A unified treatment of Wigner $\mathscr{D}$ functions, spin-weighted spherical harmonics, and monopole harmonics

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A unified, self-contained treatment of Wigner $\mathscr{D}$ functions, spin-weighted spherical harmonics, and monopole harmonics is given, both in coordinate-free language and for a particular choice of coordinates.

I. INTRODUCTION

We show in this paper that three independent generalizations of the usual spherical harmonics on S^2, namely Wigner $\mathscr{D}$ functions,[1–4] spin-weighted spherical harmonics,[5–7] and monopole harmonics[8–9] are completely equivalent.

It is well known that the Wigner $\mathscr{D}$ functions form an orthogonal basis for $L^2(S^3)$, the square-integrable functions on S^3. Monopole harmonics, on the other hand, form an orthonormal basis for (square integrable) sections of (all of the) complex line bundles over S^2. A standard result from the theory of fiber bundles (see, e.g., Ref. 10), however, asserts that these two concepts are entirely equivalent; this will be discussed in more detail in Sec. II below. Thus, monopole harmonics are equivalent to Wigner $\mathscr{D}$ functions. Finally, spin-weighted spherical harmonics can also be interpreted as sections of complex line bundles over S^2 and are therefore the same as monopole harmonics. (This was checked in coordinates by Dray.[11]) Goldberg *et al.*[6] showed directly that the spin-weighted spherical harmonics are equivalent to the Wigner $\mathscr{D}$ functions. Thus, all three of these concepts are equivalent; this paper is devoted to making this equivalence precise.

The paper is divided into two parts. In Part I (Secs. II–V) we give precise mathematical definitions in coordinate-free language of all three kinds of harmonics and establish their equivalence. In Part II (Secs. VI–VIII) we repeat the results of Part I in a particular choice of coordinates, thus establishing a direct connection between the precise mathematical definitions of Part I and the standard literature, which is mostly in the coordinate language of Part II. Parts I and II are written so as to be independent of each other; part of the purpose of this paper is to serve as a dictionary between the coordinate and coordinate-free versions of these results. Some readers may prefer to skip Part I on first reading. However, we feel that it is only in the coordinate-free language of Part I that the fundamental nature of the equivalence of the three kinds of harmonics becomes apparent.

The equivalence of the monopole harmonics to the Wigner $\mathscr{D}$ functions is at least implicitly contained in Refs. 9 and 12 while the interpretation of the spin-weighted spherical harmonics as sections of complex line bundles, and thus their equivalence to monopole harmonics, is also known. However, several features of our presentation are new. Foremost among these is the fact that the standard definition of spin-weighted spherical harmonics does *not* make explicit the fact that they are sections of a fiber bundle.[13] We interpret the standard definition as defining spin-weighted spherical harmonics to be functions on the (unit) tangent bundle to S^2; we make this precise in Sec. IV below and show the equivalence of our definition to the standard definition in Sec. VIII.

For monopole harmonics the situation is somewhat better in that the fiber bundle structure has been given explicitly.[10] However, the monopole harmonics themselves have only been given explicitly with respect to one particular trivializing cover of the complex line bundles.[8,14] We introduce the monopole harmonics in Sec. V as sections of the complex line bundles *irrespective* of local trivializations. The *explicit* coordinate version of both the monopole harmonics and the spin-weighted spherical harmonics in an *arbitrary* local trivialization ("spin-gauge"), given in Sec. VIII [(175) with (166)], is new.

The definition of the Wigner $\mathscr{D}$ functions in Sec. III as the matrix representation of SU(2) acting on irreducible representations of SU(2) in $L^2(S^3)$ is also new. This is usually done only for *integer* spin.[15] Our approach has the advantage that it can be done in a coordinate-free way, i.e., *without* introducing a parametrization in terms of Euler angles.

Finally, one motivation for this work was the desire to provide a self-contained, consistent presentation of these three types of harmonics in order to eliminate the necessity of worrying about which conventions have been used in the three different sets of literature.[16]

The paper is organized as follows. In Sec. II we introduce the mathematical concepts and notation that we will use throughout Part I. Sections III, IV, and V define, respectively, Wigner $\mathscr{D}$ functions, spin-weighted spherical harmonics, and monopole harmonics in abstract, coordinate-free language. Each definition is compared to the previous definition(s) as it arises. Finally, in Part II the results of Part I are rewritten in coordinate language and related to previous work. Section VI reproduces the notation of Sec. II in coordinate language, while Sec. VII does the same for the Wigner $\mathscr{D}$ functions of Sec. III. Section VIII then discusses both spin-weighted spherical harmonics (Sec. IV) and monopole harmonics (Sec. V) in coordinate language.

II. NOTATION

In this section we define angular momentum operators and give the basic properties of complex line bundles in order to fix our notation. The results are standard; our presentation is largely based on Kuwabara[17] and Greub and Petry.[10]

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Explicit coordinate versions of most of the results appear in Sec. VI. The generalization of many of the concepts presented here to higher-dimensional vector bundles over higher-dimensional spaces is discussed by Guillemin and Uribe.[18]

Let the isomorphism of $SU(2) := SU(2,\mathbb{C})$ and S^3 be given by

$$T\colon SU(2) \overset{\approx}{\rightarrow} S^3 \tag{1}$$

and let $B \in SU(2)$ act on S^3 on the left via

$$p \mapsto Bp := T[B(T^{-1}p)]. \tag{2}$$

The corresponding action of SU(2) on $L^2(S^3)$ is (L^2 denotes the set of square integrable functions)

$$f \mapsto D(B) f, \tag{3a}$$

$$D(B) f|_p := f|_{B^{-1}p}, \tag{3b}$$

$$D(B'B) f \equiv D(B')D(B) f. \tag{3c}$$

Similarly, let $A \in SO(3) := SO(3,\mathbb{R})$ act on S^2 on the left via

$$x \mapsto Ax; \tag{4}$$

the corresponding action of SO(3) on $L^2(S^2)$ is

$$g \mapsto D(A) g, \tag{5a}$$

$$D(a) g|_x := g|_{A^{-1}x}. \tag{5b}$$

Consider

$$U(1) = \{H(\lambda): \lambda \in [0,2\pi)\} \subset SU(2], \tag{6a}$$

$$H(a)H(b) = H(a+b), \tag{6b}$$

$$H(\lambda) = 1 \Leftrightarrow \lambda = 0 \ (\mathrm{mod}\ 2\pi).$$

The Hopf bundle is defined to be the principal bundle

$$\begin{array}{l} U(1) \rightarrow S^3 \\ \qquad\quad \downarrow \pi \\ \qquad\quad S^2 \end{array}, \tag{7}$$

where the (right) action of U(1) on S^3 is given by

$$p \mapsto pe^{i\lambda} := T[(T^{-1}p)H(\lambda)]. \tag{8}$$

Thus, $\pi(pe^{i\lambda}) = \pi(p)$ and $\pi(B pe^{i\lambda}) = \pi(Bp)$. We therefore get an induced map

$$\begin{aligned} &\hat{\pi}: SU(2) \rightarrow SO(3), \\ &\hat{\pi}(B)\pi(p) \equiv \pi(Bp). \end{aligned} \tag{9}$$

We will *assume* that the U(1) subgroup of SU(2) in (6) has been chosen so that the Chern class of the Hopf bundle is $[R]$, where $R = -(i/2)\Omega$ and Ω is the volume form on S^2. Thus the Hopf bundle has Chern number

$$\oint_{S^2} \frac{iR}{2\pi} = +1 \tag{10}$$

instead of -1 (the only other possibility). [This can always be achieved by replacing $H(\lambda)$ by $H(-\lambda)$ if necessary.]

We wish to introduce a basis $\Lambda_a \in su(2)$, the Lie algebra of SU(2), which satisfies

$$[\Lambda_a, \Lambda_b] = \epsilon_{abc}\Lambda_c, \tag{11}$$

where the indices run from 1 to 3 and ϵ_{abc} is the totally antisymmetric tensor defined by $\epsilon_{123} = +1$. However, if we define

$$\beta_a(\tau) := \exp(\tau\Lambda_a) \in SU(2), \tag{12}$$

for Λ_a satisfying (11), then β_a is periodic in τ *with period* 4π.

We define

$$\Lambda_3 := \frac{d}{d\tau}\bigg|_{\tau=0} H\left(-\frac{\tau}{2}\right) \tag{13}$$

(the minus sign is conventional) and choose Λ_1 and Λ_2 so that (11) is satisfied; note that we now have

$$H(\lambda) \equiv \beta_3(-2\lambda). \tag{14}$$

Introduce angular momentum operators on S^3 via

$$\begin{aligned} &J_a: L^2(S^3) \rightarrow L^2(S^3), \\ &f \mapsto i \frac{d}{d\tau}\bigg|_{\tau=0} D(\beta_a(\tau)) f, \end{aligned} \tag{15a}$$

i.e.,

$$J_a f|_p = -i \frac{d}{d\tau}\bigg|_{\tau=0} f|_{\beta_a(\tau)p}. \tag{15b}$$

Then we have

$$[J_a, J_b] = i\epsilon_{abc}J_c. \tag{16}$$

Define

$$\begin{aligned} &J^2 := \sum_a J_a^2, \\ &J_\pm := J_1 \pm iJ_2. \end{aligned} \tag{17}$$

Then [compare (131) below] $J^2 \equiv -\frac{1}{4}\Box_3$, where $\Box_3$ is the standard Laplacian on S^3.

If we now define angular momentum operators on S^2 by

$$\begin{aligned} &\hat{J}_a: L^2(S^2) \rightarrow L^2(S^2), \\ &g \mapsto i \frac{d}{d\tau}\bigg|_{\tau=0} D(\alpha_a(\tau)) g, \end{aligned} \tag{18}$$

where $\alpha_a(\tau) := \hat{\pi}(\beta_a(\tau))$, then we have

$$\hat{J}_a g|_{\pi(p)} \equiv J_a(g \circ \pi)|_p \quad [g \in L^2(S^2)]. \tag{19}$$

Note that J_a and $\hat{J}_a$ are Hermitian operators.

