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# A uniqueness result for a Schrödinger–Poisson system with strong singularity

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**Abstract.** In this paper, we consider the following Schrödinger–Poisson system with strong singularity

$$\begin{cases} -\Delta u + \phi u = f(x)u^{-\gamma}, & x \in \Omega, \\ -\Delta \phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\gamma > 1$, $f \in L^1(\Omega)$ is a positive function (i.e. $f(x) > 0$ a.e. in $\Omega$). A necessary and sufficient condition on the existence and uniqueness of positive weak solution of the system is obtained. The results supplement the main conclusions in recent literature.

**Keywords:** Schrödinger–Poisson system, strong singularity, uniqueness, variational method, necessary and sufficient condition.

**2010 Mathematics Subject Classification:** 35A15, 35B09, 35J75.

## 1 Introduction

In this paper, we consider the existence and uniqueness of positive solution for the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \phi u = f(x)u^{-\gamma}, & x \in \Omega, \\ -\Delta \phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{SP}$$

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where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\gamma > 1$, $f \in L^1(\Omega)$ is a positive function (i.e. $f(x) > 0$ a.e. in $\Omega$). System (SP) can be viewed as a special case of the following Schrödinger–Poisson system with singularity

$$\begin{cases} -\Delta u + \eta\phi u = f(x)u^{-\gamma} + g(x,u), & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

which has been investigated recently. When $g(x,u) = 0$, $f(x) = \mu$ is a positive parameter and $0 < \gamma < 1$ (i.e. weak singularity), Zhang [28] obtained a sufficient condition on the existence, uniqueness and multiplicity of positive solutions for system (1.1) with $\eta = \pm 1$. When $\eta = -1$, $g(x,u) = \lambda h(x)u + u^3$, $f(x) = \frac{\mu}{|x|^\beta}$ and $0 < \gamma < 1$, Wang [25] considered the existence and multiplicity of positive solutions for system (1.1) under some suitable conditions by Nehari manifold. Combining with variational method and Nehari manifold method, Lei and Liao [7] generalized a part of the results in Zhang [28] to the critical problem and obtained two positive solutions of system (1.1) with $\eta = 1$, $g(x,u) = u^5$, $f(x) = \frac{\mu}{|x|^\beta}$ and $0 < \gamma < 1$. Jiang and Zhou [5] established the existence and a priori estimate of positive solutions of non-autonomous Schrödinger–Poisson system with singular potential. In addition, Kirchhoff type of problems with singularity have been considered by many researchers, one could refer to [3, 8, 9, 14–16, 24, 26] and the references cited therein. In a more general sense, Lei, Suo and Chu [10] studied a class of Schrödinger–Newton systems with singular and critical growth terms in unbounded domains and established results on the existence and multiplicity of positive solutions. We [27] obtained the uniqueness and asymptotical behavior of solutions to a Choquard equation with singularity in unbounded domains. Mu and Lu [17], Li et al. [13] and Zhang [29] studied the existence, uniqueness and multiple results to singular Schrödinger–Kirchhoff–Poisson system.

However, investigations (see [3, 5, 7–10, 13–17, 24–29] and references therein) considered elliptic equations with singularity have mainly focused on weak singularity (i.e. $0 < \gamma < 1$ ) and seldom with strong singularity (i.e. $\gamma > 1$ ) which have been studied extensively (see [1, 2, 4, 6, 11, 12, 18–23, 30] and references therein). In 2013, Sun [20] considered the following nonlinear elliptic problem

$$\begin{cases} -\Delta u = f(x)u^{-\gamma} + k(x)u^q, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded open set with smooth boundary $\partial\Omega$, $k \in L^\infty(\Omega)$ is a non-negative function, $q \in (0,1)$, $\gamma > 1$ (i.e. strong singularity) and $f \in L^1(\Omega)$ is positive (i.e. $f(x) > 0$ a.e. in $\Omega$). By using variational method, Sun [20] has derived a compatible condition between coefficients and negative exponents, which is optimal for $H_0^1(\Omega)$-solutions of problem (1.2). The results obtained by Sun [20] supplement and improve the main conclusions in [9-13]. When $N \geq 3$ and $k(x) \equiv 0$, Sun [22] further obtained the existence of solutions of problem (1.2) and showed the reason on why 3 plays a crucial role in the study of elliptic equations with negative exponents. When $k(x) \equiv 0$ and $-\Delta u$ was replaced by $-\mathrm{div}(M(x)\nabla u)$ where $M(x)$ is a bounded elliptic matrix, Tan and Sun [23] also proved the existence of a positive $H_0^1(\Omega)$-solutions of problem (1.2). Furthermore, Cong and Han [2], Li and Gao [11] both considered the existence of positive solutions to elliptic boundary value problem with strong singularity and $p$-Laplace operator. As for Kirchhoff type equations with strong singularities, Li et al. [12], Tan and Sun [21] and Santos et al. [18] have obtained some perfect results. However, to the best of our knowledge, Schrödinger–Poisson system with strong singularity has not been studied until now. Thus, the main purpose of this paper is to consider the existence and uniqueness of positive solution for system (SP) with strong singularity. Indeed, we obtain the following results.

**Theorem 1.1.** *Assume that $f \in L^1(\Omega)$ is a positive function (i.e. $f(x) > 0$ a.e. in $\Omega$), $\gamma > 1$, then system (SP) admits a unique positive solution if and only if there exists a $u_0 \in H_0^1(\Omega)$, such that*

$$\int_\Omega f(x)|u_0|^{1-\gamma}\mathrm{d}x < +\infty. \tag{1.3}$$

As a consequence of Theorem 1.1, we also have the following.

**Theorem 1.2.** *Suppose $f_1$, $f_2 \in L^1(\Omega)$ are two positive functions (i.e. $f_i(x) > 0$, $i = 1,2$ a.e. in $\Omega$) with $\int_\Omega f_i(x)|u_0|^{1-\gamma}\mathrm{d}x < +\infty$, $i = 1,2$ and $u_1$, $u_2$ are the corresponding solutions of system (SP) obtained in Theorem 1.1, then $f_1 \geq f_2$ implies $u_1 \geq u_2$.*

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain containing 0. Suppose $0 < \alpha < 3$ and $1 < \gamma < 3$, then*

$$\begin{cases} -\Delta u + \phi u = |x|^{-\alpha}u^{-\gamma}, & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases}$$

*admits a unique positive solution $u \in H_0^1(\Omega)$.*

We then consider the property of the $H_0^1(\Omega)$-solution in Theorem 1.3 and get the following result.

