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A UNIVERSAL MAPPING PROBLEM, COVERING GROUPS AND AUTOMORPHISM GROUPS OF FINITE GROUPS

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In this note, we show that an arbitrary finite group G has a unique covering group (or representation group, in the sense of Schur, cf. [5, p. 85] or [3, p. 630]) if and only if there exists a solution to a certain universal mapping problem connected with G (Theorem 2). Then, easy diagram chases yield generalizations of recent results of Alperin [2, Assumed Result (9)] and Thompson [6, Theorem], (Corollaries 2.1 and 2.2). We also discuss the notion of a centrally closed group and obtain a sufficient condition on a finite group G that implies that any covering group of G is centrally closed. In the course of our work, we give alternate proofs of two fundamental theorems of Schur [5, Sätze III, II], (Theorems 1, 3 (i)), that utilize the methods of [3, V, § 23]. We close the paper with a result relating the automorphism group of G to the automorphism group of a covering group of G under fairly general hypotheses on G.

Our notation is fairly standard. In particular, C will denote the field of complex numbers and for a finite group G, we denote the Schur multiplier of G by $H^2(G, \mathbf{C}^\times)$, (cf. [3, V, 23.1 and 23.5(c)]).

For an arbitrary finite group G, we let $\mathcal{C}(G)$ denote the class of all ordered pairs (L, λ) such that L is a group and $\lambda : L \to G$ is an epimorphism with $\mathrm{Ker}(\lambda) \leqq L' \cap Z(L)$.

Clearly $(G, 1_G) \in \mathcal{C}(G)$ and G is perfect if and only if some and hence every group, appearing as a first component of an element of $\mathcal{C}(G)$, is perfect.

DEFINITION 1. Two elements (K, η) and (L, λ) of $\mathcal{C}(G)$ are said to be equivalent, written $(K, \eta) \sim (L, \lambda)$, if there exists an isomorphism $\alpha : K \to L$ of K onto L such that $\alpha \circ \lambda = \eta$.

Clearly this relation on $\mathcal{C}(G)$ is an equivalence relation. Moreover, by definition, G is said to be centrally closed (cf. [6]) if $\mathcal{C}(G)$ has precisely one $\sim$-equivalence class (for which $(G, 1_G)$ is a representative).

Note that a covering group of G is any element (L, λ) of $\mathcal{C}(G)$ such that $|\mathrm{Ker}(\lambda)| = |H^2(G, \mathbf{C}^\times)|$. Hence the covering groups of G com-

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prise complete $\sim$-equivalence classes in $\mathcal{C}(G)$. Also G is said to have a unique covering group if the covering groups of G comprise a unique $\sim$-equivalence class in $\mathcal{C}(G)$.

DEFINITION 2. The element (K, η) of $\mathcal{C}(G)$ is said to be a solution of the universal mapping problem (ξ) for G if, for every $(L, \lambda) \in \mathcal{C}(G)$, there exists a homomorphism $\alpha : K \rightarrow L$ such that $\alpha \circ \lambda = \eta$.

Clearly this definition has an equivalent formulation in terms of a certain subcategory of the category of short exact sequences of groups that end in G.

At this point, we require:

LEMMA 1. *The following conditions hold:*

(i) *if* $(L, \lambda) \in \mathcal{C}(G)$, *then* L *is a finite group and* $\mathrm{Ker}(\lambda)$ *is isomorphic to a subgroup of* $H^2(G, \mathbf{C}^\times)$,

(ii) *if* (L, λ) *and* (M, μ) *are elements of* $\mathcal{C}(G)$ *and* $\beta : L \rightarrow M$ *is a homomorphism such that* $\beta \circ \mu = \lambda$, *then* β *is onto, and*

(iii) *if solutions for the universal mapping problem* (ξ) *for* G *exist, then the solutions comprise a unique* $\sim$*-equivalence class in* $\mathcal{C}(G)$.

