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A Universal Reduction Procedure for Hamiltonian Group Actions*

Judith M. Arms†, Richard H. Cushman‡, Mark J. Gotay§

Abstract

We give a universal method of inducing a Poisson structure on a singular reduced space from the Poisson structure on the orbit space for the group action. For proper actions we show that this reduced Poisson structure is nondegenerate. Furthermore, in cases where the Marsden-Weinstein reduction is well-defined, the action is proper, and the preimage of a coadjoint orbit under the momentum mapping is closed, we show that universal reduction and Marsden-Weinstein reduction coincide. As an example, we explicitly construct the reduced spaces and their Poisson algebras for the spherical pendulum.

1 Introduction

Reduction of the order of mechanical systems with symmetry is a venerable topic dating back to Jacobi, Routh and Poincaré. Although reduction in the regular case is well understood [1, 2], interesting and important applications continue to arise [3, 4]. (See also [5] for a comprehensive exposition.) The singular case has received much less attention, despite the growing realization that it is the rule rather than the exception. (See for instance [6–11].) For example, the solution spaces of classical field theories are invariably singular and the ramifications of this have only recently begun to be explored [12, 13].

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†Mathematics Department, University of Washington, Seattle, WA, 98195, USA

‡Mathematics Institute, Rijksuniversiteit Utrecht, 3508 TA Utrecht, the Netherlands

§Mathematical Sciences Research Institute, Berkeley, CA 94720, USA. Supported in part as a Ford Foundation Fellow and by NSF grant DMS-8805699.

Singularities also play an increasingly critical role in understanding the behavior of simple mechanical systems, such as the photon [14], the Lagrange top [15], coupled rigid bodies [16], particles with zero angular momentum [17] and even homogeneous Yang-Mills fields [18].

The central ingredient in the reduction of a mechanical system with symmetry is the construction of a symplectic reduced space of invariant states. This reduction process can be formulated abstractly as follows. Let (M, ω) be a symplectic manifold and G a Lie group with Lie algebra $\mathcal{G}$. Suppose that there is a Hamiltonian action of G on (M, ω) with an Ad^*-equivariant momentum mapping $J : M \longrightarrow \mathcal{G}^*$. We consider the problem of inducing symplectic structures on the reduced spaces

$$M_\mu = J^{-1}(\mu)/G_\mu$$

as μ ranges over $J(M) \subseteq \mathcal{G}^*$. Here G_μ is the isotropy group of μ under the coadjoint action of G on $\mathcal{G}^*$. Marsden and Weinstein [19] solved this problem in "regular" cases. (See also [20].) Specifically they showed that if μ is a weakly regular value of J and if G_μ acts freely and properly on $J^{-1}(\mu)$, then M_μ is a symplectic quotient manifold of $J^{-1}(\mu)$ in a natural way.

This Marsden-Weinstein reduction breaks down in singular situations. In this context various reduction techniques were developed and studied in [21] and [22]. Unfortunately these alternative reductions are not always applicable, and so it is sometimes necessary to switch from one reduction technique to another as μ varies in $J(M)$. Obviously this would cause havoc in the context of perturbation theory. Even worse, when these reduction procedures apply, they do not necessarily agree. Other techniques give reduced Poisson algebras which, however, are not always function algebras on M_μ. We note that the reduction procedure of Śniatycki and Weinstein [23] suffers this difficulty.

The purpose of this paper is to present a new reduction procedure which assigns a Poisson structure to each reduced space M_μ, even in the presence of singularities. The salient features of this reduction are that:

(1) it always works;

(2) it is natural;

(3) it can be applied uniformly to every M_μ for every $\mu \in J(M)$.

One can regard it as a universal method of reduction. As such it has advantages over the other singular reduction procedures discussed above.

As this paper was being written, conversations with P. Dazord [24] indicated that he has independently discovered ideas similar to the notion of universal reduction treated here. However, his techniques and results differ from ours in several respects.

The notation and terminology are explained in the appendix.

2 Universal reduction

Let (M, ω), G and J be as in §1. Our reduction procedure is motivated by the following commutative diagram:

$$\begin{array}{ccc} J^{-1}(\mu) & \hookrightarrow & M \\ \pi_\mu \downarrow & & \downarrow \pi \\ M_\mu & \xrightarrow{i_\mu} & M/G \end{array} . \tag{1}$$

Here π and π_μ are the G- and G_μ-orbit projections. The map i_μ is defined by associating, to each point $m_\mu \in M_\mu$, the G-orbit in which $\pi_\mu^{-1}(m_\mu)$ lies. Consequently $i_\mu \circ \pi_\mu = \pi | J^{-1}(\mu)$. An easy argument using the Ad^*-equivariance of J shows that i_μ is one to one. In fact i_μ is a homeomorphism of M_μ onto $\pi(J^{-1}(\mu))$. In what follows we will identify M_μ with its image under i_μ. Note that M_μ is also $\pi(J^{-1}(\mathcal{O}_\mu))$ where $\mathcal{O}_\mu$ is the G-coadjoint orbit through μ. Observe that we make *no* assumptions regarding the regularity of J or the smoothness of either M_μ or M/G.