**Theorem 1.4.** *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain containing 0. Suppose $\alpha > 2$ and $\gamma > 0$, then*

$$\begin{cases} -\Delta u + \phi u = |x|^{-\alpha}u^{-\gamma}, & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases}$$

*admits no bounded positive solution.*

**Notations**

- $L^s(\Omega)$ is a Lebesgue space whose norm is denoted by $|u|_s = (\int_\Omega |u|^s\mathrm{d}x)^{\frac{1}{s}}$.
- $H_0^1(\Omega)$ is the usual Sobolev space equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2\mathrm{d}x$.
- $u^+ = \max\{u,0\}$ and $u^- = \min\{u,0\}$ for any function $u$.
- $\to$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.
- $B_r(x_0)$ denotes the Euclidean ball of center $x_0$ and radius $r$.
- $C$ and $C_i$ $(i = 1,2,\dots)$ denotes various positive constants, which may vary from line to line.

## 2 Proof of main results

Before proving our main results, we need the following lemma (see [28]).

**Lemma 2.1.** *For each $u \in H_0^1(\Omega)$, there exists a unique $\phi_u \in H_0^1(\Omega)$ solution of*

$$\begin{cases} -\Delta\phi = u^2, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

*Moreover,*

*(i)* $\|\phi_u\|^2 = \int_\Omega \phi_u u^2 \mathrm{d}x$;

*(ii)* $\phi_u \geq 0$. *Moreover,* $\phi_u > 0$ *when* $u \neq 0$;

*(iii) for each* $t \neq 0$, $\phi_{tu} = t^2\phi_u$;

*(iv) for any* $u \in H_0^1(\Omega)$,

$$\int_\Omega \phi_u u^2 \mathrm{d}x = \int_\Omega |\nabla\phi_u|^2 \mathrm{d}x \leq S^{-1}|u|_{12/5}^4 \leq S^{-1}|u|_4^4|\Omega|^{2/3} \leq S^{-3}\|u\|^4|\Omega|,$$

*where* $S > 0$ *is the best Sobolev embedding constant.*

*(v) assume that* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$*, then* $\phi_{u_n} \to \phi_u$ *in* $H_0^1(\Omega)$ *and* $\int_\Omega \phi_{u_n} u_n v \mathrm{d}x \to \int_\Omega \phi_u uv \mathrm{d}x$ *for any* $v \in H_0^1(\Omega)$;

*(vi) we denote* $\Psi(u) = \int_\Omega \phi_u u^2 \mathrm{d}x$*, then* $\Psi : H_0^1(\Omega) \to \mathbb{R}$ *is* $C^1$ *and for any* $v \in H_0^1(\Omega)$,

$$\langle \Psi'(u), v\rangle = 4\int_\Omega \phi_u uv \mathrm{d}x;$$

*(vii) for* $u, v \in H_0^1(\Omega)$, $\int_\Omega (\phi_u u - \phi_v v)(u - v)\mathrm{d}x \geq \frac{1}{2}\|\phi_u - \phi_v\|^2$.

According to Lemma 2.1, we substitute $\phi_u$ to the first equation of system (SP), then system (SP) transforms into the following equation

$$\begin{cases} -\Delta u + \phi_u u = f(x)u^{-\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

The energy functional corresponding to equation (2.1) given by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_\Omega \phi_u |u|^2 \mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x, \tag{2.2}$$

and a function $u$ is called a solution of equation (2.1), i.e. $(u, \phi_u)$ is a solution of system (SP) if $u \in H_0^1(\Omega)$ such that $u > 0$ in $\Omega$ and for every $\psi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla\psi \mathrm{d}x + \int_\Omega \phi_u u\psi \mathrm{d}x - \int_\Omega f(x)u^{-\gamma}\psi \mathrm{d}x = 0. \tag{2.3}$$

For the sake of simplicity, we just say $u$ instead of $(u, \phi_u)$ is a solution of system (SP). In order to motivate our results, we consider the following two constrained sets:

$$\mathcal{N}_1 = \left\{ u \in H_0^1(\Omega) \,:\, \|u\|^2 + \int_\Omega \phi_u |u|^2 \mathrm{d}x - \int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x \geq 0 \right\},$$

and

$$\mathcal{N}_2 = \left\{ u \in H_0^1(\Omega) \,:\, \|u\|^2 + \int_\Omega \phi_u |u|^2 \mathrm{d}x - \int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x = 0 \right\}.$$

We now come to prove our main results.

***Proof of Theorem 1.1.*** (Necessity). Suppose $u \in H_0^1(\Omega)$ is the solution of system (SP), then $u > 0$ and satisfies (2.3). Choosing $\psi = u$ in (2.3) leads to

$$\int_\Omega f(x)u^{1-\gamma}\mathrm{d}x = \|u\|^2 + \int_\Omega \phi_u u^2 \mathrm{d}x < +\infty,$$

and the necessity is proved.

(Sufficiency) The proof will be complete in six steps.

**Step 1.** $\boldsymbol{\mathcal{N}_i \neq \emptyset,\ i = 1, 2.}$

Fix $u \in H_0^1(\Omega)$ with $\int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x < +\infty$. For any $t > 0$, according to Lemma 2.1 (iii), we have

$$I(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_\Omega \phi_u |u|^2 \mathrm{d}x - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x.$$

Set $g(t) = t\frac{\mathrm{d}I(tu)}{\mathrm{d}t}$, then

$$g(t) = t^2\|u\|^2 + t^4\int_\Omega \phi_u |u|^2 \mathrm{d}x - t^{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x.$$

Since $\gamma > 1$, one can easily obtain that $g(t)$ is increasing on $(0, +\infty)$ with $\lim_{t\to 0^+} g(t) = -\infty$ and $\lim_{t\to+\infty} g(t) = +\infty$. Thus, there exists a unique $t(u) > 0$ such that $I(t(u)u) = \min_{t>0} I(tu)$ and $g(t(u)) = 0$, i.e.

$$t^2(u)\|u\|^2 + t^4(u)\int_\Omega \phi_u |u|^2 \mathrm{d}x - t^{1-\gamma}(u)\int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x = 0,$$

that is $t(u)u \in \mathcal{N}_2$. Specially, the assumption (1.3) implies that there exists a $t(u_0) > 0$ such that $t(u_0)u_0 \in \mathcal{N}_2 \subset \mathcal{N}_1$, and so $\mathcal{N}_i \neq \emptyset,\ i = 1, 2$.