PROOF. Let $(L, \lambda) \in \mathcal{C}(G)$. Then, since $\mathrm{Ker}(\lambda) \leqq L' \cap Z(L)$, a result of Schur (cf. [3, IV, 2.3]) implies that L is finite and [3, V, 23.3] implies that $\mathrm{Ker}(\lambda)$ is isomorphic to a subgroup of $\mathrm{H}^2(G, \mathbf{C}^\times)$. Thus (i) holds. Assuming the hypotheses of (ii), we have $M = \mathrm{Im}(\beta)\mathrm{Ker}(\mu)$. Since $\mathrm{Ker}(\mu) \leqq M' \cap Z(M) \leqq \Phi(M)$ by a result of Gaschütz (cf. [3, III, 3.12]), we have $M = \mathrm{Im}(\beta)$; this proves (ii). Now (iii) follows from (i) and (ii) and we are done.

Using the context of the proof of [3, V, 23.5], we given an alternate proof of [5, Satz III]:

THEOREM 1. (SCHUR). *Let* $(L, \lambda) \in \mathcal{C}(G)$. *Then there exists a covering group* (K, η) *of* G *and an epimorphism* $\alpha : K \rightarrow L$ *such that* $\alpha \circ \lambda = \eta$. *In particular,* G *always has a covering group.*

PROOF. Choose an epimorphism $\pi : F \rightarrow G$ where F is a free group on n generators for some positive integer n and let $R = \mathrm{Ker}(\pi) \trianglelefteq F$. Thus $[R, F] \trianglelefteq F$ and $[R, F] \leqq R \cap F'$. Set $\bar{F} = F/[R, F]$ and $B = \mathrm{Ker}(\lambda)$ and let $\bar{\pi} : \bar{F} \rightarrow G$ denote the epimorphism induced by π. Thus $\mathrm{Ker}(\bar{\pi}) = \bar{R} \leqq Z(\bar{F})$ and [3, V, 23.5(a), (b)] implies that $\bar{R}$ is a finitely generated abelian group of rank n.

As in the proof of [3, V, 23.5(e)], there exists an epimorphism $\sigma : F \rightarrow L$ such that $\sigma \circ \lambda = \pi$ and such that (a) if $f \in F$, then $f^\sigma \in B$ if and only if $f \in R$, and (b) $[R, F] \leqq \mathrm{Ker}(\sigma)$. Let $\bar{\sigma} : \bar{F} \rightarrow L$ denote the epimorphism induced by σ. Thus $\bar{\sigma} \circ \lambda = \bar{\pi}$

and if $\bar{f} \in \bar{F}$, then $\bar{f}^{\bar{\sigma}} \in B$ if and only if $\bar{f} \in \bar{R}$; hence $\bar{R}^{\bar{\sigma}} = B$. Let $\bar{T} = \mathrm{Tor}(\bar{R})$, the torsion subgroup of $\bar{R}$. Then $\bar{T} = \bar{R} \cap \bar{F}' \cong H^2(G, \mathbf{C}^{\times})$ by [3, V, 23.5(a), (c)] and hence $\bar{T}^{\bar{\sigma}} \leqq B$. Let b be an arbitrary element of $B \leqq L'$. Since $L' = (\bar{F}')^{\bar{\sigma}}$, there exists an element $\bar{f} \in \bar{F}$ such that $\bar{f}^{\bar{\sigma}} = b$. Hence $\bar{f} \in \bar{R} \cap \bar{F}' = \bar{T}$, $\bar{T}^{\bar{\sigma}} = B$ and $\bar{R} = \bar{T}\,\mathrm{Ker}(\bar{\sigma})$. Thus $\mathrm{Ker}(\bar{\sigma}) = \bar{S} \times \bar{T}_1$ where $\bar{T}_1 = \mathrm{Tor}(\mathrm{Ker}(\bar{\sigma}))$ and $\bar{S}$ is a free abelian group of rank n. Hence $\bar{R} = \bar{S} \times \bar{T}$ with $\bar{S} \leqq \mathrm{Ker}(\bar{\sigma})$ and $\bar{S}$ a free abelian group of rank n.

Letting $\tilde{F} = \bar{F}/\bar{S}$, we conclude that $\bar{\sigma}$ and $\bar{\pi}$ induce epimorphisms $\tilde{\sigma} : \tilde{F} \to L$ and $\tilde{\pi} : \tilde{F} \to G$ such that $\tilde{\sigma} \circ \lambda = \tilde{\pi}$. Since $\mathrm{Ker}(\tilde{\pi}) = \tilde{T} \cong \bar{T} \cong H^2(G, \mathbf{C}^{\times})$, the theorem follows.

