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A UNIVERSAL DIFFERENTIAL EQUATION

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Dedicated to the Memory of Walter Strodt

THEOREM. *There exists a nontrivial fourth-order algebraic differential equation*

$$* \qquad P(y', y'', y''', y'''') = 0,$$

where P is a polynomial in four variables, with integer coefficients, such that for any continuous function φ on $(-\infty, \infty)$ and for any positive continuous function $\epsilon(t)$ on $(-\infty, \infty)$, there exists a C^∞ solution y of $*$ such that

$$|y(t) - \varphi(t)| < \epsilon(t) \quad \text{for all } t \in (-\infty, \infty).$$

One such specific equation (homogeneous of degree seven, with seven terms of weight 14) *is*

$$3y'^4y''y''''^2 - 4y'^4y'''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y''^3y'''^2 + 12y''^7 = 0.$$

REMARK 1. From the proof, it will be clear that we can in addition ensure that $y(t_j) = \varphi(t_j)$ for any sequence (t_j) of distinct real numbers such that $|t_j| \longrightarrow \infty$ as $j \longrightarrow \infty$.

REMARK 2. We may moreover make y monotone if φ is monotone.

REMARK 3. Without changing the equation $*$, if φ and ϵ are only defined on an open interval I, then we can make $|y(t) - \varphi(t)| < \epsilon(t)$ for all $t \in I$, where y is a C^∞ solution of $*$ on I.

If we regard the uniform limits of solutions of $*$ as "weak solutions" (the way $y = |t|$ is a weak solution of $yy' - t = 0$ as the limit of $(t^2 + \epsilon^2)^{1/2}$ as $\epsilon \longrightarrow 0$), then a corollary of our Theorem is that every continuous function φ is a weak solution of $*$.

This Theorem may be regarded as an analogue, for analog computers, of the Universal Turing Machine (see [**R**, p. 23]), because of a theorem of Shannon (see [**S**, Theorem II]) that identifies the outputs of analog computers with the solutions of algebraic differential equations. A later paper of Pour-El requires some

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uniqueness of the solutions of the differential equations, and it is an open problem whether we can require, in our Theorem, that the solution y of $*$ that approximates φ be the unique solution for its initial data. In a similar vein, we could ask for approximation by *analytic* solutions of a suitable ADE. Also see **[J]**, where certain universal diophantine equations are written down.

We go briefly into the history of our Theorem. In 1899, Borel **[BO]** found a majorant near ∞ of all solutions of all first-order algebraic equations. He claimed a similar majorant for the nth order equation, but his proof had a gap. In **[BBV]** and **[V]** in 1932 and 1937, Vijaraghavan and others constructed a second-order algebraic differential equation that has solutions that have no a priori majorant. In 1973, Babakhanian showed that a tower of n exponentials satisfies an ADE of order n but not one of smaller order. Hence, for any ADE, $P(t, \vec{u}) = 0$, there is a solution u of some ADE, $Q(t, \vec{u}) = 0$, that is not a solution of $P(t, \vec{u}) = 0$. We use $\vec{u}$ as shorthand for $u, u', \ldots, u^{(n)}$. In 1975, Bank (see **[BA**, Theorem 4**]**) modified the **[BBV]** example to produce increasing solutions of a third-order ADE that have no a priori majorant. It is an open problem whether there are a priori bounds for *entire* solutions of algebraic differential equations in the complex domain—see **[BA]** for a partial discussion of this and related questions. One is free to speculate whether the order 4 in our Theorem is best-possible.

We thank Michael Filaseta and C. Ward Henson for helping with the computer calculations. We express special gratitude to Lawrence G. Brown for pointing out that the calculations were feasible because of heavy cancellation of terms.

PROOF OF THE THEOREM. We shall write down the polynomial P explicitly. Let

$$g(t) = e^{-1/(1-t^2)}, \qquad -1 < t < 1,$$

with $g(t) = 0$ for all other t, and let

$$f(t) = \int g(t)\,dt.$$

We shall call the (graph of) f "a primitive S-module".

Now g satisfies the first-order ADE

$$g'/g = -\frac{2t}{(1-t^2)^2}$$

so that f and every $af + b$, where a and b are constants, satisfies the second order ADE

$$f''(t)(1-t^2)^2 + f'(t)2t = 0.$$

The idea is to write down a fourth order ADE, by differentiation and elimination, that is satisfied by every function $y = Af(\alpha t + \beta) + B$, whatever the constants

A, α, β, B. Our original method, using resultants, led to a twelfth-order equation. Lawrence G. Brown has found the following simple way that leads to a seventh-order equation. We are grateful to him for his permission to give it here.