The complex line bundles E_n associated with the Hopf bundle can be defined as follows. Let U(1) act on $\mathbb{C}$ via multiplication, i.e.,

$$\begin{aligned} &\mathbb{C} \rightarrow \mathbb{C}, \\ &z \mapsto e^{i\lambda} z. \end{aligned} \tag{20}$$

Define

$$\begin{aligned} E_n &:= S^3 \times_n \mathbb{C} \\ &= \{[(p,z)]\}, \end{aligned} \tag{21}$$

where the square brackets denote equivalence classes under the relation

$$(p,z) \sim (pe^{i\lambda}, e^{in\lambda}z), \tag{22}$$

E_n is a fiber bundle over S^3 with fiber $\mathbb{C}$,

$$\begin{array}{c} E_n \\ \downarrow \pi_n, \\ S^2 \end{array} \tag{23}$$

and there is a U(1) action on E_n given by

$$[(p,z)] \mapsto [(pe^{i\lambda}, z)]. \tag{24}$$

The projection π_n is given by

$$\pi_n([(p,z)]) := \pi(p). \tag{25}$$

There is a natural connection on E_n (Ref. 19) so that the curvature of E_n is [see (155)]

$$R_n = -(in/2)\Omega, \tag{26}$$

where Ω is the volume form on S^2, so that the Chern number of E_n is [compare (10)]

$$\oint_{S^2} \frac{iR_n}{2\pi} \equiv +n. \tag{27}$$

We will assume throughout the remainder of the paper that this connection has been chosen.[20]

There is an existence and uniqueness theorem which says that a line bundle over S^2 with curvature R exists if and only if

$$\oint \frac{iR}{2\pi} \in \mathbf{Z} \tag{28}$$

and that this bundle is unique up to strong bundle isomorphism.

Let

$$\hat{F}_n := \{f \in C^\infty(S^3) : f(pe^{i\lambda}) = e^{in\lambda} f(p)\}. \tag{29}$$

Given any $f \in \hat{F}_n$ we can obtain a C^∞ section σ_f of E_n by

$$\begin{aligned} \sigma_f &: S^2 \to E_n, \\ x &\mapsto [(p, f(p))], \end{aligned} \tag{30}$$

for any p such that $\pi(p) = x$. Denote by Q the map

$$Q(f) := \sigma_f. \tag{31}$$

Note that Q is one-to-one: A C^∞ section σ_f uniquely determines a function f on S^3 via (30) which much be in $\hat{F}_n$ in order to be well-defined. We write the inverse mapping as

$$f_\sigma := Q^{-1}(\sigma). \tag{32}$$

Given any smooth (C^∞) local section

$$\begin{aligned} &U_A \subset S^2, \\ &\hat{\gamma}_A : U_A \to S^3, \\ &\pi \circ \hat{\gamma}_A \equiv \mathbf{1} \end{aligned} \tag{33}$$

of the Hopf bundle we can interpret a section $\sigma \in \Gamma_n$ of E_n as a *function*

$$g_A^\sigma := f_\sigma \circ \hat{\gamma}_A \in L^2(U_A). \tag{34}$$

For $x \in U_A$ and sections σ, τ of E_n we define the scalar product

$$\begin{aligned} \langle \sigma, \tau \rangle_x &:= \overline{f_\sigma(\hat{\gamma}_A(x))} \cdot f_\tau(\hat{\gamma}_a(x)), \\ &\equiv \overline{g_A^\sigma} g_A^\tau |_x. \end{aligned} \tag{35}$$

Note that since f_σ and f_τ are *both* elements of $\hat{F}_n$ the norm is independent of the choice of local section γ_A so long as x is in the domain of definition of γ_A. We can now define

$$\langle \sigma, \tau \rangle := \oint_{S^2} \langle \sigma, \tau \rangle_x \, dx. \tag{36}$$

Note that Q is *not* an isometry but satisfies [compare (144)]

$$\langle f, f \rangle \equiv (\pi/2) \langle Q(f), Q(f) \rangle. \tag{37}$$

Thus, if we define $F_n := \hat{F}_n \cap L^2(S^3)$ and let $\Gamma_n := L^2(S^2 \to E_n)$ denote the set of square integrable sections of E_n, then Q clearly gives a one-to-one correspondence

$$Q : F_n \to \Gamma_n. \tag{38}$$

We can now introduce angular momentum operators $\hat{L}_a$ on Γ_n via

$$\hat{L}_a \sigma := Q(J_a(Q^{-1}\sigma)). \tag{39}$$

For the connection chosen above [compare (26)] the Laplacian on Γ_n is [see (157) below][17,19]

$$\Delta_n \equiv -\hat{L}^2 + n^2/4. \tag{40}$$

III. WIGNER $\mathscr{D}$ FUNCTIONS

Wigner[1] introduced the functions $\mathscr{D}^l_{qm}$ as the matrix elements of finite rotations acting on irreducible representations of the rotation group [SO(3)]. Our presentation is largely based on that of Edmonds.[2] Other standard references are Rose[3] and the more modern treatment given by Biedenharn and Louck.[4]

We can define an irreducible representation of SU(2) on S^3 for each l as follows.

Choose $\phi_{ll} \in L^2(S^3)$ with

$$J^2 \phi_{ll} = l(l+1)\phi_{ll}, \quad J_3 \phi_{ll} = l\phi_{ll}, \tag{41}$$

and define

$$\phi_{l,l-n} = [n(2l-n+1)]^{-1/2} J_- \phi_{l,l-n+1} \quad (n = 1,2,\ldots,2l). \tag{42}$$

Then

$$J_3 \phi_{lm} = m\phi_{lm}, \tag{43}$$

$$J_\pm \phi_{lm} = [(l \mp m)(l \pm m + 1)]^{1/2} \phi_{l,m\pm 1}, \tag{44}$$

and

$$\langle \phi_{lm}, \phi_{l'm'} \rangle = \langle \phi_{ll}, \phi_{ll} \rangle \delta_{ll'} \delta_{mm'}, \tag{45}$$

where $\langle \cdot , \cdot \rangle$ denotes the L^2 norm on S^3. Note that since $[J^2, J_-] = 0$, (42) implies

$$J^2 \phi_{lm} \equiv l(l+1)\phi_{lm}; \tag{46}$$

this also follows directly from (43) and (44). From (43)–(46) we see that there is a (matrix) representation of SU(2) on the vector spaces

$$W^l := \mathrm{Span}\{\phi_{lm} : 0 \leqslant l - |m| \in \mathbf{Z}\}, \tag{47}$$

for each l. The representation is irreducible because W^l is generated by the action of J_- on ϕ_{ll} [see (42)].

We can now define the Wigner $\mathscr{D}$ functions to be the matrix representation of SU(2) acting on W^l:

$$\begin{aligned} D(B) &: W^l \to W^l, \\ &\phi_{lm} \mapsto \sum_q \phi_{lq} \mathscr{D}^l_{qm}(B) \end{aligned} \tag{48}$$

with $D(B)$ as in (3). Note that this construction is independent of the choice of ϕ_{ll} satisfying (41).

Before deriving the properties of the Wigner $\mathscr{D}$ functions we first show that, for *integer* spin ($l \in \mathbf{Z}$), our definition is the same as the usual one in terms of spherical harmonics on S^2. We can introduce the usual spherical harmonics on S^2 via

$$\begin{aligned} &\hat{J}_3 Y_{lm} = m Y_{lm},\\ &\hat{J}_\pm Y_{lm} = [(l\mp m)(l\pm m+1)]^{1/2} Y_{l,m\pm 1},\\ &\langle Y_{lm}, Y_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}\\ &\quad (l\in \mathbf{Z},\quad 0 < l-|m| \in \mathbf{Z}) \end{aligned} \tag{49}$$

and a choice of phase for each l, where $\hat{J}_\pm := \hat{J}_1 \pm i\hat{J}_2$ and the norm is now the $\mathbf{L}^2$ norm on $\mathbf{S}^2$. Defining $Y^l_{0m} := Y_{lm}\circ \pi \in \mathbf{L}^2(\mathbf{S}^3)$ we see that the Y^l_{0m} satisfy (43)–(46) and thus

$$D(B)Y^l_{0m} \equiv \sum_q Y^l_{0q}\mathscr{D}^l_{qm}(B). \tag{50}$$

But

$$\begin{aligned} D(B)Y^l_{0m}|_p &= Y_{lm}\circ\pi|_{B^{-1}p}\\ &= Y_{lm}|_{\hat{\pi}(B^{-1})\pi(p)}\\ &= D(\hat{\pi}(B))Y_{lm}\circ\pi, \end{aligned}$$

and therefore

$$D(\hat{\pi}(B))Y_{lm} \equiv \sum_q Y_{lq}\mathscr{D}^l_{qm}(B), \tag{51}$$

which is equivalent to the standard definition of the functions $\mathscr{D}^l_{qm}$ in terms of SO(3).