Corollary 1.1. *The finite group G is centrally closed if and only if $H^2(G, \mathbf{C}^{\times}) = 1$.*

Proof. If G is centrally closed and (K, η) is a covering group of G, then $|H^2(G, \mathbf{C}^{\times})| = |\mathrm{Ker}(\eta)| = 1$. The converse, of course, follows from Lemma 1 (i).

Corollary 1.2. *Let H be a centrally closed finite group, let Z be an arbitrary subgroup of $H' \cap Z(H)$ and let $\pi : H \to H/Z$ denote the canonical epimorphism. Then (H, π) is a covering group of H/Z.* However, *H/Z does not necessarily possess a unique covering group. In fact, H/Z may possess a non-centrally closed covering group.*

Proof. Since $(H, \pi) \in \mathcal{C}(H/Z)$, there exists a covering group (K, η) of H/Z and an epimorphism $\alpha : K \to H$ such $\alpha \circ \pi = \eta$. Since $\mathrm{Ker}(\alpha) \leqq \mathrm{Ker}(\alpha \circ \pi) = \mathrm{Ker}(\eta) \leqq K' \cap Z(K)$, we conclude that $\mathrm{Ker}(\alpha) = 1$. Whence $(K, \eta) \sim (H, \pi)$ and (H, π) is a covering group for H/Z. Finally, let $\mathcal{Q}$ and $\mathfrak{D}$, respectively, denote a generalized quatermon group and a dihedral group with $|\mathcal{Q}| = |\mathfrak{D}| = 2^n \geqq 2^3$. Then $Z(\mathcal{Q}) = Z(\mathcal{Q}) \cap \mathcal{Q}' \cong Z(\mathfrak{D}) \cap \mathfrak{D}'$ has order 2. Also $\mathcal{Q}/Z(\mathcal{Q}) \cong \mathfrak{D}/Z(\mathfrak{D})$ is dihedral of order 2^{n-1} and both $\mathcal{Q}$ and $\mathfrak{D}$ are covering groups of $\mathfrak{D}/Z(\mathfrak{D})$ by [3, V, 25.6]. Since $\mathcal{Q}$ is centrally closed by [3, V, 25.3a], we are done.

We are now in position to prove:

Theorem 2. *Let G be a finite group. Then the following two conditions are equivalent:*

(i) *there exists a solution of the universal mapping problem (ξ) for G;*

(ii) *G has a unique covering group.*

In that case, any covering group of G is a solution of the universal mapping problem (ξ) for G.

PROOF. Clearly, by Theorem 1, we conclude that (ii) implies that any covering group of G is a solution of the universal mapping problem (ξ) for G. Conversely, let (K, η) be a fixed solution of the universal mapping problem (ξ) for G and let (L, λ) be an arbitrary covering group of G. Then, there exists an epimorphism $\alpha : K \to L$ such that $\alpha \circ \lambda = \eta$. Since $|K| = |G|\,|\mathrm{Ker}(\eta)| \leqq |G|\,|H^2(G, C^\times)| = |L|$ by Lemma 1 (i), we conclude that α is an isomorphism and we are done.

Now, standard diagram chases yield the proofs of the next two results.

We shall see below that these results are proper generalizations of a result of Alperin (cf. [2, p. 356, assumed result (9)]) and of a result of Thompson [6, Theorem].

COROLLARY 2.1. *Let (K, η) be a covering group of the finite group G, let $\tau \in \mathrm{Aut}(G)$ and assume that G has a unique covering group. Then there exists an automorphism τ^* of K such that $\tau^* \circ \eta = \eta \circ \tau$.*

COROLLARY 2.2. *Let H be a finite centrally closed group, let Z_1, Z_2 be subgroups of $H' \cap Z(H)$ and let $\pi_i : H \to H/Z_i$ denote the canonical epimorphism for $i = i, 2$. Suppose that $\alpha : H/Z_1 \to H/Z_2$ is an isomorphism and that H/Z_1 has a unique covering group. Then there exists an automorphism α^* of H such that $\alpha^* \circ \pi_2 = \pi_1 \circ \alpha$.*

The proof of the first part of the next result is an alternate proof of [5, Satz II].

THEOREM 3. *Let G be a finite group such that $(|G/G'|, |H^2(G, C^\times)|) = 1$. Then*

(i) (Schur) *G has a unique covering group, and*

(ii) *any covering group of G is centrally closed.*

In particular, if G is perfect, then G has a unique covering group and any covering group of G is centrally closed.