The key observation is that M/G is a Poisson variety. The Poisson bracket $\{\,,\}_{M/G}$ on $C^\infty(M/G)$ is given by

$$\{f, h\}_{M/G}(\pi(m)) = \{f \circ \pi, h \circ \pi\}_M(m) \tag{2}$$

where $\{\,,\}_M$ is the Poisson bracket on M associated with ω. This definition makes sense since $C^\infty(M/G) = C^\infty(M)^G$ is a Lie subalgebra of $C^\infty(M)$. This allows the possibility of inducing a Poisson structure on each reduced space M_μ by restricting the Poisson structure on M/G. Our main result is the observation that reduction by restriction *always* works.

Theorem 1 *For each $\mu \in J(M)$, M_μ inherits the structure of a Poisson variety from M/G.*

Proof: For each pair of Whitney smooth functions $f_\mu, h_\mu \in W^\infty(M_\mu)$ set

$$\{f_\mu, h_\mu\}_\mu = \{f, h\}_{M/G} | M_\mu \tag{3}$$

where f, h are any smooth extensions of f_μ, g_μ to M/G. Equation (3) defines a Poisson bracket $\{\,,\,\}_\mu$ on $W^\infty(M_\mu)$, provided we can show that its right hand side is independent of the choice of extensions f, h. This amounts to showing that the ideal $\mathcal{I}(M_\mu)$ of smooth functions vanishing on M_μ is a *Poisson ideal* in $C^\infty(M/G)$, that is,

$$\{C^\infty(M/G), \mathcal{I}(M_\mu)\}_{M/G} \subseteq \mathcal{I}(M_\mu). \tag{4}$$

Since $C^\infty(M/G) = C^\infty(M)^G$ and $M_\mu = \pi(J^{-1}(\mathcal{O}_\mu))$, equation (4) is equivalent to

$$\{C^\infty(M)^G, \mathcal{I}(J^{-1}(\mathcal{O}_\mu))^G\}_M \subseteq \mathcal{I}(J^{-1}(\mathcal{O}_\mu))^G, \tag{5}$$

where $\mathcal{I}(J^{-1}(\mathcal{O}_\mu))^G$ is the ideal of smooth G-invariant functions which vanish on $J^{-1}(\mathcal{O}_\mu)$. Let $F, H \in C^\infty(M)^G$ with $H|J^{-1}(\mathcal{O}_\mu) = 0$. We must show that for every $m \in J^{-1}(\mathcal{O}_\mu)$,

$$\{F, H\}_M(m) = -X_F\, H(m) = 0. \tag{6}$$

Now for every $\xi \in \mathcal{G}$,

$$\{J^\xi, F\}_M = -\xi_M\, F = 0,$$

where $J^\xi(m) = J(m)(\xi)$. The first equality follows from the definition of the momentum mapping J, while the second follows from $F \in C^\infty(M)^G$. Consequently for every $\xi \in \mathcal{G}$, J^ξ is constant along the integral curves of X_F. Hence the integral curve $t \longrightarrow \varphi_t^F(m)$ lies in $J^{-1}(J(m))$. This proves

$$H(\varphi_t^F(m)) = 0. \tag{7}$$

Differentiating (7) with respect to t and evaluating at $t = 0$ gives (6). ∎

In this fashion each orbit space M_μ is naturally equipped with a Poisson structure. The corresponding reduced Poisson algebra is $(W^\infty(M_\mu), \{\ ,\ \}_\mu)$. It follows from the definitions that the underlying function space can be represented as

$$\begin{aligned} W^\infty(M_\mu) &= C^\infty(M/G)/\mathcal{I}(M_\mu) \\ &= C^\infty(M)^G/\mathcal{I}(J^{-1}(\mathcal{O}_\mu))^G \\ &= C^\infty(M)^G/\mathcal{I}(J^{-1}(\mu))^G \end{aligned}$$

where $\mathcal{I}(J^{-1}(\mu))^G$ is the ideal of smooth G-invariant functions on M which vanish on $J^{-1}(\mu)$. In the last equality we have used the fact that $J^{-1}(\mathcal{O}_\mu) = G \cdot J^{-1}(\mu)$, which is a consequence of the Ad^*-equivariance of J.

