A unifying energy-based approach to stability of power grids with market dynamics

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A unifying energy-based approach to stability of power grids with market dynamics

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***Abstract*—In this paper a unifying energy-based approach is provided to the modeling and stability analysis of power systems coupled with market dynamics. We consider a standard model of the power network with a third-order model for the synchronous generators involving voltage dynamics. By applying the primal-dual gradient method to a social welfare optimization, a distributed dynamic pricing algorithm is obtained, which can be naturally formulated in port-Hamiltonian form. By interconnection with the physical model a closed-loop port-Hamiltonian system is obtained, whose properties are exploited to prove asymptotic stability to the set of optimal points. This result is extended to the case that also general nodal power constraints are included into the social welfare problem. Additionally, the case of line congestion and power transmission costs in acyclic networks is covered. Finally, a dynamic pricing algorithm is proposed that does not require knowledge about the power supply and demand.**

***Index Terms*—port-Hamiltonian, frequency regulation, optimal power dispatch, dynamic pricing, social welfare, distributed control, convex optimization.**

I. Introduction

PROVISIONING energy has become increasingly complicated due to several reasons, including the increased share of renewables. As a result, the generators operate more often near their capacity limits and transmission line congestion occurs more frequently.

One effective approach to alleviate some of these challenges is to use real-time dynamic pricing as a control method [1]. This feedback mechanism can be used to encourage the consumers to change their usage when in some parts of the grid (*control areas*) it is difficult for the generators and the network to match the demand.

Real-time dynamic pricing also allows producers and consumers to fairly share utilities and costs associated with the generation and consumption of energy among the different control areas. The challenge of achieving this in an optimal manner while the grid operates within its capacity limits, is called the *social welfare problem* [2], [3].

Many of the existing dynamic pricing algorithms focus on the economic part of optimal supply-demand matching [2], [4]. However, if market mechanisms are used to determine the optimal power dispatch (with near real-time updates of the dispatch commands) dynamic coupling occurs between the market update process and the physical response of the physical power network dynamics [5].

Consequently, under the assumption of market-based dispatch, it is essential to consider the stability of the coupled system incorporating both market operation and electromechanical power system dynamics simultaneously.

While on this subject a vast literature is already available, the aim of this paper is to provide a rigorous and unifying passivity-based stability analysis. We focus on a more accurate and higher order model for the physical power network than conventionally used in the literature. In particular, we use a third-order model for the synchronous generators including voltage dynamics. As a result, market dynamics, frequency dynamics and voltage dynamics are considered simultaneously.

Finally, we propose variations of the basic controller design that, among other things, allow the incorporation of capacity constraints on the generation and demand of power and on the transmission lines, and enhance the transient dynamics of the closed-loop system. The approach taken in this paper is to model both the dynamic pricing controller as well as the physical network in a port-Hamiltonian way, emphasizing energy storage and power flow. This provides a unified framework for the modeling, analysis and control of power networks with market dynamics, with possible extensions to more refined models of the physical power network, including for example turbine dynamics.

A. Literature review

The coupling between a high-order dynamic power network and market dynamics has been studied before in [5]. Here a fourth-order model of the synchronous generator is used in conjunction with turbine and exciter dynamics, which is coupled to a simple model describing the market dynamics. The results established in [5] are based on an eigenvalue analysis of the linearized system.

It is shown in [6] that the third-order model (often called the *flux-decay model*) for describing the power network, as used in the present paper, admits a useful passivity property that allows for a rigorous stability analysis of the interconnection with optimal power dispatch controllers, even in the presence of time-varying demand.

A common way to solve a general optimization problem like the social welfare problem is by applying the primal-dual gradient method [7], [8], [9]. Also in power grids this is a commonly used approach to design optimal distributed controllers, see e.g. [10], [11], [12], [13], [14], [15]. The

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problem formulations vary in these papers, with the focus being on either the generation side [10], [12], the load side [11], [16], [17] or both [13], [14], [18], [15]. We will elaborate on these references in the following two paragraphs.

A vast literature focuses on linear power system models coupled with gradient-method-based controllers [10], [12], [11], [17], [13], [19]. In these references the property that the linear power system dynamics can be formulated as a gradient method applied to a certain optimization problem is exploited. This is commonly referred to as *reverse-engineering* of the power system dynamics [13], [10], [12]. However, this approach falls short in dealing with models involving nonlinear power flows.

Nevertheless, [14], [18], [15], [16] show the possibility to achieve optimal power dispatch in power networks with nonlinear power flows using gradient-method-based controllers. On the other hand, the controllers proposed in [14], [18], [15] have restrictions in assigning the controller parameters and in addition require that the topology of the physical network is a tree.

B. Main contributions

The contribution of this paper is to propose a novel energy-based approach to the problem that differs substantially from the aforementioned works. We proceed along the lines of [20], [21], where a port-Hamiltonian approach to the design of gradient-method-based controllers in power networks is proposed. In those papers it is shown that both the power network as well as the controller designs admit a port-Hamiltonian representation which are then interconnected to obtain a closed-loop port-Hamiltonian system.

After showing that the third-order dynamical model describing the power network admits a port-Hamiltonian representation, we provide a systematic method to design gradient-method-based controllers that is able to balance power supply and demand while maximizing the social welfare at steady state. This design is carried out first by establishing the optimality conditions associated with the social welfare problem. Then the continuous-time gradient method is applied to obtain the port-Hamiltonian form of the dynamic pricing controller. Then, following [20], [21], the market dynamics is coupled to the physical power network in a *power-preserving* manner so that all the trajectories of the closed-loop system converge to the desired synchronous solution and to optimal power dispatch.

Although the proposed controllers share similarities with others presented in the literature, the way in which they are interconnected to the physical network, which is based on passivity, is to the best of our knowledge new. Moreover, they show several advantages.

1) Physical model: Since our approach is based on passivity and does not require to reverse-engineer the power system dynamics as a primal-dual gradient dynamics, it allows to deal with more complex nonlinear models of the power network. More specifically, the physical model for describing the power network in this paper admits nonlinear power flows and time-varying voltages, and is more accurate and reliable than the classical second-order model [22], [23], [24].

In addition, most of the results that are established in the present paper are valid for the case of nonlinear power flows and cyclic networks, in contrast to e.g. [13], [10], [12], [19], [19], where the power flows are linearized and e.g. [15], [18], [14] where the physical network topology is a tree. Moreover, in the aforementioned references the voltages are assumed to be constant. While the third-order model for the power network as considered in this paper has been studied before using passivity based techniques [6], the combination with gradient method based controllers is novel. In addition, the stability analysis does not rely on linearization and is based on energy functions which allow us to establish rigorous stability results.

2) State transformation: In [10], [12] it is shown how a state transformation of the closed-loop system can be used to eliminate the information about the demand from the controller dynamics, which improves implementation of the resulting controller. We pursue this idea and show that the same kind of state transformation can also be used for more complex physical models as considered in this paper. This avoids the requirement of knowing the demand to determine the market price.

3) Controller parameters: In the present paper we show that both the physical power network as well as the dynamic pricing controllers admit a port-Hamiltonian representation, and in particular are passive systems. As a result, the interconnection between the controller and the nonlinear power system is *power-preserving*, implying passivity of the closed-loop system as well. Consequently, we do not have to impose any condition on controller design parameters for guaranteeing asymptotic stability, contrary to [14], [18], [15].

4) Port-Hamiltonian framework: Because of the use of the port-Hamiltonian framework, the proposed controller designs have the potential to deal with more complex models for the power network compared to the model described in this paper. As long as the more complex model remains port-Hamiltonian, the results continue to be valid. This may lead to inclusion of, for example, turbine dynamics or automatic voltage regulators in the analysis, although this is beyond the scope of the present paper. Furthermore, higher order models for the synchronous generator could be considered.

In addition, we propose various extensions to the basic controller design that have not been investigated in the aforementioned references.

5) Transmission costs: In addition to nodal power constraints and line congestion, we also consider the possibility of including power transmission costs into the social welfare problem. Including such costs may in particular be useful in reducing energy losses or the risk of a breakdown of certain transmission lines.

6) Non-strict convex objective functions: By relaxing the conditions on the objective function, we show that also non-strict convex/concave cost/utility functions can be considered respectively. In addition, the proposed technique allows to add damping in the gradient method based controller which may improve the convergence rate of the closed-loop system.