**Step 2.** $\boldsymbol{\mathcal{N}_1}$ **is an unbounded closed set in** $\boldsymbol{H_0^1(\Omega)}$ **and there exists a positive constant** $\boldsymbol{C_1}$**, such that** $\boldsymbol{\|u\| \geq C_1}$ **for all** $\boldsymbol{u \in \mathcal{N}_1}$**.**

According to Step 1, $tu \in \mathcal{N}_1$ for any $t \geq t(u_0)$, so $\mathcal{N}_1$ is unbounded in $H_0^1(\Omega)$. The closeness of $\mathcal{N}_1$ follows easily from Lemma 2.1 (v) and Fatou's lemma. We claim that there exists a positive constant $C_1$, such that $\|u\| \geq C_1$ for all $u \in \mathcal{N}_1$. Arguing by contradiction, there exists a sequence $\{u_n\} \subset \mathcal{N}_1$ satisfying $u_n \to 0$ in $H_0^1(\Omega)$. Since $\gamma > 1$ and $u_n \in \mathcal{N}_1$, by the reverse form of Hölder's inequality and Lemma 2.1 (v), one can get

$$\left(\int_\Omega f^{\frac{1}{\gamma}}(x)\mathrm{d}x\right)^\gamma \left(\int_\Omega |u_n|\mathrm{d}x\right)^{1-\gamma} \leq \int_\Omega f(x)|u_n|^{1-\gamma}\mathrm{d}x \leq \|u_n\|^2 + \int_\Omega \phi_{u_n}|u_n|^2\mathrm{d}x \to 0.$$

Since $\int_\Omega f^{\frac{1}{\gamma}}(x)\mathrm{d}x > 0$, we have $\int_\Omega |u_n|\mathrm{d}x \to \infty$, which is impossible. So there exists a positive constant $C_1$, such that $\|u\| \geq C_1$ for all $u \in \mathcal{N}_1$.

**Step 3. Properties of the minimizing sequence $\{u_n\}$.**

For any $u \in \mathcal{N}_1$, according to Step 2, there exists a positive constant $C_1$ such that $\|u\| \geq C_1$, then by (2.2), $\gamma > 1$ and Lemma 2.1 (ii), one has

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_\Omega \phi_u |u|^2 \mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}\mathrm{d}x \geq \frac{1}{2}\|u\|^2,$$

therefore, $I(u)$ is coercive and bounded from below on $\mathcal{N}_1$ and so $\inf_{\mathcal{N}_1} I$ is well defined. Since $\mathcal{N}_1$ is closed, applying the Ekeland variational principle to construct a minimizing sequence $\{u_n\} \subset \mathcal{N}_1$ satisfying:

(1) $I(u_n) < \inf_{\mathcal{N}_1} I + \frac{1}{n}$;

(2) $I(z) \geq I(u_n) - \frac{1}{n}\|u_n - z\|, \forall z \in \mathcal{N}_1$.

The coerciveness of $I$ on $\mathcal{N}_1$ shows that $\|u_n\| \leq C_2$ uniformly for some suitable positive constant $C_2$. Hence, $C_1 \leq \|u_n\| \leq C_2$ and then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and a function $u_* \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_* \quad \text{in } H_0^1(\Omega), \\ u_n &\to u_* \quad \text{in } L^p(\Omega),\ p \in [1,6), \\ u_n &\to u_* \quad \text{a.e. in } \Omega. \end{aligned}$$

Since $I(|u|) = I(u)$, we could assume that $u_n \geq 0$. By $\{u_n\} \subset \mathcal{N}_1$, Lemma 2.1 (iv) and the boundness of $\{u_n\}$, we have $\int_\Omega f(x)u_n^{1-\gamma}\mathrm{d}x < +\infty$ which implies that $u_n(x) > 0$ a.e. in $\Omega$ since $f(x) > 0$ a.e. in $\Omega$, and $\gamma > 1$. Therefore, $u_*(x) \geq 0$. Furthermore, by Fatou's Lemma, we get $\int_\Omega f(x)u_*^{1-\gamma}\mathrm{d}x < +\infty$ which in turn implies $u_*(x) > 0$ a.e. in $\Omega$.

**Step 4. $u_* \in \mathcal{N}_2$, $\inf_{\mathcal{N}_1} I = I(u_*)$, $u_* > 0$ in $\Omega$ and for any $0 \leq v \in H_0^1(\Omega)$,**

$$\int_\Omega \nabla u_* \nabla v \mathrm{d}x + \int_\Omega \phi_{u_*} u_* v \mathrm{d}x - \int_\Omega f(x)u_*^{-\gamma} v \mathrm{d}x \geq 0.$$

To prove the above statements, we consider the following two cases regarding whether $\{u_n\}$ belongs to $\mathcal{N}_1 \setminus \mathcal{N}_2$ or $\mathcal{N}_2$.

**Case 1. Suppose that $\{u_n\} \subset \mathcal{N}_1 \setminus \mathcal{N}_2$ for all $n$ large.**

For any $0 \leq v \in H_0^1(\Omega)$, since $\{u_n\} \subset \mathcal{N}_1 \setminus \mathcal{N}_2$, $f(x) > 0$ a.e. in $\Omega$ and $\gamma > 1$, we can derive

$$\int_\Omega f(x)(u_n + tv)^{1-\gamma}\mathrm{d}x \leq \int_\Omega f(x)u_n^{1-\gamma}\mathrm{d}x < \|u_n\|^2 + \int_\Omega \phi_{u_n} u_n^2 \mathrm{d}x,\ \forall t \geq 0.$$

Therefore, we could choose $t > 0$ small enough such that

$$\int_\Omega f(x)(u_n + tv)^{1-\gamma}\mathrm{d}x < \|u_n + tv\|^2 + \int_\Omega \phi_{u_n+tv}(u_n + tv)^2 \mathrm{d}x,$$

that is $u_n + tv \in \mathcal{N}_1$. Applying condition (2) with $z = u_n + tv$ leads to

$$\begin{aligned} \frac{\|tv\|}{n} &\geq I(u_n) - I(u_n + tv) \\ &= \frac{1}{2}(\|u_n\|^2 - \|u_n + tv\|^2) + \frac{1}{4}\int_\Omega [\phi_{u_n} u_n^2 - \phi_{u_n+tv}(u_n + tv)^2]\mathrm{d}x \\ &\quad + \frac{1}{1-\gamma}\int_\Omega f(x)[(u_n + tv)^{1-\gamma} - u_n^{1-\gamma}]\mathrm{d}x. \end{aligned}$$