PROOF. Let $\alpha = |G/G'|$ and $\beta = |H^2(G, C^\times)|$, so that $(\alpha, \beta) = 1$. Let $F, \pi : F \to G$, n, R, $\bar{F}$, $\bar{R} \leqq Z(\bar{F})$, $\bar{\pi} : \bar{F} \to G$ and $\bar{R} \cap \bar{F}' = \bar{T} = \mathrm{Tor}\ (\bar{R}) \cong H^2(G, C^\times)$ be as in the proof of Theorem 1. Choose $\bar{S} \leqq \bar{R}$ with $\bar{S}$ a free abelian group of rank n such that $\bar{R} = \bar{S} \times \bar{T}$. Then $\bar{R}^\beta = \bar{S}^\beta$ is independent of the choice of $\bar{S}$, since $|\bar{T}| = |H^2(G, C^\times)| = \beta$. Set $\tilde{F} = \bar{F}/\bar{R}^\beta$. Since $\bar{R} = \mathrm{Ker}(\bar{\pi})$, we conclude that $\bar{\pi}$ induces the epimorphism $\tilde{\pi} : \tilde{F} \to G$ such that $\mathrm{Ker}(\tilde{\pi}) = \tilde{R} = \tilde{T} \times \tilde{S} \leqq Z(\tilde{F})$ where $\tilde{T} \cong \bar{T}$ and $\tilde{S} = \bar{S}/\bar{S}^\beta$ has order β^n. Thus $\tilde{F}$ is a finite group and $\tilde{F}' \cap \tilde{R} = \tilde{T}$ since $\bar{R} \cap (\bar{F}'\bar{R}^\beta) = (\bar{R} \cap \bar{F}')\bar{R}^\beta = \bar{T} \times \bar{R}^\beta$. Thus $\alpha = |G/G'| = |\tilde{F}/\tilde{F}'\tilde{R}| = (|\tilde{F}/\tilde{F}'|)\beta^{-n}$ and hence $\alpha\beta^n = |\tilde{F}/\tilde{F}'|$. Since $(\alpha, \beta) = 1$, there exists a unique subgroup K of F containing $F'R^\beta$ with $|F/K| = |\bar{F}/\bar{K}| = |\tilde{F}/\tilde{K}| = \beta^n$ and $|\tilde{K}/\tilde{F}'|$

$= \alpha$. Since $|S| = \beta^n$, we have $\tilde{S} \cap \tilde{K} \leqq \tilde{S} \cap \tilde{F}' = \tilde{S} \cap (\tilde{R} \cap \tilde{F}') = \tilde{S} \cap \tilde{T} = 1$. Hence $\tilde{F}' = \tilde{K}'$ and $\tilde{R} \cap \tilde{K} = \tilde{T} \leqq Z(\tilde{K}) \cap \tilde{K}'$. Letting $\tilde{\eta} = \tilde{\pi}|_K : \tilde{K} \to G$, we conclude that $\tilde{\eta}$ is onto and that $H^2(G, C^\times) \cong \tilde{T} = \mathrm{Ker}(\tilde{\eta}) \leqq Z(\tilde{K}) \cap \tilde{K}'$. Thus $(\tilde{K}, \tilde{\eta})$ is a covering group of G.

Next, let $(L, \lambda) \in \mathcal{C}(G)$ and set $B = \mathrm{Ker}(\lambda)$. Then, as in the proof of Theorem 1, there exists an epimorphism $\bar{\sigma} : \bar{F} \to L$ such that $\bar{\sigma} \circ \lambda = \bar{\pi}$ and $\bar{R}^{\bar{\sigma}} = B = \bar{T}^{\bar{\sigma}}$. But $B = \mathrm{Ker}(\lambda)$ is isomorphic to a subgroup of $H^2(G, C^\times)$ by Lemma 1 (i), so that $\bar{R}^\beta \leqq B^\beta = 1$. Thus $\bar{R}^\beta \leqq \mathrm{Ker}(\bar{\sigma})$, whence $\bar{\sigma}$ induces the epimorphism $\tilde{\sigma} : \tilde{F} \to L$ such that $\tilde{\sigma} \circ \lambda = \tilde{\pi}$. Then $\tilde{\gamma} = \tilde{\sigma}|_K : \tilde{K} \to L$ is such that $\tilde{\gamma} \circ \lambda = \tilde{\eta}$ and hence $(\tilde{K}, \tilde{\eta})$ is a solution of the universal mapping problem (ξ) for G. Thus (i) holds.