7) Barrier functions: We highlight the possibility to use barrier functions to enforce the trajectories to stay withing the feasible region, which allows operation within the capacity constraints for all time, even during transients. This permits a more realistic application of the proposed controller design.

The remainder of this paper is organized as follows. In Section II the preliminaries are stated. Then the basic dynamic pricing algorithm is discussed in Section III and convergence of the closed-loop system is proven. Variations of the basic controller design are discussed in Section IV where in Section IV-A nodal power congestion is included into the social welfare problem, and in Section IV-B the case line congestion for the acyclic power networks is discussed. A dynamic pricing algorithm is proposed in Section IV-C which does not require knowledge about the power supply and demand. In Section IV-D the possibility to relax the convexity assumption and to improve the transient dynamics of the basic controller is discussed. Finally, the conclusions and suggestions for future research are discussed in Section V.

II. Preliminaries

A. Notation

Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we write $A > 0$ ($A \geq 0$) to indicate that A is a positive (semi-)definite matrix. The set of positive real numbers is denoted by $\mathbb{R}_{>0}$ and likewise the set of vectors in $\mathbb{R}^n$ whose elements are positive by $\mathbb{R}^n_{>0}$. We write $A \vee B$ to indicate that either condition A or condition B holds, e.g., $a > 0 \vee b > 0$ means that either $a > 0$ or $b > 0$ holds. Given $v \in \mathbb{R}^n$ then $v \succeq 0$ denotes the element-wise inequality. The notation $\mathbb{1} \in \mathbb{R}^n$ is used for the vector whose elements are equal to 1. Given a twice-differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ then the Hessian of f evaluated at x is denoted by $\nabla^2 f(x)$. Given a vector $\eta \in \mathbb{R}^m$, we denote by $\mathbf{sin}\,\eta \in \mathbb{R}^m$ the element-wise sine function. Given a differentiable function $f(x_1, x_2), x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$ then $\nabla f(x_1, x_2)$ denotes the gradient of f with respect to x_1, x_2 evaluated at x_1, x_2 and likewise $\nabla_{x_1} f(x_1, x_2)$ denotes the gradient of f with respect to x_1. Given a solution x of $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lebesgue measurable function and locally bounded, the *omega-limit set* $\Omega(x)$ is defined as [25]

$$\Omega(x) := \Big\{ \bar{x} \in \mathbb{R}^n \mid \exists \{t_k\}_{k=1}^{\infty} \subset [0, \infty) \text{ with } \lim_{k\to\infty} t_k = \infty \text{ and } \lim_{k\to\infty} x(t_k) = \bar{x} \Big\}.$$

B. Power network model

Consider a power grid consisting of n buses. The network is represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the nodes, $\mathcal{V} = \{1, ..., n\}$, is the set of buses and the edges, $\mathcal{E} = \{1, \ldots, m\} \subset \mathcal{V} \times \mathcal{V}$, is the set of transmission lines connecting the buses. The k-th edge connecting nodes i and j is denoted as $k = (i, j) = (j, i)$. The ends of edge k are arbitrary labeled with a '+' and a '-', so that the incidence matrix D of the resulting directed graph is given by

$$D_{ik} = \begin{cases} +1 & \text{if } i \text{ is the positive end of edge } k \\ -1 & \text{if } i \text{ is the negative end of edge } k \\ 0 & \text{otherwise.} \end{cases}$$

Each bus represents a control area and is assumed to have a controllable power supply and demand. The dynamics at each bus is assumed to be given by [22], [6]

$$\begin{aligned} \dot{\delta}_i &= \omega_i \\ M_i \dot{\omega}_i &= -\sum_{j\in\mathcal{N}_i} B_{ij} E'_{qi} E'_{qj} \sin\delta_{ij} - A_i \omega_i + P_{gi} - P_{di} \\ T'_{di} \dot{E}'_{qi} &= E_{fi} - (1 - (X_{di} - X'_{di}) B_{ii}) E'_{qi} \\ &\quad - (X_{di} - X'_{di}) \sum_{j\in\mathcal{N}_i} B_{ij} E'_{qj} \cos\delta_{ij}, \end{aligned} \tag{1}$$

which is commonly referred to as the *flux-decay model*. Here we use a similar notation as used in established literature on power systems [22], [23], [26], [24]. See Table I for a list of symbols used in the model (1) and throughout the paper.

δ_i	Voltage angle
ω_i^b	Frequency
ω^n	Nominal frequency
ω_i	Frequency deviation $\omega_i := \omega_i^b - \omega^n$
E'_{qi}	q-axis transient internal voltage
E_{fi}	Excitation voltage
P_{di}	Power demand
P_{gi}	Power generation
M_i	Moment of inertia
$\mathcal{N}_i$	Set of buses connected to bus i
A_i	Asynchronous damping constant
B_{ij}	Negative of the susceptance of transmission line (i, j)
B_{ii}	Self-susceptance
X_{di}	d-axis synchronous reactance of generator i
X'_{di}	d-axis transient reactance of generator i
T'^{di}_{di}	d-axis open-circuit transient time constant

Table I
Parameters and state variables of model (1).

Assumption 1. By using the power network model (1) the following assumptions are made, which are standard in a broad range of literature on power network dynamics [22].

- Lines are purely inductive, i.e., the conductance is zero. This assumption is generally valid for the case of high voltage lines connecting different control areas.
- The grid is operating around the synchronous frequency which implies $\omega_i^b \approx \omega^n$ for each $i \in \mathcal{V}$.
- In addition, we assume for simplicity that the excitation voltage E_{fi} is constant for all $i \in \mathcal{V}$.

Define the voltage angle differences between the buses by $\eta = D^T \delta$. Further define the angular momenta by $p := M\omega$, where $\omega = \omega^b - \mathbb{1}\omega^n$ are the (aggregated) frequency deviations and $M = \mathrm{diag}_{i\in\mathcal{V}}\{M_i\}$ are the moments of inertia. Let $\Gamma(E'_q) = \mathrm{diag}_{k\in\mathcal{E}}\{\gamma_k\}$ and $\gamma_k = B_{ij} E'_{qi} E'_{qj} = B_{ji} E'_{qi} E'_{qj}$

where k corresponds to the edge between node i and j. Then we can write (1) more compactly as [6]

$$\begin{aligned} \dot{\eta} &= D^T\omega \\ M\dot{\omega} &= -D\Gamma(E'_q)\,\mathbf{sin}\,\eta - A\omega + P_g - P_d \\ T'_d\dot{E}'_q &= -F(\eta)E'_q + E_f \end{aligned} \tag{2}$$

where $A = \mathrm{diag}_{i\in\mathcal{V}}\{A_i\}, P_g = \mathrm{col}_{i\in\mathcal{V}}\{P_{gi}\}, P_d = \mathrm{col}_{i\in\mathcal{V}}\{P_{di}\}, T'_d = \mathrm{diag}_{i\in\mathcal{V}}\{T'_{di}\}, E'_q = \mathrm{col}_{i\in\mathcal{V}}\{E'_{qi}\}, E_f = \mathrm{col}_{i\in\mathcal{V}}\{E_{fi}\}$. For a given η, the components of the matrix $F(\eta)\in\mathbb{R}^{n\times n}$ are defined as

$$\begin{aligned} F_{ii}(\eta) &= \frac{1}{X_{di}-X'_{di}} + B_{ii}, \qquad i\in\mathcal{V} \\ F_{ij}(\eta) &= -B_{ij}\cos\eta_k = F_{ji}(\eta), \quad k=(i,j)\in\mathcal{E} \end{aligned} \tag{3}$$

and $F_{ij}(\eta)=0$ otherwise. Since for realistic power networks $X_{di} > X'_{di}$, and $B_{ii} = \sum_{j\in\mathcal{N}_i} B_{ij} > 0$ for all $i\in\mathcal{V}$, it follows that $F(\eta)>0$ for all η [22], [23].

Considering the physical energy[1] stored in the generator and the transmission lines respectively, we define the Hamiltonian as

$$\begin{aligned} H_p = &\frac{1}{2}\sum_{i\in\mathcal{V}}\left(M_i^{-1}p_i^2 + \frac{(E'_{qi}-E_{fi})^2}{X_{di}-X'_{di}}\right) \\ &+ \frac{1}{2}\sum_{k=(i,j)\in\mathcal{E}} B_{ij}\left((E'_{qi})^2 + (E'_{qj})^2 - 2E'_{qi}E'_{qj}\cos\eta_k\right) \end{aligned} \tag{4}$$

where $\eta_k = \delta_i - \delta_j$. The first term of the Hamiltonian H_p represents the shifted kinetic energy stored in the rotors of the generators and the second term corresponds to the magnetic energy stored in the generator circuits. Finally, the last term of H_p corresponds to the magnetic energy stored in the inductive transmission lines.