Dividing by $t > 0$ and passing to the liminf as $t \to 0^+$, then we obtain from Fatou's Lemma that

$$\begin{aligned}\frac{\|v\|}{n} + \int_\Omega \nabla u_n \cdot \nabla v \mathrm{d}x + \int_\Omega \phi_{u_n} u_n v \mathrm{d}x &\geq \liminf_{t\to 0^+} \frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_n+tv)^{1-\gamma}-u_n^{1-\gamma}}{t}\mathrm{d}x \\ &\geq \int_\Omega \liminf_{t\to 0^+} \frac{f(x)}{1-\gamma}\frac{(u_n+tv)^{1-\gamma}-u_n^{1-\gamma}}{t}\mathrm{d}x \\ &= \int_\Omega f(x) u_n^{-\gamma} v \mathrm{d}x,\ \ (\text{since } u_n > 0 \text{ a.e. in } \Omega).\end{aligned}$$

Letting $n \to \infty$, according to Lemma 2.1 (v) and Fatou's Lemma again, one can get

$$\int_\Omega \nabla u_* \nabla v \mathrm{d}x + \int_\Omega \phi_{u_*} u_* v \mathrm{d}x \geq \int_\Omega f(x) u_*^{-\gamma} v \mathrm{d}x \text{ and } \int_\Omega f(x) u_*^{-\gamma} v \mathrm{d}x < +\infty. \tag{2.4}$$

Choose $v = u_*$ in (2.4), we get $u_* \in \mathcal{N}_1$, $\int_\Omega f(x) u_*^{1-\gamma}\mathrm{d}x < +\infty$ and then Step 1 shows the existence of unique $t(u_*) > 0$ satisfying $t(u_*)u_* \in \mathcal{N}_2$ and $I(t(u_*)u_*) = \min_{t>0} I(tu_*)$. Hence, according to the weakly lower semi-continuity of the norm, Lemma 2.1 (v) and Fatou's Lemma, one has

$$\begin{aligned}\inf_{\mathcal{N}_1} I &= \lim_{n\to\infty} I(u_n) \\ &= \liminf_{n\to\infty}\left[\frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_\Omega \phi_{u_n} u_n^2 \mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega f(x) u_n^{1-\gamma}\mathrm{d}x\right] \\ &\geq \liminf_{n\to\infty}\left[\frac{1}{2}\|u_n\|^2\right] + \liminf_{n\to\infty}\left[\frac{1}{4}\int_\Omega \phi_{u_n} u_n^2 \mathrm{d}x\right] + \liminf_{n\to\infty}\left[\frac{1}{\gamma-1}\int_\Omega f(x) u_n^{1-\gamma}\mathrm{d}x\right] \\ &\geq \frac{1}{2}\|u_*\|^2 + \frac{1}{4}\int_\Omega \phi_{u_*} u_*^2 \mathrm{d}x + \frac{1}{\gamma-1}\int_\Omega f(x) u_*^{1-\gamma}\mathrm{d}x \\ &= I(u_*) \geq I(t(u_*)u_*) \geq \inf_{\mathcal{N}_2} I \geq \inf_{\mathcal{N}_1} I.\end{aligned}$$

Thus, the above inequalities are actually equalities. By the uniqueness of $t(u_*)$, we have $t(u_*) = 1$, which implies that

$$u_* \in \mathcal{N}_2, \qquad \inf_{\mathcal{N}_1} I = I(u_*). \tag{2.5}$$

Moreover, we can also obtain that $\liminf_{n\to\infty}\|u_n\|^2 = \|u_*\|^2$ and a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), such that $\lim_{n\to\infty}\|u_n\|^2 = \|u_*\|^2$. This together with the weak convergence of $\{u_n\}$ in $H_0^1(\Omega)$ implies $u_n \to u_*$ strongly in $H_0^1(\Omega)$.

**Case 2. There exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) which belongs to $\mathcal{N}_2$.**

For any $0 \leq v \in H_0^1(\Omega)$, according to $\gamma > 1$, the boundness of $\{u_n\}$, Lemma 2.1 (iv), we have

$$\int_\Omega f(x)(u_n+tv)^{1-\gamma}\mathrm{d}x \leq \int_\Omega f(x) u_n^{1-\gamma}\mathrm{d}x = \|u_n\|^2 + \int_\Omega \phi_{u_n} u_n^2 \mathrm{d}x < +\infty, \qquad \forall t \geq 0,$$

then Step 1 shows the existence of some functions $h_{n,v}(t) : [0,+\infty) \to (0,+\infty)$ corresponding to $u_n + tv$ such that

$$h_{n,v}(0) = 1, \qquad h_{n,v}(t)(u_n+tv) \in \mathcal{N}_2, \qquad \forall t \geq 0.$$

The continuity of $h_{n,v}(t)$ with respect to $t$ follows from Lemma 2.1 (v) and the dominated convergence theorem since $\gamma > 1$ and $\int_\Omega f(x)|u_n|^{1-\gamma}\mathrm{d}x < +\infty$. However, we have no idea whether or not $h_{n,v}(t)$ is differentiable. For the sake of proof, we set

$$h'_{n,v}(0) = \lim_{t\to 0^+} \frac{h_{n,v}(t)-1}{t} \in [-\infty, +\infty].$$

If the above limit does not exist, we choose $t_k \to 0$ (instead of $t \to 0$) with $t_k > 0$ such that $h'_{n,v}(0) = \lim_{k\to\infty} \frac{h_{n,v}(t_k)-1}{t_k} \in [-\infty,+\infty]$. According to $u_n \in \mathcal{N}_2$, $h_{n,v}(t)(u_n+tv) \in \mathcal{N}_2$ and Lemma 2.1 (iii), we have

$$\|u_n\|^2 + \int_\Omega \phi_{u_n} u_n^2 \mathrm{d}x - \int_\Omega f(x) u_n^{1-\gamma}\mathrm{d}x = 0,$$

$$h^2_{n,v}(t)\|u_n+tv\|^2 + h^4_{n,v}(t)\int_\Omega \phi_{u_n+tv}(u_n+tv)^2\mathrm{d}x - h^{1-\gamma}_{n,v}(t)\int_\Omega f(x)(u_n+tv)^{1-\gamma}\mathrm{d}x = 0.$$