For (ii), let (K, η) be a covering group for G, let $(M, \mu) \in \mathcal{C}(K)$, $Y = \mathrm{Ker}(\mu)$ and let $X = \mathrm{Ker}(\mu \circ \eta)$ where $\mu \circ \eta : M \to G$ is an epimorphism. Clearly $\mu|_X : X \to \mathrm{Ker}(\eta)$ is onto and $Y = \mathrm{Ker}(\mu) = \mathrm{Ker}(\mu|_X) \leqq M' \cap Z(M)$ since $Y \leqq X$. Also $[M, X]^\mu = [K, X^\mu] = [K, \mathrm{Ker}(\eta)] = 1$ since $\mathrm{Ker}(\eta) \leqq Z(K)$. Hence $[M, X] \leqq Y$ and $[M, X, M] = 1 = [X, M, M]$. Thus M stabilizes the chain $1 \leqq Y \leqq X$ and M' centralizes X by the three subgroup lemma ([3, III, 1.10(b))]). Moreover $Y \leqq Z(M)$, $X/Y \cong H^2(G, C^\times)$ has order β and $M/M' \cong K/K' \cong G/G'$ has order α. Hence, as is well known, it follows from a theorem of Burnside ([3, IV, 2.6]) that there exists characteristic subgroups X_1, X_2 of X such that $X = X_1 \times X_2$, $(|X_1|, \beta) = 1$, $X_1 \leqq Y \leqq Z(M)$, X_2 is a $\pi(\beta)$-group and such that $X_2/(X_2 \cap Y) \cong X/Y$. But $[M', X] = 1 = [X_1, M]$, M stabilizes the chain $1 \leqq X_2 \cap Y \leqq X_2$ and $(|X_2|, |M/M'|) = 1$. We conclude that $X \leqq Z(M)$. Also, setting $\bar{M} = M/Y$, we have $|\bar{X}| = \beta$ while $\bar{M}/\bar{M}' \cong M/M'$ has order α, since $Y \leqq M'$. Thus $X \leqq M' \cap Z(M)$ and hence $(M, \mu \circ \eta) \in \mathcal{C}(G)$. Now Lemma 1 (i) implies that $|X| \leqq |H^2(G, C^\times)| = |X/Y|$. Hence $Y = \mathrm{Ker}(\mu) = 1$, μ is an isomorphism and (ii) holds.

An alternate proof to (ii) can be given by using [4, Theorem 2.1]. For, under our hypotheses, it is easy to prove that the abelian group of all pairings $P(G, H^2(G, C^\times); C^\times)$, (cf. [4, p. 131]) is trivial.

Note that, in general, covering groups are not necessarily centrally closed. In fact, as we have seen in Corollary 1.2, a group may possess both a centrally closed and a non-centrally closed covering group.

We shall close our discussion with another example.

Let $S = Sz(8)$, let $P \in \mathrm{Syl}_2(S)$ and let $G = N_S(P)$. Then, as is well known, $P = O_2(G)$ is special with $P/P' \cong P' = Z(P)$ elementary abelian of order 8, G/P of order 7, G/P acting irreducibly on P/P' and $Z(P)$ and with $G' = P$. Also [3, V, 25.1 and 25.3(a)] implies that $H^2(G, C^\times)$ is a 2-group and hence G has a unique covering group. Letting (K, η)

