

A Universal Inequality for CFT and Quantum Gravity

Simeon Hellerman

*Institute for the Physics and Mathematics of the Universe
The University of Tokyo
Kashiwa, Chiba 277-8582, Japan*

Abstract

We prove that every unitary two-dimensional conformal field theory (with no extended chiral algebra, and with $c, \tilde{c} > 1$) contains a primary operator with dimension Δ_1 that satisfies $0 < \Delta_1 < \frac{c+\tilde{c}}{12} + 0.473695$. Translated into gravitational language using the AdS₃/CFT₂ dictionary, our result proves rigorously that the lightest massive excitation in *any* theory of 3D gravity with cosmological constant $\Lambda < 0$ can be no heavier than $1/(4G_N) + o(\sqrt{-\Lambda})$. In the flat-space approximation, this limiting mass is twice that of the lightest BTZ black hole. The derivation applies at finite central charge for the boundary CFT, and does not rely on an asymptotic expansion at large central charge. Neither does our proof rely on any special property of the CFT such as supersymmetry or holomorphic factorization, nor on any bulk interpretation in terms of string theory or semiclassical gravity. Our only assumptions are unitarity and modular invariance of the dual CFT. Our proof demonstrates for the first time that there exists a universal center-of-mass energy beyond which a theory of "pure" quantum gravity can never consistently be extended.

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1 Introduction

Quantum gravity in three dimensions [1, 2, 3, 4, 5] has long been a subject of much interest. Particularly interesting is the case of three dimensional quantum gravity with negative cosmological constant, which has anti-de Sitter space (AdS) as a maximally symmetric solution. Studying the case of negative cosmological constant allows us to confront specific treatments of the quantum dynamics with a set of general principles [8] which any consistent theory of quantum gravity in AdS is believed to obey. Namely, any quantum mechanical model of gravity in AdS must have its dynamics encoded by a dual theory on the boundary of the spacetime. Furthermore this dual theory should be a conformal field theory in one dimension less than that of the bulk spacetime, which satisfies the usual axioms of unitarity, locality, the existence of an operator product expansion, and so on.

This duality, known as the AdS/CFT correspondence, has had many applications, but in this paper we wish to exploit a particular one of its virtues, namely its role as a *universal set of rules* for consistent quantum gravity. The correspondence reduces a very complicated, badly understood and seemingly ill-defined set of theories – namely, models of quantum gravity in D dimensions – to the precisely defined set of (D-1)-dimensional CFT. This allows us in principle to make definite statements about models of quantum gravity, and in particular to rule out the possibility of quantum gravity theories with certain hypothetical properties. For instance, unitary CFT in two dimensions with central charge greater than 1 must contain an infinite number of conformal families. On the quantum gravity side, this tells us that a consistent theory of quantum gravity, the spectrum of states cannot be accounted for solely by excitations of the metric. There must exist massive states in addition to the boundary graviton gas. In the limit where the energy is high compared to the Planck mass $1/G_N$, the density of such states is predicted by Cardy’s formula [11] and agrees with the geometric prediction for the Bekenstein-Hawking entropy of the AdS/Schwarzschild black hole in the régime where the approximation by semiclassical general relativity is valid [13]. These constraints are central to the study of the fundamental degrees of freedom of quantum gravity and of their dynamics.

Nonetheless, the predictions of Cardy’s formula are in some sense unpalatably weak: massive states in AdS will appear – eventually, at some energy – and assume a particular entropy – approximately, in an approximation that will eventually be good at high enough temperatures. The predictions derived from Cardy’s formula suffer from the problem of *asymptoticity* – they are asymptotic predictions that can never be falsified by performing experiments at a given energy scale or a given temperature. Cardy’s formula is not sufficient to falsify the existence of a dual CFT, no matter how high the energy scale of an experiment: the formula gives precise information about the behavior of the level densities at sufficiently high energies, but remains completely silent as to the energy threshold at which the asymptotic predictions begin to apply.

Meanwhile, if we would like to derive a firm prediction for the lowest center-of-mass

energy at which new states must appear in a particular theory, we cannot usefully apply methods of effective field theory in the bulk: Planck-scale black holes have quantum corrections to their masses that are necessarily of the same order of magnitude as the semiclassical prediction.

Recently, an intriguing paper appeared [6] (based on earlier work [9, 10]) proposing a context in which this question can be addressed with great precision. The paper [6] examines the gravity duals of two dimensional CFTs in which the partition function is holomorphically factorized as a function of the complex structure τ of the torus. The theories examined in [6] have an operator algebra that decomposes completely into a tensor product of a right-moving and left-moving operator algebra, with no further projections or additional sectors. Any such theory must necessarily have a partition function that factorizes into a product of holomorphic and antiholomorphic factors, from which it follows directly that $c, \tilde{c} \in 24\mathbf{Z}$. In this class of CFT, it can be shown that the lowest primary operator is either purely left- or right-moving, and can have a weight no larger than $1 + \min(\frac{c}{24}, \frac{\tilde{c}}{24})$. There exists a unique candidate partition function in which this bound is saturated, though it is not clear that this partition function necessarily corresponds to an actual conformal field theory [26].

Other recent work [25] considers the case of theories with extended (2,2) supersymmetry, which allows the authors to exploit the power of holomorphic dependence on the complex structure without assuming holomorphic factorization of the full partition function. Study of a certain subclass of (2,2) SUSY CFT's suggests a bound that goes as $\Delta_1 \leq \frac{c}{24}$ for large central charge. However it has not been possible so far to demonstrate this bound conclusively within this special class of SCFT nor to generalize the conjecture to all (2,2) SCFT, let alone to CFT with reduced supersymmetry or none at all.

In this paper we will derive a completely general lower bound on the weight of the lowest primary operator in a completely general two dimensional conformal field theory, assuming only the basic properties of unitarity, modular invariance, and a discrete operator spectrum.¹ Our bound will refer only to the energy of a *single state*, namely the lowest excited primary state, rather than to the behavior of states at high temperature. Furthermore we will not use any sort of reasoning that refers to the bulk three-dimensional spacetime, nor make use of any asymptotic expansion at large central charge. Our methods thus circumvent the asymptoticity problem, and prove a general upper bound on the lowest-weight primary in a general CFT.

Our upper bound translates directly into an upper bound on the mass of the lightest massive state in a theory of gravity and matter in three dimensions. The bound we derive applies to *all* theories of gravity with an AdS₃ ground state. In particular, we do not assume holomorphic factorization, exact or approximate supersymmetry, or any

¹Even this last condition can be weakened substantially: Our conclusions will also apply to CFT with continuous spectrum that can be realized as limits of CFT with discrete spectrum. The "singular points" in the moduli space of D1-D5 CFT [27, 28], for instance, are of this type [29, 30, 31].

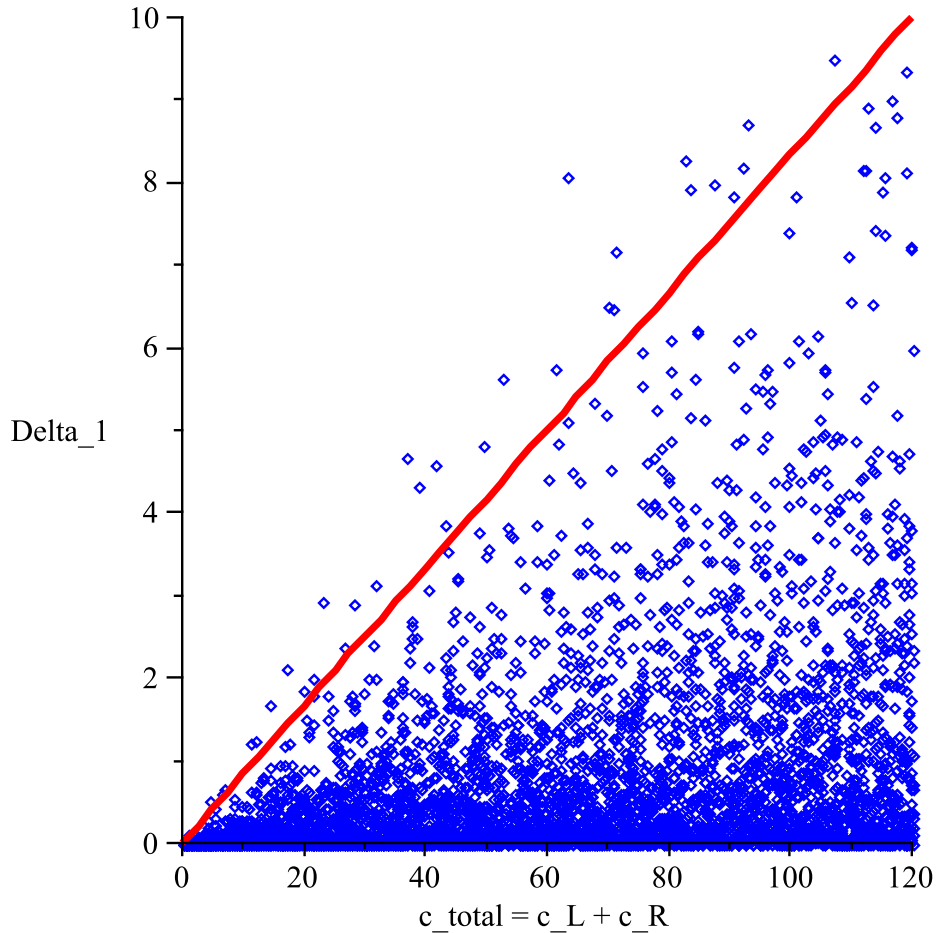


Figure 1: One logical possibility is that there exists no "sharp" bound on $\Delta_1 / c_{\text{total}}$, but rather only a statistical falloff at large values of the ratio.

other special property.²

2 Inequalities from modular invariance

In this section we will use unitarity and modular invariance to derive constraints on the energy levels of a general conformal field theory in two dimensions. The techniques

²We will assume that the Hilbert space has a positive definite norm, and that the spectrum the Hamiltonian is discrete. Positivity is necessary for a consistent interpretation of quantum mechanics, and discreteness is necessary in order for the system to have well-behaved thermodynamic properties. The assumption of discrete spectrum does not really count as a "special" property, as it holds on open sets of the moduli spaces of CFT that come in infinite families.

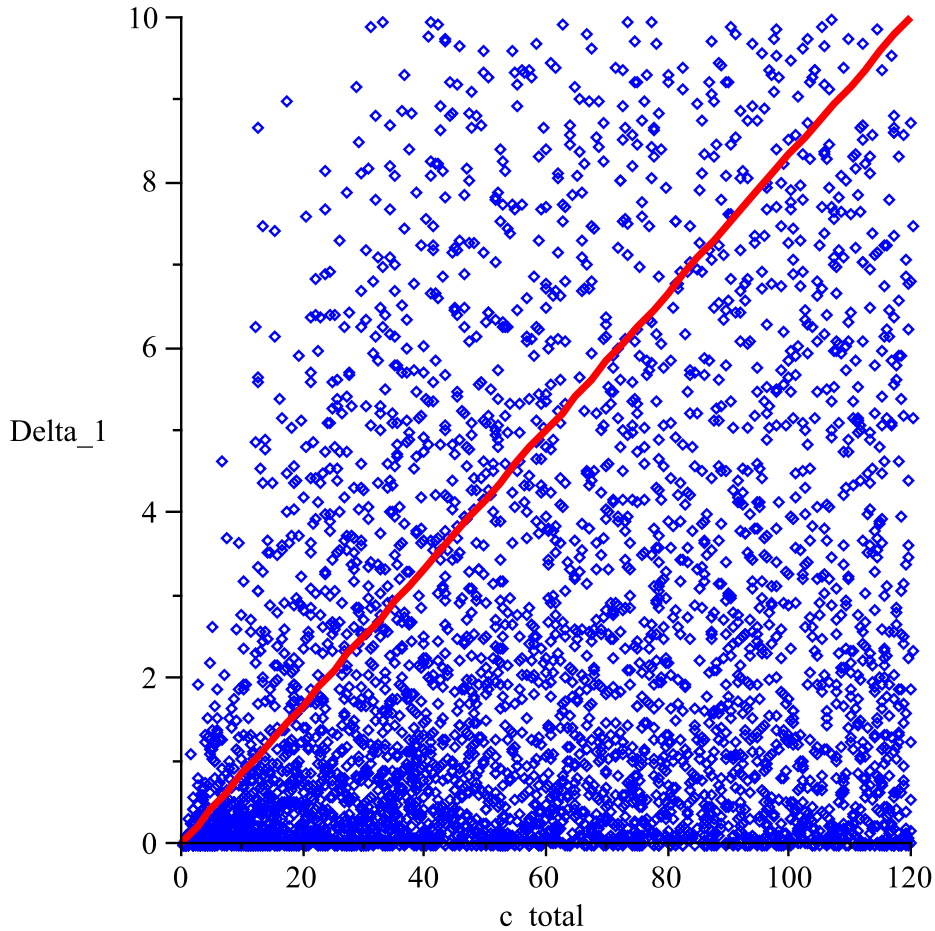


Figure 2: One logical possibility is that there is no limit on Δ_1 whatsoever for any given central charge, even in the "statistical" sense.

described in this section were invented previously in [11, 12], where they were used to estimate dimensions of operators in special cases.³ More recently, related techniques have been used to bound certain operator dimensions in conformal field theories in $D=4$ [15]. In this section, we apply a similar method to bound the dimension Δ_1 of the lowest primary operator in a general 2D CFT with $\tilde{c}, c < 9.135$. This will serve as a warm-up to demonstrate the general ideas at work. In the next section we will derive an improved bound that applies to CFT with arbitrary central charges greater than 1.