We now establish the various properties of the $\mathscr{D}^l_{\rm qm}$. From (45) we have

$$\begin{aligned} &\langle\phi_{ll},\phi_{ll}\rangle\delta_{ll'}\delta_{mm'}\\ &\quad = \oint_{\mathbf{S}^3} \overline{\phi_{lm}(p)}\phi_{l'm'}(p)dS\\ &\quad \equiv \oint_{\mathbf{S}^3} \overline{\phi_{lm}(B^{-1}p)}\phi_{l'm'}(B^{-1}p)dS\\ &\quad \equiv \sum_{q,q'}\oint_{\mathbf{S}^3} \overline{\phi_{lq}(p)\mathscr{D}^l_{qm}(B)}\phi_{l'q'}(p)\mathscr{D}^{l'}_{q'm'}(B)dS\\ &\quad \equiv \sum_q \overline{\mathscr{D}^l_{qm}(B)}\mathscr{D}^{l'}_{qm'}(B)\langle\phi_{ll},\phi_{ll}\rangle \end{aligned}$$

and therefore

$$\sum_q \overline{\mathscr{D}^l_{qm}(B)}\mathscr{D}^l_{qm'}(B) \equiv \delta_{mm'}. \tag{52}$$

From (3c) we also have

$$\mathscr{D}^l_{qm}(B'B) = \sum_n \mathscr{D}^l_{qn}(B')\mathscr{D}^l_{nm}(B). \tag{53}$$

Setting $B' = B^{-1}$ and relabeling indices we get

$$\delta_{mm'} = \sum_q \mathscr{D}^l_{mq}(B^{-1})\mathscr{D}^l_{qm'}(B), \tag{54}$$

so that we finally obtain

$$\overline{\mathscr{D}^l_{qm}(B)} \equiv \mathscr{D}^l_{mq}(B^{-1}). \tag{55}$$

In other words, the matrix $(\mathscr{D}^l)_{qm} = \mathscr{D}^l_{qm}$ is unitary; $(\mathscr{D}^l)^{-1} = \overline{(\mathscr{D}^l)}^t$.

Define the operators L_a and K_a on $\mathbf{L}^2(\mathrm{SU}(2))$ via

$$L_a h|_B := -i\frac{d}{d\tau}\bigg|_{\tau=0} h|_{\beta_a(\tau)B}, \tag{56a}$$

$$K_a h|_B := +i\frac{d}{d\tau}\bigg|_{\tau=0} h|_{B\beta_a(\tau)}, \tag{56b}$$

where $h\in \mathbf{L}^2(\mathrm{SU}(2))$. We then have

$$\begin{aligned} \phi_{lq}(L_a\mathscr{D}^l_{qm}(B)) &= -i\frac{d}{d\tau}\bigg|_{\tau=0}\phi_{lq}\mathscr{D}^l_{qm}(\beta_a(\tau)B)\\ &= -i\frac{d}{d\tau}\bigg|_{\tau=0} D(\beta_a(\tau)B)\phi_{lq}\\ &\equiv -J_a[D(B)\phi_{lq}], \end{aligned} \tag{57}$$

whereas

$$\begin{aligned} \phi_{lq}(K_a\mathscr{D}^l_{qm}(B)) &= i\frac{d}{d\tau}\bigg|_{\tau=0}\phi_{lq}\mathscr{D}^l_{qm}(B\beta_a(\tau))\\ &= i\frac{d}{d\tau}\bigg|_{\tau=0} D(B\beta_a(\tau))\phi_{lq}\\ &\equiv D(B)[J_a\phi_{lq}]. \end{aligned} \tag{58}$$

Using (43) and (44) it is now easy to compute

$$\begin{aligned} L_3\mathscr{D}^l_{qm} &= -q\mathscr{D}^l_{qm},\\ L_\pm\mathscr{D}^l_{qm} &= -[(l\pm q)(l\mp q+1)]^{1/2}\mathscr{D}^l_{q\mp 1,m}; \end{aligned} \tag{59}$$

$$\begin{aligned} K_3\mathscr{D}^l_{qm} &= m\mathscr{D}^l_{qm},\\ \mathrm{K}_\pm\mathscr{D}^l_{qm} &= [(l\mp m)(l\pm m+1)]^{1/2}\mathscr{D}^l_{q,m\pm 1}; \end{aligned} \tag{60}$$

where $L_\pm := L_1 \pm iL_2$ and $K_\pm := K_1 \pm iK_2$. Note that *both* L_a and K_a satisfy the usual commutation relations, namely those satisfied by J_a, and that $L^2 \equiv K^2$.

From (59) and (60) we see that the $\mathscr{D}^l_{qm}$ are orthogonal, i.e.,

$$\langle\mathscr{D}^l_{qm},\mathscr{D}^{l'}_{q'm'}\rangle = \delta_{ll'}\delta_{qq'}\delta_{mm'}C_l, \tag{61}$$

where the norm is the $\mathbf{L}^2$ norm on SU(2) and where $C_l := \langle\mathscr{D}^l_{ll},\mathscr{D}^l_{ll}\rangle$. [Note that by (59) and (60) the normalization depends only on l.] But from (52) we have

$$\sum_q \langle\mathscr{D}^l_{qm},\mathscr{D}^l_{qm}\rangle \equiv 2\pi^2, \tag{62}$$

so that

$$C_l = 2\pi^2/(2l+1) \tag{63}$$

[which shows that $\mathscr{D}^l_{qm}\in \mathbf{L}^2(\mathrm{SU}(2))$].

Note that $\mathbf{L}^2(\mathbf{S}^3)$ is of course isomorphic to $\mathbf{L}^2(\mathrm{SU}(2))$ via

$$\begin{aligned} &T^*: \mathbf{L}^2(\mathbf{S}^3)\to \mathbf{L}^2(\mathrm{SU}(2)),\\ &\qquad f\mapsto f\circ T. \end{aligned} \tag{64}$$

We thus define

$$\begin{aligned} \hat{\mathscr{D}}^l_{qm} &:= T_*\mathscr{D}^l_{qm},\\ &\equiv \mathscr{D}^l_{qm}\circ T^{-1}\in \mathbf{L}^2(\mathbf{S}^3). \end{aligned} \tag{65}$$

Under this isomorphism we have

$$L_a(f\circ T) \equiv J_a f\circ T, \tag{66}$$

and we see that the matrix representation of SU(2) on the space spanned by the $\hat{\mathscr{D}}^l_{qm}$ for fixed l and m [given by (59)] is *not* the same as the matrix representation of SU(2) on W^l [given by (43) and (44)]. We can fix this by defining

$$\mathscr{Y}^l_{qm}(B) := \sqrt{(2l+1)/4\pi}\ \mathscr{D}^l_{-q,m}(B^{-1}), \\ \equiv \sqrt{(2l+1)/4\pi}\ \overline{\mathscr{D}^l_{m,-q}(B)}, \tag{67}$$

and

$$\hat{\mathscr{Y}}^l_{qm} := \mathscr{Y}^l_{qm} \circ T^{-1} \in \mathbf{L}^2(\mathbf{S}^3), \tag{68}$$

where the factor $\sqrt{(2l+1)/4\pi}$ has been added for convenience.

Note that

$$\begin{aligned} L_a \mathscr{Y}^l_{qm}|_B &= -i \frac{d}{d\tau}\bigg|_{\tau=0} \mathscr{Y}^l_{qm}|_{\beta_a(\tau)B} \\ &= -i\sqrt{\frac{2l+1}{4\pi}} \frac{d}{d\tau}\bigg|_{\tau=0} \mathscr{D}^l_{-qm}|_{[\beta_a(\tau)B]^{-1}} \\ &= +i\sqrt{\frac{2l+1}{4\pi}} \frac{d}{d\tau}\bigg|_{\tau=0} \mathscr{D}^l_{-qm}|_{B^{-1}\beta_a(\tau)}, \end{aligned}$$

so that

$$L_a \mathscr{Y}^l_{qm}|_B \equiv \sqrt{(2l+1)/4\pi}\ K_a \mathscr{D}^l_{-qm}|_{B^{-1}} \tag{69}$$

and similarly

$$K_a \mathscr{Y}^l_{qm}|_B \equiv \sqrt{(2l+1)/4\pi}\ L_a \mathscr{D}^l_{-qm}|_{B^{-1}}. \tag{70}$$

Thus

$$L_3 \mathscr{Y}^l_{qm} = m\mathscr{Y}^l_{qm}, \\ L_\pm \mathscr{Y}^l_{qm} = [(l\mp m)(l\pm m+1)]^{1/2}\mathscr{Y}^l_{q,m\pm 1}; \tag{71}$$

$$K_3 \mathscr{Y}^l_{qm} = +q\mathscr{Y}^l_{qm}, \\ K_\pm \mathscr{Y}^l_{qm} = -[(l\mp q)(l\pm q+1)]^{1/2}\mathscr{Y}^l_{q\pm 1,m}, \tag{72}$$

and

$$\langle \mathscr{Y}^l_{qm}, \mathscr{Y}^{l'}_{q'm'}\rangle = (\pi/2)\delta_{ll'}\delta_{qq'}\delta_{mm'}. \tag{73}$$

The matrix representation of SU(2) on the spaces

$$W^l_q := \mathrm{span}\{\hat{\mathscr{Y}}_{qm} : 0 < l - |m| \in \mathbf{Z}\}, \tag{74}$$

for each q ($0 < l - |q| \in \mathbf{Z}$) is now precisely the same as on W^l and there are $2l+1$ spaces W^l_q for each l. Using the Peter–Weyl theorem[21] we conclude that

$$\mathbf{L}^2(\mathbf{S}^3) \equiv \bigoplus_{l,q} W^l_q, \tag{75}$$

and that $\{\sqrt{2/\pi}\,\hat{\mathscr{Y}}^l_{qm}\}$ therefore forms an *orthonormal* basis for $\mathbf{L}^2(\mathbf{S}^3)$. The Wigner $\mathscr{D}$ functions $\{\hat{\mathscr{D}}^l_{qm}\}$ thus form an *orthogonal* basis for $\mathbf{L}^2(\mathbf{S}^3)$.