Since $\gamma > 1$, the above two equalities yield

$$\begin{aligned} 0 = \; & [h_{n,v}(t)-1]\Big\{ [h_{n,v}(t)+1]\,\|u_n+tv\|^2 - \frac{h^{1-\gamma}_{n,v}(t)-1}{h_{n,v}(t)-1}\int_\Omega f(x)(u_n+tv)^{1-\gamma}\mathrm{d}x \\ & + \left[h^2_{n,v}(t)+1\right][h_{n,v}(t)+1]\int_\Omega \phi_{u_n+tv}(u_n+tv)^2\mathrm{d}x\Big\} + \left[\|u_n+tv\|^2 - \|u_n\|^2\right] \\ & + \int_\Omega \left[\phi_{u_n+tv}(u_n+tv)^2 - \phi_{u_n}u_n^2\right]\mathrm{d}x - \int_\Omega f(x)\left[(u_n+tv)^{1-\gamma} - u_n^{1-\gamma}\right]\mathrm{d}x \end{aligned}$$

Dividing by $t > 0$ and passing to the limit as $t \to 0^+$, using Lemma 2.1 (vi), the continuity of $h_{n,v}(t)$ and $u_n \in \mathcal{N}_2$, we obtain

$$\begin{aligned} 0 \ge\; & h'_{n,v}(0)\Big\{2\|u_n\|^2 + (\gamma-1)\int_\Omega f(x)u_n^{1-\gamma}\mathrm{d}x + 4\int_\Omega \phi_{u_n}u_n^2\mathrm{d}x\Big\} \\ & + 2\int_\Omega \nabla u_n \nabla v\mathrm{d}x + 4\int_\Omega \phi_{u_n}u_n v\mathrm{d}x \\ =\; & h'_{n,v}(0)\Big\{(\gamma+1)\|u_n\|^2 + (\gamma+3)\int_\Omega \phi_{u_n}u_n^2\mathrm{d}x\Big\} + 2\int_\Omega \nabla u_n\nabla v\mathrm{d}x + 4\int_\Omega \phi_{u_n}u_n v\mathrm{d}x \end{aligned}$$

We claim that there exists $C_3 > 0$, such that $h'_{n,v}(0) \le C_3$ uniformly in $n$. Fix $n$, either $h'_{n,v}(0)$ is nonnegative, or $h'_{n,v}(0)$ is negative. If $h'_{n,v}(0) \ge 0$, then from the above inequality and Lemma 2.1 (ii), one can get

$$0 \ge (\gamma+1)h'_{n,v}(0)\|u_n\|^2 + 2\int_\Omega \nabla u_n\nabla v\mathrm{d}x.$$

Since $C_1 \le \|u_n\| \le C_2$ by Step 3, we can conclude that

$$h'_{n,v}(0) \le C_3 \quad \text{uniformly in } n \tag{2.6}$$

for some suitable constant $C_3 > 0$ and

$$\frac{\|u_n\|}{n} - \frac{(\gamma+1)C_1^2}{\gamma-1} < 0$$

for $n$ large enough. We also claim that there exists a constant $C_4$, such that $h'_{n,v}(0) \ge C_4$ uniformly in all $n$ large. If $h'_{n,v}(0) < 0$, then $h_{n,v}(t) < 1$ for $t > 0$ small. Applying condition (2) with $z = h_{n,v}(t)(u_n+tv)$ leads to

$$\begin{aligned} \frac{1}{n}[1-h_{n,v}(t)]\|u_n\| + \frac{t}{n}h_{n,v}(t)\|v\| &\ge \frac{1}{n}\|u_n - h_{n,v}(t)(u_n+tv)\| \\ &\ge I(u_n) - I[h_{n,v}(t)(u_n+tv)]. \end{aligned} \tag{2.7}$$

Since $u_n \in \mathcal{N}_2$, Lemma 2.1 (iii) together with (2.7) leads to

$$\begin{aligned}\frac{\|v\|}{n}h_{n,v}(t) \geq \frac{h_{n,v}(t)-1}{t}\Big\{&\frac{\|u_n\|}{n} - \left(\frac{1}{2}+\frac{1}{\gamma-1}\right)[h_{n,v}(t)+1]\|u_n+tv\|^2 \\ &- \left(\frac{1}{4}+\frac{1}{\gamma-1}\right)\left[h^2_{n,v}(t)+1\right]\left[h_{n,v}(t)+1\right]\int_\Omega \phi_{u_n+tv}(u_n+tv)^2\mathrm{d}x\Big\} \\ &- \left(\frac{1}{2}+\frac{1}{\gamma-1}\right)\frac{\|u_n+tv\|^2-\|u_n\|^2}{t} \\ &- \left(\frac{1}{4}+\frac{1}{\gamma-1}\right)\int_\Omega \frac{\phi_{u_n+tv}(u_n+tv)^2-\phi_{u_n}u_n^2}{t}\mathrm{d}x.\end{aligned}$$

Letting $t\to 0^+$, using Lemma 2.1 (vi), the continuity of $h_{n,v}(t)$ and $C_1 \leq \|u_n\| \leq C_2$, we obtain

$$\begin{aligned}\frac{\|v\|}{n} &\geq h'_{n,v}(0)\left\{\frac{\|u_n\|}{n} - 2\left(\frac{1}{2}+\frac{1}{\gamma-1}\right)\|u_n\|^2 - 4\left(\frac{1}{4}+\frac{1}{\gamma-1}\right)\int_\Omega \phi_{u_n}u_n^2\mathrm{d}x\right\} \\ &\quad - 2\left(\frac{1}{2}+\frac{1}{\gamma-1}\right)\int_\Omega \nabla u_n\nabla v\mathrm{d}x - 4\left(\frac{1}{4}+\frac{1}{\gamma-1}\right)\int_\Omega \phi_{u_n}u_nv\mathrm{d}x \\ &= h'_{n,v}(0)\left\{\frac{\|u_n\|}{n} - \frac{1}{\gamma-1}\left((\gamma+1)\|u_n\|^2+(\gamma+3)\int_\Omega \phi_{u_n}u_n^2\mathrm{d}x\right)\right\} \\ &\quad - \left(1+\frac{2}{\gamma-1}\right)\int_\Omega \nabla u_n\nabla v\mathrm{d}x - \left(1+\frac{4}{\gamma-1}\right)\int_\Omega \phi_{u_n}u_nv\mathrm{d}x \\ &\geq h'_{n,v}(0)\left\{\frac{\|u_n\|}{n} - \frac{(\gamma+1)C_1^2}{\gamma-1}\right\} - \left(1+\frac{2}{\gamma-1}\right)\int_\Omega \nabla u_n\nabla v\mathrm{d}x \\ &\quad - \left(1+\frac{4}{\gamma-1}\right)\int_\Omega \phi_{u_n}u_nv\mathrm{d}x\end{aligned}$$