³In particular, see appendix A of [12]. We thank J. Cardy for correspondence relating to these and other results in the literature.

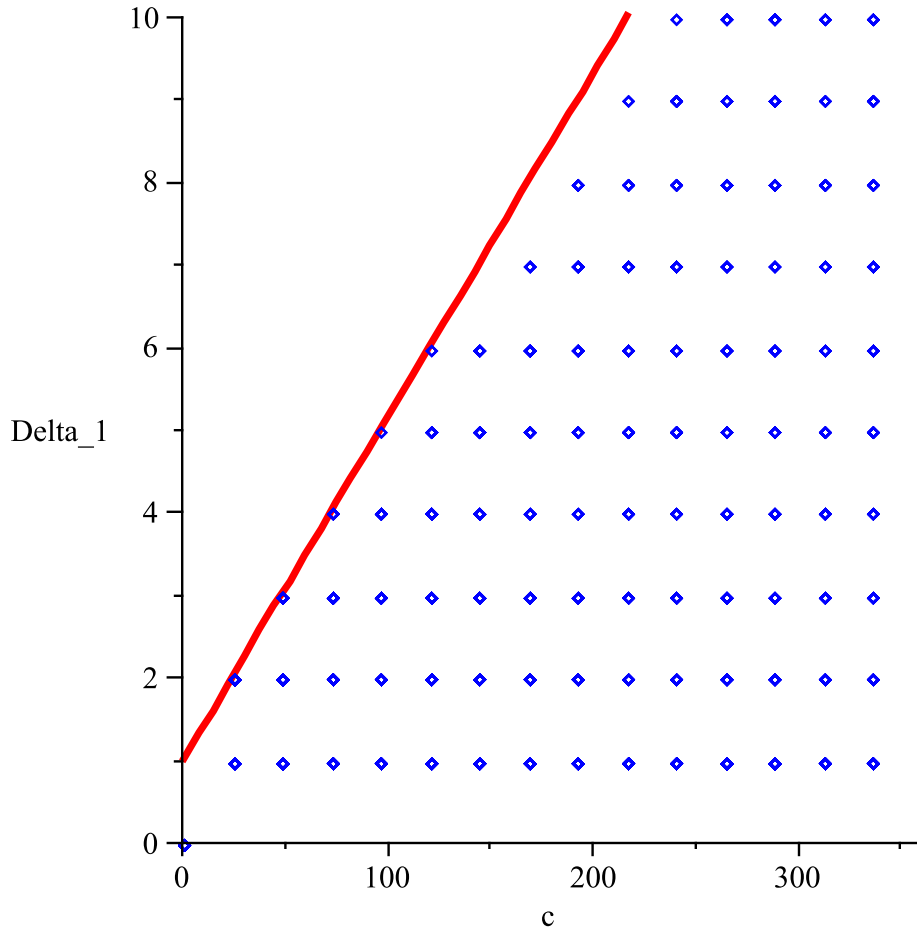


Figure 3: In the case where the Hilbert space factorizes completely as a product of purely left- and right-moving CFTs, it is possible to show that Δ_1 can never be greater than $\frac{c_{\text{total}}}{24} + 1$. This is the case described by Höhn-Witten's conjectured "extremal" CFT. It is unknown whether or not CFT exist that saturate this bound for c equal to any positive integer multiple of 24.

2.1 Conformal invariance and modular invariance

Let us consider a general CFT in two dimensions with positive norm and discrete spectrum. When the spatial direction σ^1 of the theory is compactified on a circle of length 2π , the partition function of the theory at temperature $\frac{1}{\beta}$ is given by

$$Z(\beta) \equiv \text{Tr} \left(\exp \{-\beta H\} \right) = \sum_n a(n) \exp \{-\beta E_n\} ,$$

where H is the Hamiltonian on a circle of length 2π , E_n is the n^{th} energy eigenvalue, and $a(n)$ is the degeneracy at the energy E_n . Unitarity and discreteness of the spectrum

imply that the $a(n)$ are positive integers. (We will sometimes suppress the degeneracy $a(n)$ henceforth.) The partition function can be refined by adding a thermodynamic potential K^1 for momentum P_1 in the compact spatial direction σ^1 :

$$Z(\beta, K^1) \equiv \text{Tr} \left(\exp \{iK^1 P_1 - \beta H\} \right)$$

Defining $\tau \equiv (K^1 + iH)/2\pi$, the partition function can be written in the familiar form

$$Z(\tau, \bar{\tau}) \equiv \text{Tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right),$$

where $q \equiv \exp\{2\pi i\tau\}$, c and \tilde{c} are the right- and left-moving central charges, $L_0 = \frac{1}{2}(H + P_1) + \frac{c}{24}$ and $\tilde{L}_0 = \frac{1}{2}(H - P_1) + \frac{\tilde{c}}{24}$ are the right- and left-moving conformal weight operators, and $q \equiv 2\pi i\tau$, which fit into a Virasoro algebra

$$\left[L_m, L_n \right] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$$\left[\tilde{L}_m, \tilde{L}_n \right] = (m - n)\tilde{L}_{m+n} + \frac{\tilde{c}}{12}(m^3 - m)\delta_{m,-n}$$

$$[L_m, \tilde{L}_n] = 0$$

The Virasoro generators obey the Hermiticity condition $L_m^\dagger = L_{-m}$, and it follows from unitarity that every primary operator has nonnegative weight, with weight zero if and only if the operator is the identity.

The partition function can be realized as the path integral of the conformal field theory on a torus of complex structure τ , with no operator insertions. Large coordinate transformations of the torus have the structure of the modular group $\mathbf{PSL}(2, \mathbf{Z})$, with the generator $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting as $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$. The group is generated by the transformations $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which act as $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\frac{1}{\tau}$, respectively.

Invariance of the partition function under the T transformation is completely equivalent to the condition that every state have $h - \tilde{h} \in \mathbf{Z}$, where h, \tilde{h} are the state's eigenvalues under L_0, \tilde{L}_0 . By contrast, invariance of the partition function under the modular S transformation gives a condition that far less transparent as a set of constraints on the spectrum of the theory. The rest of this section is devoted to extracting useful information from the invariance of $Z(\tau)$ under this transformation.

2.2 The medium-temperature expansion

The most basic constraint is Cardy's formula [11], which has to do with the asymptotic density of energy levels. By relating the low-temperature limit $\tau_2 \rightarrow \infty$ directly to

the high-temperature limit $\tau_2 \rightarrow 0$, the modular S -transformation implies that the asymptotic density of states goes as

$$\rho(\Delta, J) \equiv \frac{d^2 n}{d\Delta dJ} \simeq \exp \left\{ 2\pi \left(\sqrt{\frac{\tilde{c}(\Delta-J)}{12}} + \sqrt{\frac{c(\Delta+J)}{12}} \right) \right\} ,$$

where $\Delta = h + \tilde{h}$ is the scaling dimension of the operator, $J \equiv h - \tilde{h}$ is the spin of the operator, and we take Δ to be large. Setting to zero the chemical potential K^1 for spin, we maximize the exponent with respect to J , which gives

$$\rho(\Delta) \equiv \frac{dn}{d\Delta} \simeq \exp \left\{ +2\pi \sqrt{\frac{c_{\text{total}}\Delta}{6}} \right\}$$

for large Δ at a given total central charge

$$c_{\text{total}} \equiv c + \tilde{c} . \tag{2.1}$$

Since the level density increases slower than any geometric progression, this implies that the sum $\sum_n (q\bar{q})^{E_n}$ converges for any $|q| < 1$, by the ratio test. This will be useful for us in what follows, but by itself it does not make a statement about the spectrum that can be tested at finite energies or temperatures – Cardy’s formula is limited by its status as a formula that applies only asymptotically.

We would like to devise a test that allows us to look at a finite number of energy levels in a candidate spectrum for a CFT, and to decide whether that set of energy levels can actually be the spectrum of a consistent theory. And we’d like to do it by using as few inputs as possible – in particular, we would like to see what can be accomplished just using unitarity and modular invariance, and not having to use other consistency constraints on CFT, such as the existence of an associative operator product expansion.

For instance, the considerations in [9, 6] yield a constraint on the spectrum of a CFT in the case that the partition function of the CFT factorizes as a product of holomorphic and anti-holomorphic functions. Using only unitarity, holomorphy and modular invariance, one can show that in a holomorphically factorized CFT, the weight of the lowest-lying primary state (other than the identity) can be no higher than $\Delta_1 \leq \min(\frac{\tilde{c}}{24}, \frac{c}{24}) + 1$.

Similarly, in [25] the authors examined the case of CFT with $(2, 2)$ superconformal invariance, and used the holomorphic properties of the elliptic genus suggest a conjectural bound of $\Delta_1 \leq \frac{1}{24} \min(c, \tilde{c}) + o(c_{\text{total}}^0)$ at large central charge, under some special conditions. (In [25], the bound contains more information than the one in [9, 6], in that the terms subleading in \tilde{c}, c have precise coefficients depending on the $U(1)$ R-charge of the state of interest.)

To derive – and fully prove – a bound in the general case, let us consider the nature of the inputs we are using. If we use only unitarity and modular invariance, what we are really studying is just the set of modular invariant functions with a Fourier

expansion that is discrete, with positive integer Fourier coefficients. What are the general properties of this set of functions?

Functions $f(\tau, \bar{\tau})$ on the upper half plane \mathbb{H} that are modular invariant are in one to one correspondence with functions on the fundamental domain $\mathbb{H}/\mathbf{PSL}(2, \mathbb{Z})$. But if we further assume that $f(\tau, \bar{\tau})$ is smooth on the covering space, this gives us extra information, because the set of modular-invariant *smooth* functions $f(\tau, \bar{\tau})$ is *not* in correspondence with the set of smooth functions on the fundamental region. A smooth function on the fundamental region lifts to a modular-invariant smooth function on the covering space if and only if it satisfies certain conditions on its derivatives at the special points on the fundamental region that correspond to fixed points of elliptic elements of the modular group [11, 12]. Cardy's formula implies that $Z(\tau, \bar{\tau})$ and all its derivatives are continuous in the entire upper half plane, so we can indeed apply this reasoning to the partition function.

We will focus here on the point $\tau = +i$, which is fixed under the modular transformation $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This point in complex structure moduli space corresponds to a torus that is square – that is, the metric on the torus has no shear (off-diagonal components), and both radii are equal. The path integral on a square torus corresponds to the thermal partition function of a CFT compactified on a circle, at a temperature equal to the inverse circumference of the circle. The partition function at higher temperatures can be expressed, using the modular S -transformation, in terms of the partition function at lower temperatures. So the complex structure $\tau = +i$, corresponding to $\beta = \frac{1}{k_B T} = 2\pi$, can be thought of as lying exactly between the high-temperature and low-temperature régimes, or equivalently between the large- and small-complex structure limits of the moduli space of the torus.