We now derive a property of the $\mathscr{Y}^l_{qm}$ that will be crucial in what follows. Note that

$$L_a \mathscr{Y}^l_{qm}|_{BH(\lambda)} \equiv L_a(\mathscr{Y}^l_{qm}(BH(\lambda))), \tag{76}$$

but that

$$K_a \mathscr{Y}^l_{qm}|_{BH(\lambda)} = i\frac{d}{d\tau}\bigg|_{\tau=0} \mathscr{Y}^l_{qm}|_{BH(\lambda)\beta_a(\tau)}, \tag{77}$$

which, in general, is *not* equal to

$$K_a(\mathscr{Y}^l_{qm}(BH(\lambda))) = i\frac{d}{d\tau}\bigg|_{\tau=0} \mathscr{Y}^l_{qm}|_{B\beta_a(\lambda)H(\tau)}. \tag{78}$$

However, since $[\beta_3(\tau), H(\lambda)] = 0$ we see that we *do* have

$$K_3 \mathscr{Y}^l_{qm}|_{BH(\lambda)} \equiv K_3(\mathscr{Y}^l_{qm}(BH(\lambda))). \tag{79}$$

But since the $\mathscr{Y}^l_{qm}$ are fully determined by their eigenvalues with respect to L^2, L_3, and K_3 and since they form a basis for $\mathbf{L}^2(\mathrm{SU}(2))$, we conclude that

$$\mathscr{Y}^l_{qm}(BH(\lambda)) \equiv c(\lambda)\mathscr{Y}^l_{qm}(B), \tag{80}$$

where c may depend on (l,q,m). Finally, using (14) we have

$$\begin{aligned} q\mathscr{Y}^l_{qm}|_B &= K_3\mathscr{Y}^l_{qm}|_B \\ &= i\frac{d}{d\tau}\bigg|_{\tau=0} \mathscr{Y}^l_{qm}|_{B\beta_3(\lambda)} \\ &= -\frac{i}{2}\frac{d}{d\lambda}\bigg|_{\lambda=0} \mathscr{Y}^l_{qm}|_{BH(\lambda)}. \end{aligned}$$

Thus

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} c(\lambda) = 2iq, \tag{81}$$

which, together with $c(0) = 1$ and $c(a+b) = c(a)c(b)$ [which follows from (6b)], implies $c(\lambda) = e^{2iq\lambda}$, so that

$$\mathscr{Y}^l_{qm}(BH(\lambda)) \equiv e^{2iq\lambda}\mathscr{Y}^l_{qm}(B) \tag{82}$$

and therefore

$$\hat{\mathscr{Y}}^l_{qm}(pe^{i\lambda}) \equiv e^{2iq\lambda}\hat{\mathscr{Y}}^l_{qm}(p), \tag{83}$$

i.e., $\hat{\mathscr{Y}}^l_{qm} \in F_{2q}$. In fact $\{\sqrt{2/\pi}\,\mathscr{Y}^l_{qm}\}$ for fixed q clearly forms an orthonormal basis for F_{2q}. We can now write (77) as (79) together with

$$K_\pm \mathscr{Y}^l_{qm}|_{BH(\lambda)} \equiv e^{\pm 2i\lambda}K_\pm(\mathscr{Y}^l_{qm}(BH(\lambda))). \tag{84}$$

IV. SPIN-WEIGHTED SPHERICAL HARMONICS

Newman and Penrose[5] introduced spin weighted spherical harmonics in a particular choice of spin gauge. The (trivial) generalization to an arbitrary spin gauge (for a particular choice of coordinates) can be found in Dray.[11] (A spinorial definition has been given by Penrose and Rindler.[7] See also Refs. 22 and 23.) Consider the complexified tangent bundle

$$\begin{array}{l} T_{\mathbf{C}}\mathbf{S}^2 := T\mathbf{S}^2 \otimes \mathbf{C} \\ \big\downarrow \\ \mathbf{S}^2 \end{array}, \tag{85}$$

and let m be a (complex) vector field on $\mathbf{S}^2$, i.e., a *local* section of the tangent bundle

$$m : U \to T_{\mathbf{C}}\mathbf{S}^2 \quad (U \subset \mathbf{S}^2), \tag{86}$$

which satisfies

$$\langle m,m\rangle = 0, \quad \langle m,\bar{m}\rangle = 2, \tag{87}$$

at each point of U.

A quantity Q is said to have[5] *spin weight* s [we write $\mathrm{sw}(Q) = s$] if under the transformation

$$m \to e^{i\kappa}m, \tag{88a}$$

Q transforms according to

$$Q \mapsto e^{is\kappa}Q. \tag{88b}$$

What does this mean? We interpret this imprecise definition as follows.

Consider the space V consisting of all elements v of $T_{\mathbf{C}}\mathbf{S}^2$ satisfying (87). There is a natural decomposition

$$V=V_0\cup\overline{V}_0, \tag{89a}$$

where for *any* $v_0\in V_0$ we have

$$V_0=\{e^{i\chi}v_0;\quad \chi\colon S^2\to[0,2\pi)\}. \tag{89b}$$

Note that V_0 is a subbundle of the tangent bundle (85), i.e.,

$$\begin{array}{c} V_0 \\ \big\downarrow \tilde{\pi} \\ S^2 \end{array}. \tag{90}$$

Furthermore, since we have a natural U(1) action on V_0, namely

$$V_0\to V_0,\quad v\mapsto e^{i\lambda}v, \tag{91a}$$

and since

$$\tilde{\pi}(e^{i\lambda}v)\equiv\tilde{\pi}(v), \tag{91b}$$

V_0 is clearly a circle bundle over S^2. But since $V_0\otimes\mathbb{R}$ is equivalent to the *real* tangent bundle TS^2 and since TS^2 is bundle isomorphic to E_1, there is a fiber-preserving isomorphism[24]

$$\eta\colon V_0\overset{\approx}{\longrightarrow}S^3,\quad \eta(e^{i\lambda}v)\equiv\eta(v)e^{i\lambda}. \tag{92}$$

We therefore interpret "quantities of spin weight s" to be functions on V_0, i.e., elements of $L^2(S^3)$, with a particular behavior under the circle action. We must, however, be extremely careful here: The vector field m in the usual definition of spin-weighted spherical harmonics has a definite behavior under the circle action, so we are *not* free to specify this independently. We claim that the correct choice is to require m to behave in the same way as K_+ under the circle action, namely [compare (84)]

$$m\mapsto e^{+2i\lambda}m \tag{93}$$

[so that $\kappa=2\lambda$ in (88)]. We will see below [compare (177c)] that this correctly reproduces the standard definition in coordinates.

We thus define a "quantity of spin weight s" to be a function

$$\tilde{f}\colon V\to\mathbb{C} \tag{94a}$$

satisfying

$$\tilde{f}(e^{i\lambda}v)\equiv e^{+2is\lambda}\tilde{f}(v). \tag{94b}$$

We can turn $\tilde{f}$ into a function $f:=\tilde{f}\circ\eta^{-1}$ on S^3, and we therefore define *spin-weighted functions on* S^3 to be elements f of $L^2(S^3)$ satisfying

$$f(pe^{i\lambda})\equiv e^{+2is\lambda}f(p), \tag{95}$$

i.e., $f\in F_{2s}$ for some s, and define s to be the *spin weight* of $f(\mathrm{sw}(f)=s)$. But note that from (83) we have

$$\mathrm{sw}(\hat{\mathscr{Y}}^l_{qm})\equiv q, \tag{96}$$

so that $\{\hat{\mathscr{Y}}^l_{qm}\}$ for fixed q forms a basis for the functions of spin weight q for $2q\in\mathbb{Z}$. We call the $\hat{\mathscr{Y}}^l_{qm}$ *spin-weighted spherical harmonics on* S^3.

Note further that, for *integer* spin ($l\in\mathbb{Z}$), (72) implies

$$\mathscr{Y}^l_{qm}=\begin{cases}\left[\dfrac{(l-q)!}{(l+q)!}\right]^{1/2}(-1)^q(K_+)^q\mathscr{Y}^l_{0m} & (0<q<l),\\[2ex] \left[\dfrac{(l+q)!}{(l-q)!}\right]^{1/2}(-1)^q(K_-)^{|q|}\mathscr{Y}^l_{0m} & (-l<q<0),\\[2ex] 0 & (l<|q|),\end{cases} \tag{97}$$

which is equivalent to the standard definition of spin-weighted spherical harmonics for integer spin.

V. MONOPOLE HARMONICS

Greub and Petry[10] were the first to introduce the idea of using a Hilbert space of sections of complex line bundles to obtain a description of the Dirac magnetic monopole which is free of string singularities. Wu and Yang[8] independently discovered the same idea and gave an orthonormal basis for this Hilbert space (with respect to a particular trivialization of the bundles) which they called *monopole harmonics*. (A combined treatment of these two approaches can be found in Biedenharn and Louck.[9])

Consider the electromagnetic field $\hat{F}$ of a Dirac magnetic monopole of strength g located at the origin. The field $\hat{F}$ is a spherically symmetric, time-independent two-form over $\mathbb{R}^4$ so it is sufficient to consider the pullback F of $\hat{F}$ to S^2. Then we have

$$F=g\Omega, \tag{98}$$

where Ω is the volume form on S^2, i.e.,

$$\oint_{S^2}\Omega=4\pi. \tag{99}$$

Maxwell's equations imply that F is closed, i.e., $dF=0$, but we do *not* assume that F is exact, i.e., we do not assume that there exists a *globally* defined vector potential A satisfying $dA=F$.

The Schrödinger equation for a particle with electric charge e and mass m moving in this field can be written

$$i\,\partial_t\psi=-\frac{1}{2m}\left(\partial_r^2+\frac{2}{r}\partial_r+\frac{\Delta}{r^2}\right)\psi, \tag{100}$$

which we interpret as follows. Make the ansatz

$$\psi=e^{-iEt}\rho(r)\sigma,$$

where σ is a section of the line bundle over S^2 with curvature $R=-ieF$ (see Ref. 25). Then Δ represents the natural Laplacian [compare (26) and (40)] on this bundle. The Schrödinger equation now becomes

$$r^2(2mE\rho+\rho''+2\rho'/r)\sigma=-\rho(\Delta\sigma), \tag{101}$$

so that the angular part of the wave function, σ, must be an eigensection of Δ.