By (4), the system (2) can be written in port-Hamiltonian form [27] as

$$\begin{aligned} \dot{x}_p &= \begin{bmatrix} 0 & D^T & 0 \\ -D & -A & 0 \\ 0 & 0 & -R_q \end{bmatrix}\nabla H_p + \begin{bmatrix} 0 & 0 \\ I & -I \\ 0 & 0 \end{bmatrix} u_p \\ y_p &= \begin{bmatrix} 0 & I & 0 \\ 0 & -I & 0 \end{bmatrix}\nabla H_p = \begin{bmatrix} \omega \\ -\omega \end{bmatrix} \end{aligned} \tag{5}$$

where $x_p = \mathrm{col}(\eta, p, E'_q)$, $u_p = \mathrm{col}(P_g, P_d)$ and

$$\begin{aligned} R_q &= (T'_d)^{-1}(X_d - X'_d) > 0, \\ T'_d &= \mathrm{diag}_{i\in\mathcal{V}}\{T'_{di}\} > 0, \\ X_d - X'_d &= \mathrm{diag}_{i\in\mathcal{V}}\{X_{di}-X'_{di}\} > 0. \end{aligned}$$

For a study on the stability and equilibria of the flux-decay model (5), based on the Hamiltonian function (4), we refer to [6]. The stability results established in [6] rely on the following assumption.

Assumption 2. Given a constant input $u_p = \bar{u}_p$. There exists an equilibrium $(\bar{\eta}, \bar{p}, \bar{E}'_q)$ of (5) that satisfies $\bar{\eta}\in\mathrm{im}\,D^T$, $\bar{\eta}\in(-\pi/2,\pi/2)^m$ and $\nabla^2 H(\bar{\eta},\bar{p},\bar{E}'_q) > 0$.

[1]For aesthetic reasons we define the Hamiltonian H_p as ω^n times the physical energy as the factor $1/\omega^n$ appears in each of the energy functions. As a result, H_p has the dimension of power instead of energy.

The assumption $\bar{\eta}\in(-\pi/2,\pi/2)^m$ is standard in studies on power grid stability and is also referred to as a security constraint [6]. In addition, the Hessian condition guarantees the existence of a local storage function around the equilibrium. The following result, which establishes decentralized conditions for checking the positive definiteness of the Hessian, was proven in [28]:

Proposition 1. *Let $\bar{E}'_{qi}\in\mathbb{R}^n_{>0}$ and $\bar{\eta}\in(-\pi/2,\pi/2)^m$. If for all $i\in\mathcal{V}$ we have*

$$\begin{aligned} &\frac{1}{X_{di}-X'_{di}} + B_{ii} + \sum_{k=(i,j)\in\mathcal{E}} B_{ij}\frac{E'_{qj}\sin^2\bar{\eta}_k}{E'_{qi}\cos\bar{\eta}_k} \\ &> \sum_{k=(i,j)\in\mathcal{E}} B_{ij}\cos\bar{\eta}_k\left(1 + \frac{\bar{E}_{qi}}{\bar{E}_{qj}}\tan^2\bar{\eta}_k\right) > 0 \end{aligned}$$

then $\nabla^2 H_p(\bar{x}_p) > 0$.

It can be verified that the condition stated in Proposition 1 is satisfied if the following holds [28]:

- the generator reactances are small compared to the transmission line reactances
- the voltage (angle) differences are small.

Remarkably, these conditions hold for a typical operation point in power transmission networks.

C. Social welfare problem

We define the social welfare by $S(P_g,P_d) := U(P_d) - C(P_g)$, which consists of a utility function $U(P_d)$ of the power consumption P_d and the cost $C(P_g)$ associated to the power production P_g. We assume that $C(P_g), U(P_d)$ are strictly convex and strictly concave functions respectively.

Remark 1. It is also possible to include mutual costs and utilities among the different control areas, provided that the convexity/concavity assumption is satisfied.

The objective is to maximize the social welfare while achieving zero frequency deviation. By analyzing the equilibria of (1), it follows that a necessary condition for zero frequency deviation is $\mathbb{1}^T P_d = \mathbb{1}^T P_g$ [6], i.e., the total supply must match the total demand. It can be noted that (P_g, P_d) is a solution to the latter equation if and only if there exists a $v\in\mathbb{R}^{m_c}$ satisfying $D_c v - P_g + P_d = 0$ where $D_c\in\mathbb{R}^{n\times m_c}$ is the incidence matrix of some connected *communication* graph with m_c edges and n nodes. Because of the latter equivalence, we consider the following convex minimization problem:

$$\min_{P_g,P_d,v} \quad -S(P_g,P_d) = C(P_g) - U(P_d) \tag{6a}$$

$$\text{s.t.} \quad D_c v - P_g + P_d = 0. \tag{6b}$$

Remark 2. Although this paper focuses on optimal *active* power sharing, we stress that it is also possible to consider (optimal) *reactive* power sharing simultaneously, see e.g. [28] for more details.

The Lagrangian corresponding to (6) is given by

$$\mathcal{L} = C(P_g) - U(P_d) + \lambda^T(D_c v - P_g + P_d) \tag{7}$$

with Lagrange multipliers $\lambda \in \mathbb{R}^n$. The resulting first-order optimality conditions are given by the Karush–Kuhn–Tucker (KKT) conditions

$$\begin{aligned} \nabla C(\bar{P}_g) - \bar{\lambda} &= 0, \\ -\nabla U(\bar{P}_d) + \bar{\lambda} &= 0, \\ D_c^T \bar{\lambda} &= 0, \\ D_c \bar{v} - \bar{P}_g + \bar{P}_d &= 0. \end{aligned} \tag{8}$$

Since the minimization problem is convex, strong duality holds and it follows that $(\bar{P}_g, \bar{P}_d, \bar{v})$ is an optimal solution to (6) if and only if there exists an $\bar{\lambda} \in \mathbb{R}^n$ that satisfies (8) [29].

III. Basic primal-dual gradient controller

In this section we design the basic dynamic pricing algorithm which will be used as the starting point for the controllers designs discussed in Section IV. Its dynamics is obtained by applying the primal-dual gradient method [7], [10], [14] to the minimization problem (6), resulting in

$$\tau_g \dot{P}_g = -\nabla C(P_g) + \lambda + u_c^g \tag{9a}$$
$$\tau_d \dot{P}_d = \nabla U(P_d) - \lambda + u_c^d \tag{9b}$$
$$\tau_v \dot{v} = -D_c^T \lambda \tag{9c}$$
$$\tau_\lambda \dot{\lambda} = D_c v - P_g + P_d. \tag{9d}$$

Here we introduce additional inputs $u_c = \mathrm{col}(u_c^g, u_c^d)$ which are to be specified later on, and $\tau_c := \mathrm{blockdiag}(\tau_g, \tau_d, \tau_v, \tau_\lambda) > 0$ are controller design parameters. Recall from Section II-C that there is freedom in choosing a communication network and the associated incidence matrix. Depending on the application, one may prefer all-to-all communication where the underlying graph is complete, or communication networks where its associated graph is a star, line or cycle graph. In addition, τ_c determines the converge rate of the dynamics (9); a large τ_c gives a slow convergence rate whereas a small τ_c gives a fast convergence rate.

Observe that the dynamics (9) has a clear economic interpretation [1], [2], [5]: each power producer aims at maximizing their own profit, which occurs whenever their individual marginal cost is equal to the local price $\lambda_i + u_{ci}^g$. At the same time, each consumer maximizes its own utility but is penalized by the local price $\lambda_i - u_{ci}^d$.

The equations (9c), (9d) represent the distributed dynamic pricing mechanism where the quantity v represents a *virtual* power flow along the edges of the communication graph with incidence matrix D_c. We emphasize *virtual*, since v may not correspond to the real physical power flow as the communication graph may be different than the physical network topology. Equation (9d) shows that the local price λ_i rises if the power demand plus power outflow at node $i \in \mathcal{V}$ is greater than the local power supply plus power inflow of power at node i and vice versa. The inputs u_c^g, u_c^d are interpreted as additional penalties or prices that are assigned to the power producers and consumers respectively. These inputs can be chosen appropriately to compensate for the frequency deviation in the physical power network as we will show now.