Formally we see that reduction in singular cases works exactly as it does in regular cases. We caution, however, that the results of singular reduction may seem a bit strange. For example, in the regular case, one knows that the reduced Poisson algebra is *nondegenerate* in the sense that it contains no nontrivial elements which Poisson commute with everything, that is, there are no nontrivial Casimirs. This is not necessarily so in the singular case without additional assumptions. Likewise in the regular case M_μ is a finite union of symplectic leaves of M/G. This also is no longer true if the reduced algebra has nontrivial Casimirs. We illustrate these remarks with the following example. Lift the irrational flow on T^2 to an $\mathbf{R}$-action on T^*T^2. Here M_μ may be identified with $T^2/\mathbf{R} \times \{\mu\}$. The Poisson algebra consists entirely of Casimirs, some nontrivial. Certainly M_μ is not "symplectic". Note that the symplectic leaves of $T^*T^2/\mathbf{R}$ are just points. (See [21, example 3.4] for more details.) These kinds of behavior cannot occur if the action is proper. When this is the case, we will show in §3 that the reduced Poisson bracket is nondegenerate.

If G is noncompact, we encounter another phenomenon which can arise even in the regular case. For example, let $M = T^*G$ with G acting on M by the lift of left translation. Let $J : M \longrightarrow \mathcal{G}^*$ be the momentum map given by $J(\alpha_g) = R_g^*\alpha_g$ for $\alpha_g \in M$. Every $\mu \in \mathcal{G}$ is a regular value of J and $M_\mu = J^{-1}(\mu)/G_\mu$ is symplectically diffeomorphic to $\mathcal{O}_\mu$ with its Kostant-Kirillov symplectic structure [1]. On the other hand, T^*G/G is isomorphic as a Poisson manifold to $\mathcal{G}^*$ with its Lie-Poisson structure [4]. Thus the basic commutative diagram (1) becomes

$$\begin{array}{ccc} J^{-1}(\mu) & \hookrightarrow & T^*G \\ \pi_\mu \downarrow & & \downarrow \pi \\ \mathcal{O}_\mu & \overset{i_\mu}{\hookrightarrow} & \mathcal{G}^* \end{array}$$

where i_μ is the inclusion mapping. Now suppose that $\mathcal{O}_\mu$ is not closed in $\mathcal{G}^*$. Then the Marsden-Weinstein reduction, which views $\mathcal{O}_\mu$ abstractly as a quotient space, differs slightly from the universal reduction, which views $\mathcal{O}_\mu$ as a subset of $\mathcal{G}^*$. This distinction is apparent on the level of function algebras. The Marsden-Weinstein reduced function algebra is $C^\infty(\mathcal{O}_\mu)$, whereas the corresponding function algebra for the universal reduction is $W^\infty(i_\mu(\mathcal{O}_\mu))$. In applications there are plausible arguments for the appropriateness of both function spaces. In general relativity and other field theories, the momentum for the gauge group is constrained to vanish for all physically admissible states. (See [25] for more details.) This situation strongly suggests using the Marsden-Weinstein reduced function algebra $C^\infty(\mathcal{O}_\mu)$: the extension of functions to nearby, physically unrealizable values of momentum is irrelevant. In mechanics, however, momentum usually is a parameter which may be varied. In such cases, Hamiltonians should be smoothly extendable to nearby values, and therefore should lie in the universal reduced function algebra $W^\infty(i_\mu(\mathcal{O}_\mu))$.

A concrete example of the above situation is obtained by taking G to be the subgroup of $Sl(2, \mathbf{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad a > 0.$$

Let

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the standard basis for $\mathcal{G}$. Viewing (ξ_1, ξ_2) as coordinates on $\mathcal{G}^*$, the coadjoint orbits can be pictured as

Figure 1. The coadjoint orbits of G on $\mathcal{G}^*$.

There are two classes of orbits: (1) the open upper and lower half planes, where $G_\mu = \{e\}$, and (2) points along the ξ_1-axis where $G_\mu = G$.

3 Universal reduction for proper actions

In this section we give conditions which eliminate the unusual features in the reduction process illustrated by the examples in §2. We will show that if the action of G is proper, then the universal reduced Poisson algebras $(W^\infty(M_\mu), \{\, ,\, \}_\mu)$ are nondegenerate and hence give "symplectic structures" on the reduced spaces M_μ. If in addition to having a proper action we know that $J^{-1}(\mathcal{O}_\mu)$ is closed in M and that the Marsden-Weinstein reduction procedure applies, then the Marsden-Weinstein and universal reductions agree. To obtain these results requires some machinery. As a byproduct we will show that the singularities in $J^{-1}(\mu)$ are quadratic.