For the line bundle with curvature $R=-ieF$ to exist we must have [see (28)]

$$\oint\frac{eF}{2\pi}\in\mathbb{Z}, \tag{102}$$

which is just the Dirac quantization condition

$$2eg\in\mathbb{Z}; \tag{103}$$

σ is thus a section of E_{2q} for $q=eg$. We thus have [see (40)]

$$\Delta\equiv\Delta_{2q}\equiv-\hat{L}^2+q^2 \tag{104}$$

so that eigensections of Δ are also eigensections of $\hat{L}^2$. We are only interested in sections σ that are square integrable, so that $\sigma \in \Gamma_{2q}$, and thus $Q^{-1}\sigma \in F_{2q}$. But we have seen that $\{\hat{\mathscr{Y}}^l_{qm}\}$ forms a basis for F_{2q}. We are thus led to define the *monopole harmonics*

$$\mathscr{Y}_{qlm} := Q(\hat{\mathscr{Y}}^l_{qm}) \in \Gamma_{2q}. \tag{105}$$

The monopole harmonics can also be defined intrinsically by the conditions

$$\begin{aligned} &\mathscr{Y}_{qlm} \in \Gamma_{2q},\\ &\hat{L}^2\mathscr{Y}_{qlm} = l(l+1)\mathscr{Y}_{qlm},\\ &\hat{L}_3\mathscr{Y}_{qlm} = m\mathscr{Y}_{qlm},\\ &\langle \mathscr{Y}_{qlm}, \mathscr{Y}_{q'l'm'}\rangle = \delta_{ll'}\delta_{qq'}\delta_{mm'}; \end{aligned} \tag{106}$$

these follow from the definition (105) together with (71), (73), and (37).

We thus see that the monopole harmonics are completely equivalent to the spin-weighted spherical harmonics on S^3, where the equivalence is given by the mapping Q.

VI. COORDINATE NOTATION

In this section we introduce a particular coordinatization of SU(2) [and thus also of SO(3) and S^3] in terms of Euler angles. We then introduce the complex line bundles over S^2 and angular momentum operators in terms of these coordinates.

The group $SU(2) := SU(2,\mathbb{C})$ can be defined as the set of 2×2 complex matrices satisfying

$$B^{-1} = \overline{B^t}, \quad \det B = 1, \tag{107a}$$

or equivalently

$$B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \quad (a\bar{a} + b\bar{b} = 1). \tag{107b}$$

Similarly, $SO(3) := SO(3,\mathbb{R})$ can be defined as the set of 3×3 real matrices satisfying

$$A^{-1} = A^t, \quad \det A = 1. \tag{108}$$

We choose the parametrization for SU(2) given by

$$\begin{aligned} a &= \cos(\beta/2)\, e^{-i(\gamma+\alpha)/2},\\ b &= \sin(\beta/2)\, e^{+i(\gamma-\alpha)/2}, \end{aligned} \tag{109a}$$

where

$$\beta \in [0,\pi], \quad \alpha \in [0,2\pi), \quad \gamma \in [0,4\pi); \tag{109b}$$

we write $B(\alpha,\beta,\gamma)$ for the matrix so determined.[26]

We consider S^3 to be the subspace of $\mathbb{C}^2$ defined by

$$S^3 := \left\{\begin{pmatrix} u \\ v\end{pmatrix} \in \mathbb{C}^2 : u\bar{u} + v\bar{v} = 1\right\}. \tag{110}$$

We choose the parametrization

$$\begin{aligned} u &= \cos(\theta/2) e^{-i(\psi+\phi)/2},\\ v &= -\sin(\theta/2) e^{-i(\psi-\phi)/2}, \end{aligned} \tag{111a}$$

where

$$\theta \in [0,\pi], \quad \phi \in [0,2\pi), \quad \psi \in [0,4\pi); \tag{111b}$$

we write (θ,ϕ,ψ) for the point so determined.[26] The isomorphism between SU(2) and S^3 can be given as

$$\begin{aligned} T:\ & SU(2) \xrightarrow{\approx} S^3,\\ & B(\alpha,\beta,\gamma) \mapsto (\beta,\alpha,\gamma), \end{aligned} \tag{112a}$$

or equivalently

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \mapsto \begin{pmatrix} a \\ -\bar{b}\end{pmatrix}. \tag{112b}$$

The metric on S^3 is now

$$\begin{aligned} ds^2 &:= du\, d\bar{u} + dv\, d\bar{v}\\ &\equiv \tfrac{1}{4}(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta\, d\phi\, d\psi), \end{aligned} \tag{113}$$

so that the Laplacian on S^3 is given by

$$\begin{aligned} \Box_3 = 4\Big(&\partial^2_\theta + \cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial^2_\phi + \frac{1}{\sin^2\theta}\partial^2_\psi\\ &- \frac{2\cos\theta}{\sin^2\theta}\partial_\psi\,\partial_\phi\Big). \end{aligned} \tag{114}$$

We consider S^2 to be the subspace of $\mathbb{R}^3$ defined by

$$S^2 := \left\{\begin{pmatrix} x \\ y \\ z\end{pmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\} \tag{115}$$

and we choose the usual parametrization in terms of spherical coordinates

$$x + iy = e^{i\phi}\sin\theta, \quad z = \cos\theta, \tag{116}$$

where θ and ϕ have the same ranges as in (111b). Consider the elements $H(\lambda)$ of SU(2) defined by

$$H(\lambda) := \begin{pmatrix} e^{+i\lambda} & 0 \\ 0 & e^{-i\lambda}\end{pmatrix} \equiv B(0,0,-2\lambda). \tag{117}$$

Then $\{H(\lambda) : \lambda \in [0,2\pi)\}$ is isomorphic to U(1) so we can define the Hopf bundle via

$$\begin{array}{ccc} U(1) \to & S^3 & \\ & \downarrow \pi, & \\ & S^2 & \end{array} \tag{118a}$$

where the projection is the obvious map

$$\pi(\theta,\phi,\psi) := (\theta,\phi) \tag{118b}$$

or equivalently

$$\begin{pmatrix} u \\ v\end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Re}(-2\bar{u}v) \\ \mathrm{Im}(-2\bar{u}v) \\ u\bar{u} - v\bar{v}\end{pmatrix} \tag{118c}$$

and the circle action is defined by

$$\begin{aligned} (\theta,\phi,\psi)e^{i\lambda} &:= T(B(\theta,\phi,\psi)H(\lambda))\\ &\equiv (\theta,\phi,\psi - 2\lambda). \end{aligned} \tag{118d}$$

From (118c) we see that we get an induced map

$$\hat{\pi} : SU(2) \to SO(3) \tag{119a}$$

which can be defined by

$$\hat{\pi}(B)\pi\left(\begin{pmatrix} p \\ q\end{pmatrix}\right) \equiv \pi\left(B\begin{pmatrix} p \\ q\end{pmatrix}\right). \tag{119b}$$

Direct calculation shows that under this map we have

$$\hat{\pi}:\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}\mapsto\begin{pmatrix} \mathrm{Re}(a^2-b^2) & \mathrm{Im}(a^2+b^2) & \mathrm{Re}(2ab)\\ -\mathrm{Im}(a^2-b^2) & \mathrm{Re}(a^2+b^2) & -\mathrm{Im}(2ab)\\ -\mathrm{Re}(2\bar{a}b) & \mathrm{Im}(2\bar{a}b) & a\bar{a}-b\bar{b}\end{pmatrix}, \tag{119c}$$

so that

$$A(\alpha,\beta,\gamma):=\hat{\pi}(B(\alpha,\beta.\gamma))$$

$$\equiv\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta\\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta\\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta\end{pmatrix}. \tag{119d}$$

Noting that $A(\alpha,\beta,\gamma)\equiv A(\alpha,\beta,\gamma+2\pi)$ we can choose the parametrization of SO(3) to be given by (119d), where[26]

$$\beta\in[0,\pi],\quad \alpha\in[0,2\pi),\quad \gamma\in[0,2\pi). \tag{119e}$$

Define $\alpha_a(\tau)\in\mathrm{SO}(3)$, $a=1,2,3$, to be the matrix which rotates S^2 about the ath axis counterclockwise through an angle τ. Thus

$$\alpha_1(\tau)=\begin{pmatrix}1&0&0\\0&\cos\tau&-\sin\tau\\0&\sin\tau&\cos\tau\end{pmatrix}\equiv A\left(\frac{3\pi}{2},\tau,\frac{\tau}{2}\right),$$
$$\alpha_2(\tau)=\begin{pmatrix}\cos\tau&0&\sin\tau\\0&1&0\\-\sin\tau&0&\cos\tau\end{pmatrix}\equiv A(0,\tau,0), \tag{120}$$
$$\alpha_3(\tau)=\begin{pmatrix}\cos\tau&-\sin\tau&0\\\sin\tau&\cos\tau&0\\0&0&1\end{pmatrix}\equiv A(0,0,\tau).$$

Note that

$$A(\alpha,\beta,\gamma)\equiv\alpha_3(\alpha)\alpha_2(\beta)\alpha_3(\gamma), \tag{121}$$

so that the parametrization (119d) of SO(3) is just the usual one in terms of Euler angles: $A(\alpha,\beta,\gamma)$ is the element of SO(3) that rotates the sphere S^2 first by γ around the z axis, then by β around the (original) y axis, and finally by α around the (original) z axis. We wish to find matrices $\beta_a\in\mathrm{SU}(2)$, $a=1,2,3$, satisfying

$$\hat{\pi}(\beta_a(\tau))\equiv\alpha_a(\tau). \tag{122a}$$

Although $\hat{\pi}$ is a *two*-to-one mapping, the additional requirement that

$$\beta_a(0)\equiv I \tag{122b}$$

determines the $\beta_a(\tau)$ uniquely as

$$\beta_1(\tau)=B\left(\frac{3\pi}{2},\tau,\frac{5\pi}{2}\right)\equiv\begin{pmatrix}\cos(\tau/2)&i\sin(\tau/2)\\i\sin(\tau/2)&\cos(\tau/2)\end{pmatrix},$$
$$\beta_2(\tau)=B(0,\tau,0)\equiv\begin{pmatrix}\cos(\tau/2)&\sin(\tau/2\\-\sin(\tau/2)&\cos(\tau/2)\end{pmatrix}, \tag{123}$$
$$\beta_3(\tau)=B(0,0,\tau)\equiv\begin{pmatrix}e^{-i\tau/2}&0\\0&e^{+i\tau/2}\end{pmatrix}.$$

From (109) we see that the general element of SU(2) can be written

$$B(\alpha,\beta,\gamma)\equiv\beta_3(\alpha)\beta_2(\beta)\beta_3(\gamma). \tag{124}$$

Note that (121) and (124) imply

$$A(\alpha,\beta,\gamma)^{-1}\equiv A(-\gamma,-\beta,-\alpha),$$
$$B(\alpha,\beta,\gamma)^{-1}\equiv B(-\gamma,-\beta,-\alpha). \tag{125}$$

Define

$$\Lambda_a:=\left.\frac{d}{d\tau}\right|_{\tau=0}\beta_a(\tau), \tag{126a}$$

$$\hat{\Lambda}_a:=\left.\frac{d}{d\tau}\right|_{\tau=0}\alpha_a(\tau) \tag{126b}$$

and notice that both Λ_a and $\hat{\Lambda}_a$ satisfy the commutation relations (11).[27] Note that (13) and (14) are also satisfied.