To this end, define the variables $x_c = (x_g, x_d, x_v, x_\lambda) = (\tau_g P_g, \tau_d P_d, \tau_v v, \tau_\lambda \lambda) = \tau_c z_c$ and note that, in the sequel, we interchangeably write the system dynamics in terms of both x_c and z_c for ease of notation. In these new variables the dynamics (9) admits a natural port-Hamiltonian representation [20], which is given by

$$\begin{aligned} \dot{x}_c &= \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & -D_c^T \\ -I & I & D_c & 0 \end{bmatrix} \nabla H_c(x_c) + \nabla S(z_c) \\ &\quad + \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u_c \end{aligned} \tag{10}$$

$$y_c = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \nabla H_c(x_c) = \begin{bmatrix} P_g \\ P_d \end{bmatrix},$$

$$H_c(x_c) = \frac{1}{2} x_c^T \tau_c^{-1} x_c. \tag{11}$$

Note that the system (10) is indeed a port-Hamiltonian system[2] since S is concave and therefore satisfies the incremental passivity property

$$(z_1 - z_2)^T (\nabla S(z_1) - \nabla S(z_2)) \leq 0, \ \forall z_1, z_2 \in \mathbb{R}^{3n+m_c}.$$

The port-Hamiltonian controller (10) is interconnected to the physical network (5) by taking $u_c = -y_p, u_p = y_c$. Define the extended vectors of variables by

$$x := \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_\lambda \end{bmatrix} \begin{bmatrix} \eta \\ \omega \\ E_q' \\ P_g \\ P_d \\ v \\ \lambda \end{bmatrix} =: \tau z. \tag{12}$$

Then the closed-loop port-Hamiltonian system takes the form

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & D^T & 0 & 0 & 0 & 0 & 0 \\ -D & -A & 0 & I & -I & 0 & 0 \\ 0 & 0 & -R_q & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & 0 & 0 & -D_c^T \\ 0 & 0 & 0 & -I & I & D_c & 0 \end{bmatrix} \nabla H(x) \\ &\quad + \nabla S(z), \end{aligned} \tag{13}$$

where $H = H_p + H_c$ is equal to the sum of the energy function (4) corresponding to the physical model, and the controller Hamiltonian (11). In the sequel we write (13) more compactly as

$$\dot{x} = (J - R)\nabla H(x) + \nabla S(z),$$

where $R = R^T \geq 0, J = -J^T$. We define the equilibrium set of (13), expressed in the variable z, by

$$\mathcal{Z}_1 = \{\bar{z} \mid \bar{z} \text{ is an equilibrium of (13)}\}. \tag{14}$$

Note that each $\bar{z} \in \mathcal{Z}_1$ satisfies the optimality conditions (8) and simultaneously the zero frequency constraints of the

[2]Strictly speaking, (10) is an *incremental* port-Hamiltonian system [27].

physical network (5) given by $\bar{\omega} = 0$. Hence, $\mathcal{Z}_1$ corresponds to the desired equilibria, and the next theorem states the convergence to this set of optimal points.

Theorem 1. *For every $\bar{z} \in \mathcal{Z}_1$ satisfying Assumption 2 there exists a neighborhood Υ around $\bar{z}$ where all trajectories z satisfying* (13) *with initial conditions in Υ converge to the set $\mathcal{Z}_1$. In addition, the convergence of each such trajectory is to a point.*

Proof. Let $\bar{z} \in \mathcal{Z}_1$ and define the shifted Hamiltonian $\bar{H}$ around $\bar{x} := \tau\bar{z}$ as [27], [6]

$$\bar{H}(x) = H(x) - (x - \bar{x})^T \nabla H(\bar{x}) - H(\bar{x}). \tag{15}$$

After rewriting, the closed-loop port-Hamiltonian system (13) is equivalently described by

$$\dot{x} = (J - R)\nabla\bar{H}(x) + \nabla S(z) - \nabla S(\bar{z}).$$

The shifted Hamiltonian $\bar{H}$ satisfies

$$\begin{aligned}\dot{\bar{H}} = &-\omega^T A\omega + (z - \bar{z})^T(\nabla S(z) - \nabla S(\bar{z})) \\ &- (\nabla_{E'_q}\bar{H})^T R_q \nabla_{E'_q}\bar{H} \leq 0,\end{aligned} \tag{16}$$

where equality holds if and only if $\nabla_{E'_q}\bar{H}(x) = \nabla_{E'_q}H(x) = 0, \omega = 0, P_g = \bar{P}_g, P_d = \bar{P}_d$ since $S(z)$ is strictly concave in P_g and P_d. Bearing in mind Assumption 2, it is observed that $\nabla^2 H(x) = \nabla^2\bar{H}(x) > 0$ for all x in a sufficiently small open neighborhood around $\bar{x}$. Hence, as $\dot{\bar{H}} \leq 0$, there exists a compact sublevel set Υ of $\bar{H}$ around $\bar{z}$ contained in such neighborhood, which is forward invariant. By LaSalle's invariance principle, each the solution with initial conditions in Υ converges to the largest invariant set $\mathcal{S}$ contained in $\Upsilon \cap \{z \mid \nabla_{E'_q}H(x) = 0, \omega = 0,\ P_g = \bar{P}_g,\ P_d = \bar{P}_d\}$. On such invariant set $\lambda = \bar{\lambda}$ and η, v, E'_q are constant. Hence, z converges to $\mathcal{S} \subset \mathcal{Z}_1$ as $t \to \infty$.

Finally, we prove that the convergence of each solution of (13) initializing in Υ is to a point. This is equivalent to proving that its omega-limit set $\Omega(x)$ is a singleton. Since the solution x is bounded, $\Omega(x) \neq \emptyset$ by the Bolzano-Weierstrass theorem [30]. By contradiction, suppose now that there exist two distinct point in $\Omega(x)$, say $\bar{x}_1, \bar{x}_2 \in \Omega(x), \bar{x}_1 \neq \bar{x}_2$. Then there exists $\bar{H}_1(x), \bar{H}_2(x)$ defined by (15) with respect to $\bar{x}_1, \bar{x}_2$ respectively and scalars $c_1, c_2 \in \mathbb{R}_{>0}$ such that $\bar{H}_1^{-1}(\leq c_1) := \{x \mid \bar{H}_1(x) \leq c_1\}, \bar{H}_2^{-1}(\leq c_2) := \{x \mid \bar{H}_2(x) \leq c_2\}$ are disjoint and compact as the Hessian of $\bar{H}_1, \bar{H}_2$ is positive definite in the neighborhood Υ. Since each trajectory z converges to $\mathcal{Z}_1$ as proven above, it follows that $\tau^{-1}\bar{x}_1, \tau^{-1}\bar{x}_2 \in \mathcal{Z}_1$. Together with $\bar{x}_1 \in \Omega(x)$, this implies that there exists a finite time $t_1 > 0$ such that $x(t) \in \bar{H}_1^{-1}(\leq c_1)$ for all $t \geq t_1$ as the set $\bar{H}_1^{-1}(\leq c_1)$ is invariant by the dissipation inequality (16). Similarly, there exists a finite time $t_2 > 0$ such that $x(t) \in \bar{H}_2^{-1}(\leq c_2)$ for all $t \geq t_2$. This implies that the solution $x(t)$ satisfies $x(t) \in \bar{H}_1^{-1}(\leq c_1) \cap \bar{H}_1^{-1}(\leq c_1) = \emptyset$ for $t \geq \max(t_1, t_2)$ which is a contradiction. This concludes the proof. □

IV. Variations in the controller design

In this section we propose several variations and extensions of the controller designed in the previous section. These include, among other things, the possibility to incorporate nodal power constraints, and line congestion in conjunction with transmission costs into the social welfare problem.

A. Including nodal power constraints

The results of Section III can be extended to the case where nodal constraints on the power production and consumption are included into the optimization problem (6). To this end, consider the social welfare problem

$$\min_{P_g, P_d, v} \quad -S(P_g, P_d) := C(P_g) - U(P_d) \tag{17a}$$

$$\text{s.t.} \quad D_c v - P_g + P_d = 0, \tag{17b}$$

$$g(P_g, P_d) \preceq 0 \tag{17c}$$

where $g : \mathbb{R}^{2n} \to \mathbb{R}^l$ is a convex function.