since $\gamma > 1$ and $h'_{n,v}(0) < 0$. Then, from the construction of coefficient we see that $h'_{n,v}(0) \neq -\infty$ and cannot diverge to $-\infty$ as $n\to\infty$, that is,

$$h'_{n,v}(0) \neq -\infty \text{ and } h'_{n,v}(0) \geq C_4 \text{ uniformly in } n \text{ large} \tag{2.8}$$

for some suitable constant $C_4$. So, it follows from (2.6) and (2.8) that

$$h'_{n,v}(0) \in (-\infty,+\infty) \text{ and } |h'_{n,v}(0)| \leq C \text{ uniformly in } n \text{ large},$$

where $C = \max\{C_3, |C_4|\}$ is independent of $n$. Furthermore, applying condition (2) with $z = h_{n,v}(t)(u_n+tv)$ again leads to

$$\begin{aligned}&\frac{|1-h_{n,v}(t)|}{t}\frac{\|u_n\|}{n} + \frac{\|v\|}{n}h_{n,v}(t) \\ &\geq \frac{1}{nt}\|u_n - h_{n,v}(t)(u_n+tv)\| \geq \frac{1}{t}\left[I(u_n) - I(h_{n,v}(t)(u_n+tv))\right] \\ &\geq \frac{h_{n,v}(t)-1}{t}\Big\{-\frac{h_{n,v}(t)+1}{2}\|u_n+tv\|^2 + \frac{h^{1-\gamma}_{n,v}(t)-1}{(1-\gamma)[h_{n,v}(t)-1]}\int_\Omega f(x)(u_n+tv)^{1-\gamma}\mathrm{d}x \\ &\quad - \frac{1}{4}\left[h^2_{n,v}(t)+1\right]\left[h_{n,v}(t)+1\right]\int_\Omega \phi_{u_n+tv}(u_n+tv)^2\mathrm{d}x\Big\} - \frac{1}{2}\frac{\|u_n+tv\|^2-\|u_n\|^2}{t} \\ &\quad - \frac{1}{4}\int_\Omega \frac{\phi_{u_n+tv}(u_n+tv)^2-\phi_{u_n}u_n^2}{t}\mathrm{d}x + \frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_n+tv)^{1-\gamma}-u_n^{1-\gamma}}{t}\mathrm{d}x\end{aligned}$$

Passing to the liminf as $t \to 0^+$, then we get from Lemma 2.1 (vi), the continuity of $h_{n,v}(t)$ and Fatou's Lemma that

$$\begin{aligned}
&\frac{|h'_{n,v}(0)| \cdot \|u_n\|}{n} + \frac{\|v\|}{n} \\
&\geq h'_{n,v}(0)\Big\{ -\|u_n\|^2 + \int_\Omega f(x)u_n^{1-\gamma}\mathrm{d}x - \int_\Omega \phi_{u_n}u_n^2\mathrm{d}x\Big\} \\
&\quad - \int_\Omega \nabla u_n \nabla v \mathrm{d}x - \int_\Omega \phi_{u_n}u_n v\mathrm{d}x + \liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_n+tv)^{1-\gamma}-u_n^{1-\gamma}}{t}\mathrm{d}x \\
&\geq -\int_\Omega \nabla u_n \nabla v \mathrm{d}x - \int_\Omega \phi_{u_n}u_n v\mathrm{d}x + \int_\Omega \frac{f(x)}{1-\gamma}\liminf_{t\to 0^+}\frac{(u_n+tv)^{1-\gamma}-u_n^{1-\gamma}}{t}\mathrm{d}x \\
&= -\int_\Omega \nabla u_n \nabla v \mathrm{d}x - \int_\Omega \phi_{u_n}u_n v\mathrm{d}x + \int_\Omega f(x)u_n^{-\gamma}v\mathrm{d}x,
\end{aligned}$$

since $u_n \in \mathcal{N}_2$. Furthermore, by Lemma 2.1 (iv), for $n$ large, we have

$$\begin{aligned}
\int_\Omega f(x)u_n^{-\gamma}vdx &\leq \frac{|h'_{n,v}(0)| \cdot \|u_n\|}{n} + \frac{\|v\|}{n} + \int_\Omega \nabla u_n \nabla v\mathrm{d}x + \int_\Omega \phi_{u_n}u_n v\mathrm{d}x \\
&\leq \frac{C\cdot C_2 + \|v\|}{n} + \int_\Omega \nabla u_n \nabla v\mathrm{d}x + \int_\Omega \phi_{u_n}u_n v\mathrm{d}x < +\infty,
\end{aligned}$$

thanks to $C_1 \leq \|u_n\| \leq C_2$ and $|h'_{n,v}(0)| \leq C$ uniformly in $n$ large. Passing to the limit as $n \to \infty$ with using Lemma 2.1 (v) and Fatou's Lemma again leads to

$$\int_\Omega f(x)u_*^{-\gamma}v\mathrm{d}x \leq \liminf_{n\to\infty}\int_\Omega f(x)u_n^{-\gamma}v\mathrm{d}x \leq \int_\Omega \nabla u_* \nabla v\mathrm{d}x + \int_\Omega \phi_{u_*}u_* v\mathrm{d}x < +\infty, \tag{2.9}$$

for any $0 \leq v \in H_0^1(\Omega)$. By the same argument as in Case 1, we can also obtain that

$$u_* \in \mathcal{N}_2, \qquad \inf_{\mathcal{N}_1} I = I(u_*). \tag{2.10}$$

in Case 2. Therefore, Combining (2.4), (2.5), (2.9) and (2.10), we could conclude that in either case, up to subsequence, $u_n \to u_*$ strongly in $H_0^1(\Omega)$, $u_* \in \mathcal{N}_2$, $\inf_{\mathcal{N}_1} I = I(u_*)$ and

$$\int_\Omega \nabla u_* \nabla v\mathrm{d}x + \int_\Omega \phi_{u_*}u_* v\mathrm{d}x - \int_\Omega f(x)u_*^{-\gamma}v\mathrm{d}x \geq 0, \tag{2.11}$$

for any $0 \leq v \in H_0^1(\Omega)$. Hence, $-\Delta u_* + \phi_{u_*}u_* \geq 0$ in the week sense. By Step 3, $u_*(x) > 0$ a.e. in $\Omega$ and similar to the proof in [28], we get $u_* > 0$ in $\Omega$.