We first recall some definitions. Throughout this section we will assume that the action of G on M is *proper*. This means that

$$\Phi : G \times M \longrightarrow M \times M : (g, m) \longrightarrow (g \cdot m, m)$$

is a proper mapping, that is, the preimage of any compact set is compact. For $m \in M$, let G_m be the isotropy group of m. A (smooth) *slice* at m for the G-action is a submanifold $S_m \subseteq M$ such that

(i) $G \cdot S_m$ is an open neighborhood of the orbit $\mathcal{O}_m$; and
(ii) there is a smooth equivariant retraction

$$r : G \cdot S_m \longrightarrow G \cdot m$$

such that $S_m = r^{-1}(m)$.

From these properties one easily derives the following (cf. [26], propositions 2.1.2 and 2.1.4):

(a) S_m is closed in $G \cdot S_m$, which we call the *sweep* of S_m.
(b) m belongs to S_m.
(c) S_m is invariant under G_m.
(d) For $g \notin G_m$, the sets S_m and $g \cdot S_m$ are disjoint.
(e) Let $\sigma : \mathcal{U} \longrightarrow G$ be a local cross-section of G/G_m. Then

$$F : \mathcal{U} \times S_m \longrightarrow M : (u, s) \longrightarrow \sigma(u) \cdot s$$

is a diffeomorphism onto an open subset of M.

In the literature a slice is sometimes defined by requiring (a-e) or (i) and (a-d); these definitions are equivalent to that given above.

Proposition 1 *There is a slice for the G-action at each $m \in M$. Each isotropy group G_m is compact, and M/G is Hausdorff.*

Proof: The first statement follows immediately from proposition 2.2.2 and remark 2.2.3 of [26]. The fact that G_m is compact is a direct consequence of the properness of the action, since $G_m = \rho(\Phi^{-1}(m, m))$ where $\rho(g, m) = g$. Finally, since Φ is proper, it is a closed mapping. Now apply proposition 4.1.19 of [1] to conclude that M/G is Hausdorff. ∎

Remark: Palais' definition of proper action [26] differs from ours, but a straightforward exercise in point set topology shows that the two definitions are equivalent.

The existence of slices for the G-action has several important consequences. One is that it guarantees that there are enough smooth G-invariant functions to separate orbits. This is a corollary of the following proposition.

Proposition 2 *Let N be any G-invariant subset of M and suppose that $f \in W^\infty(N)$ is constant on each G-orbit. Then there is a smooth G-invariant extension F of f with $F \in C^\infty(M)^G$ and $F|N = f$.*

Proof: The main ideas of this proof appear in Palais' argument showing the existence of an invariant Riemannian metric for a smooth proper action ([26], theorem 4.3.1). We will pick a set of local slices whose sweeps are a locally finite open cover of M, and then construct a G-invariant partition of unity subordinated to this cover. The properties of slices allow us to extend f as desired on each sweep. Then we patch these local extensions together using the partition of unity.

As M/G is locally compact, σ-compact, and Hausdorff, we can choose a sequence of points m_i and slices S_i at these points such that $\{\pi(S_i)\}$ is a locally finite cover of M/G. Then $\{G \cdot S_i\}$ is a locally finite cover of M. Furthermore M/G is normal, so we may assume that the slices were chosen so that each m_i has a relatively compact neighborhood $N_i \subseteq S_i$ such that $\{\pi(N_i)\}$ also covers M/G. On each slice S_i construct a function $\tilde{h}_i$ which is positive on N_i and has compact support in S_i. As G_{m_i} is compact, we may average over its orbits to get an invariant function on S_i. (Here we have used both proposition 1 and property (c) of the slice.) By properties (d) and (i) of the slice, we may extend $\tilde{h}_i$ to the sweep and then to all of M by requiring it to be G-invariant and to vanish outside of the sweep. Now by the local finiteness of the covering by sweeps, the $\tilde{h}_i$ can be normalized to a G-invariant partition of unity h_i. That is, let $h_i = \tilde{h}_i / \sum_j \tilde{h}_j$, so that $\sum_j h_j = 1$ at all points of M.

To extend f, we first define a function F_i on each sweep $G \cdot S_i$. If $N \cap S_i$ is empty, set $F_i = 0$. Otherwise, we can extend $f|S_i \cap N$ smoothly to S_i because f belongs to $W^\infty(N)$ and S_i is a submanifold of M. As above we can average over the orbits of the isotropy group on S_i and extend to a G-invariant function f_i on the sweep. Now define F_i belonging to $C^\infty(M)^G$ by setting $F_i = h_i \cdot f_i$ on the sweep and $F_i = 0$ elsewhere. It is clear that $F = \sum_i h_i \cdot F_i$ has the desired properties. ∎

Corollary 1 *$C^\infty(M)^G$ separates G-orbits in M.*

Proof: Let $N = \mathcal{O}_n \cup \mathcal{O}_m$ and let $f = 1$ on $\mathcal{O}_m$ and $f = 0$ on $\mathcal{O}_n$; such a function exists and belongs to $W^\infty(N)$ since these orbits are closed, embedded submanifolds of M (because the action is proper). Thus by proposition 2, f has an invariant extension F which separates $\mathcal{O}_m$ and $\mathcal{O}_n$. ∎

Another consequence of the existence of slices is that the singularities of J must have a particularly simple form.