It is in the neighborhood of $\tau = i$ that modular invariance of the partition function, which is obscure in the large- and small-complex structure limits, becomes manifest. To parametrize this neighborhood conveniently, write

$$\tau \equiv +i \cdot \exp \{s\} .$$

In terms of the variable s , the modular S -transformation $\tau \rightarrow -\frac{1}{\tau}$ acts as

$$s \rightarrow -s .$$

In terms of the variable s , then, the condition of invariance under the modular S -transformation can be written as

$$Z(+i \exp \{s\}, -i \exp \{\bar{s}\}) = Z(+i \exp \{-s\}, -i \exp \{-\bar{s}\}) .$$

Scaling $s \rightarrow 0$ and examining the behavior of the partition function in that régime is what we shall refer to as the *medium complex structure expansion*, or *medium temperature expansion* when we restrict ourselves to real values of s .

Taking derivatives at $s = 0$, this gives

$$\left(\frac{\partial}{\partial s} \right)^{N_R} \left(\frac{\partial}{\partial \bar{s}} \right)^{N_L} Z(+i \exp \{s\}, -i \exp \{\bar{s}\}) \Big|_{s=0} = 0 \text{ for } N_R + N_L \text{ odd} .$$

In terms of the usual variable τ , this means

$$\left(\tau \frac{\partial}{\partial \tau} \right)^{N_R} \left(\bar{\tau} \frac{\partial}{\partial \bar{\tau}} \right)^{N_L} Z(\tau, \bar{\tau}) \Big|_{\tau=+i} = 0 \text{ for } N_R + N_L \text{ odd} .$$

For purely imaginary complex structures $\tau = i\beta/2\pi$, this condition implies

$$\left(\beta \frac{\partial}{\partial \beta} \right)^N Z(\beta) \Big|_{\beta=2\pi} = 0 \text{ for } N \text{ odd} .$$

Before proceeding to derive our inequalities, let us make a few remarks on the medium temperature expansion. First of all, we want to emphasize that this expansion really is quite powerful – the N^{th} order of the medium temperature expansion generates a new constraint on the partition function, independent from all the previous ones.

Secondly, the medium temperature expansion really contains complementary information to that of the low- and high-temperature limits. In the low-temperature régime, unitarity is manifest because the function decomposes into a sum of exponentials with positive integer coefficients, but modular invariance is completely invisible. The high-temperature régime is redundant with the low-temperature régime, and Cardy’s formula exploits this relation in order to derive asymptotic formulæ for level densities. But still modular invariance constrains neither of these two limits separately, it only relates them to one another. In the medium temperature régime, unitarity is obscure, but modular invariance is manifest and imposes an infinite number of separate constraints on the derivatives of the partition function.

Thirdly, we will see in the next section that the medium temperature expansion is *useful*, when combined with the unitarity constraints that are visible in the low-temperature expansion and the convergence properties inferred *via* Cardy’s formula for the high-temperature expansion. We will see that one need not use the full hierarchy of differential constraints on $Z(\beta)$ for all odd N : there is a useful inequality implied just by the constraints at $N = 1, 3$ alone. We now turn to the derivation of this inequality.

2.3 Warm-up : the case of $c, \tilde{c} < 9.135$

In this subsection we will perform a “warm-up” derivation, where we show that every compact, unitary CFT (with $c > 1$) has a local operator of some kind – not necessarily a primary operator – whose scaling dimension $\Delta \equiv h + \tilde{h}$ satisfies

$$\Delta \leq \Delta_+^{(\text{warm-up})} \equiv \frac{c_{\text{total}}}{12} + \frac{3}{2\pi} . \quad (2.2)$$

Since we include descendants in our partition function along with primary operators, the bound (2.2) gives us interesting information only for low values of the total central charge: for $c_{\text{total}} > 24 - \frac{36}{\pi} \simeq 18.270$ the right hand side of (2.2) lies above $\Delta = 2$, where there must always be a stress tensor anyway. When $c_{\text{total}} < 18.270$, then $\Delta_+^{(\text{warm-up})} < 2$, which means that the operator with dimension Δ_1 cannot be a descendent of the identity. It must either be primary, or else it must be the L_{-1} or \tilde{L}_{-1} descendent of a primary with dimension $\Delta_1 - 1$. In either case, this means there is a nontrivial primary with dimension less than or equal to $\Delta_1 < \Delta_+$.

Our method for deriving the inequality can be refined to deal with primary operators only, and we will do this later in the following sections. For now, we will derive our weaker bound in order to illustrate the basic method involved.

First, we take the expression for the full partition function

$$Z(\beta) = \sum_n \exp\{-\beta E_n\}$$

and decompose it as

$$Z(\beta) = Z^{(\text{vac})}(\beta) + Z^{(\text{excited})}(\beta) \tag{2.3}$$

where

$$Z^{(\text{vac})}(\beta) \equiv \exp\{-\beta E_0\} \tag{2.4}$$

$$Z^{(\text{excited})}(\beta) \equiv \sum_{n \geq 1} \exp\{-\beta E_n\} .$$

The basic idea behind the inequality we want to derive is to show that with every derivative $\beta \frac{\partial}{\partial \beta}$ that we take at $\beta = 2\pi$, the higher energy levels tend to gain in importance and also to contribute negatively to derivatives of odd order. If we assume the degeneracies at high levels are large enough that the first derivative with respect to β will vanish at $\beta = 2\pi$, then those high levels will tend to contribute *even more negatively* to the third derivative at $\beta = 2\pi$, and the only way they can be balanced out is by the lower positive energy levels.

So the conclusion will be that the lowest of the excited energy level must be low enough that it does not make an overwhelmingly large negative contribution to the third derivative, given that it makes a *sufficiently large* contribution to the first derivative. Let us now make this intuition precise. We define a *relative importance* $I(E)$ associated with each energy level $|E\rangle$. This function is designed to measure the importance of $|E\rangle$'s contribution to the third derivative of the partition function at medium temperature ($\beta = 2\pi$), compared to $|E\rangle$'s contribution to the first derivative at medium temperature. The energy level $|E\rangle$ contributes to $Z(\beta)$ as $\exp\{-\beta E\}$ at temperature $1/\beta$, so we

define

$$\begin{aligned}
I(E) &\equiv \frac{\left(\beta \frac{\partial}{\partial \beta}\right)^3 \exp\{-\beta E\}}{\left(\beta \frac{\partial}{\partial \beta}\right)^1 \exp\{-\beta E\}} \Bigg|_{\beta=2\pi} \\
&= 4\pi^2 E^2 - 6\pi E + 1
\end{aligned} \tag{2.5}$$

At $\beta = 2\pi$, the derivatives of $Z^{(\text{vac})}$ and $Z^{(\text{excited})}$ will be equal and opposite, by virtue of the medium-temperature expansion at $N = 1$:

$$\left(\beta \frac{\partial}{\partial \beta}\right)^1 Z^{(\text{vac})}(\beta) \Bigg|_{\beta=2\pi} = - \left(\beta \frac{\partial}{\partial \beta}\right)^1 Z^{(\text{excited})}(\beta) \Bigg|_{\beta=2\pi} . \tag{2.6}$$

Likewise by virtue of the medium-temperature expansion at order $N = 3$, we have:

$$\left(\beta \frac{\partial}{\partial \beta}\right)^3 Z^{(\text{vac})}(\beta) \Bigg|_{\beta=2\pi} = - \left(\beta \frac{\partial}{\partial \beta}\right)^3 Z^{(\text{excited})}(\beta) \Bigg|_{\beta=2\pi} . \tag{2.7}$$

(The zero energy states make no contribution to any of the derivatives.)

Now define the ratios

$$\mathcal{R}_{31}^{(\text{vac})} \equiv \frac{\left(\beta \frac{\partial}{\partial \beta}\right)^3 Z^{(\text{vac})}(\beta)}{\left(\beta \frac{\partial}{\partial \beta}\right)^1 Z^{(\text{vac})}(\beta)} \Bigg|_{\beta=2\pi} = I(E_0)$$

and

$$\mathcal{R}_{31}^{(\text{excited})} \equiv \frac{\left(\beta \frac{\partial}{\partial \beta}\right)^3 Z^{(\text{high})}(\beta)}{\left(\beta \frac{\partial}{\partial \beta}\right)^1 Z^{(\text{high})}(\beta)} \Bigg|_{\beta=2\pi} \tag{2.8}$$

The ratio $\mathcal{R}_{31}^{(\text{vac})}$ is equal to $I(E_0)$ because the partition function $Z^{(\text{vac})}(\beta)$ contains only a single exponential $\exp\{-\beta E_0\}$.

The ratios (2.8) are necessarily equal in any modular invariant theory, by virtue of relations (2.6) and (2.7):

$$\mathcal{R}_{31}^{(\text{excited})} = \mathcal{R}_{31}^{(\text{vac})} = I(E_0) . \tag{2.9}$$

Now we will use the relative importance factor $I(E)$ to show that (2.9) can never be satisfied if the bound (2.2) is violated.

To do this, write the ratio $\mathcal{R}_{31}^{(\text{excited})}$ as follows:

$$\mathcal{R}_{31}^{(\text{excited})} = \frac{\sum_{m=1}^{\infty} I(E_m) E_m \exp\{-2\pi E_m\}}{\sum_{n=1}^{\infty} E_n \exp\{-2\pi E_n\}} . \quad (2.10)$$

Now let us compare the individual factors $I(E_n)$ to the ratio $I(E_0) = \mathcal{R}_{31}^{(\text{vac})}$. The equation $I(E) = I(E_0)$ has two roots, namely E_0 itself, and

$$E_+^{(\text{warm-up})} \equiv \frac{3}{2\pi} - E_0 = \frac{3}{2\pi} + \frac{c+\tilde{c}}{24} . \quad (2.11)$$

The larger root $E_+^{(\text{warm-up})}$ is positive, since E_0 is negative in any unitary theory. So there are two possible ranges for the relative importance function $I(E)$, namely

$$I(E) \leq (E_0) \quad \text{for} \quad E_0 \leq E \leq E_+^{(\text{warm-up})} \quad (2.12)$$

and

$$I(E) > (E_0) \quad \text{for} \quad E > E_+^{(\text{warm-up})} . \quad (2.13)$$

Now we will use proof by contradiction to show that the lowest excited energy level E_1 can be no greater than E_+ . Suppose E_1 lies in the second range, (2.13). Then so does every excited level $E_n, n \geq 1$. This would give us the inequalities

$$E_n \geq E_1 > E_+^{(\text{warm-up})} > 0 , \quad (2.14)$$

$$I(E_n) > I(E_0) > 0$$

for all $n \geq 1$. These inequalities lead to a contradiction: subtracting the two sides of (2.9) and using the identity (2.10), we obtain

$$0 = \mathcal{R}_{31}^{(\text{excited})} - I(E_0) = \frac{\sum_{m=1}^{\infty} (I(E_m) - I(E_0)) \cdot E_m \cdot \exp\{-2\pi E_m\}}{\sum_{n=1}^{\infty} E_n \cdot \exp\{-2\pi E_n\}} . \quad (2.15)$$

If $E_1 > E_+^{(\text{warm-up})}$, then every term in the numerator and denominator of (2.15) is positive, by virtue of the inequalities (2.14), and equation (2.15) cannot be satisfied. We conclude that

$$E_1 \leq \frac{c_{\text{total}}}{24} + \frac{3}{2\pi} , \quad \text{Q.E.D.} \quad (2.16)$$

The proof above establishes a general upper bound for the lowest excited energy level in any unitary CFT with discrete spectrum. Written in terms of operator dimensions $\Delta \equiv E - E_0$, we have a lower bound on the scaling dimension of the lowest-dimension operator other than the identity:

$$\Delta_1 \leq \Delta_+^{(\text{warm-up})} , \quad (2.17)$$

with $\Delta_+^{(\text{warm-up})} \equiv \frac{c_{\text{total}}}{12} + \frac{3}{2\pi}$. For $c + \tilde{c} \leq 24 - \frac{18}{\pi} \simeq 18.270$, we have $\Delta_+^{(\text{warm-up})} < 2$, so the lowest operator satisfying the bound, other than the identity, must be a primary operator. For $c_{\text{total}} > 18.270$, the value of $\Delta_+^{(\text{warm-up})}$ is greater than 2, so the bound yields no information, since any CFT always contains a stress tensor, which has dimension two.