**Step 5. $u_*$ is a solution of system (SP).**

For any $\psi \in H_0^1(\Omega) \setminus \{0\}$ and $\varepsilon > 0$. Since $0 < u_* \in \mathcal{N}_2$, applying inequality (2.11) with $v = (u_* + \varepsilon\psi)^+$ leads to

$$
\begin{aligned}
0 &\leq \frac{1}{\varepsilon}\Big\{ \int_{\Omega} \nabla u_* \nabla (u_* + \varepsilon\psi)^+ \mathrm{d}x + \int_{\Omega} \phi_{u_*} u_* (u_* + \varepsilon\psi)^+ \mathrm{d}x - \int_{\Omega} f(x) u_*^{-\gamma} (u_* + \varepsilon\psi)^+ \mathrm{d}x \Big\} \\
&= \frac{1}{\varepsilon} \int_{[u_* + \varepsilon\psi \geq 0]} \Big\{ \nabla u_* \nabla (u_* + \varepsilon\psi) + \phi_{u_*} u_* (u_* + \varepsilon\psi) - f(x) u_*^{-\gamma} (u_* + \varepsilon\psi) \Big\} \mathrm{d}x \\
&= \frac{1}{\varepsilon} \Big( \int_{\Omega} - \int_{[u_* + \varepsilon\psi < 0]} \Big) \Big\{ \nabla u_* \nabla (u_* + \varepsilon\psi) + \phi_{u_*} u_* (u_* + \varepsilon\psi) - f(x) u_*^{-\gamma} (u_* + \varepsilon\psi) \Big\} \mathrm{d}x \\
&\leq \frac{1}{\varepsilon} \Big\{ \|u_*\|^2 + \int_{\Omega} \phi_{u_*} u_*^2 \mathrm{d}x - \int_{\Omega} f(x) u_*^{1-\gamma} \mathrm{d}x \Big\} \\
&\quad + \Big\{ \int_{\Omega} \nabla u_* \nabla \psi \mathrm{d}x + \int_{\Omega} \phi_{u_*} u_* \psi \mathrm{d}x - \int_{\Omega} f(x) u_*^{-\gamma} \psi \mathrm{d}x \Big\} \\
&\quad - \frac{1}{\varepsilon} \int_{[u_* + \varepsilon\psi < 0]} \Big[ \nabla u_* \nabla (u_* + \varepsilon\psi) + \phi_{u_*} u_* (u_* + \varepsilon\psi) \Big] \mathrm{d}x \\
&\quad + \frac{1}{\varepsilon} \int_{[u_* + \varepsilon\psi < 0]} f(x) u_*^{-\gamma} (u_* + \varepsilon\psi) \mathrm{d}x \\
&\leq \Big\{ \int_{\Omega} \nabla u_* \nabla \psi \mathrm{d}x + \int_{\Omega} \phi_{u_*} u_* \psi \mathrm{d}x - \int_{\Omega} f(x) u_*^{-\gamma} \psi \mathrm{d}x \Big\} \\
&\quad - \frac{1}{\varepsilon} \int_{[u_* + \varepsilon\psi < 0]} \Big[ \nabla u_* \nabla u_* + \phi_{u_*} u_*^2 \Big] \mathrm{d}x - \int_{[u_* + \varepsilon\psi < 0]} \Big[ \nabla u_* \nabla \psi + \phi_{u_*} u_* \psi \Big] \mathrm{d}x \\
&\leq \Big\{ \int_{\Omega} \nabla u_* \nabla \psi \mathrm{d}x + \int_{\Omega} \phi_{u_*} u_* \psi \mathrm{d}x - \int_{\Omega} f(x) u_*^{-\gamma} \psi \mathrm{d}x \Big\} - \int_{[u_* + \varepsilon\psi < 0]} \Big[ \nabla u_* \nabla \psi + \phi_{u_*} u_* \psi \Big] \mathrm{d}x.
\end{aligned}
$$

Letting $\varepsilon \to 0^+$ to the above inequality and using the fact that $\mathrm{meas}[u_* + \varepsilon\psi < 0] \to 0$ as $\varepsilon \to 0^+$, we have

$$
\int_{\Omega} \nabla u_* \nabla \psi \mathrm{d}x + \int_{\Omega} \phi_{u_*} u_* \psi \mathrm{d}x - \int_{\Omega} f(x) u_*^{-\gamma} \psi \mathrm{d}x \geq 0, \qquad \forall \psi \in H_0^1(\Omega).
$$

This inequality also holds for $-\psi$, hence we obtain

$$
\int_{\Omega} \nabla u_* \nabla \psi \mathrm{d}x + \int_{\Omega} \phi_{u_*} u_* \psi \mathrm{d}x - \int_{\Omega} f(x) u_*^{-\gamma} \psi \mathrm{d}x = 0, \qquad \forall \psi \in H_0^1(\Omega). \tag{2.12}
$$

Thus $u_* \in H_0^1(\Omega)$ is a solution of system (SP).

**Step 6. $u_*$ is a unique solution of system (SP).**

Suppose $v_* \in H_0^1(\Omega)$ is also a solution of system (SP), then for any $\psi \in H_0^1(\Omega)$, we have

$$
\int_{\Omega} \nabla v_* \nabla \psi \mathrm{d}x + \int_{\Omega} \phi_{v_*} v_* \psi \mathrm{d}x - \int_{\Omega} f(x) v_*^{-\gamma} \psi \mathrm{d}x = 0, \qquad \forall \psi \in H_0^1(\Omega). \tag{2.13}
$$

Taking $\psi = u_* - v_*$ in both equations (2.12)–(2.13) and subtracting term by term, we obtain

$$
\begin{aligned}
0 &\geq \int_{\Omega} f(x) (u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*) \mathrm{d}x \\
&= \|u_* - v_*\|^2 + \int_{\Omega} (\phi_{u_*} u_* - \phi_{v_*} v_*)(u_* - v_*) \mathrm{d}x \\
&\geq \|u_* - v_*\|^2 + \frac{1}{2} \|\phi_{u_*} - \phi_{v_*}\|^2 \geq \|u_* - v_*\|^2 \geq 0,
\end{aligned}
$$

where we use Lemma 2.1 (vii). So $\|u_* - v_*\|^2 = 0$, then $u_* = v_*$ and $u_*$ is the unique solution of system (SP). □