Proposition 3 *For each $\mu \in J(M)$, the singularities of $J^{-1}(\mu)$ are quadratic.*

Proof: We first reduce to the case $\mu = 0$ by the following standard construction (see theorem 4.1 of [13]). Define an action of G on $M \times \mathcal{O}_\mu$ by $(g, (m, \nu)) \longrightarrow (g \cdot m, Ad^*_{g^{-1}}\nu)$. This action is proper because the original action on M is. Then give $M \times \mathcal{O}_\mu$ the symplectic structure $\Theta = \pi_1^*\omega - \pi_2^*\Omega$ where ω is the symplectic structure on M, Ω is the Kostant-Kirillov symplectic structure on $\mathcal{O}_\mu$, and π_i is the projection of $M \times \mathcal{O}_\mu$ on the i^{th} factor. Note that on $(M \times \mathcal{O}_\mu, \Theta)$ the G-action is Hamiltonian with Ad^*-equivariant momentum mapping $\mathcal{J} : M \times \mathcal{O}_\mu \longrightarrow \mathcal{G}^*$ given by $\mathcal{J}(m, \nu)(\xi) = J^\xi(m) - \nu(\xi)$ for all $\xi \in \mathcal{G}$.

The level sets $J^{-1}(\mu)$ and $\mathcal{J}^{-1}(0)$ are closely related. In fact we can construct a local diffeomorphism

$$\alpha : M \times \mathcal{U} \longrightarrow M \times \mathcal{U},$$

where $\mathcal{U}$ is a neighborhood of $\mu \in \mathcal{O}_\mu$, such that

$$\alpha(J^{-1}(\mu) \times \mathcal{U}) = \mathcal{J}^{-1}(0) \cap (M \times \mathcal{U}).$$

To do this use a local section for G/G_μ to construct a map $\beta : \mathcal{U} \longrightarrow G$ such that $\beta(\mu)$ is the identity element of G and $Ad^*{}_{\beta(\nu)^{-1}}\mu = \nu$. Then define $\alpha(m, \nu) = (\beta(\nu) \cdot m, \nu)$. It follows easily that α is a local diffeomorphism. The equivariance of the action implies that $J(m) = \mu$ if and only if $\mathcal{J}(\alpha(m, \nu)) = 0$ for all $\nu \in \mathcal{U}$. Thus $J^{-1}(\mu)$ has a quadratic singularity if and only if $\mathcal{J}^{-1}(0)$ has.

For the case $\mu = 0$, the result follows immediately from [8]; we only need to confirm that the hypotheses of that paper are satisfied. By theorem 4.3.1 of [26], a smooth proper action admits an invariant Riemannian metric. The usual polarization construction allows us to assume that the metric and symplectic structures are compatible in the sense of being related by means of a G-invariant almost complex structure (cf. p.8 of [27]). By proposition 1, there is a slice at each point. Also by proposition 1, the isotropy group of each point is compact, so we can construct a metric on the dual Lie algebra which is invariant under the coadjoint action of the isotropy group. Thus by theorem 5 of [8] the singularities of $J^{-1}(0)$ are quadratic. ∎

Since the singularities in $J^{-1}(\mu)$ are quadratic, we have

Corollary 2 *Each $J^{-1}(\mu)$ is locally path connected.* in the proof of the proposition, it case $\mu = 0$. Now connected as quadratic

In fact proposition 3 shows that nearby points of $J^{-1}(\mu)$ can be connected by piecewise smooth curves in M lying in $J^{-1}(\mu)$. With these results in hand, we are ready to prove the main results of this section.

Theorem 2 *Let G act properly on M. Then for each $\mu \in J(M)$ the reduced Poisson algebra $(W^\infty(M_\mu), \{\ ,\ \}_\mu)$ is nondegenerate.*

In other words, each Casimir in $W^\infty(M_\mu)$ is locally constant. Thus in the context of universal reduction we are able to answer a question of Weinstein [4, p.429].