Remark 3. Note that (17c) captures the convex inequality constraints considered in the existing literature. For example, by choosing g as

$$g(P_g, P_d) = \begin{bmatrix} g_1(P_g, P_d) \\ g_2(P_g, P_d) \\ g_3(P_g, P_d) \\ g_4(P_g, P_d) \end{bmatrix} = \begin{bmatrix} P_g - P_g^{\max} \\ P_g^{\min} - P_g \\ P_d - P_d^{\max} \\ P_d^{\min} - P_d \end{bmatrix},$$

the resulting inequality constraints (17c) become $P_g^{\min} \preceq P_g \preceq P_g^{\max},\ P_d^{\min} \preceq P_d \preceq P_d^{\max}$ which, among others, are used in [13], [14], [18].

In the sequel, we assume that (17) satisfies Slater's condition [29]. As a result, $(\bar{P}_g, \bar{P}_d, \bar{v})$ is an optimal solution to (17) if and only if there exists $\bar{\lambda} \in \mathbb{R}^n, \bar{\mu} \in \mathbb{R}^l_{\geq 0}$ satisfying the following KKT optimality conditions:

$$\begin{aligned} \nabla C(\bar{P}_g) - \bar{\lambda} + \frac{\partial g}{\partial P_g}(\bar{P}_g, \bar{P}_d)\bar{\mu} = 0, \\ -\nabla U(\bar{P}_d) + \bar{\lambda} + \frac{\partial g}{\partial P_d}(\bar{P}_g, \bar{P}_d)\bar{\mu} = 0, \\ D_c\bar{v} - \bar{P}_g + \bar{P}_d = 0, \qquad D_c^T\bar{\lambda} = 0, \\ g(\bar{P}_g, \bar{P}_d) \preceq 0, \qquad \bar{\mu} \succeq 0, \qquad \bar{\mu}^T g(\bar{P}_g, \bar{P}_d) = 0. \end{aligned} \tag{18}$$

Next, we introduce the following subsystems [20], [9]

$$\begin{aligned} \dot{x}_{\mu_i} = (g_i(w_i))^+_{\mu_i} := \begin{cases} g_i(w_i) & \text{if } \mu_i > 0 \\ \max\{0, g_i(w)\} & \text{if } \mu_i = 0 \end{cases} \\ y_{\mu_i} = \nabla g_i(w_i)\nabla H_{\mu_i}(x_{\mu_i}),\ H_{\mu_i}(x_{\mu_i}) = \frac{1}{2}x_{\mu_i}^T\tau_{\mu_i}^{-1}x_{\mu_i} \end{aligned} \tag{19}$$

with state $x_{\mu_i} := \tau_{\mu_i}\mu_i \in \mathbb{R}_{\geq 0}$, outputs $y_{\mu_i} \in \mathbb{R}^l$, inputs $w_i \in \mathbb{R}^{2n}$, and $i \in \mathcal{I} := \{1, \ldots, l\}$. Here $g_i(.)$ is the i'th entry of the vector-valued function $g(.) = \text{col}_{i\in\mathcal{V}}\{g_i(.)\}$. Note that, for a given $i \in \mathcal{I}$ and for a constant input $\bar{w}_i$, the equilibrium set $\mathcal{Z}_{\mu_i}$ of (19) is characterized by all $(\bar{\mu}_i, \bar{w}_i)$ satisfying

$$g_i(\bar{w}_i) \leq 0, \qquad \bar{\mu}_i \geq 0, \qquad \bar{\mu}_i = 0 \vee g_i(\bar{w}_i) = 0. \tag{20}$$

More formally, for $i \in \mathcal{I}$ the equilibrium set $\mathcal{Z}_{\mu_i}$ of (19) is given by

$$\mathcal{Z}_{\mu_i} := \{(\bar{\mu}_i, \bar{w}_i) \mid (\bar{\mu}_i, \bar{w}_i) \text{ satisfies (20)}\}.$$

Remark 4. In case the inequality constraints of Remark 3 (e.g. $P_g \preceq P_g^{\max}$) are considered, the subsystems (19) take the decentralized form

$$\dot{x}_{\mu_i} = (P_{gi} - P_{gi}^{\max})^+_{\mu_i} = \begin{cases} P_{gi} - P_{gi}^{\max} & \text{if } \mu_i > 0 \\ \max\{0, P_{gi} - P_{gi}^{\max}\} & \text{if } \mu_i = 0 \end{cases}$$
$$y_{\mu_i} = \nabla H_{\mu_i}(x_{\mu_i}), \quad H_{\mu_i}(x_{\mu_i}) = \frac{1}{2} x_{\mu_i}^T \tau_{\mu_i}^{-1} x_{\mu_i}, \quad i \in \mathcal{V}, \tag{21}$$

and similar expressions can be given for the remaining inequalities $P_g^{\min} \preceq P_g,\ P_d^{\min} \preceq P_d \preceq P_d^{\max}$.

The subsystems (19) have the following passivity property [20].

Proposition 2 ([20])**.** *Let $i \in \mathcal{I}, (\bar{\mu}_i, \bar{w}_i) \in \mathcal{Z}_{\mu_i}$ and define $\bar{y}_{\mu_i} := \nabla g_i(\bar{w}_i)\bar{\mu}_i$. Then* (19) *is passive with respect to the shifted external port-variables $\tilde{w}_i := w_i - \bar{w}_i, \tilde{y}_{\mu_i} := y_{\mu_i} - \bar{y}_{\mu_i}$. Additionally, $(\mu_i, w_i) \to \mathcal{Z}_{\mu_i}$ as $t \to \infty$ for $(\mu_i, w_i), w_i = \bar{w}_i$ satisfying* (19)*.*

Consider again system (13)

$$\dot{x} = (J - R)\nabla H(x) + \nabla S(z) + G^T u$$
$$y = G\nabla H(x) = \begin{bmatrix} P_g \\ P_d \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} \tag{22}$$

with the state x defined by (12), where we introduce an additional input $u \in \mathbb{R}^{2n}$ and output $y \in \mathbb{R}^{2n}$.

Remark 5. Note that for any steady state $(\bar{x}, \bar{u})$ of (22), the latter system is passive with respect to the shifted external port-variables $\tilde{u} := u - \bar{u},\ \tilde{y} = y - \bar{y},\ \bar{y} := G\nabla H(\bar{x})$, using the storage function

$$\bar{H}(x) := H(x) - (x - \bar{x})^T \nabla H(\bar{x}) - H(\bar{x}). \tag{23}$$

We interconnect the subsystems (19) to (22) in a power-preserving way by

$$w_i = w = y \quad \forall i \in \mathcal{I}, \qquad u = -\sum_{i \in \mathcal{I}} y_{\mu_i}$$

to obtain the closed-loop system

$$\dot{\eta} = D^T \omega \tag{24a}$$
$$M\dot{\omega} = -D\Gamma(E_q')\,\mathbf{sin}\,\eta - A\omega + P_g - P_d \tag{24b}$$
$$T_d'\dot{E}_q' = -F(\eta)E_q' + E_f \tag{24c}$$
$$\tau_g \dot{P}_g = -\nabla C(P_g) + \lambda - \frac{\partial g}{\partial P_g}(P_g, P_d)\mu - \omega \tag{24d}$$
$$\tau_d \dot{P}_d = \nabla U(P_d) - \lambda - \frac{\partial g}{\partial P_d}(P_g, P_d)\mu + \omega \tag{24e}$$
$$\tau_v \dot{v} = -\nabla C_T(v) - D^T\lambda \tag{24f}$$
$$\tau_\lambda \dot{\lambda} = Dv - P_g + P_d \tag{24g}$$
$$\tau_{\mu_i}\dot{\mu}_i = (g_i(P_g, P_d))^+_{\mu_i}, \qquad i \in \mathcal{I}. \tag{24h}$$

Observe that the equilibrium set $\mathcal{Z}_2$ of (24), if expressed in the co-energy variables, is characterized by all $(\bar{z}, \bar{\mu})$ that satisfy (18) in addition to $\bar{\omega} = 0,\ -D\Gamma(\bar{E}_q')\,\mathbf{sin}\,\bar{\eta} + \bar{P}_g - \bar{P}_d = 0, -F(\bar{\eta})\bar{E}_q' + E_f = 0$, and therefore corresponds to the desired operation points.