In the range $c_{\text{total}} \leq 18.270$, the individual central charges c and \tilde{c} must be equal, since both are positive (by unitarity) and their difference is an integer multiple of 24 (by modular invariance). So the theorem states that there exists a primary operator (other than the identity) of scaling dimension less than $\frac{c}{6} + \frac{3}{2\pi}$ for any unitary, modular invariant CFT with $c, \tilde{c} < 9.135$. This is a somewhat limited range of central charge, but it contains many interesting theories, including any supersymmetric sigma model on a Calabi-Yau threefold (with diagonal GSO projection).

3 A general inequality for primary operators

In order to derive a useful inequality for theories with $c_{\text{total}} > 18.270$, we would like to separate primaries from descendants, so that we can find an upper bound on the weight of the lowest *primary* state. In this section, we will adapt the methods of the previous section to focus on primary operators alone.

3.1 Strategy

Our strategy to derive such an upper bound, in parallel with the derivation of the previous section, is to write the partition function explicitly in terms of weights of primaries, rather than in terms of weights of general operators. To do this, we proceed as follows:

1. Restrict our attention to the case where both left- and right-moving central charges are greater than 1. In this case, the Virasoro representations of primary states are particularly simple. We will simplify the analysis further by assuming that the CFT has no chiral algebra other than the Virasoro algebra.
2. Write down a partition function for primary operators alone. For the cases we consider, this partition function is related by a simple equation to the full partition function.
3. Use the relation between the full partition function and the partition function for primaries to derive a transformation law for the latter under the modular S -operation $\tau \rightarrow -\frac{1}{\tau}$, given the modular invariance of the former.
4. Express modular invariance as an infinite sequence of differential identities on the partition function for primaries at medium temperature $\beta^{-1} = \frac{1}{2\pi}$.

5. Show that the first two of these identities are not compatible with one another if the energy of the lowest primary state is too large.

3.2 Review of Virasoro representations

For the rest of this section we will assume c and \tilde{c} are both greater than 1. This has well-known and useful implications for the structure of representations of the Virasoro algebra [14, 16, 17]. In particular, for $c > 1$ the unitary highest-weight representations of the Virasoro algebra are of two types, characterized by the weight h of the primary state $|h\rangle$ on which the representation is built. For the first type of representation, the primary state $|h\rangle$ has weight $h > 0$, and all each ordered monomial of Virasoro raising operators creates an independent state. That is, for $h > 0$ there are no linear relations among the states

$$L_{-n_1} \cdot L_{-n_2} \cdots L_{-n_k} \cdot |h\rangle$$

for any collection (possibly empty) of n_i with $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$. The second type of Virasoro representation is the one in which the primary state has $h = 0$. In this case the linearly independent states of the representation are given by

$$L_{-n_1} \cdot L_{-n_2} \cdots L_{-n_k} \cdot |0\rangle$$

for any collection (possibly empty) of n_i with $n_1 \geq n_2 \geq \cdots \geq n_k \geq 2$.

3.3 Decomposition of the partition function

By assuming the theory has no chiral algebra beyond the Virasoro algebra, we eliminate from consideration primaries with $h = 0, \tilde{h} \neq 0$ or vice versa. So the only primaries in our theory have $h = \tilde{h} = 0$ or $h, \tilde{h} > 0$. By cluster decomposition, the CFT can contain only one operator with $h = \tilde{h} = 0$, namely the identity operator. So we can decompose our partition function $Z(\tau)$ into a sum over conformal families, including the identity family:

$$Z(\tau) = Z_{\text{id}}(\tau) + \sum_A Z_A(\tau) ,$$

where $Z_{\text{id}}(\tau)$ is the sum over states in the conformal family of the identity, and $Z_A(\tau)$ is the sum over all states in the conformal family of the A^{th} primary, which has conformal weights h_A, \tilde{h}_A . By the structure theorem for Virasoro representations with $c > 1$ referred to above [14, 16, 17], the partition function $Z_A(\tau)$ is

$$Z_A(\tau) = q^{h_A - \frac{c}{24}} \bar{q}^{\tilde{h}_A - \frac{\tilde{c}}{24}} \prod_{m=1}^{\infty} (1 - q^m)^{-1} \prod_{n=1}^{\infty} (1 - \bar{q}^n)^{-1}$$

Likewise, the partition function $Z_{\text{id}}(\tau)$ is

$$Z_{\text{id}}(\tau) = q^{-\frac{c}{24}} \bar{q}^{-\frac{\tilde{c}}{24}} \prod_{m=2}^{\infty} (1 - q^m)^{-1} \prod_{n=2}^{\infty} (1 - \bar{q}^n)^{-1}$$

So we can write the full partition function as

$$Z(\tau) = q^{-\frac{c}{24}} \bar{q}^{-\frac{\bar{c}}{24}} \left[\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right] \cdot \left[\prod_{n=1}^{\infty} (1 - \bar{q}^n)^{-1} \right] \cdot \left[(1 - q)(1 - \bar{q}) + Y(\tau) \right]$$

where

$$Y(\tau) \equiv \sum_{A=1}^{\infty} q^{-h_A} \bar{q}^{-\tilde{h}_A}$$

is a sum over primary states only, with the vacuum omitted.

We can simplify this expression by using the definition of the Dedekind eta function

$$\eta(\tau) \equiv q^{+\frac{1}{24}} \cdot \prod_{n=1}^{\infty} (1 - q^n)^{+1}$$

so

$$\prod_{n=1}^{\infty} (1 - q^n)^{-1} = \frac{q^{+\frac{1}{24}}}{\eta(\tau)} .$$

So we write the full partition function as

$$Z(\tau) = q^{-\frac{c-1}{24}} \bar{q}^{-\frac{\bar{c}-1}{24}} \cdot |\eta(\tau)|^{-2} \cdot \left[(1 - q)(1 - \bar{q}) + Y(\tau) \right]$$

Restricting to the imaginary axis $\tau \equiv i\beta/(2\pi)$, with β real, we have $q = \bar{q} = \exp\{-\beta\}$, and

$$Z(\beta) = M(\beta)Y(\beta) + B(\beta) \tag{3.18}$$

with

$$M(\beta) \equiv \frac{\exp\{-(E_0 + \frac{1}{12})\beta\}}{\eta(i\beta/2\pi)^2} \tag{3.19}$$

and

$$B(\beta) \equiv M(\beta) \cdot \left(1 - \exp\{-\beta\} \right)^{+2} , \tag{3.20}$$

where $E_0 \equiv -\frac{c_{\text{total}}}{24}$. For real β , the partition function over primaries $Y(\beta)$ becomes

$$Y(\beta) \equiv \sum_A \exp\{-\beta\Delta_A\} , \tag{3.21}$$

where $\Delta_A \equiv h_A + \tilde{h}_A$ is the weight of the primary operator \mathcal{O}_A .

In what follows, it will be convenient to define the polynomials $f_p(x)$ by the equation

$$\left(\beta \partial_\beta \right)^p \left[\frac{\exp\{-z\beta\}}{\eta(i\beta/2\pi)^2} \right] \Big|_{\beta=2\pi} \equiv (-1)^p \cdot \eta(i)^{-2} \cdot \exp\{-2\pi z\} \cdot f_p(z) . \tag{3.22}$$

The first few polynomials are

$$\begin{aligned}
f_0(z) &= 1 \\
f_1(z) &= (2\pi z) - \frac{1}{2} \\
f_2(z) &= (2\pi z)^2 - 2(2\pi z) + \left(\frac{7}{8} + 2r_{20} \right)
\end{aligned} \tag{3.23}$$

$$f_3(z) = (2\pi z)^3 - \frac{9}{2}(2\pi z)^2 + \left(\frac{41}{8} + 6r_{20} \right) (2\pi z) - \left(\frac{17}{16} + 3r_{20} \right)$$

where r_{20} is a numerical constant we have defined as

$$\begin{aligned}
r_{20} \equiv \frac{\eta''(i)}{\eta(i)} &= \left(\frac{\eta'(i)}{\eta(i)} \right)^2 + \partial_\tau^2 \left[\ln \left(\eta(\tau) \right) \right] \Big|_{\tau=i} = -\frac{1}{16} + \sum_{n=1}^{\infty} \frac{\pi^2 n^2}{\sinh^2(\pi n)} \\
&= 0.0120528 + o\left(10^{-8}\right) .
\end{aligned} \tag{3.24}$$

In deriving these polynomials we have used the identities

$$\begin{aligned}
\eta'(i) &= \frac{i}{4}\eta(i) \\
\eta'''(i) &= \frac{15i}{32} \left(\eta(i) + 8\eta''(i) \right) .
\end{aligned} \tag{3.25}$$

The identities (3.25) follow from the modular transformation law of the eta function,

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \cdot \eta(\tau) ,$$

whose medium-complex-structure expansion yields

$$\left(\tau \partial_\tau + \frac{1}{4} \right)^p \eta(i) \Big|_{\tau=i} = 0, \quad p \text{ odd} .$$

The $p = 1, 3$ identities yield eqns. (3.25) directly.

Now we would like to take derivatives of the two terms $B(\beta)$ and $M(\beta)Y(\beta)$ at medium temperature. For $M(\beta)Y(\beta)$, equations (3.19),(3.21) together with the formula (3.22) give

$$\begin{aligned}
&\left(\beta \partial_\beta \right)^p M(\beta)Y(\beta) \Big|_{\beta=2\pi} = \\
&(-1)^p \eta(i)^{-2} \exp \left\{ -2\pi \left(E_0 + \frac{1}{12} \right) \right\} \sum_{A=1}^{\infty} f_p \left(\Delta_A + E_0 + \frac{1}{12} \right) \exp \left\{ -2\pi \Delta_A \right\}
\end{aligned} \tag{3.26}$$

As for $B(\beta)$, applying the formula (3.22) to eq. (3.20) gives

$$\left(\beta \partial_\beta \right)^p B(\beta) \Big|_{\beta=2\pi} = (-1)^p \eta(i)^{-2} \exp \left\{ -2\pi \left(E_0 + \frac{1}{12} \right) \right\} \cdot \quad (3.27)$$

$$\left[f_p \left(E_0 + \frac{1}{12} \right) - 2 \exp \{ -2\pi \} f_p \left(E_0 + \frac{13}{12} \right) + \exp \{ -4\pi \} f_p \left(E_0 + \frac{25}{12} \right) \right]$$

For simplicity, we now define $\hat{E}_0 \equiv E_0 + \frac{1}{12} = \frac{2-c_{\text{total}}}{24}$. Then we have

$$\begin{aligned} \left(\beta \partial_\beta \right)^p M(\beta) Y(\beta) \Big|_{\beta=2\pi} &= (-1)^p \eta(i)^{-2} \exp \left\{ -2\pi \hat{E}_0 \right\} \sum_{A=1}^{\infty} f_p(\Delta_A + \hat{E}_0) \exp \{ -2\pi \Delta_A \} \\ \left(\beta \partial_\beta \right)^p B(\beta) \Big|_{\beta=2\pi} &= (-1)^p \eta(i)^{-2} \exp \left\{ -2\pi \hat{E}_0 \right\} \cdot b_p(\hat{E}_0) \end{aligned} \quad (3.28)$$

$$b_p(x) \equiv f_p(x) - 2 \exp \{ -2\pi \} f_p(x+1) + \exp \{ -4\pi \} f_p(x+2)$$

Thus the medium-temperature equations for modular invariance of $Z(\beta)$ at $p = 1, 3$ reduce to

$$\begin{aligned} \sum_{A=1}^{\infty} f_1(\Delta_A + \hat{E}_0) \exp \{ -2\pi \Delta_A \} &= -b_1(\hat{E}_0) \\ \text{and} & \end{aligned} \quad (3.29)$$

$$\sum_{A=1}^{\infty} f_3(\Delta_A + \hat{E}_0) \exp \{ -2\pi \Delta_A \} = -b_3(\hat{E}_0)$$

Now we will proceed in parallel with our "warm-up" proof. Define the relative importance factor

$$I_{31}(x) \equiv \frac{f_3(x)}{f_1(x)} \quad (3.30)$$

and the coefficient

$$K_{31}(\hat{E}_0) \equiv \frac{b_3(\hat{E}_0)}{b_1(\hat{E}_0)}. \quad (3.31)$$

As before, we will take the scaling dimensions Δ_n of our primary operators to be indexed in order so that Δ_n is increasing:

$$0 = \Delta_0 < \Delta_1 \leq \Delta_2 \leq \dots$$

To derive an upper bound on the dimension Δ_1 of the lowest primary operator other than the identity, divide the two equations (3.29). We then obtain

$$\frac{\sum_{A=1}^{\infty} I_{31}(\Delta_A + \hat{E}_0) f_1(\Delta_A + \hat{E}_0) \exp \{ -2\pi \Delta_A \}}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0) \exp \{ -2\pi \Delta_B \}} = K_{31}(\hat{E}_0) \quad (3.32)$$

Now we will prove that (for $c, \tilde{c} > 1$) the value of Δ_1 must always be less than or equal to Δ_+ , which we define as the largest solution Δ to the equation

$$I_{31}(\Delta + \hat{E}_0) = K_{31}(\hat{E}_0) , \quad (3.33)$$

which is equivalent to the cubic equation

$$f_3(\Delta + \hat{E}_0) - K_{31}(\hat{E}_0)f_1(\Delta + \hat{E}_0) = 0 . \quad (3.34)$$

Note that Δ_+ is implicitly a function of \hat{E}_0 , though we will not always indicate the \hat{E}_0 dependence in our notation.