We now intoduce angular mometum operators. Using the chain rule the definition (18) of angular momentum on S^2 is equivalent to

$$\hat{J}_a g=+i(x\quad y\quad z)\hat{\Lambda}_a\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}g, \tag{127}$$

which yields the familiar result

$$\hat{J}_3=-i\,\partial_\phi,$$
$$\hat{J}_\pm=e^{\pm i\phi}(\pm\partial_\theta+i\cot\theta\,\partial_\phi). \tag{128}$$

Similarly, the definition (15) of angular momentum on S^3 is equivalent to

$$J_a f=+i(u\quad v)\bar{\Lambda}_a\begin{pmatrix}\partial_u\\\partial_v\end{pmatrix}f+i(\bar{u}\quad\bar{v})\Lambda_a\begin{pmatrix}\partial_{\bar{u}}\\\partial_{\bar{v}}\end{pmatrix}f, \tag{129}$$

which yields

$$J_3=-i\,\partial_\phi,$$
$$J_\pm=e^{\pm i\phi}(\pm\partial_\theta+i\cot\theta\,\partial_\phi-(i/\sin\theta)\partial_\psi) \tag{130}$$

so that

$$J^2:=J_3^2-J_3+J_+J_-\equiv-\tfrac{1}{4}\Box_3. \tag{131}$$

Both of these operators satisfy the standard commutation relations (16), e.g.,

$$[J_3,J_\pm]=\pm J_\pm,\quad [J_+,J_-]=2J_3 \tag{132}$$

(all others zero).

We now introduce the complex line bundles E_n over S^2 associated with the Hopf bundle (118). The points of E_n are equivalence classes

$$[(\theta,\phi,\psi;z)]\in S^3\times_n\mathbb{C} \tag{133a}$$

under the equivalence relation

$$(\theta,\phi,\psi;z) \sim (\theta,\phi,\psi - 2\lambda;e^{in\lambda}z). \tag{133b}$$

We thus obtain the line bundle

$$\begin{array}{c} E_n \\ \big\downarrow \pi_n \,, \\ S^2 \end{array} \tag{134a}$$

where the projection π_n is given by

$$\pi_n([(\theta,\phi,\psi;z)]) := (\theta,\phi). \tag{134b}$$

There is a one-to-one correspondence Q between (smooth) functions on S^3 satisfying

$$f(\theta,\phi,\psi) \equiv e^{-in\psi/2}F(\theta,\phi) \tag{135}$$

and (smooth) sections

$$\sigma: S^2 \to E_n \tag{136}$$

of the line bundles E_n, which is given by

$$Q(f)(\theta,\phi) := [(\theta,\gamma,\psi;f(\theta,\phi,\psi))]. \tag{137}$$

We will use the notation

$$Q(f) =: \sigma_f, \quad Q^{-1}(\sigma) =: f_\sigma. \tag{138}$$

Given any (smooth) local section[28]

$$\begin{aligned} &U_A \subset S^2, \\ &\hat{\gamma}_A : U_A \to S^3, \\ &\quad (\theta,\phi) \mapsto (\theta,\phi,\gamma_A(\theta,\phi)) \end{aligned} \tag{139}$$

of the Hopf bundle we can interpret any section (136) of E_n as a *function* on U_A via

$$\begin{aligned} g_A^\sigma &:= f_\sigma \circ \hat{\gamma}_A \\ &\equiv e^{-i(n/2)\gamma_A}F_\sigma, \end{aligned} \tag{140}$$

where F_σ is defined from f_σ as in (135). There is thus a one-to-one correspondence Q_A between (smooth) functions on U_A and (smooth) sections of E_n (restricted to U_A), which is given by

$$Q_A^{-1}(\sigma) := Q^{-1}(\sigma) \circ \hat{\gamma}_A \equiv g_A^\sigma, \tag{141}$$

i.e.,

$$Q_A(F) = [(\theta,\phi,\psi;e^{-i(n/2)(\psi-\gamma_A)}F)]. \tag{142}$$

There is a natural norm on the space of sections (136) of E_n given by

$$\langle \sigma,\tau \rangle := \oint_{S^2} \overline{F}_\sigma F_\tau \, dS. \tag{143}$$

Note that for σ,τ both sections of E_n we have [compare (37)]

$$\begin{aligned} \langle f_\sigma, f_\tau \rangle &= \int_{S^3} \bar{f}_\sigma f_\tau \, dS \\ &\equiv \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{\psi=0}^{4\pi} \overline{F}_\sigma F_\tau \frac{1}{8} \sin\theta \, d\theta \, d\phi \, d\psi \\ &\equiv (\pi/2)\langle \sigma,\tau \rangle. \end{aligned} \tag{144}$$

Given any operator Z on the space Γ_n of square-integrable sections of E_n we can obtain an operator on $L^2(U_A)$ defined by

$$Z^A := Q_A^{-1} \circ Z \circ Q_A \tag{145}$$

and this is clearly a one-to-one mapping of the corresponding operator spaces. Here Z^A will be referred to as the operator a *with respect to* (the section of the Hopf bundle) $\hat{\gamma}_A$.

We can introduce angular momentum operators $\hat{L}_a$ on Γ_n via

$$\hat{L}_a \sigma := Q(J_a(Q^{-1}\sigma)) \tag{146}$$

and thus obtain the operators[29]

$$\begin{aligned} L_3^A &= -i\partial_\phi + (n/2)\,\partial_\phi \gamma_A, \\ L_\pm^A &= e^{\pm i\phi}(\pm\partial_\theta + i\cot\theta\,\partial_\phi \\ &\qquad - (n/2)((1/\sin\theta) \mp i\,\partial_\theta\gamma_A + \cot\theta\,\partial_\phi\gamma_A)), \end{aligned} \tag{147}$$

$$(L^2)^A = -\square_2^A + \frac{n\cos\theta}{\sin^2\theta}L_3^A + \frac{n^2}{4\sin^2\theta},$$

on $L^2(U_A)$, where

$$\begin{aligned} \square_2^A &:= \square_2 + i(n/2)(\square_2\gamma_A) \\ &\quad + in[(\partial_\theta\gamma_A)\,\partial_\theta + (\partial_\phi\gamma_A)\partial_\phi/\sin^2\theta] \\ &\quad - (n^2/4)[(\partial_\theta\gamma_A)^2 + (\partial_\phi\gamma_A)^2/\sin^2\theta] \end{aligned} \tag{148}$$

is the operator obtained from $\square_2$ by the substitutions

$$\partial_\theta \mapsto \partial_\theta + i(n/2)(\partial_\theta\gamma_A), \tag{149}$$

$$\partial_\phi \mapsto \partial_\phi + i(n/2)(\partial_\phi\gamma_A), \tag{150}$$

where $\square_2$ denotes the Laplacian on S^2.

The natural connection on E_n is given by[10,17]

$$\tilde{d}\sigma \equiv Q(e^{-i(n/2)\psi}\,dF_\sigma + i(n/2)f_\sigma\cos\theta\,d\phi), \tag{151}$$

where d denotes the exterior derivative on S^2. The connection one-form ω_n^A of the bundle E_n with respect to $\hat{\gamma}_A$ is defined by

$$\tilde{d}[(\theta,\phi,\gamma_A;1)] =: [(\theta,\phi,\gamma_A;\omega_n^A]. \tag{152}$$

But

$$Q^{-1}([(\theta,\phi,\gamma_A;1)]) \equiv e^{-i(n/2)(\gamma_A-\psi)} \tag{153}$$

so that

$$\omega_n^A \equiv i(n/2)(\cos\theta\,d\phi + d\gamma_A). \tag{154}$$

The curvature E_n is thus

$$\begin{aligned} R_n &= d\omega_n^A \\ &\equiv -i(n/2)\sin\theta\,d\theta\wedge d\phi \\ &\equiv -i(n/2)\Omega \end{aligned} \tag{155}$$

as desired [compare (26)].

The Laplacian Δ_n on E_n associated with the connection (151) can now be defined as follows:

$$\Delta_n := Q_A \circ \Delta_n^A \circ Q_A^{-1}, \tag{156a}$$

where[17]

$$\begin{aligned} \Delta_n^A &:= \square_2 + 2g^{ab}(\omega_n^A)_a\nabla_b + g^{ab}(\nabla_a(\omega_n^A)_b \\ &\qquad + (\omega_n^A)_a(\omega_n^A)_b), \end{aligned} \tag{156b}$$

where g_{ab} is the standard metric on S^2 and ∇_a denotes covariant differentiation on S^2 (so that $\square_2 \equiv g^{ab}\nabla_a\nabla_b$). Inserting our choice (154) for the connection ω_n^A in (156) we obtain

$$\Delta_n^A = +\Box_2^A - \frac{n\cos\theta}{\sin^2\theta} L_3^A - \frac{n^2}{4}\frac{\cos^2\theta}{\sin^2\theta}, \tag{157a}$$

so that

$$\Delta_n \equiv -\hat{L}^2 + n^2/4. \tag{157b}$$

VII. WIGNER $\mathscr{D}$ FUNCTIONS

We now introduce the Wigner $\mathscr{D}$ functions[1-4] as the matrix elements of finite rotations acting on irreducible representations of the group SU(2).