Since both the subsystems (19) and the system (13) admit an *incrementally passivity* property with respect to their steady states, the closed-loop system inherits the same property provided that an equilibrium of (24) exists.

Theorem 2. *For every $(\bar{z}, \bar{\mu}) \in \mathcal{Z}_2$ satisfying Assumption 2 there exists a neighborhood Υ of $(\bar{z}, \bar{\mu})$ where all trajectories z satisfying* (24) *with initial conditions in Υ converge to the set $\mathcal{Z}_2$ and the convergence of each such trajectory is to a point.*

Proof. Let $(\bar{z}, \bar{\mu}) \in \mathcal{Z}_2$ and consider the shifted Hamiltonian $\bar{H}_e$ around $(\bar{x}, \bar{x}_\mu) = (\tau\bar{z}, \tau_\mu\bar{\mu})$ defined by

$$\bar{H}_e(x, x_\mu) := \bar{H}(x) + \sum_{i \in \mathcal{I}} \bar{H}_{\mu_i}(x_{\mu_i}) = \bar{H}(x) + \frac{1}{2}\tilde{x}_\mu^T \tau_{x_\mu}^{-1} \tilde{x}_\mu$$

where $\tilde{x}_\mu := x_\mu - \bar{x}_\mu$ and $\bar{H}$ is defined by (23). By Proposition 2 and Remark 5, the time-derivative of $\bar{H}_e$ satisfies

$$\dot{\bar{H}}_e \le \tilde{u}^T\tilde{y} + \tilde{w}^T \sum_{i \in \mathcal{I}} \tilde{y}_{\mu_i} = \tilde{u}^T\tilde{y} - \tilde{u}^T\tilde{y} = 0$$

where equality holds only if $P_g = \bar{P}_g, P_d = \bar{P}_d, \omega = 0, \nabla_{E_q'} H(x) = 0$. On the largest invariant set where $\dot{\bar{H}}_e = 0$ it follows by the second statement of Proposition 2 that $\mu = \bar{\mu}$. As a result, $\lambda = \bar{\lambda}$ and v, η, E_q' are constant on this invariant set. Since the right-hand side of (19) is discontinuous and takes the same form as in [25], we can apply the invariance principle for discontinuous Caratheodory systems [25, Proposition 2.1] to conclude that $(z, \mu) \to \mathcal{Z}_2$ as $t \to \infty$. By following the same line of arguments as in the proof of Theorem 1, convergence of each trajectory to a point is proven. □

Remark 6. Theorem 2 uses the Caratheodory variant of the Invariance Principle which requires that the Caratheodory solution of (24) is unique and that its omega-limit set is invariant [25]. These requirements are indeed satisfied by extending Lemmas 4.1-4.4 of [25] to the case where equality constraints and *nonstrict* convex/concave (utility) functions are considered in the optimization problem [25, equation (3)], noting that these lemmas only require convexity/concavity instead of their strict versions. In particular, by adding a quadratic function of the Lagrange multipliers associated with the equality constraints to the Lyapunov function, it can be proven that monotonicity of the primal-dual dynamics with respect to primal-dual optimizers as stated in [25, Lemma 4.1] holds for this more general case as well, see also [20], [31].

Remark 7. Instead of using the hybrid dynamics (19) for dealing with the inequality constraints (17c), we can instead introduce the so called *barrier functions* $B_i = -\nu\log(-g_i(P_g, P_d))$ that are added to the objective function [29]. Simultaneously, the corresponding inequalities in the social welfare problem (17) are removed to obtain the modified convex optimization problem

$$\begin{aligned} \min_{P_g, P_d, v} \quad & -S(P_g, P_d) - \nu \sum_{i \in \mathcal{V}} \log(-g_i(P_g, P_d)) \\ \text{s.t.} \quad & D_c v - P_g + P_d = 0. \end{aligned} \tag{25}$$

Here $\nu > 0$ is called the *barrier parameter* and is usually chosen small. By applying the primal-dual gradient method to

(25) it can be shown that, if the system is initialized in the interior of the feasible region, i.e. where (17c) holds, then the trajectories of the resulting gradient dynamics remain within the feasible region and the system converges to a suboptimal value of the social welfare [7], [29], [32]. However, if Slater's condition holds, this suboptimal value which depends on ν converges to the optimal value of the social welfare problem as $\nu \to 0$ [29]. The particular advantage of using barrier functions is to avoid the use of an hybrid controller and to enforce that the trajectories remain within the feasible region for all future time.

B. Including line congestion and transmission costs

The previous section shows how to include nodal power constraints into the social welfare problem. In case the network is acyclic, line congestion and power transmission costs can be incorporated into the optimization problem as well.

To this end, define the (modified) social welfare by $U(P_d) - C(P_g) - C_T(v)$ where the convex function $C_T(v)$ corresponds to the power transmission cost. If security constraints on the transmission lines are included as well, the optimization problem (6) modifies to

$$\begin{aligned} \min_{P_g, P_d, v} \quad & -S(P_g, P_d, v) := C(P_g) + C_T(v) - U(P_d) && \text{(26a)} \\ \text{s.t.} \quad & Dv - P_g + P_d = 0 && \text{(26b)} \\ & -\kappa \preceq v \preceq \kappa, && \text{(26c)} \end{aligned}$$

where $\kappa \in \mathbb{R}^m$ satisfies the element-wise inequality $\kappa \succ 0$. Note that in this case the communication graph is chosen to be identical with the topology of the physical network, i.e., $D_c = D$. As a result, the additional constraints (26c) bound the (virtual) power flow along the transmission lines as $|v_k| \leq \kappa_k, k \in \mathcal{E}$. The corresponding Lagrangian is given by

$$\begin{aligned} \mathcal{L} = {} & C(P_g) + C_T(v) - U(P_d) + \lambda^T (Dv - P_g + P_d) \\ & + \mu_+^T (v - \kappa) + \mu_-^T (-\kappa - v) \end{aligned}$$

with Lagrange multipliers $\lambda \in \mathbb{R}^n$, μ_+, $\mu_+ \in \mathbb{R}^m_{\geq 0}$. The resulting KKT optimality conditions are given by

$$\begin{aligned} \nabla C(\bar{P}_g) - \bar{\lambda} = 0, \qquad & -\nabla U(\bar{P}_d) + \bar{\lambda} = 0, \\ \nabla C_T(\bar{v}) + D^T \bar{\lambda} + \bar{\mu}_+ - \bar{\mu}_- = 0, \qquad & \\ -\kappa \preceq \bar{v} \preceq \kappa, \qquad & D\bar{v} - \bar{P}_g + \bar{P}_d = 0, \\ \bar{\mu}_+, \bar{\mu}_- \succeq 0, \quad \bar{\mu}_+^T (\bar{v} - \kappa) = 0, \quad & \bar{\mu}_-^T (-\kappa - \bar{v}) = 0. \end{aligned} \tag{27}$$

Suppose that Slater's condition holds. Then, since the optimization problem (26) is convex, it follows that $(\bar{P}_g, \bar{P}_d, \bar{v})$ is an optimal solution to (26) if and only if there exists $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\mu} = \mathrm{col}(\bar{\mu}_+, \bar{\mu}_-) \in \mathbb{R}^{2m}_{\geq 0}$ satisfying (27) [29].

By applying the gradient method to (26) in a similar manner as before and connecting the resulting controller with the physical model (2), we obtain the following closed-loop system:

$$\begin{aligned} \dot{\eta} &= D^T \omega && \text{(28a)} \\ M\dot{\omega} &= -D\Gamma(E'_q) \sin \eta - A\omega + P_g - P_d && \text{(28b)} \\ T'_d \dot{E}'_q &= -F(\eta) E'_q + E_f && \text{(28c)} \\ \tau_g \dot{P}_g &= -\nabla C(P_g) + \lambda - \omega && \text{(28d)} \\ \tau_d \dot{P}_d &= \nabla U(P_d) - \lambda + \omega && \text{(28e)} \\ \tau_v \dot{v} &= -\nabla C_T(v) - D^T \lambda - \mu_+ + \mu_- && \text{(28f)} \\ \tau_\lambda \dot{\lambda} &= Dv - P_g + P_d && \text{(28g)} \\ \tau_+ \dot{\mu}_+ &= (v - \kappa)^+_{\mu_+} && \text{(28h)} \\ \tau_- \dot{\mu}_- &= (-\kappa - v)^+_{\mu_-}. && \text{(28i)} \end{aligned}$$

The latter system can partially be put into a port-Hamiltonian form, since equations (28a)-(28g) can be rewritten as

$$\begin{aligned} \dot{x} &= (J - R)\nabla H(x) + \nabla S(z) + N\mu \\ N &= \begin{bmatrix} 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}^T, \end{aligned} \tag{29}$$

where the variables x, z and the Hamiltonian H are respectively defined by (12) and (13) as before, and $\mu = \mathrm{col}(\mu_+, \mu_-)$.