We will prove our desired result by contradiction. Suppose $\Delta_1 > \Delta_+$. Now subtract the two sides of (2.9), to give

$$0 = \frac{\sum_{A=1}^{\infty} \left(I_{31}(\Delta_A + \hat{E}_0) - K_{31}(\hat{E}_0) \right) \cdot f_1(\Delta_A + \hat{E}_0) \cdot \exp\{-2\pi\Delta_A\}}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0) \cdot \exp\{-2\pi\Delta_B\}} \quad (3.35)$$

By definition of Δ_+ , the function $I_{31}(\Delta + \hat{E}_0)$ is greater than $K_{31}(\hat{E}_0)$ for $\Delta > \Delta_+$. So if it were the case that $\Delta_1 > \Delta_+$, then we would have the inequalities

$$\Delta_n \geq \Delta_1 > \Delta_+, \quad \text{all } n \geq 1 , \quad (3.36)$$

$$I_{31}(\Delta_n + \hat{E}_0) > K_{31}(\hat{E}_0), \quad \text{all } n \geq 1 .$$

We have one additional lemma to establish: that Δ_+ is necessarily greater than $\frac{1}{4\pi} - \hat{E}_0$ if c and \tilde{c} are greater than 1. It is straightforward to check this property of Δ_+ numerically, and we establish it with an analytic proof in one subsection of the Appendix. This property of Δ_+ means that

$$f_1(\Delta_n + \hat{E}_0) = 2\pi\Delta_n + 2\pi\hat{E}_0 - \frac{1}{2} > 0, \quad \text{all } n \geq 1 . \quad (3.37)$$

From the inequalities (3.36) and (3.37), we could infer that every term in the numerator and denominator of the right hand side of (3.35) would have to be positive, so the equation (3.35) would be inconsistent. Therefore our hypothesis cannot be true, and we conclude that

$$\Delta_1 \leq \Delta_+, \text{ Q.E.D.} \quad (3.38)$$

3.4 Extended chiral algebras, and $c \leq 1$

We will comment briefly on the special cases we have excluded from our considerations.

CFT with extended chiral algebras

Our proof assumes that there does not exist an extended chiral algebra in the CFT – that is, that there does not exist an operator other than the identity with $h = 0$ or $\tilde{h} = 0$. Relaxing our assumptions to include such operators may lead to further interesting results. In particular, the case $h = 0, \tilde{h} = 1$ (or vice versa) is interesting to consider: this is the case in which the CFT carries continuous global current algebra symmetries. In this case there are necessarily a tower of Virasoro primaries which have low dimension by virtue of being current algebra descendants of the identity. (In the bulk interpretation, these states correspond to a gas of chiral gauge field excitations confined to the boundary). The interesting calculation in this case would be to derive an upper bound on the weight of the lowest nontrivial operator that is primary with respect to the *full* chiral algebra – the Virasoro and current algebra pieces simultaneously. When the current algebra group is abelian, such a derivation could be interpreted as an upper bound on the mass of the lightest charged state in a theory of gravity, and could amount to a rigorous proof of the “weak gravity conjecture” [18] of Arkani-Hamed *et al.* for the case of negative cosmological constant.

Note also that the holomorphically factorized CFT of [6, 9, 10] also contain very large chiral algebras, and the methods of our proof are not directly relevant to these. In general, though, CFT with chiral algebras tend to have low-dimension operators (for instance, the chiral algebra itself), and it is likely that the best possible bound for Δ_1 will tend to be lower in a theory with a nontrivial chiral algebra than in a theory without, of the same central charge. This is an interesting direction for further investigation.

The special case of $c \leq 1$

Our removal of gravitational descendants in the case $c > 1$ relied on the structure theorem for unitary representations of the Virasoro algebra. For $c \leq 1$ there are unitary Virasoro modules with nontrivial structure that one must take into account if one wants to extend the proof.

Conformal field theories with $c \leq 1$ but $\tilde{c} > 1$ are not classified, and may come in an infinite variety. However we would expect that there should always be a large left-moving chiral algebra in such theories, and the special issues involving extended chiral algebras would become relevant.

The gravitational interpretation of theories with $0 < c \leq 1$ and \tilde{c} very large is still open. If we would like to describe the bulk theory with a local Lagrangian, we would seem to need to add a gravitational Chern-Simons term [32, 33], of the kind used in [34]. The paper [34] only deals with the case in which one of the two central charges strictly vanishes. It is not clear how to generalize [34] to the case where the left- and right-moving central charges are unequal but both nonzero. Other AdS/CFT dualities exist [35] in which the left- and right-moving central charges are nonzero and unequal, and described in the bulk by a perturbative string theory rather than a local action. However in all such examples the left- and right-moving central charges are both large; neither is between 0 and 1.

In the case where both c and \tilde{c} are less than or equal to 1, our understanding of the CFT is complete: compact, unitary CFT with $c < 1$ are completely classified⁴ and it is possible to inspect the operator spectra of these theories directly rather than deriving a bound by abstract methods. The range $0 < c_{\text{total}} \leq 2$ represents AdS_3 spaces with Planck-scale curvatures, so these cases are exotic at best as theories of gravity in three dimensions.

4 The gravitational interpretation of the upper bound on Δ_1

In this section we turn to the gravitational interpretation of our CFT results. We have derived an inequality that is completely universal in the set of two-dimensional conformal field theories, with some mild conditions: unitarity, discreteness of the spectrum, and the condition that $c, \tilde{c} > 1$, as well as the absence of purely left- or right-moving operators other than the stress tensor.

Such a universal inequality may have many interesting applications. In particular, our bound is relevant for the physics of gravity with negative cosmological constant. The virtue of our approach is that we can derive a nontrivial, non-asymptotic constraint on the spectrum of a theory of gravity with negative cosmological constant. In order to express our bound in this form, we need only express the cosmological constant of the theory in terms of the central charge of the corresponding CFT, and the mass of a state in terms of the dimension of the corresponding local operator.

4.1 Central charge and AdS radius

In the case of the AdS_3/CFT_2 correspondence, the matching between the central charge of the CFT and the (negative) cosmological constant predates the understanding of the AdS/CFT correspondence as a dynamical principle. Brown and Henneaux [19] were able to identify the central charge in the Virasoro algebra of AdS_3 symmetries, based purely on the structure of the classical Poisson bracket algebra, leading to the identification

$$c + \tilde{c} = \frac{3}{G_N \sqrt{|\Lambda|}} . \tag{4.39}$$

It was later verified [20] that this is indeed the correct central charge for the CFT that corresponds to the theory in the sense of [21].

⁴For a review of the classification of modular-invariant theories with $c, \tilde{c} < 1$, see [36].

4.2 Dimensions, masses, and rest energies

We also wish to match the spectrum of massive objects with the spectrum of primary operators. A primary state should be thought of as corresponding to a state at rest with respect to the global time coordinate of AdS, because its energy cannot be lowered by acting with boost generators. As always, we simplify the situation by assuming the absence of holomorphic primary operators. (These would have a little group different from that of a massive particle in the bulk of AdS; therefore for small $\Lambda = -L^{-2}$ they can only correspond to massless states, which do not have a rest frame, or else to states which do not propagate into the bulk of AdS at all.)

So we have the correspondence

$$E^{(rest)} = \frac{\Delta}{L} , \tag{4.40}$$

where $E^{(rest)}$ is the rest energy of an object in the bulk of AdS, and Δ is the dimension of the primary operator.

For a minimally coupled massive scalar field of mass m_s , we could use the dictionary of [8], [22]:

$$\begin{aligned} \Delta &= 1 + \sqrt{1 + m_s^2 L^2} \\ &\Leftrightarrow \end{aligned} \tag{4.41}$$

$$m_s = \frac{1}{L} \cdot \sqrt{\Delta^2 - 2\Delta} .$$

The formula (4.41) gives

$$m_s \simeq \frac{\Delta}{L} + o\left(\Delta^0\right)$$

when Δ is large. In the limit where $L \rightarrow \infty$ with m_s held fixed, the order $\frac{1}{\Delta}$ terms in the difference between m_s and the rest energy of a massive excitation can be thought of as the coupling of the massive field to the AdS curvature.

4.3 Primaries and descendants

As for the bulk interpretation of descendants of the primary with dimension Δ , we follow [6] in interpreting these as the original massive state in the bulk with boundary metric excitations added.

To be more precise, the states obtained by acting with L_{-n}, \tilde{L}_{-n} with $n \geq 2$ correspond with creation operators for quadrupole and higher modes of the metric; these are localized at spatial infinity and can be thought of loosely as "boundary gravitons" or "boundary metric excitations". (No graviton states ever propagate in the bulk in

three dimensions.) Acting with L_{-1} and \tilde{L}_{-1} , on the other hand, can be thought of as exciting the dipole mode of the metric. In other words, the raising operators L_{-1} and \tilde{L}_{-1} boost the massive object in the bulk to a state of motion with higher energy. So the primary states of the CFT correspond one to one with massive states in the bulk that are at rest, in which no boundary gravitons are excited. Their descendants correspond to objects either in a nonzero state of motion, or with some boundary gravitons excited, or both.

4.4 Bulk interpretation of the upper bound on Δ_1

We do not wish to assume that the lightest massive state is a scalar, nor that it is necessarily described by a minimally coupled local field, so we will use the formula (4.40) to interpret our bound (3.38) in terms of the bulk physics in the flat-space limit. Using formulæ (4.39) and (4.40), we can interpret (3.38) as saying that every consistent theory of quantum gravity with negative cosmological constant $\Lambda = -L^{-2}$ must necessarily have a massive state in the bulk (with no boundary gravitons excited), with center-of-mass energy equal to M_1 , where

$$M_1 < M_+ \tag{4.42}$$

$$M_+ \equiv \frac{1}{L} \Delta_+ \Big|_{c_{\text{total}} = \frac{3L}{G_N}} .$$

Of course we are assuming, as always, that the AdS radius L is not so small that $c, \tilde{c} \leq 1$.

We can now use our best linear upper bound on Δ_+ , as derived in the Appendix and stated in (A.6). There, we show that

$$\Delta_+ < \frac{c_{\text{total}}}{12} + \delta_0 , \tag{4.43}$$

where $\delta_0 \equiv 0.473695$. In gravitational terms, this means that

$$M_1 \leq M_+ < \frac{1}{4G_N} + \frac{\delta_0}{L} . \tag{4.44}$$

In the flat-space limit $\Lambda \rightarrow 0$, this says that

$$M_1 \leq \frac{1}{4G_N} . \tag{4.45}$$

This value of the mass M_1 is suggestive. The rest energy of the lightest BTZ black hole is $\frac{1}{8G_N}$ above the energy of the vacuum.⁵ The maximum possible value of M_1 is

⁵We should think of $\frac{1}{8G_N}$ as the rest mass of the lightest object in the spectrum of classical black holes, despite the fact that the lightest BTZ black hole is often referred to as the "zero mass" BTZ black hole.

twice that amount, so intuitively we may say that, since the BTZ black hole exists as a state in every theory of 3D gravity and matter, then there should always be a massive state at about that energy scale, even when quantum corrections are taken into account.