Proof: Suppose that $c_\mu \in W^\infty(M_\mu)$ is a Casimir. This means that for all $H \in C^\infty(M)^G$

$$\{C, H\}_M | J^{-1}(\mathcal{O}_\mu) = 0,$$

where C is any smooth G-invariant function M with $C|J^{-1}(\mathcal{O}_\mu) = c_\mu \circ \pi$. Since by Ad^*-equivariance $J^{-1}(\mathcal{O}_\mu) = G \cdot J^{-1}(\mu)$, this is equivalent to

$$\{C, H\}_M | J^{-1}(\mu) = -(X_C\, H)|J^{-1}(\mu) = 0. \tag{8}$$

By corollary 1 the G-invariant smooth functions separate G-orbits on M. Thus (8) implies that $X_C|J^{-1}(\mu)$ is tangent to the G-orbits in $J^{-1}(\mu)$, that is, at each $m \in J^{-1}(\mu)$ there is $\xi \in \mathcal{G}$ such that $X_C(m) = \xi_M(m)$. Equivalently,

$$dC(m) = dJ^\xi(m). \tag{9}$$

Since by corollary 2 $J^{-1}(\mu)$ is locally path connected, equation (9) implies that C is locally constant on $J^{-1}(\mu)$ and hence that c_μ is locally constant on M_μ. ∎

Theorem 3 *Suppose that G acts properly on M, $J^{-1}(\mathcal{O}_\mu)$ is closed in M, and the Marsden-Weinstein reduction is well-defined. Then the Marsden-Weinstein and universal reductions are the same.*

Proof: The reduced spaces are the same and both Poisson brackets descend from that on M, so the only question is whether the reduced function algebras are the same. For the Marsden-Weinstein reduction, the function algebra is

$$C^\infty(M_\mu) = C^\infty(J^{-1}(\mu))^{G_\mu} = C^\infty(J^{-1}(\mathcal{O}_\mu))^G.$$

As $J^{-1}(\mathcal{O}_\mu)$ is closed, any smooth function on $J^{-1}(\mathcal{O}_\mu)$ has a smooth extension, and by proposition 2 we may assume that the extension is G-invariant. Thus

$$C^\infty(J^{-1}(\mathcal{O}_\mu))^G = C^\infty(M)^G / \mathcal{I}(J^{-1}(\mathcal{O}_\mu))^G.$$

But the left hand side of the preceding equation is $C^\infty(M_\mu)$ while the right hand side is $W^\infty(M_\mu)$, so the Marsden-Weinstein and universal reduced function algebras coincide. ∎

The results of this section demonstrate that proper actions are rather well behaved as far as reduction is concerned. In particular theorem 2 shows that the orbit space M/G for such an action is *always* partitioned into symplectic leaves, these being the connected components of the reduced spaces M_μ.

4 Universal reduction for compact groups

In this section we briefly discuss universal reduction when G is a compact Lie group. This is the nicest of all possible situations, because theorems 2 and 3 apply.

According to a recent result of Gotay and Tuynman [28], every symplectic manifold of finite type which admits a Hamiltonian action of a compact (connected) Lie group G can be obtained by equivariant reduction from a linear symplectic action of G on $\mathbf{R}^{2n}$ with its standard symplectic structure ω. Therefore, without loss of generality, we may suppose that $M = \mathbf{R}^{2n}$ with its standard symplectic form ω and that G is a subgroup of the group of linear symplectic mappings of $\mathbf{R}^{2n}$ onto itself. This symplectic G-action is Hamiltonian because the mapping $J : \mathbf{R}^{2n} \longrightarrow \mathcal{G}^*$ given by $J(m)\xi = \frac{1}{2}\omega(\xi_{\mathbf{R}^{2n}}(m), m)$ for every $\xi \in \mathcal{G}$ is an Ad^*-equivariant momentum mapping [29].

To construct the G-orbit space $\mathbf{R}^{2n}/G$ we use invariant theory. Since G is compact and acts linearly on $\mathbf{R}^{2n}$, the algebra of G-invariant polynomials is finitely generated [30]. Let $\sigma_1, \ldots, \sigma_k$ be a set of generators. Define the Hilbert map for the G-action by

$$\sigma : \mathbf{R}^{2n} \longrightarrow \mathbf{R}^k : m \longrightarrow (\sigma_1(m), \ldots, \sigma_k(m)) .$$

Since G is compact, σ separates G-orbits [31]. Therefore $\sigma(\mathbf{R}^{2n})$ is the G-orbit space $\mathbf{R}^{2n}/G$. Because σ is a polynomial mapping, $\sigma(\mathbf{R}^{2n})$ is a semialgebraic subset of $\mathbf{R}^k$, by the Tarski-Seidenberg theorem [32].