Since the network topology is a tree (i.e. $\ker D = \{0\}$), the equilibrium of (28) satisfies $\bar{v} = \Gamma(\bar{E}'_q) \sin \bar{\eta}$. Hence, the controller variable v corresponds to the *physical* power flow of the network if the closed-loop system is at steady state. Consequently, the constraints and costs on v correspond to constraints and costs of the physical power flow if the system converges to an equilibrium.

Theorem 3. *Let the network topology be acyclic and let $(\bar{z}, \bar{\mu})$ be an (isolated) equilibrium of* (28) *satisfying Assumption 2. Then all trajectories (z, μ) of* (28) *initialized in a sufficiently small neighborhood around $(\bar{z}, \bar{\mu})$ converge asymptotically to $(\bar{z}, \bar{\mu})$*

Proof. Let $(\bar{z}, \bar{\mu})$ be the equilibrium of (28). By defining the shifted Hamiltonian $\bar{H}(x)$ around $\bar{x} := \tau \bar{z}$ by

$$\bar{H}(x) = H(x) - (x - \bar{x})^T \nabla H(\bar{x}) - H(\bar{x})$$

one can rewrite (29) as

$$\dot{x} = (J - R)\nabla \bar{H}(x) + \nabla S(z) - \nabla S(\bar{z}) + N\tilde{\mu} \tag{30}$$

where $\tilde{\mu} := \mu - \bar{\mu}$. Consider candidate Lyapunov function

$$V(x, \mu) = \bar{H}(x) + \frac{1}{2} \tilde{\mu}_+ \tau_{\mu_+} \tilde{\mu}_+ + \frac{1}{2} \tilde{\mu}_-^T \tau_{\mu_-} \tilde{\mu}_-$$

and observe that

$$\begin{aligned} \tilde{\mu}_+^T (v - \kappa)^+_{\mu_+} &\leq \tilde{\mu}_+^T (v - \kappa) = \tilde{\mu}_+^T (\bar{v} - \kappa + \tilde{v}) \leq \tilde{\mu}_+^T \tilde{v} \\ \tilde{\mu}_-^T (-\kappa - v)^+_{\mu_-} &\leq \tilde{\mu}_-^T (-\kappa - v) \\ &= \tilde{\mu}_-^T (-\kappa - \bar{v} - \tilde{v}) \leq -\tilde{\mu}_-^T \tilde{v}. \end{aligned}$$

Bearing in mind (30), the time-derivative of V amounts to

$$\begin{aligned}\dot{V} &= -\omega^T A\omega - (\nabla_{E_q'} H(x))^T R_q \nabla_{E_q'} H(x)\\ &+ (z-\bar{z})^T(\nabla S(z) - \nabla S(\bar{z}))\\ &- \tilde{v}^T\tilde{\mu}_+ + \tilde{v}^T\tilde{\mu}_- + \tilde{\mu}_+^T(v-\kappa)_{\mu_+}^+ + \tilde{\mu}_-^T(-\kappa - v)_{\mu_-}^+\\ &\leq -\omega^T A\omega + (z-\bar{z})^T(\nabla S(z) - \nabla S(\bar{z}))\\ &- (\nabla_{E_q'} H(x))^T R_q \nabla_{E_q'} H(x) \leq 0\end{aligned}$$

where equality holds only if $\nabla_{E_q'} H(x) = 0, \omega = 0, P_g = \bar{P}_g, P_d = \bar{P}_d$. On the largest invariant set $\mathcal{S}$ where $\nabla_{E_q'} H(x) = 0, \omega = 0, P_g = \bar{P}_g, P_d = \bar{P}_d$ it follows that, since the graph contains no cycles $\lambda = \bar{\lambda}, v = \bar{v}, \mu = \bar{\mu}$ and that η, E_q' are constant, which corresponds to an equilibrium. In particular $\nabla V(x,\mu) = 0$ for all $(z,\mu) \in \mathcal{S}$ and $(\bar{z},\bar{\mu}) \in \mathcal{S}$. Since by Assumption 2 we have $\nabla^2 V(\bar{x},\bar{\mu}) > 0$, it follows that $(\bar{z},\bar{\mu})$ is isolated. By the invariance principle for discontinuous Caratheodory systems [25] all trajectories (z,μ) of (28) initializing in a sufficiently small neighborhood around $(\bar{z},\bar{\mu})$ satisfy $\mu \to \bar{\mu}$, $z \to \bar{z}$ as $t \to \infty$.

□

Remark 8. It is possible to include nodal power constraints, line congestion and transmission costs simultaneously. However, as the results in this section are only valid for acyclic graphs, it should also be assumed for the more general case that the physical network is a tree.

C. State transformation

Consider again the minimization problem (6). As shown before, by applying the gradient method to the social welfare problem, the closed-loop system (13) is obtained.

Note that in the λ-dynamics the demand P_d appears, which in practice is often uncertain. A possibility to eliminate the demand from the controller dynamics is by a state transformation [10], [12]. To this end, define the new variables

$$\hat{x} := \begin{bmatrix}\eta\\ p\\ E_q'\\ x_g\\ x_d\\ x_v\\ x_\theta\end{bmatrix} = \begin{bmatrix}I&0&0&0&0&0&0\\ 0&I&0&0&0&0&0\\ 0&0&I&0&0&0&0\\ 0&0&0&I&0&0&0\\ 0&0&0&0&I&0&0\\ 0&0&0&0&0&I&0\\ 0&I&0&0&0&0&I\end{bmatrix} x = \hat{\tau}\begin{bmatrix}\eta\\ p\\ E_q'\\ P_g\\ P_d\\ v\\ \theta\end{bmatrix} =: \hat{\tau}\hat{z},$$

i.e., $x_\theta := \tau_\theta\theta = p + x_\lambda$. Then the port-Hamiltonian system (13) transforms to

$$\begin{aligned}\dot{\hat{x}} = &\begin{bmatrix}0&D^T&0&0&0&0&D^T\\ -D&-A&0&I&-I&0&-A\\ 0&0&-R_q&0&0&0&0\\ 0&-I&0&0&0&0&0\\ 0&I&0&0&0&0&0\\ 0&0&0&0&0&0&-D_c^T\\ -D&-A&0&0&0&D_c&-A\end{bmatrix}\nabla\hat{H}(\hat{x})\\ &+\nabla S(\hat{z})\end{aligned} \tag{31}$$

with Hamiltonian

$$\begin{aligned}\hat{H}(\hat{x}) = H_p &+ \frac{1}{2}x_g^T\tau_g^{-1}x_g + \frac{1}{2}x_d^T\tau_d^{-1}x_d\\ &+ \frac{1}{2}x_v^T\tau_v^{-1}x_v + \frac{1}{2}(x_\theta - p)\tau_\lambda^{-1}(x_\theta - p).\end{aligned}$$

By writing the system of differential equations (31) explicitly we obtain

$$\begin{aligned}\dot{\eta} &= D^T\omega\\ M\dot{\omega} &= -D\Gamma(E_q')\,\mathbf{sin}\,\eta - A\omega + P_g - P_d\\ T_d'\dot{E}_q' &= -F(\eta)E_q' + E_f\\ \tau_g\dot{P}_g &= -\nabla C(P_g) + \tau_\lambda^{-1}(\tau_\theta\theta - M\omega) - \omega\\ \tau_d\dot{P}_d &= \nabla U(P_d) - \tau_\lambda^{-1}(\tau_\theta\theta - M\omega) + \omega\\ \tau_v\dot{v} &= -D_c^T\tau_\lambda^{-1}(\tau_\theta\theta - M\omega)\\ \tau_\theta\dot{\theta} &= D_c v - D\Gamma\,\mathbf{sin}\,\eta - A\omega.\end{aligned} \tag{32}$$

Define $\mathcal{Z}_4$ as the set of all $\hat{z}^* := (\bar{\eta},\bar{\omega},\bar{P}_g,\bar{P}_d,\bar{v},\bar{\theta})$ that are an equilibrium of (32). Using the previous established tools we can prove asymptotic stability to the set of optimal points $\mathcal{Z}_4$.