We introduce irreducible representations of SU(2) on S^3 as follows. For $\binom{u}{v} \in S^3$ define

$$\phi_{lm} := \frac{u^{l-m} v^{l+m}}{\sqrt{(l-m)!(l+m)!}} \quad (0 \leqslant 2l \in \mathbf{Z},\ 0 \leqslant l - |m| \in \mathbf{Z}). \tag{158}$$

It is easy to check that $\phi_{lm} \in L^2(S^3)$. The operators J_a [(129) and (130)] when acting on ϕ_{lm} take the form

$$J_3 = \tfrac{1}{2}(v\,\partial_v - u\,\partial_u), \quad J_+ = v\,\partial_u, \quad J_- = u\,\partial_v, \tag{159}$$

and direct calculation shows that Eqs. (43)–(46) are satisfied so that there is an irreducible matrix representation of SU(2) on the vector spaces W^l defined in (47). We define the Wigner $\mathscr{D}$ functions by (3) and (48) and write

$$\mathscr{D}^l_{qm}(\alpha,\beta,\gamma) := \mathscr{D}^l_{qm}(B(\alpha,\beta,\gamma)). \tag{160}$$

Then the $\mathscr{D}^l_{qm}(\alpha,\beta,\gamma)$ of course satisfy properties (52) through (55), in particular,

$$\overline{\mathscr{D}^l_{qm}(\alpha,\beta,\gamma)} \equiv \mathscr{D}^l_{mq}(-\gamma,-\beta,-\alpha), \tag{161}$$

where we have used (125).

We now turn to the angular momentum operators L_a and K_a defined in (56). From (66) and the expressions (130) for J_a it is clear that

$$L_3 = -i\,\partial_\alpha,$$
$$L_\pm = e^{\pm i\alpha}(\pm\partial_\beta + i\cot\beta\,\partial_\alpha - (i/\sin\beta)\,\partial_\gamma). \tag{162}$$

We derive expressions for the K_a as differential operators by noticing that, using the chain rule, definition (56b) is equivalent to

$$K_a h \equiv +i(a \quad b)\Lambda_a\begin{pmatrix}\partial_a\\ \partial_b\end{pmatrix}h + i(\bar{a} \quad \bar{b})\bar{\Lambda}_a\begin{pmatrix}\partial_{\bar{a}}\\ \partial_{\bar{b}}\end{pmatrix}h, \tag{163}$$

which yields

$$K_3 = +i\,\partial_\gamma,$$
$$K_\pm = -e^{\mp i\gamma}(\pm\partial_\beta + (i/\sin\beta)\,\partial_\alpha - i\cot\beta\,\partial_\gamma). \tag{164}$$

Using (59)–(61) and (63) one can show that[30]

$$\mathscr{D}^l_{qm}(\alpha,\beta,\gamma) = \left[\frac{(l+m)!(l-m)!}{(l+q)!(l-q)!}\right]^{1/2}\left(\sin\frac{\beta}{2}\right)^{2l} \times \sum_{n=n_{\min}}^{n_{\max}} \binom{l+q}{n}\binom{l-q}{n-q-m}(-1)^{l+m-n} \times e^{-iq\alpha}(\cot(\beta/2))^{2n-m-q}e^{-im\gamma}, \tag{165a}$$

where

$$n_{\min} = \max(0,m+q), \quad n_{\max} = \min(l+q,l+m). \tag{165b}$$

Defining $\hat{\mathscr{Y}}^l_{qm}$ via (67) and (68) we thus obtain

$$\hat{\mathscr{Y}}^l_{qm}(\theta,\phi,\psi) = \left[\frac{(l+q)!(l-q)!(2l+1)}{(l+m)!(l-m)!(4\pi)}\right]^{1/2}\left(\sin\frac{\theta}{2}\right)^{2l} \times \sum_{k=k_{\min}}^{k_{\max}} \binom{l+m}{k}\binom{l-m}{k+q-m}(-1)^{l-q-k} \times e^{+im\phi}(\cot(\theta/2))^{2k+q-m}e^{-iq\psi}, \tag{166a}$$

where

$$k_{\min} = \max(0,m-q), \quad k_{\max} = \min(l+m,l-q). \tag{166b}$$

The properties of the $\mathscr{Y}^l_{qm}$ are easily obtained from (71)–(73) using the isomorphism (64). Define the operator $\eth$ ("edth") on $L^2(S^3)$ by

$$\eth f := -K_+(f\circ T)\circ T^{-1} \tag{167a}$$

[compare (66); the minus sign is conventional] so that

$$\bar{\eth} f \equiv K_-(f\circ T)\circ T^{-1}. \tag{167b}$$

Explicitly, we have

$$\eth = e^{-i\psi}(\partial_\theta + (i/\sin\theta)\partial_\phi - i\cot\theta\,\partial_\psi), \tag{168}$$
$$\bar{\eth} = e^{+i\psi}(\partial_\theta - (i/\sin\theta)\partial_\phi + i\cot\theta\,\partial_\psi)$$

and

$$[\eth,\bar{\eth}] = -2i\,\partial_\psi. \tag{169}$$

Then we have

$$J_3\hat{\mathscr{Y}}^l_{qm} = m\hat{\mathscr{Y}}^l_{qm},$$
$$J_\pm\hat{\mathscr{Y}}^l_{qm} = [(l\mp m)(l\pm m+1)]^{1/2}\hat{\mathscr{Y}}^l_{q,m\pm1}; \tag{170}$$

$$i\,\partial_\psi\hat{\mathscr{Y}}^l_{qm} = q\hat{\mathscr{Y}}^l_{qm},$$
$$\eth\hat{\mathscr{Y}}^l_{qm} = [(l-q)(l+q+1)]^{1/2}\hat{\mathscr{Y}}^l_{q+1,m}, \tag{171}$$
$$\bar{\eth}\hat{\mathscr{Y}}^l_{qm} = -[(l+q)(l-q+1)]^{1/2}\hat{\mathscr{Y}}^l_{q-1,m},$$

and

$$\langle\hat{\mathscr{Y}}^l_{qm},\hat{\mathscr{Y}}^{l'}_{q'm'}\rangle = (\pi/2)\delta_{ll'}\delta_{qq'}\delta_{mm'}, \tag{172}$$

so that $\{\sqrt{2/\pi}\hat{\mathscr{Y}}^l_{qm}\}$ forms an orthonormal basis for $L^2(S^3)$.

Comparing (128), (130), (170), and (172) we see that we can *define* the usual spherical harmonics on S^2 to be [compare (49)]

$$Y_{lm}(\theta,\phi) := \hat{\mathscr{Y}}^l_{0m}(\theta,\phi,0) \quad (l\in\mathbf{Z},\ 0\leqslant l-|m|\in\mathbf{Z}). \tag{173}$$

VIII. SPIN-WEIGHTED SPHERICAL HARMONICS AND MONOPOLE HARMONICS

Having defined the functions $\hat{\mathscr{Y}}^l_{qm} \in L^2(S^3)$ in terms of Wigner $\mathscr{D}$ functions we now show how to obtain the usual coordinate definitions of both spin-weighted spherical harmonics and monopole harmonics. We first note that by comparing (166) and (135) we see that we can define a section $\mathscr{Y}_{qlm}$ of the line bundle E_{2q} by (105) so that [compare (137)]

$$\mathscr{Y}_{qlm}(\theta,\phi) := [(\theta,\phi,\psi;\hat{\mathscr{Y}}^l_{qm}(\theta,\phi,\psi))]. \tag{174}$$

The $\mathscr{Y}_{qlm}$ of course satisfy (106).

Given a local section of the Hopf bundle defined by the function $\gamma_A \in L^2(U_A)$ and (139), we thus obtain the functions [compare (141)]

$$\mathscr{Y}^A_{qlm}(\theta,\phi) := Q_A^{-1}(\mathscr{Y}_{qlm})$$
$$\equiv \hat{\mathscr{Y}}^l_{qm}(\theta,\phi,\gamma_A(\theta,\phi))$$
$$\equiv e^{-iq\gamma_A(\theta,\phi)}\hat{\mathscr{Y}}^l_{qm}(\theta,\phi,0) \in \mathbf{L}^2(U_A). \quad (175)$$

The properties of the $\mathscr{Y}^A_{qlm}$ are completely analogous to those of the $\hat{\mathscr{Y}}^A_{qlm}$, i.e., (170)–(172). Before giving them explicitly, however, we need to introduce an operator on sections of E_n analogous to $\eth$. We do this by defining [compare (146)]

$$\hat{\eth}: \quad \Gamma_n \to \Gamma_{n+2}, \qquad \hat{\eth}\sigma := -Q(K_+(Q^{-1}\sigma)) \quad (176a)$$

and

$$\hat{\bar{\eth}}: \quad \Gamma_n \to \Gamma_{n-2}, \qquad \hat{\bar{\eth}}\sigma := +Q(K_-(Q^{-1}\sigma)), \quad (176b)$$

so that [see (145)][31]

$$\eth^A = e^{-i\gamma_A}\left[\partial_\theta + \frac{i}{\sin\theta}\partial_\phi + \frac{n}{2}\left(-\cot\theta + i\,\partial_\theta\gamma_A - \frac{1}{\sin\theta}\partial_\phi\gamma_A\right)\right], \quad (177a)$$

$$\bar{\eth}^A = e^{+i\gamma_A}\left[\partial_\theta - \frac{i}{\sin\theta}\partial_\phi - \frac{n}{2}\left(-\cot\theta - i\,\partial_\theta\gamma_A - \frac{1}{\sin\theta}\partial_\phi\gamma_A\right)\right]. \quad (177b)$$