Put $y = Af(\alpha t + \beta) + B$ so that

(1) $y' = A\alpha f'$,

(2) $y'' = A\alpha^2 f''$,

(3) $y''' = A\alpha^3 f'''$,

(4) $y'''' = A\alpha^4 f''''$,

where (for $|s| < 1$)

$$\text{(i)} \qquad f'(s) = e^{-1/(1-s^2)},$$

$$\text{(ii)} \qquad f''(s) = \frac{-2s}{(1-s^2)^2}\, e^{-1/(1-s^2)},$$

$$\text{(iii)} \qquad f'''(s) = \frac{6s^4-2}{(1-s^2)^4}\, e^{-1/(1-s^2)},$$

$$\text{(iv)} \qquad f''''(s) = \frac{-24s^7 - 12s^5 + 40s^3 - 12s}{(1-s^2)^6}\, e^{-1/(1-s^2)}.$$

In principle, one could solve for A, α, β in terms of y', y'', y''', and t and substitute into the expression for y''''. In practice, we let $s = \alpha t + \beta$, $\widetilde{A} = A \exp(-1/(1-s^2))$, and solve for $\widetilde{A}$, α, s. A surprising amount of cancellation takes place, so that the resulting ADE is simpler than one would expect in advance. The computations are not without tedium.

From (1), $\widetilde{A}\alpha = y'$ so that (2) becomes

$$y'' = \alpha y' \left(\frac{-2s}{(1-s^2)^2}\right) \tag{2'}$$

and (3) becomes

$$y''' = \alpha^2 y' \frac{6s^4-2}{(1-s^2)^4}. \tag{3'}$$

From (2′), we get

$$\alpha = -\frac{y''}{y'} \frac{(1-s^2)^2}{2s},$$

so that (3′) becomes

$$y''' = \frac{y''^2}{y'} \frac{3s^4-1}{2s^2}.$$

From this we get

$$3y''s^4 - 2y'y'''s^2 - y''^2 = 0,$$

so that

$$s^2 = \frac{y'y''' + \sqrt{y'^2y'''^2 + 3y''^4}}{3y''^2}.$$

Substituting into (4), we get

$$y'''' = \frac{-y''^3}{y'^2}\,\frac{-6s^6 - 3s^4 + 10s^2 - 3}{2s^2}.$$

Putting in the expression for s^2 and rationalizing the denominator, we get

$$y'''' = \frac{1}{3y'^2y''}\,[2y'^2y'''^2 - 12y''^4 - 3y'y''^2y''' + (6y''^2 + 2y'y''')\sqrt{y'^2y'''^2 + 3y''^4}].$$

Clearing fractions, isolating the square root on one side of the equation, and squaring both sides, we obtain an ADE of degree 8, divisible by $3y''$. We get

$$3y'^4y''y''''^2 - 4y'^4y'''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y''^3y'''^2 + 12y''^7 = 0,$$

which is the announced equation, after dividing out the $3y''$ term.

It remains to prove that the C^∞ solutions y of $P = 0$ approximate the given function φ within $\epsilon(t)$. Since φ can be so approximated by a piecewise affine function, there is no harm in taking φ itself to be a piecewise affine function.

Now call an "S-module" any function of the form $F = af(\alpha t + \beta) + b$ on a closed interval J, that is constant in neighborhoods of the endpoints of J. If $J = [a, b]$ then an S-module $\sigma(t)$ is a C^∞ function that takes some value A at a, some value B at b, is constant for $a \leqslant t \leqslant a + \delta$, is constant for $b - \delta \leqslant t \leqslant b$ and is a particular monotone function on $a + \delta \leqslant t \leqslant b - \delta$, for some small constant δ. It is clear that every S-module satisfies $*$. Moreover, any "S-chain" is also a solution of $*$, where by S-chain we mean any C^∞ function that consists of S-modules pieced together, possibly countably many.

Let us take any finite interval K on which φ is affine. Cut K into a large number N of equal pieces (depending on the infimum of $\epsilon(t)$ over K and on the slope of φ on K) and sew together N small S-modules that interpolate φ at the endpoints of the N subintervals. Since φ is monotone, and since the S-modules are likewise monotone, they differ by less than $\epsilon(t)$ on K if N is large enough. Now proceed with the next affine piece of φ, and join all the S-modules into an infinite S-chain. This will be the graph of a C^∞ solution of $*$, and the result is proved.

Note. It has just come to my attention that R. C. Buck has obtained universal *partial* algebraic differential equations using Kolmogorov's solution of Hilbert's Thirteenth Problem; see R. C. Buck, *The solutions to a smooth PDE can be dense in* $C[I]$, J. Differential Equations (to appear).

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