According to Mather's refinement of Schwarz's theorem [33], the mapping $\sigma^* : C^\infty(\mathbf{R}^k) \longrightarrow C^\infty(\mathbf{R}^{2n})^G : f \longrightarrow f \circ \sigma$ is split surjective. An easy diagram chase shows that the space of smooth functions on $\mathbf{R}^{2n}/G$ is isomorphic as a Fréchet space to the space of Whitney smooth functions on $\mathbf{R}^{2n}/G \subseteq \mathbf{R}^k$. We know that the space of smooth functions on the orbit space has a Poisson bracket $\{\, ,\}_{\mathbf{R}^{2n}/G}$. The invariant theoretic construction of $\mathbf{R}^{2n}/G$ gives rise to the question: is there a Poisson bracket $\{\, ,\}_{\mathbf{R}^k}$ on $C^\infty(\mathbf{R}^k)$ such that $C^\infty(\mathbf{R}^{2n}/G)$ is a Poisson subalgebra? In all known examples, the answer is yes.

We can identify the reduced space M_μ with $\sigma(J^{-1}(\mathcal{O}_\mu))$. Moreover, $J^{-1}(\mathcal{O}_\mu)$ is an algebraic variety, since J is a polynomial mapping and $\mathcal{O}_\mu \subseteq \mathcal{G}^*$ is an algebraic variety. Consequently, $M_\mu = \sigma(J^{-1}(\mathcal{O}_\mu))$ is a semialgebraic variety, which is a subvariety of the orbit space $\mathbf{R}^{2n}/G$. By theorem 1 we know that we have a Poisson bracket on the space of Whitney smooth functions on M_μ. Comparing this with [21], in particular propositions 5.5, 5.6, and 5.7, we see that in the present case universal reduction agrees with geometric reduction and also Dirac reduction, when the latter applies. It then follows from either theorem 2 or proposition 5.9 of [21] that the Poisson structure on M_μ is nondegenerate. In [21] one can find an example which shows that universal reduction can differ from the Śniatycki-Weinstein reduction even in the case of a compact linear group action.

5 The spherical pendulum

To make this somewhat abstract discussion more concrete, we use invariant theory to construct the reduced spaces and Poisson algebras occurring in the spherical pendulum. For other treatments see [34, 35].

Consider the linear symplectic action of S^1 on $(T\mathbf{R}^3, \omega)$ defined by

$$\Phi : S^1 \times T\mathbf{R}^3 \longrightarrow T\mathbf{R}^3 : (t, (x, y)) \longrightarrow (R_t\, x, R_t\, y)$$

where

$$R_t = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Φ leaves the subspace defined by $\{x_1 = x_2 = y_1 = y_2 = 0\}$ fixed and is the diagonal action of $SO(2, \mathbf{R})$ on $\mathbf{R}^2 \times \mathbf{R}^2$, which is the subspace defined by $\{x_3 = y_3 = 0\}$. A theorem of Weyl [36] states that the algebra of S^1-invariant polynomials is generated by

$$\begin{array}{lll} \tau_1 = x_3 & \tau_3 = y_1^2 + y_2^2 & \tau_5 = x_1^2 + x_2^2 \\ \tau_2 = y_3 & \tau_4 = x_1y_1 + x_2y_2 & \tau_6 = x_1y_2 - x_2y_1. \end{array}$$

Later calculations are simplified by using the equivalent set of generators

$$\begin{array}{lll} \sigma_1 = x_3 & \sigma_3 = y_1^2 + y_2^2 + y_3^2 & \sigma_5 = x_1^2 + x_2^2 \\ \sigma_2 = y_3 & \sigma_4 = x_1y_1 + x_2y_2 & \sigma_6 = x_1y_2 - x_2y_1. \end{array}$$

The Hilbert map for the S^1-action is

$$\sigma : T\mathbf{R}^3 \longrightarrow \mathbf{R}^6 : m \longrightarrow (\sigma_1(m), \ldots, \sigma_6(m)).$$

Here the σ_i satisfy the relation

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2) \tag{10}$$

together with

$$\sigma_3 \geq 0 \;\&\; \sigma_5 \geq 0. \tag{11}$$

As is shown in [36] these are the only relations. Thus the S^1-orbit space $T\mathbf{R}^3/S^1$ is the semialgebraic variety $\sigma(T\mathbf{R}^3)$ definedby (10) and (11).

Now observe that

$$TS^2 = \{(x, y) \in T\mathbf{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1 \;\&\; x_1y_1 + x_2y_2 + x_3y_3 = 0\}$$

is invariant under Φ. Moreover $\omega|TS^2$ is a symplectic form on TS^2. Therefore the S^1-orbit space TS^2/S^1 of Φ is the semialgebraic variety $\sigma(TS^2)$ defined by (10), (11) and

$$\begin{aligned} \sigma_5 + \sigma_1^2 &= 1 \\ \sigma_4 + \sigma_1\sigma_2 &= 0. \end{aligned} \tag{12}$$