We cannot, however, find any independent bulk reasoning that could predict the coefficient of $\frac{1}{G_N}$ in such an upper bound. The tree-level mass of a Planck-scale black hole need not have any particular significance at the quantum level: small black holes would be expected to receive $o(1)$ multiplicative renormalizations to their masses from virtual matter particles. It is not apparent how one could use bulk reasoning to prove any upper bound on the quantum mass renormalization of the lightest black hole, for a general theory of gravity coupled to matter.

The mass $\frac{1}{4G_N}$ has been argued [37] to have a special significance when $\Lambda = 0$, as the maximum value of the total energy of a collection of matter coupled to gravity in 2+1 dimensions.⁶ The reason is simple to understand: viewed from a long enough distance away, any collection of matter with energy M looks like a point particle, which creates a conical deficit in the metric of $\Delta\phi = 16\pi G_N M$. For $M = \frac{1}{4G_N}$ this means the space closes off entirely into a sphere, and for $M > \frac{1}{4G_N}$ there is no consistent geometry at all.

For small but negative Λ the closing-off of the space can be avoided if the collection of matter diffuse enough that the negative vacuum energy in any region cancels or overcancels the positive energy carried by the matter. This cancellation cannot be achieved if the matter is made of point particles of mass $M = \frac{1}{4G_N}$ but it could be achieved if the objects of $M = \frac{1}{4G_N}$ were composite objects such as solitons or strings.

We are not entirely certain how to relate the result of [37] to the bound (4.44) the limit $\Lambda \rightarrow 0$. It may be that any theory of gravity and matter saturating (4.44) is necessarily very degenerate in the flat-space limit, with dynamics that break up into a product of disjoint systems with a small number of states in each one. Alternately, the result of [37] may be a sign that a better upper bound on Δ_1 can be proven that is lower than Δ_+ by some numerical factor, in the limit $c_{\text{total}} \rightarrow \infty$. Such a conclusion would fit well with the results [9, 10, 6, 25].

5 Conclusions

In this note we have derived a rigorous upper bound on the scaling dimension of the lowest primary operator (other than the identity) in a two-dimensional conformal field theory. This bound is universal among all unitary conformal field theories satisfying some very mild conditions: $c, \tilde{c} > 1$ and the absence of purely left- or right-moving operators beyond the components of the stress tensor itself.

As a warm-up, we also derived an upper bound for the scaling dimension of the

⁶We thank Lee Smolin for making us aware of this paper.

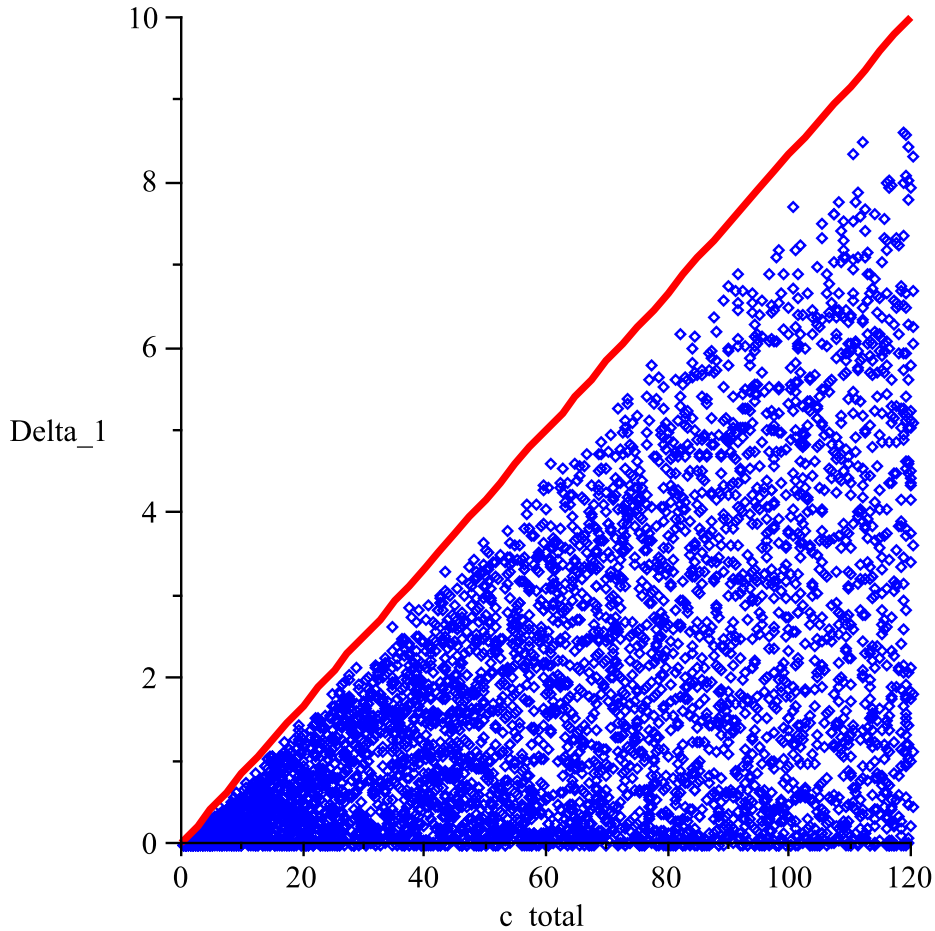


Figure 4: In this paper we have proven that the distribution of unitary conformal field theories in two dimensions looks something like the scatter plot above, where Δ_1 is the weight of the lowest primary operator. It is an open question whether there exist CFT that saturate the bound at leading order in c_{total} , or whether further considerations could reduce the slope of the red bounding line from $\frac{1}{12}$, perhaps to as low as $\frac{1}{24}$.

lowest operator of any kind – primary or descendant – other than the identity. For sufficiently large values of the total central charge, this version of the bound provides no information, since every theory contains a stress tensor, which has scaling dimension 2. But for $\tilde{c} + c$ less than $24 - \frac{18}{\pi} \simeq 18.270$, even this rudimentary version of the bound does predict the existence of a primary of dimension less than 2.

One compelling open question is to what extent $\Delta_+ \sim \frac{c_{\text{total}}}{12}$ is the best bound possible at large central charge, in the full set of 2D conformal field theories satisfying our conditions.

On the one hand, there is some reason to suspect that our bound could be improved.

In the holomorphically factorized case [6], which includes the biggest gap in the spectrum of primaries of any known class of examples, the bound $\Delta_1 \leq \frac{c}{24} + 1$ is a factor of two lower than the one we have derived, for large c . The work of [25] suggests the same bound in another special class of theories with $(2, 2)$ supersymmetry. If these special classes of CFT are typical of 2D CFT as a whole, then it should be possible to cut our upper bound Δ_+ in half.

We also observe that in the familiar cases of $\text{AdS}_3/\text{CFT}_2$ duality, there is not only a single primary operator lying below Δ_+ , but many. In fact, in every known example, there are an infinite number of primary operators with $\Delta < \Delta_+$ in the limit $c_{\text{total}} \rightarrow \infty$. In terms of the bulk, these operators can be realized as strings, Kaluza-Klein modes of a decompactifying internal dimension, or some other states of energy lower than $\frac{1}{4G_N}$. This pattern also suggests the possibility of improving on our inequality.

On the other hand, it may well be that the holomorphically factorized case is a misleading guide to the upper bound on Δ_1 in the general case. Holomorphically factorized CFT have many special properties that are highly atypical of the set of CFT as a whole. As an obvious example, we note that factorized CFT always lack the chaos and thermalizing behavior associated with true black holes.⁷ The extreme specialness of holomorphically factorized CFT, in this respect and others, suggests that they may not ever realize the largest gap Δ_1 in primary operator dimensions that is achievable in general.

The other prominent examples [25] in which there is good evidence for a bound of $\Delta_1 \simeq \frac{c_{\text{total}}}{24}$, also have atypical properties, by virtue of their $(2, 2)$ -extended supersymmetry. The states of dimension $\Delta_1 \simeq \frac{c_{\text{total}}}{24}$, that realize the proposed bound of [25], are BPS operators – they are chiral primaries of the $(2, 2)$ superalgebra. Their dimensions, and the masses of the corresponding bulk states, are protected by supersymmetry from any renormalizations. For generic deformations of these theories breaking all of their SUSY, we might expect that the mass of the state could receive an $o(1)$ multiplicative renormalization. Such a renormalization could push the mass upwards by some factor from its a tree-level value of $\frac{1}{8G_N}$, possibly to as high as $\frac{1}{4G_N}$.

Thus it is not clear at present whether our bound $\Delta_1 \leq \Delta_+ \simeq \frac{c_{\text{total}}}{12}$ is the lowest achievable in the set of generic 2D CFT. It would be interesting to know with confidence what the value best possible bound is for large c_{total} . It may be possible to learn this optimal value, by deriving an inequality for Δ_1 in the general case together with a series of examples where Δ_1 saturates the inequality in the limit where c_{total} is large.

The goal of our work was to understand how the rigorous holographic definition of quantum gravity in terms of CFT might generate universal predictions that would hold

⁷In a factorized CFT, amplitudes are automatically periodic in time with period 2π , in units where the radius of the spatial circle is 1. Translated into gravitational language, this means that every correlation function would return precisely to itself after shifting the Lorentzian time coordinate of any one of the operator insertions by the AdS time $L = |\Lambda|^{-\frac{1}{2}}$. Thermal correlators therefore cannot decay exponentially as they would in the presence of black hole, for a generic theory of gravity coupled to matter [38, 39, 40].

among *all* theories with an AdS ground state of a given size. We have actually exceeded that goal: we have derived an upper bound on the mass of the lightest massive state that is *independent of the boundary conditions* in the limit where the AdS becomes large. That is, the universal upper bound on the lightest mass approaches a finite limit in Planck units, giving us a universal, falsifiable prediction about local bulk physics, that does not refer at all to the regulating AdS boundary condition.

The desirability of such robust predictions has grown increasingly acute as our understanding of quantum gravity has developed: just as we have come to understand that holography provides the "genetic code" of quantum gravity, we have simultaneously discovered an unimaginably vast and complex jungle of theories [41] realizing that underlying code in myriad ways. We are only now beginning to learn what family resemblances the flora and fauna of this vast ecosystem have in common.

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Appendix

A Properties of the function $\Delta_+(\hat{E}_0)$

In this Appendix we will establish some facts about the function $\Delta_+(\hat{E}_0)$. We only deal with CFT for which $c, \tilde{c} > 1$, so we will always restrict the domain of definition of $\Delta_+(\hat{E}_0)$ to $\hat{E}_0 \leq 0$, or equivalently $c_{\text{total}} > 2$.

A.1 Definition of Δ_+

Recall that we have defined $\Delta_+(\hat{E}_0)$ to be the largest real root Δ of the cubic polynomial

$$P_{31}(\Delta) \equiv f_3(\Delta + \hat{E}_0) - K_{31}(\hat{E}_0)f_1(\Delta + \hat{E}_0) = \frac{1}{b_1(\hat{E}_0)} F_{31}(\Delta) ,$$

$$F_{31}(\Delta) \equiv b_1(\hat{E}_0)f_3(\Delta + \hat{E}_0) - b_3(\hat{E}_0)f_1(\Delta + \hat{E}_0) .$$

We have defined K_{31} in (3.31), in the main body of the paper.