***Proof of Theorem 1.2.*** Since $u_1,\, u_2 \in H_0^1(\Omega)$ are two positive solutions of system (SP) corresponding to $f_1$ and $f_2$ respectively, then for any $\psi \in H_0^1(\Omega)$, we have

$$\int_\Omega \nabla u_1 \nabla \psi \mathrm{d}x + \int_\Omega \phi_{u_1} u_1 \psi \mathrm{d}x - \int_\Omega f_1(x) u_1^{-\gamma} \psi \mathrm{d}x = 0,$$
$$\int_\Omega \nabla u_2 \nabla \psi \mathrm{d}x + \int_\Omega \phi_{u_2} u_2 \psi \mathrm{d}x - \int_\Omega f_2(x) u_2^{-\gamma} \psi \mathrm{d}x = 0.$$

Set $\Omega_1 = \{x | u_2(x) \geq u_1(x),\, x \in \Omega\}$, then subtracting the above two equations and choosing $\psi = (u_2 - u_1)^+ \in H_0^1(\Omega)$ yield

$$\begin{aligned} 0 &\geq \int_\Omega (f_2(x) u_2^{-\gamma} - f_1(x) u_1^{-\gamma})(u_2 - u_1)^+ \mathrm{d}x \\ &= \|(u_2 - u_1)^+\|^2 + \int_\Omega (\phi_{u_2} u_2 - \phi_{u_1} u_1)(u_2 - u_1)^+ \mathrm{d}x \\ &= \|(u_2 - u_1)^+\|^2 + \int_{\Omega_1} (\phi_{u_2} u_2 - \phi_{u_1} u_1)(u_2 - u_1) \mathrm{d}x \\ &\geq \|(u_2 - u_1)^+\|^2 \geq 0, \end{aligned}$$

where we use $f_1 \geq f_2$, $\gamma > 1$ and Lemma 2.1 (vii). So $(u_2 - u_1)^+ \equiv 0$ and hence $u_1 \geq u_2$. □

***Proof of Theorem 1.3.*** The proof is exactly the same as Sun and Tan [21]. We omit the details here. □

***Proof of Theorem 1.4.*** We prove Theorem 1.4 by contradiction that $\sup_\Omega u < +\infty$. Motivated by Sun and Tan [21], Choose a sequence of test functions $\{\varphi_\delta\} \subset C_0^\infty(\Omega)$ satisfying $0 \leq \varphi_\delta \leq 1$, $\varphi_\delta \equiv 0$ in $B_\delta(0)$, $\varphi_\delta \equiv 1$ in $B_{5\delta/3}(0) \setminus B_{4\delta/3}(0)$, $\varphi_\delta \equiv 0$ in $\Omega \setminus B_{2\delta}(0)$ and $|\Delta \varphi_\delta| \leq \frac{C_5}{\delta^2}$ in $\Omega$. Thus, we have

$$\int_\Omega \nabla u \nabla \varphi_\delta \mathrm{d}x + \int_\Omega \phi_u u \varphi_\delta \mathrm{d}x - \int_\Omega |x|^{-\alpha} u^{-\gamma} \varphi_\delta \mathrm{d}x = 0. \tag{2.14}$$

According to the definition of $\varphi_\delta(x)$ and $\gamma > 0$, we have

$$\begin{aligned} \int_\Omega |x|^{-\alpha} u^{-\gamma} \varphi_\delta \mathrm{d}x &= \int_{B_{2\delta}(0) \setminus B_\delta(0)} |x|^{-\alpha} u^{-\gamma} \varphi_\delta \mathrm{d}x \\ &\geq \left(\sup_\Omega u\right)^{-\gamma} \int_{B_{2\delta}(0) \setminus B_\delta(0)} |x|^{-\alpha} \varphi_\delta \mathrm{d}x \\ &\geq \left(\sup_\Omega u\right)^{-\gamma} \int_{B_{5\delta/3}(0) \setminus B_{4\delta/3}(0)} |x|^{-\alpha} \mathrm{d}x \\ &= \left(\sup_\Omega u\right)^{-\gamma} \frac{4\pi}{3 - \alpha} \left[ \left(\frac{5}{3}\right)^{3-\alpha} - \left(\frac{4}{3}\right)^{3-\alpha} \right] \delta^{3-\alpha}. \end{aligned}$$

On the other hand, by Sobolev inequalities and Lemma 2.1 (i), (iv), we have

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi_\delta \mathrm{d}x + \int_\Omega \phi_u u \varphi_\delta \mathrm{d}x &= -\int_\Omega u \Delta \varphi_\delta \mathrm{d}x + \int_\Omega \phi_u u \varphi_\delta \mathrm{d}x \\ &\leq \int_\Omega u |\Delta \varphi_\delta| \mathrm{d}x + \int_\Omega \phi_u u \varphi_\delta \mathrm{d}x \\ &\leq \left(\sup_\Omega u\right) \left[ \int_\Omega |\Delta \varphi_\delta| \mathrm{d}x + \int_\Omega \phi_u \varphi_\delta \mathrm{d}x \right] \\ &\leq \left(\sup_\Omega u\right) \left[ \int_{B_{2\delta}(0) \setminus B_\delta(0)} |\Delta \varphi_\delta| \mathrm{d}x + \int_{B_{2\delta}(0) \setminus B_\delta(0)} \phi_u \mathrm{d}x \right] \\ &\leq \left(\sup_\Omega u\right) \left[ \frac{28\pi C_5 \delta}{3} + C_6 \|u\|^2 \delta^{5/2} \right]. \end{aligned}$$

Therefore

$$\left(\sup_{\Omega} u\right)^{1+\gamma} \geq \frac{12\pi}{(3-\alpha)\left[28\pi C_5 + 3C_6\|u\|^2\delta^{3/2}\right]}\left[\left(\frac{5}{3}\right)^{3-\alpha} - \left(\frac{4}{3}\right)^{3-\alpha}\right]\delta^{2-\alpha} \to +\infty$$

a contradiction as $\delta \to 0^+$ since $\alpha > 2$ and this ends the proof of Theorem 1.4. □

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