Theorem 4. *For every $\hat{z}^* \in \mathcal{Z}_4$ satisfying Assumption 2 there exists a neighborhood Υ around $\hat{z}^*$ where all trajectories $\hat{z}$ satisfying* (32) *(or equivalently* (31)*) and initializing in Υ converge to $\mathcal{Z}_4$. In addition, the convergence of each such trajectory is to a point.*

Proof. We proceed along the same lines as in the proof of Theorem 1. Since the stability result of Theorem 1 is preserved after a state transformation, the proof is concluded. □

Note that the latter result holds for all $\tau_g, \tau_d, \tau_v, \tau_\lambda, \tau_\theta > 0$. The controller appearing in (32) can be simplified by choosing $\tau_\lambda = \tau_\theta = M$. As a result, the controller dynamics is described by

$$\tau_g\dot{P}_g = -\nabla C(P_g) + \theta - 2\omega \tag{33a}$$

$$\tau_d\dot{P}_d = \nabla U(P_d) - \theta + 2\omega \tag{33b}$$

$$\tau_v\dot{v} = -D_c^T(\theta - \omega) \tag{33c}$$

$$M\dot{\theta} = D_c v - D\Gamma(E_q')\,\mathbf{sin}\,\eta - A\omega. \tag{33d}$$

The main advantage of controller design (33) is that no information about the power supply and demand is required in the dynamic pricing algorithm (33c), (33d), where we observe that the quantity $\theta - 2\omega$ acts here as the electricity price for the producers and consumers. Another benefit of the proposed dynamic pricing algorithm is that, contrary to [16], no information is required about $\dot{\omega}$.

On the other hand, knowledge about the physical power flows and the power system parameters M, A is required. Determining the radius of uncertainty of these parameters under which asymptotic stability is preserved remains an open question [10]; see [17] for results in a similar setting where only the damping term A is assumed to be uncertain.

D. Relaxing the strict convexity assumption

By making a minor modification to the social welfare problem (6), it is possible to relax the condition that the

functions C, U are *strictly* convex and concave respectively. To this end, consider the optimization problem

$$\min_{P_g, P_d, v} \quad C(P_g) - U(P_d) + \frac{1}{2}\rho||D_c v - P_g + P_d||^2 \tag{34a}$$

$$\text{s.t.} \quad D_c v - P_g + P_d = 0, \tag{34b}$$

where $\rho > 0$, $C(P_g)$ is convex and $U(P_d)$ is concave, which makes the optimization problem (34) convex. Suppose that there exists a feasible solution to the minimization problem, then the set of optimal points of (34) is identical with the set of optimal points of (6) which is characterized by set of points satisfying the KKT conditions (8). The corresponding augmented Lagrangian of (34) is given by

$$\begin{aligned}\mathcal{L}_p = C(P_g) - U(P_d) - \lambda^T(D_c v + P_g - P_d) \\ + \frac{1}{2}\rho||D_c v + P_g - P_d||^2.\end{aligned}$$

Consequenctly, the distributed dynamics of the primal-dual gradient method applied to (34) amounts to

$$\begin{aligned}\tau_g \dot{P}_g &= -\nabla C(P_g) + \lambda - \rho(D_c v + P_g - P_d) \\ \tau_d \dot{P}_d &= \nabla U(P_d) - \lambda + \rho(D_c v + P_g - P_d) \\ \tau_v \dot{v} &= D_c^T \lambda - \rho D_c^T(D_c v + P_g - P_d) \\ \tau_\lambda \dot{\lambda} &= -D_c v - P_g + P_d,\end{aligned} \tag{35}$$

which can be written in the same port-Hamiltonian form as (13) where in this case

$$S(P_g, P_d, v) = U(P_d) - C(P_g) - \frac{1}{2}\rho||D_c v - P_g + P_d||^2. \tag{36}$$

This leads to the following result.

Theorem 5. *Consider the system* (13) *where S is given by* (36) *and suppose that C, U are convex and concave functions respectively. Then for every $\bar{z} \in \mathcal{Z}_1$ satisfying Assumption 2, where $\mathcal{Z}_1$ is defined by* (14)*, there exists a neighborhood Υ around $\bar{z}$ wherein each trajectory z satisfying* (13) *converges to a point in $\mathcal{Z}_1$.*

Proof. Let $\bar{z} \in \mathcal{Z}_1$. By the proof of Theorem 1 it follows that

$$\begin{aligned}\dot{\bar{H}} = -\omega^T A \omega + (z - \bar{z})^T(\nabla S(z) - \nabla S(\bar{z})) \\ - (\nabla_{E_q'} \bar{H})^T R_q \nabla_{E_q'} \bar{H},\end{aligned}$$

where the second term can be written as

$$\begin{aligned}&\tilde{P}_d^T(\nabla U(P_d) - \nabla U(\bar{P}_d)) - \tilde{P}_g^T(\nabla C(P_g) - \nabla C(\bar{P}_g)) \\ &- \rho \begin{bmatrix}\tilde{P}_g \\ \tilde{P}_d \\ \tilde{v}\end{bmatrix}^T \begin{bmatrix}-I & I & -D_c \\ I & -I & D_c \\ -D_c^T & D_c^T & -D_c^T D_c\end{bmatrix} \begin{bmatrix}\tilde{P}_g \\ \tilde{P}_d \\ \tilde{v}\end{bmatrix} \leq 0\end{aligned} \tag{37}$$

where $\tilde{P}_g = P_g - \bar{P}_g, \tilde{P}_d = P_d - \bar{P}_d, \tilde{v} = v - \bar{v}$. Hence, we obtain that $\dot{\bar{H}} \leq 0$ where equality holds only if $\omega = 0, \nabla_{E_q'} \bar{H}(x) = 0$ and $D_c \tilde{v} + \tilde{P}_g - \tilde{P}_d = D_c v + P_g - P_d = 0$. On the largest invariant set $\mathcal{S}$ where $\dot{\bar{H}}_c = 0$ we have $\omega = 0$ and η, E_q' are constant and (P_g, P_d, v, λ) satisfy the KKT optimality conditions (8). Therefore $\mathcal{S} \subset \mathcal{Z}_1$ and by LaSalle's invariance principle there exists a neighborhood Υ around $\bar{z}$ where all trajectories z satisfying (13) converge to the set $\mathcal{S} \subset \mathcal{Z}_1$. By continuing along the same lines as the proof of Theorem 1, convergence of each trajectory to a point is proven. □

Remark 9. Adding the quadratic term in the social welfare problem as done in (34a) provides an additional advantage. As this introduces more damping in the resulting gradient-method-based controller, see (37), it may improve the convergence properties of the closed-loop dynamics [33], [34]. Moreover, the amount of damping injected into the system depends on parameter ρ, which can be chosen freely.

V. Conclusions and future research

In this paper a unifying and systematic energy-based approach in modeling and stability analysis of power networks has been established. Convergence of the closed-loop system to the set of optimal points using gradient-method-based controllers have been proven using passivity based arguments. This result is extended to the case where nodal power constraints are included into the problem as well. However, for line congestion and power transmission cost the power network is required to be acyclic to prove asymptotic stability to the set of optimal points.

The results established in this paper lend themselves to many possible extensions. One possibility is to design an additional (passive) controller that regulates the voltages to the desired values or achieves alternative objectives like (optimal) reactive power sharing. This could for example be realized by continuing along the lines of [28].

Recent observations, see [35], suggest that the port-Hamiltonian framework also lends itself to consider higher-dimensional models for the synchronous generator than the third-order model used in this paper, while the same controllers as designed in the present paper can be used in this case as well. In addition, current research includes extending the results of the present paper to network-preserving models where a distinction is made between generator and load nodes.

One of the remaining open questions is how to deal with line congestion and power transmission costs in cyclic power networks with nonlinear power flows. In addition, all of the results established for the nonlinear power network only provide *local* asymptotic stability to the set of optimal points. Future research includes determining the region of attraction.

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