The explicit, analytic expressions for the polynomials $f_{1,3}$ and $b_{1,3}$ are:

$$f_1(z) = (2\pi z) - \frac{1}{2} ,$$

$$f_3(z) = (2\pi z)^3 - \frac{9}{2}(2\pi z)^2 + \left(\frac{41}{8} + 6r_{20} \right) (2\pi z) - \left(\frac{17}{16} + 3r_{20} \right) ,$$

$$b_1(z) = f_1(z) - 2 \exp \{-2\pi\} f_1(z+1) + \exp \{-4\pi\} f_1(z+2) ,$$

$$b_3(z) = f_3(z) - 2 \exp \{-2\pi\} f_3(z+1) + \exp \{-4\pi\} f_3(z+2) .$$

It is a straightforward exercise to establish that the constant r_{20} defined in (3.24) does not appear in the expression $b_1(x)f_3(y) - b_3(x)f_1(y)$ for general x, y . The expression $F_{31}(\Delta)$ is of this general form, so we will never need to use the value of r_{20} . Thought of as a polynomial in two variables (Δ, \hat{E}_0) , the coefficients in expression F_{31} involve only π and $\exp \{-2\pi\}$.

For completeness, we will write the full expression for F_{31} , for general \hat{E}_0 .

$$F_{31}(\Delta) = \sum_{m=0}^3 \sum_{n=0}^{4-m} A_{mn} \hat{E}_0^n \Delta^m$$

with the coefficients A_{mn} as follows:

$$A_{31} = 16\pi^4 (1 - \exp \{-2\pi\})^2$$

$$A_{30} = -4\pi^3 (1 - \exp \{-2\pi\}) \left[1 + (8\pi - 1)\exp \{-2\pi\} \right]$$

$$A_{22} = 48\pi^4(1 - \exp \{-2\pi\})^2$$

$$A_{21} = -48\pi^3 (1 - \exp \{-2\pi\}) \left[1 + (2\pi - 1)\exp \{-2\pi\} \right]$$

$$A_{20} = 9\pi^2 (1 - \exp \{-2\pi\}) \left[1 + (8\pi - 1) \exp \{-2\pi\} \right]$$

$$A_{13} = 32\pi^4 (1 - \exp \{-2\pi\})^2$$

$$A_{12} = -48\pi^3 (1 - \exp \{-2\pi\})^2$$

$$A_{11} = 6\pi^2 \left[3 + 2(8\pi^2 - 3) \exp \{-2\pi\} - (32\pi^2 - 3) \exp \{-4\pi\} \right]$$

$$A_{10} = -\pi \left[3 - (32\pi^3 - 72\pi^2 + 6) \exp \{-2\pi\} + (128\pi^3 - 144\pi^2 + 3) \exp \{-4\pi\} \right]$$

$$A_{04} = 0$$

$$A_{03} = 64\pi^4 \exp \{-2\pi\} (1 - \exp \{-2\pi\})$$

$$A_{02} = 96\pi^3 \exp \{-2\pi\} \left[(\pi - 1) - (2\pi - 1) \exp \{-2\pi\} \right]$$

$$A_{01} = 4\pi^2 \exp \{-2\pi\} \left[(8\pi^2 - 24\pi + 9) - (32\pi^2 - 48\pi + 9) \exp \{-2\pi\} \right]$$

$$A_{00} = -2\pi \exp \{-2\pi\} \left[(4\pi^2 - 9\pi + 3) - (16\pi^2 - 18\pi + 3) \exp \{-2\pi\} \right]$$

A.2 Δ_+ is a smooth function of c_{total}

We will now show that Δ_+ is continuous as a function of c_{total} in the range of interest, $c_{\text{total}} \in [2, \infty)$. Δ_+ is defined implicitly as the largest root Δ of the polynomial $P_{31}(\Delta, \hat{E}_0)$, with $\hat{E}_0 \equiv \frac{2-c_{\text{total}}}{24}$. The value of a root of a polynomial depends smoothly

on its coefficients, except when two roots become coincident. (This assumes the leading coefficient of the polynomial is constant, as in this case.) So in order to show that Δ_+ depends smoothly on c_{total} , we need to show that the coefficients in P_{31} are smooth functions of c_{total} , and that the roots of P_{31} are all distinct.

The coefficients of P_{31} depend smoothly on \hat{E}_0 as long as the rational function $K_{31}(\hat{E}_0) = \frac{b_3(\hat{E}_0)}{b_1(\hat{E}_0)}$ does. The denominator $b_1(\hat{E}_0)$, is given explicitly by

$$b_1(\hat{E}_0) = 2\pi(1 - \exp\{-2\pi\})^2 \hat{E}_0 - \frac{1}{2}(1 - \exp\{-2\pi\})^2 - 4\pi \exp\{-2\pi\} (1 - \exp\{-2\pi\}).$$

The second and third terms are negative, and the first is negative for $c_{\text{total}} > 2$ and vanishes for $c_{\text{total}} = 2$. So $b_1(\hat{E}_0)$ can never vanish, and the coefficients of P_{31} depend smoothly on c_{total} for $c_{\text{total}} \in [2, \infty]$.

So the only possible way Δ_+ could be non-smooth would be if P_{31} had coincident roots for some value of c_{total} . Fortunately P_{31} is only cubic as a function of Δ , so it is straightforward to check for coincident roots. Let us put P_{31} into canonical form for a cubic polynomial by changing variables so that the quadratic term vanishes and the coefficient of the leading term is 1.

The two leading terms of P_{31} are

$$P_{31}(\Delta, \hat{E}_0) = 8\pi^3 \left[\Delta^3 + \left(3\hat{E}_0 - \frac{9}{4\pi} \right) \Delta^2 + o(\Delta) \right]$$

so let us take

$$x \equiv 2\pi(\Delta + \hat{E}_0) - \frac{3}{2} \qquad \Delta = \frac{x}{2\pi} - \hat{E}_0 + \frac{3}{4\pi}.$$

Then

$$\begin{aligned} P_{31}(\Delta, \hat{E}_0) &= f_3\left(\frac{x+\frac{3}{2}}{2\pi}\right) - K_{31}(\hat{E}_0)f_1\left(\frac{x+\frac{3}{2}}{2\pi}\right) \\ &= x^3 + C_1(\hat{E}_0)x + C_0(\hat{E}_0) \end{aligned}$$

with

$$C_1(\hat{E}_0) \equiv -K_{31}(\hat{E}_0) + 6r_{20} - \frac{13}{8}$$

$$C_0(\hat{E}_0) \equiv -K_{31}(\hat{E}_0) + 6r_{20} - \frac{1}{8} = C_1(\hat{E}_0) + \frac{3}{2}$$

Note that the constant r_{20} drops out of the combinations $C_1(\hat{E}_0)$ and $C_0(\hat{E}_0)$.

Now we make use of the well-known formula for the discriminant of a cubic polynomial with leading term equal to 1 and vanishing quadratic term. The discriminant is given by

$$\text{Disc}(\hat{E}_0) \equiv 4C_1^3(\hat{E}_0) + 27C_0^2(\hat{E}_0)$$

As a function of x , the roots of $P_{31}(\Delta, \hat{E}_0)$ are distinct when $\text{Disc}(\hat{E}_0)$ vanishes. Using $C_1 = C_0 - \frac{3}{2}$, we can write

$$\text{Disc}(\hat{E}_0) = \mathbf{D}(C_0(\hat{E}_0)) ,$$

where

$$\mathbf{D}(y) \equiv 4y^3 + 9y^2 + 27y - \frac{27}{2}$$

Using Cardano's formula, we can find all the roots of $\mathbf{D}(y)$. Two of the roots are complex, and the real root is

$$y^* \equiv \frac{3}{4} \left[-1 + \left(6\sqrt{3} + 9 \right)^{\frac{1}{3}} - \left(6\sqrt{3} - 9 \right)^{\frac{1}{3}} \right] \simeq 0.427505 + o\left(10^{-7} \right) .$$

So the polynomial $P_{31}(\Delta, \hat{E}_0)$ can have a double zero only if

$$C_0(\hat{E}_0) = -K_{31}(\hat{E}_0) + 6r_{20} - \frac{1}{8} = y^*$$

$$\Leftrightarrow$$

$$\left[b_3(\hat{E}_0) - 6r_{20}b_1(\hat{E}_0) \right] + (y^* + \frac{1}{8})b_1(\hat{E}_0) = 0$$

This equation has three real roots, at

$$\hat{E}_0^{*(1)} = 0.0821971 + o\left(10^{-8} \right) ,$$

$$\hat{E}_0^{*(2)} = 0.184241 + o\left(10^{-7} \right) ,$$

$$\hat{E}_0^{*(3)} = 0.460984 + o\left(10^{-7} \right) .$$

In particular, all roots \hat{E}_0^* are positive. But we are only interested in $c_{\text{total}} > 2$, in which case \hat{E}_0 is negative or zero. It follows that the polynomial $P_{31}(\Delta, \hat{E}_0)$ has no multiple roots (as a function of Δ) for $c_{\text{total}} \in [2, \infty)$. We conclude that in this range the function $\Delta_+(\hat{E}_0)$ depends smoothly on c_{total} , *Q. E. D.*

A.3 The function Δ_+ is greater than $\frac{1}{4\pi} - \hat{E}_0$ for $c_{\text{total}} > 2$

Now we wish to prove a lemma of which we make use in the body of the paper. We would like to show that $\Delta_+(\hat{E}_0) + \hat{E}_0 > \frac{1}{4\pi}$ for all $c_{\text{total}} > 2$. In order to show this, it

will be convenient to work in terms of the variable $x \equiv 2\pi(\Delta + \hat{E}_0) - \frac{3}{2}$, in terms of which the polynomial $P_{31}(\Delta, \hat{E}_0)$ takes the canonical form

$$P_{31}\left(\frac{x+\frac{3}{2}}{2\pi} - \hat{E}_0, \hat{E}_0\right) = x^3 + C_1(\hat{E}_0)x + C_0(\hat{E}_0) \quad (\text{A.1})$$

Define x_+ to be the largest root of this polynomial. In terms of x_+ , we need to establish that $x_+ \geq -1$ at any local minimum of Δ_+ . The function $x_+(\hat{E}_0)$ could develop a critical point in one of two ways: either the coefficient functions $C_0(\hat{E}_0)$ and $C_1(\hat{E}_0)$ could have a critical point as a function of \hat{E}_0 , or the largest root x_+ polynomial $x^3 + C_1x + C_0$ could have a critical point as a function of C_0 . We will show that the former possibility cannot occur for $c_{\text{total}} > 2$, and the latter possibility can never occur at all.

The condition for the functions $C_{0,1}(\hat{E}_0)$ to have a critical point is

$$\begin{aligned} K'_{31}(\hat{E}_0) = 0 & \quad \Leftrightarrow b'_3(\hat{E}_0)b_1(\hat{E}_0) - b_1'(\hat{E}_0)b_3(\hat{E}_0) \\ & \simeq 3093.87\hat{E}_0^3 - 1511.95\hat{E}_0^2 + 187.515\hat{E}_0 - 6.9832 . \end{aligned}$$

The coefficients of this polynomial are strictly alternating, so it can have no negative roots. It follows that $C_0(\hat{E}_0), C_1(\hat{E}_0)$ never have critical points as a function of c_{total} for $c_{\text{total}} > 2$.

Next we will show that x_+ can never have a critical point as a function of C_0 . Let $x_{1,2,3}$ be the three solutions to the equation

$$\begin{aligned} & = x^3 + C_1x + C_0 \\ & = x^3 + \left(C_0 - \frac{3}{2} \right) x + C_0 . \end{aligned}$$

These roots can be thought of as implicit functions of C_0 , and they satisfy

$$\begin{aligned} x_1 + x_2 + x_3 & = 0 \\ x_1x_2 + x_1x_3 + x_2x_3 & = C_0 - \frac{3}{2} \\ x_1x_2x_3 & = -C_0 . \end{aligned}$$

Now suppose there is some value C_0^* of C_0 such that one of the roots, say x_3 , has a

critical point as a function of C_0 . Then at $C_0 = C_0^*$, we have

$$\dot{x}_3 = 0$$

$$\dot{x}_1 + \dot{x}_2 = 0$$

$$x_1 \dot{x}_2 + x_2 \dot{x}_1 = 1$$

$$(x_1 \dot{x}_2 + x_2 \dot{x}_1) x_3 = -1 .$$

We conclude that any root x_3 of (A.1) that is a local extremum as a function of C_0 must necessarily take the value $x_3 = -1$. But if we plug this value back into the defining equation, we find

$$0 = x_3^3 + C_1 x_3 + C_0 = (C_0 - C_1) - 1 = \frac{3}{2} - 1 = \frac{1}{2} ,$$

leading to a contradiction. So no root of the defining polynomial can never be equal to -1 , and therefore x_+ is monotonic as a function of C_0 . Taken together with the result that C_0 is monotonic as a function of c_{total} for $c_{\text{total}} > 2$, this implies that x_+ is monotonic as a function of c_{total} in the same range. We conclude that $\Delta_+(\hat{E}_0) + \hat{E}_0$ is monotonic as a function of c_{total} for $c_{\text{total}} > 2$.