The vector m of Sec. IV is given by[11]

$$m = e^{-i\gamma_A}(\partial_\theta + (i/\sin\theta)\partial_\phi), \quad (177c)$$

where the choice of the function $\gamma_A(\theta,\phi)$ is referred to as the choice of a *spin gauge*. Note that since

$$\sigma\in\Gamma_n \Rightarrow \bar{\sigma}\in\Gamma_{-n}, \quad (178a)$$

we have

$$\overline{\hat{\eth}\sigma} \equiv \hat{\bar{\eth}}\bar{\sigma}. \quad (178b)$$

The properties of the $\mathscr{Y}^A_{qlm}$ are thus [compare (169)–(171)]

$$L^A_3\mathscr{Y}^A_{qlm} = m\mathscr{Y}^A_{qlm}, \qquad L^A_\pm\mathscr{Y}^A_{qlm} = [(l\mp m)(l\pm m+1)]^{1/2}\mathscr{Y}_{q,l,m\pm 1}; \quad (179)$$

$$\eth^A\mathscr{Y}^A_{qlm} = [(l-q)(l+q+1)]^{1/2}\mathscr{Y}^A_{q+1,l,m},$$
$$\bar{\eth}^A\mathscr{Y}^A_{qlm} = -[(l+q)(l-q+1)]^{1/2}\mathscr{Y}^A_{q-1,l,m}, \quad (180)$$
$$[\eth^A,\bar{\eth}^A]\mathscr{Y}^A_{qlm} = -2q\mathscr{Y}^A_{qlm}.$$

Furthermore, if γ_A is chosen so that its domain of definition U_A is dense in $\mathbf{S}^2$, then (172) becomes

$$\oint_{\mathbf{S}^2}\overline{\mathscr{Y}^A_{qlm}(\theta,\phi)}\,\mathscr{Y}^A_{q'l'm'}(\theta,\phi) = \delta_{ll'}\delta_{qq'}\delta_{mm'}. \quad (181)$$

Finally, note that for *integer* spin ($l\in\mathbf{Z}$), (173) and (175) imply

$$\mathscr{Y}^A_{0lm}(\theta,\phi) \equiv Y_{lm}(\theta,\phi) \quad (182)$$

and that from (180) we now obtain [compare (97)]

$$\mathscr{Y}^A_{qlm}(\theta,\phi) = \begin{cases} \left[\dfrac{(l-q)!}{(l+q)!}\right]^{1/2}(\eth^A)^q Y_{lm}(\theta,\phi) & (0\leqslant q\leqslant l), \\ \left[\dfrac{(l+q)!}{(l-q)!}\right]^{1/2}(-1)^q(\bar{\eth}^A)^{|q|}Y_{lm}(\theta,\phi) & (-l\leqslant q\leqslant 0), \\ 0 & (l<|q|), \end{cases} \quad (183)$$

where defined [i.e., for $(\theta,\phi)\in U_A$]. But the standard coordinate definition of the spin-weighted spherical harmonics for *integer* spin is just[6,11] (183) in the "standard" spin gauge[32]

$$\gamma_0(\theta,\phi) := 0, \quad (184a)$$

so that[33]

$${}_qY_{lm} := \mathscr{Y}^0_{qlm}(\theta,\phi) = \hat{\mathscr{Y}}^l_{qm}(\theta,\phi,0)$$
$$\equiv \sqrt{\frac{2l+1}{4\pi}}\;\overline{\mathscr{D}^l_{m,-q}(\theta,\phi,0)}. \quad (184b)$$

In an arbitrary "spin gauge," (175) implies[34]

$$\mathscr{Y}^A_{qlm}(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}}\;\overline{\mathscr{D}^l_{m,-q}(\theta,\phi,\gamma_A(\theta,\phi))}$$
$$\equiv e^{-iq\gamma_A(\theta,\phi)}\mathscr{Y}^0_{qlm}(\theta,\phi). \quad (185)$$

One normally *defines* the spin-weighted spherical harmonics in standard gauge for half-integer spin by (184). Thus, the standard spin-weighted spherical harmonics are just the $\mathscr{Y}^A_{qlm}$ in a particular (dense) trivialization of the line bundles E_{2q} [namely the one induced by (184a)].

The monopole harmonics Y_{qlm} are even easier. They are defined by[8,11] (179) and (181) (and a choice of phase for each l) in the gauges[32]

$$\gamma_a(\theta,\phi) := -\phi, \quad \gamma_b(\theta,\phi) := +\phi. \quad (186a)$$

With an appropriate choice of phase we have[35]

$$Y^a_{qlm} \equiv \mathscr{Y}^a_{qlm}, \quad Y^b_{qlm} \equiv \mathscr{Y}^b_{qlm}, \quad (186b)$$

so that the monopole harmonics Y_{qlm} of Ref. 8 are just the $\mathscr{Y}_{qlm}$ in a particular trivializing cover of E_{2q} [namely the one defined by (186) (Ref. 32)].

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[13]The interpretation of spin-weighted *functions* (95) as sections of complex line bundles was given in Ref. 22. The standard definition of spin-weighted spherical harmonics (for integrer spin) is (183), which is given in terms of the differential operator ð. The interpretation of ð as an operator on sections of line bundles was given in Ref. 23. However, Ref. 22 does not discuss spin-weighted spherical *harmonics* at all, and although Ref. 23 does give a precise definition of them it does not discuss them in any detail. The author wishes to thank Ted Newman for providing these two references.

[14]This is to be contrasted with the "standard spin gauge" for spin-weighted spherical harmonics [see Ref. 11 and (184a) below], which amounts to giving only a *local* trivialization of the complex line bundles (which of course does *not* cover S^2 but only a dense subspace of S^2).

[15]The standard procedure (see, e.g., Ref. 9) for *half-integer* spin is to exponentiate the angular momentum operators.

[16]A previous effort along these lines, namely Ref. 6, unfortunately uses an internally inconsistent choice of conventions.

[17]R. Kuwabara, J. Math. Tokushima Univ. **16**, 1 (1982).

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[19]See, e.g. Ref. 10. Equation (40) can be thought of as the *definition* of this preferred connection, which will be given in coordinates in Sec. VI.

[20]As *Riemannian manifolds* the spaces E_n ($n \neq 0$) can be thought of as the lens spaces $S^3/Z_{|n|}$. In particular, $E_{\pm n}$ are isomorphic and the Hopf bundle thought of in this way is isomorphic to S^3.

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[24]We could just as well have used $\overline{V}_0$ in constructing the bundle (90). Only one of these bundles is *strong* bundle isomorphic to the Hopf bundle (7) but this does not affect the argument leading up to (92).

[25]The minus sign comes about because one usually writes the momentum as $p - eA$, i.e., as the operator $-i(\nabla - ieA)$, so that the connection is $-ieA$ and the curvature is $-ie\,dA \equiv -ieF$.

[26]Note that the coordinates (α,β,γ) and (θ,ϕ,ψ) are not well defined at the poles $\beta = 0, \pi$ and $\theta = 0,\pi$, respectively.

[27]This follows immediately from the definition of $\alpha_a(\tau)$ as the usual rotation matrices.

[28]Note that $\hat{\gamma}_A$ is a *section*, whereas γ_A is a *function*.

[29]These agree with (24) of Ref. 11 with $\gamma_A \equiv -\gamma$ and $n \equiv 2s$. We have omitted the hats for simplicity.

[30]This agrees with (4.1.12) and (4.1.15) of Ref. 2 if we note that Edmonds defines

$$\begin{aligned} {}_E\mathscr{D}^l_{qm}(\alpha,\beta,\gamma) :&= \mathscr{D}^l_{qm}(\beta_3^{-1}(\alpha)\,\beta_2^{-1}(\beta)\,\beta_3^{-1}(\gamma)) \\ &\equiv \mathscr{D}^l_{qm}(B(\gamma,\beta,\alpha)^{-1}) \\ &\equiv \mathscr{D}^l_{qm}(-\alpha,-\beta,-\gamma) \\ &\equiv \overline{\mathscr{D}^l_{qm}(\alpha,-\beta,\gamma)}. \end{aligned}$$

However, if we interpret (3.4) of Ref. 6 (see also our Ref. 16) as defining

$$\begin{aligned} {}_G\mathscr{D}^l_{qm}(\alpha,\beta,\gamma) :&= \mathscr{D}^l_{qm}(B(\alpha,\beta,\gamma)^{-1}) \\ &\equiv \mathscr{D}^l_{qm}(-\gamma,-\beta,-\alpha) \\ &\equiv \overline{\mathscr{D}^l_{mq}(\alpha,\beta,\gamma)} \\ &\equiv {}_E\mathscr{D}_{mq}(\alpha,-\beta,\gamma), \end{aligned}$$

then we are forced to conclude that (3.9) of Ref. 6 is missing a factor $(-1)^{m+m'}$. Finally, note that although no explicit expression analogous to (165) is given in Ref. 4 the functions $D_{lqm}(\alpha,\beta,\gamma)$ defined there are identical to our functions $\mathscr{D}^l_{qm}(\alpha,\beta,\gamma)$.

[31]This agrees with (19) of Ref. 11, where $\gamma_A \equiv -\gamma$ and $n \equiv +2s$. We have omitted the hats for simplicity.

[32]Note that the section of the Hopf bundle (139) induced by γ_0 is defined everywhere on S^2 except at the poles $\theta = 0,\pi$, whereas the sections induced by γ_a,γ_b are each defined everywhere on S^2 except at *one* pole (namely $\theta = 0$ and $\theta = \pi$, respectively).

[33]This agrees with the ${}_qY_{lm}$ of Ref. 11 in the standard spin gauge but differs from Ref. 6 by a factor $(-1)^q$.

[34]The factor $(-1)^s$ in (28) of Ref. 11 is thus incorrect.

[35]This agrees with both Ref. 8 and Ref. 11.