denote a covering group of G, we conclude that K is centrally closed and, from [1, pp. 518–519], that $\mathrm{Ker}(\eta) \cong H^2(G, \mathbf{C}^\times)$ is a four-group. Also G, K possess automorphisms τ, τ^*, respectively, of order 3 such that $\tau^* \circ \eta = \eta \circ \tau$ and such that τ^* is transitive on the three involutions of $\mathrm{Ker}(\eta) \cong H^2(G, \mathbf{C}^\times)$. Let Z denote an arbitrary proper subgroup of $\mathrm{Ker}(\eta)$, let $\bar{K} = K/Z$ and let $\pi : K \to K/Z = \bar{K}$ denote the canonical epimorphism. Then $\bar{K}' = O_2(\bar{K})$, $|\bar{K}/\bar{K}'| = 7$, $Z \cong H^2(\bar{K}, \mathbf{C}^\times)$ and $\bar{K}$ has a unique covering group. Letting $Z_1 = Z^{\tau^*}$ and π_1 denote the canonical epimorphism $\pi_1 : K \to K/Z_1$, we conclude that τ^* induces an isomorphism $\bar{\tau} : K/Z \to K/Z_1$, such that $\pi \circ \bar{\tau} = \tau^* \circ \pi_1$. We have illustrated both Corollaries 2.1 and 2.2. Moreover, this example and Theorem 3 show that these corollaries are proper generalizations of the results of Alperin and Thompson mentioned above.

Our final results are obtained by applying Corollary 2.1 and Theorem 3(i).

THEOREM 4. *Let G be a finite group such that* $(|G/G'|, |H^2(G, \mathbf{C}^\times)|) = 1$. *Let (K, κ) be a covering group of G, let $J = \mathrm{Ker}(\kappa)$, let $\bar{K} = K/J$ and let $A = \{\alpha \in \mathrm{Aut}(K) | J^\alpha = J\}$. Clearly $\bar{K} \cong G$, $\mathrm{Aut}(\bar{K}) \cong \mathrm{Aut}(G)$ and A acts on $\bar{K}$. Let $\gamma : A \to \mathrm{Aut}(\bar{K})$ denote the induced homomorphism. Then* (i) *$\gamma : A \to \mathrm{Aut}(\bar{K})$ is an isomorphism, and* (ii) $Z(\bar{K}) = Z(K)/J$.

PROOF. Let $\alpha \in A$ be such that $[K, \alpha] \leqq J$. Note that $J \leqq K' \cap Z(K)$. Let k_1, k_2 be arbitrary elements of K. Thus $(k_1k_2)^{-1}(k_1k_2)^\alpha = k_2^{-1}k_1^{-1}k_1^\alpha k_2^\alpha = (k_1^{-1}k_1^\alpha)(k_2^{-1}k_2^\alpha)$ and hence the mapping $\beta : K \to J$ defined by $k\beta = k^{-1}k^\alpha$ for all $k \in K$ is a homomorphism. Since $J = \mathrm{Ker}(\kappa) \cong H^2(G, \mathbf{C}^\times)$ and $G/G' \cong K/K'$, we have $(|K/K'|, |J|) = 1$. Thus β is trivial, γ is a monomorphism and (ii) holds.

Suppose that $t \in \mathrm{Aut}(\bar{K})$. Letting $\pi : K \to \bar{K} = K/J$ denote the natural epimorphism, it follows from Theorem 3(i) and Corollary 2.1 that there is a $t^* \in \mathrm{Aut}(K)$ such that $t^* \circ \pi = \pi \circ t$. Hence $t^* \in A$, $t^* \circ \gamma = t$ and the proof is complete.

COROLLARY 4.1. *Let G be a finite group such that $G = G'$ and $Z(G) = 1$. Let (K, κ) be a covering group of G. Then $Z(K) = \mathrm{Ker}(\kappa)$ and* $\mathrm{Aut}(K) \cong \mathrm{Aut}(G)$.

We conclude the paper with an example of Theorem 4 in which A is a proper subgroup of $\mathrm{Aut}(K)$.

To this effect, let q denote an odd prime power such that $q \notin \{3, 9\}$ and let K be a group that is the direct product of two distinct normal subgroups H_1, H_2 such that $H_1 \cong H_2 \cong SL(2, q)$. Clearly $H_i' = H_i$ and $|Z(H_i)| = 2$ for $i = 1, 2$. Let $\bar{K} = K/Z(H_1)$ and let $\kappa : K \to \bar{K}$ denote the natural epimorphism. Clearly $K = K' \cong SL(2, q) \times SL(2, q)$, $\bar{K} = \bar{K}' \cong \mathrm{PSL}(2, q) \times SL(2, q)$ and (K, κ) is a covering group of $\bar{K}$ by [3, V, 25.5, 25.7 and 25.10]. Since Aut(K) contains an involution λ such that $H_1{}^\lambda = H_2$, it follows that $|\mathrm{Aut}(\bar{K})| < |\mathrm{Aut}(K)|$ and we are done.

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