We will see in the next subsection that $\Delta_+(\hat{E}_0)$ goes as $-2\hat{E}_0 + o(c_{\text{total}}^0) = \frac{c_{\text{total}}}{12} + o(c_{\text{total}}^0)$ at large c_{total} , so Δ_+ is monotonically increasing as a function of c_{total} in the range of interest.

So we have learned that $\Delta_+(\hat{E}_0)$ is always greater than $\frac{1}{4\pi}$ in the range $c_{\text{total}} > 2$. It follows that $\Delta_+(\hat{E}_0) + \hat{E}_0$ is an increasing function of c_{total} for $c_{\text{total}} > 2$. The value of $\Delta_+(\hat{E}_0) + \hat{E}_0$ at $c_{\text{total}} = 2$ is $\Delta_+(0) = 0.615286 + o(10^{-7})$, so we conclude

$$\Delta_+(\hat{E}_0) + \hat{E}_0 > \Delta_+(0) > \frac{1}{4\pi}, \text{ Q. E. D.}$$

A.4 Behavior of Δ_+ for large central charge

Now let us take the total central charge c_{total} to be large and positive. Taking $c_{\text{total}} \rightarrow +\infty$ means taking $\hat{E}_0 \equiv E_0 + \frac{1}{12} = \frac{2-c_{\text{total}}}{24}$ to $-\infty$. In this limit it is easy to see that Δ_+ is proportional to c_{total} , plus corrections of order c_{total}^0 . To see this, it is useful to expand Δ_+ as a series at large central charge:

$$\Delta_+ \equiv \sum_{a=-1}^{\infty} \delta_{-a} \left(\frac{c_{\text{total}}}{24} \right)^{-a} . \tag{A.2}$$

The defining property of Δ_+ is that it satisfies

$$F_{31}(\Delta_+, \hat{E}_0) = 0 ,$$

and that Δ_+ is the largest real value with that property, for a given value of c_{total} . Using the definition of \hat{E}_0 in terms of c_{total} and the expansion (A.2) or Δ_+ , we can expand $F_{31}(\Delta_+, \hat{E}_0)$ to arbitrary order in $\frac{1}{c_{\text{total}}}$, and solve for the universal numerical coefficients δ_{-a} . To leading order in c_{total} , we thus obtain:

$$F_{31}(\Delta_+, \hat{E}_0) = -\frac{\pi^4(1-\exp\{-2\pi\})^2}{20736} \left(\delta_1^2 - 3\delta_1 + 2 \right) c_{\text{total}}^4 + o\left(c_{\text{total}}^3 \right). \quad (\text{A.3})$$

We conclude that δ_1 is the larger of the two roots of $\delta_1^3 - 3\delta_1^2 + 2\delta_1 = \delta(\delta_1 - 1)(\delta_1 - 2)$. So $\delta_1 = +2$, which means

$$\Delta_+ = \frac{c_{\text{total}}}{12} + \delta_0 + o\left(c_{\text{total}}^{-1} \right).$$

To determine δ_0 we expand F_{31} to order c_{total}^3 . Fixing $\delta_1 = +2$, we find

$$F_{31} = -\left(\frac{\pi^4(1-\exp\{-2\pi\})^2}{432} \right) c_{\text{total}}^3 \left[\delta_0 - \frac{(12-\pi)+(13\pi-12)\exp\{-2\pi\}}{6\pi(1-\exp\{-2\pi\})} \right] + o\left(c_{\text{total}}^2 \right).$$

This determines the coefficient δ_0 to be

$$\delta_0 \equiv \frac{(12-\pi)+(13\pi-12)\exp\{-2\pi\}}{6\pi(1-\exp\{-2\pi\})} \simeq 0.473695 + o\left(10^{-7} \right).$$

We could easily determine the higher coefficients $\delta_{-1}, \delta_{-2}, \dots$ to arbitrary order. However we will not bother to derive any coefficients beyond δ_0 , for a simple reason: There is no guarantee that the bound we have derived is the best possible, even asymptotically at large c_{total} . If the c_{total}^{-a} term in Δ_+ is not the lowest possible value that can be obtained by any method, then the $c_{\text{total}}^{-(a+1)}$ term will not be relevant at all. We are not even certain if our leading expression $\Delta_+ \simeq \frac{c_{\text{total}}}{12}$ is the best possible upper bound at large c_{total} , which means the finite correction δ_0 may not be meaningful. Even if it is, it seems quite unlikely that $\frac{c_{\text{total}}}{12} + \delta_0$ is the best possible upper bound to order c_{total}^0 . Unless we have some reason to believe that there exist actual CFT with total central charge c_{total} that can attain the values $\Delta_1 = \frac{c_{\text{total}}}{12} + \delta_0 + o(c_{\text{total}}^{-1})$, there is nothing to be gained in carrying the expansion of Δ_+ to order c_{total}^{-1} .

A.5 The function Δ_+ is bounded above by $\frac{c_{\text{total}}}{12} + 0.473695$.

To extract the simplest possible conclusions from our inequality, we would like to find a linear function $a c_{\text{total}} + k$ that is always greater than or equal to Δ_+ whenever c_{total} is greater than 2. We would also like this function to be a good approximation to Δ_+ at large c_{total} , so we will take a to be equal to $+\frac{1}{12}$.

Let us find the lowest possible number k (independent of c_{total}) such that $\Delta_+(\hat{E}_0) \leq \frac{c_{\text{total}}}{12} + k$ for all $c_{\text{total}} > 2$. It is convenient to rewrite this condition as $\Delta_+ \leq -2\hat{E}_0 + k'$, with $k' = k + \frac{1}{6}$, so that we can use the variable \hat{E}_0 .

If k' is the smallest possible number such that the inequality is satisfied, then one of three possibilities holds:

- The inequality could be saturated asymptotically as $\hat{E}_0 \rightarrow -\infty$.
- The inequality could be saturated at $\hat{E}_0 = 0$.
- The inequality could be saturated at some value of \hat{E}_0 in between 0 and $-\infty$.

Let us eliminate the third possibility. Suppose there were some value $\hat{E}_0^{(sat)}$ between 0 and $-\infty$ such that the inequality is saturated. Then the line $\Delta = -2\hat{E}_0 + k'$ must be tangent to the curve $\Delta = \Delta_+(\hat{E}_0)$ at the point $(\hat{E}_0^{(sat)}, \Delta_+(\hat{E}_0^{(sat)}))$. That is, the derivative $\Delta'_+(\hat{E}_0)$ must be equal to -2 at the point $\hat{E}_0 = \hat{E}_0^{(sat)}$.

In terms of the variable $x_+ \equiv 2\pi(\Delta_+ + \hat{E}_0) - \frac{3}{2}$, we have the defining equation

$$x_+^3 + C_1 x_+ + C_0 = 0, \quad (\text{A.4})$$

and we wish to search for values of \hat{E}_0 such that $x'_+(\hat{E}_0) = -2\pi$. The functions x_+, C_0, C_1 all vary with \hat{E}_0 ; using a prime to denote differentiation with respect to \hat{E}_0 we have

$$3x_+^2 x'_+ + C_1' x_+ + C_1 x_+' + C_0' = 0.$$

For the particular values $x_+^{(sat)}, \hat{E}_0^{(sat)}$ where the equation $x' = -2\pi$, so we have

$$-6\pi x_+^2 + C_1' x_+ - 2\pi C_1 + C_0' \Big|_{\hat{E}_0 = \hat{E}_0^{(sat)}} = 0. \quad (\text{A.5})$$

Multiplying by x_+ and using the defining equation (A.4), we obtain

$$C_1' x_+^2 + \left(4\pi C_1 + C_0'\right) x_+ + 6\pi C_0 \Big|_{\hat{E}_0 = \hat{E}_0^{(sat)}} = 0.$$

Combining the two to cancel the x_+^2 term, and solving for x_+ , we get

$$x_+^{(sat)} = \frac{2\pi C_1 C_1' - C_0' C_1' - 36\pi^2 C_0}{C_1'^2 + 6\pi C_0' + 24\pi^2 C_1} \Big|_{\hat{E}_0 = \hat{E}_0^{(sat)}}$$

Plugging back into (A.5), we obtain

$$\begin{aligned} & -C_0'^3 + C_0 C_1'^3 - 6\pi C_0'^2 C_1 + 216\pi^3 C_0^2 \\ & + C_0' C_1' \left(-C_1 C_1' + 18\pi C_0 \right) + 2\pi C_1^2 \left(C_1'^2 + 16\pi^2 C_1 \right) = 0 \end{aligned}$$

Substituting the actual values of $C_1(\hat{E}_0)$ and $C_0(\hat{E}_0)$ into this equation, the polynomial

above can be factorized as

$$\begin{aligned}
& 1.82307 \times 10^{19} \cdot (6716.62 \hat{E}_0 - 559.624)^{-7} \\
& \cdot (-1.02204 + \hat{E}_0)(-0.538217 + \hat{E}_0)(-0.0833194 + \hat{E}_0) \\
& \cdot (0.00793973 - 0.172323\hat{E}_0 + \hat{E}_0^2)(0.00701421 - 0.167214\hat{E}_0 + \hat{E}_0^2) \\
& \cdot (0.0069072 - 0.165724\hat{E}_0 + \hat{E}_0^2)(0.0164746 - 0.0886746\hat{E}_0 + \hat{E}_0^2) \\
& \cdot \left[3.32249 \times 10^{13} + (\hat{E}_0 + 0.0163377)^2 \right]
\end{aligned}$$

For $c_{\text{total}} > 2$, we have $\hat{E}_0 \leq 0$ and every term in the above expression is nonvanishing. We conclude that the function $\Delta_+(\hat{E}_0) + 2\hat{E}_0$ has no critical points for $c > 2$.

So the coefficient k that optimizes the bound $\Delta_+ \leq \frac{c_{\text{total}}}{12} + k$ in the range $c_{\text{total}} \in [2, +\infty)$ must occur at one of the endpoints – either at $c_{\text{total}} = 2$ or in the limit $c_{\text{total}} \rightarrow +\infty$.

So that means

$$k = \max \left(\Delta_+(0) - \frac{1}{6} , \lim_{c_{\text{total}} \rightarrow \infty} \Delta_+\left(\frac{2-c_{\text{total}}}{24}\right) - \frac{c_{\text{total}}}{12} \right)$$

From the previous section, we know that $\lim_{c_{\text{total}} \rightarrow \infty} \Delta_+\left(\frac{2-c_{\text{total}}}{24}\right) - \frac{c_{\text{total}}}{12} = \delta_0 = 0.473695$. We can check that the value of $\Delta_+(\hat{E}_0)$ at $c_{\text{total}} = 2$ is

$$\Delta_+(0) = 0.615286$$

so

$$\Delta_+(0) - \frac{1}{6} = 0.448619 ,$$

which is less than δ_0 . So indeed $k = \delta_0$, and our best linear bound is

$$\Delta_1 \leq \Delta_+(\hat{E}_0) \leq \frac{c_{\text{total}}}{12} + \delta_0 = \frac{c_{\text{total}}}{12} + 0.473695 . \quad (\text{A.6})$$

In other words, retaining only the first two terms in the asymptotic expansion of Δ_+ at large c_{total} gives a linear function that bounds Δ_+ above uniformly on the semi-infinite interval of interest, $c_{\text{total}} \in [2, \infty)$.

We also point out that the value of $\frac{3}{2\pi}$ is 0.477465, so our bound on primaries is very slightly better than the warm-up bound we derived for the lowest-dimension nontrivial operator in general.

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