account for the sudden arrival of investment opportunities or changes in the cost of external financing. Nor does it include other risks that firms are exposed to in practice such as litigation risk. Our model also does not have any endogenous leverage decision. Finally, our analysis only looks at risk management from the shareholders' perspective. In practice, however, risk management decisions are made by

self-interested managers. We believe that all of these features can be incorporated into our basic model while still retaining a tractable and operational model. We leave the incorporation of these elements into a richer model to future research.

Appendix

Boundary conditions We begin by showing that $P_W(K,W) \geq 1$. The intuition is as follows. The firm always can distribute cash to investors. Given $P(K,W)$, paying investors $\zeta > 0$ in cash changes firm value from $P(K,W)$ to $P(K,W-\zeta)$. Therefore, if the firm chooses not to distribute cash to investors, firm value $P(K,W)$ must satisfy

$$P(K,W) \geq P(K,W-\zeta) + \zeta,$$

where the inequality describes the implication of the optimality condition. With differentiability, we have $P_W(K,W) \geq 1$ in the accumulation region. In other words, the marginal benefit of retaining cash within the firm must be at least unity due to costly external financing. Let $\overline{W}(K)$ denote the threshold level for cash holding, where $\overline{W}(K)$ solves

$$P_W\left(K, \overline{W}(K)\right) = 1. \tag{40}$$

The above argument implies the following payout policy:

$$dU_t = \max\{W_t - \overline{W}(K_t), 0\},$$

where $\overline{W}(K)$ is the endogenously determined payout boundary. Note that paying cash to investors reduces cash holding W and involves a linear cost. The following standard condition, known as *super contact* condition, characterizes the endogenous upper cash payout boundary (see e.g. Dumas, 1991 or Dixit, 1993):

$$P_{WW}(K, \overline{W}(K)) = 0. \tag{41}$$

When the firm's cash balance is sufficiently low ($W \leq \underline{W}$), under-investment becomes too costly. The firm may thus rationally increase its internal funds to the amount $\underline{W}$ by raising total amount of external funds $(1+\gamma)(\underline{W} - W)$. Optimality implies that

$$P(K,W) = P(K,\underline{W}) - (1+\gamma)(\underline{W} - W), \quad W \leq \underline{W}. \tag{42}$$

Taking the limit by letting $W \to \underline{W}$ in (42), we have

$$P_W(K, \underline{W}(K)) = 1 + \gamma. \tag{43}$$

Risk-neutral probability Assuming market completeness, the linkage between the physical measure $\mathcal{P}$ and the risk-adjusted measure $\mathcal{Q}$ is determined by the stochastic discount factor Λ_t (see e.g. Duffie (2001)). For simplicity, we assume that the CAPM holds, so that Λ_t follows

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta d\widehat{B}_t,$$

where $\widehat{B}_t$ is a standard Brownian motion under $\mathcal{P}$ and is the source of shocks to market returns, and η is the Sharpe ratio of the market portfolio. Then, B_t is a standard Brownian motion under Q satisfying $dB_t = d\widehat{B}_t + \eta dt$. We calibrate the Sharpe ratio of the market portfolio $\eta = 0.3$, and assume that the correlation between the technology shocks Z_t and market returns is $\rho = 0.8$.

The stationary distributions of cash-capital ratio w in this paper are computed under the physical measure. The dynamics of w under $\mathcal{P}$ is

$$dw_t = ((r - \lambda)w_t + \mu + \rho\eta\sigma - i(w) - g(i(w)))\, dt + \sigma d\widehat{Z}_t, \tag{44}$$

where $\widehat{Z}_t$ is a standard Brownian motion under $\mathcal{P}$.

Numerical Procedure We use the following procedure to solve the free boundary problem specified by ODE (13) and the boundary conditions associated with the different cases. First, we postulate the value of the free (upper) boundary $\bar{w}$, and solve the corresponding initial value problem using the Runge-Kutta method. For each value of $\bar{w}$ we can compute the value of $p(w)$ over the interval $[0, \bar{w}]$. We can then search for the $\bar{w}$ that will satisfy the boundary condition for p at $w = 0$. In the cases with additional free boundaries, including Case III and the model of hedging with margin requirements, we search for $\bar{w}$ jointly with the other free boundaries by imposing additional conditions at the free boundaries.

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