Solving (12) for σ_5 and σ_4, and substituting the result into (10) gives

$$\begin{aligned}\sigma(TS^2) &= \{(\sigma_1, \sigma_2, \sigma_3, \sigma_6) \in \mathbf{R}^4 | \\ &\qquad \sigma_1^2\sigma_2^2 + \sigma_6^2 = (1-\sigma_1^2)(\sigma_3 - \sigma_2^2), \ \ |\sigma_1| \le 1 \ \& \ \sigma_3 \ge 0\}.\end{aligned}$$

The S^1 action Φ has the angular momentum mapping

$$L : T\mathbf{R}^3 \longrightarrow \mathbf{R} : (x, y) \longrightarrow x_1y_2 - x_2y_1 = \sigma_6.$$

Set $J = L|TS^2$. Then the reduced space

$$M_\ell = \sigma(J^{-1}(\ell)) = \sigma(L^{-1}(\ell) \cap TS^2)$$

is the semialgebraic subvariety of $\sigma(TS^2)$ given by

$$\sigma_6 = \ell. \tag{13}$$

Equivalently M_ℓ is the semialgebraic variety in $\mathbf{R}^3$ defined by

$$(1-\sigma_1^2)\sigma_3 = \sigma_2^2 + \ell^2$$

with $|\sigma_1| \le 1$ & $\sigma_3 \ge 0$. When $\ell \ne 0$, M_ℓ is diffeomorphic to $\mathbf{R}^2$, being the graph of the function

$$\sigma_3 = \frac{\sigma_2^2 + \ell^2}{1-\sigma_1^2}, \quad |\sigma_1| < 1.$$

When $\ell = 0$, M_0 is not the graph of a function, because it contains the vertical lines $\{(\pm 1, 0, \sigma_3) \in \mathbf{R}^3 | \sigma_3 \ge 0\}$ (see figure 2). However, this singular space is still homeomorphic to $\mathbf{R}^2$.

Figure 2. The reduced spaces M_ℓ.

Next we compute the Poisson structure on TS^2/S^1. We could do this using (2) and the symplectic form on TS^2. However, we proceed in a slightly different (but equivalent) way. A straightforward calculation shows that $C^\infty(\mathbf{R}^6)$ with coordinates (σ_i) has a Poisson bracket whose structure matrix is the skew symmetric matrix, half of which is given in table 1.

$\{A,B\}_{\mathbf{R}^6}$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	B
σ_1	0	1	$2\sigma_2$	0	0	0	
σ_2		0	0	0	0	0	
σ_3			0	$-2(\sigma_3-\sigma_2^2)$	$-4\sigma_4$	0	
σ_4				0	$-2\sigma_5$	0	
σ_5					0	0	
σ_6						0	
A							

Table 1. Structure matrix for the Poisson bracket on $\mathbf{R}^6$.

Another calculation shows that $C_1 = \sigma_4^2 + \sigma_6^2 - \sigma_5(\sigma_3 - \sigma_2^2)$ and $C_2 = \sigma_6$ are Casimirs. Since $\sigma(T\mathbf{R}^3) \subseteq \mathbf{R}^6$ is defined by $C_1 = 0$, the Poisson bracket on $T\mathbf{R}^3/S^1$ has the same structure matrix as the Poisson bracket on $\mathbf{R}^6$. Because

$$\{\sigma_5 + \sigma_1^2, \sigma_4 + \sigma_1\sigma_2\}_{\sigma(T\mathbf{R}^3)} |\sigma(TS^2) = 2(\sigma_5 + \sigma_1^2)|\sigma(TS^2) = 2,$$

$\sigma(TS^2)$ is a cosymplectic subvariety of $\sigma(T\mathbf{R}^3)$. Consequently, the Poisson bracket on $\sigma(TS^2) = TS^2/S^1$ may be computed using the Dirac process [37]. The structure matrix of $\{\ ,\ \}_{TS^2/S^1}$ is given in table 2.

$\{A,B\}_{TS^2/S^1}$	σ_1	σ_2	σ_3	σ_6	B
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$	0	
σ_2		0	$-2\sigma_1\sigma_3$	0	
σ_3			0	0	
σ_6				0	
A					

Table 2. Structure matrix for the Poisson bracket on TS^2/S^1.

Since the reduced space $M_\ell \subseteq \sigma(TS^2)$ is defined by $C_2 = \ell$, the Poisson bracket on M_ℓ has the same structure matrix as given in table 2 (with the last row and column deleted). If we set

$$\psi(\sigma_1,\sigma_2,\sigma_3) = \sigma_3(1-\sigma_1^2) - \sigma_2^2 - \ell^2, \tag{14}$$

then a careful look at table 2 shows that

$$\{\sigma_i,\sigma_j\}_\ell = \sum_k \epsilon_{ijk}\frac{\partial\psi}{\partial\sigma_k} \tag{15}$$

$$\{f_\ell,g_\ell\}_\ell = (\nabla f_\ell \times \nabla g_\ell)\cdot\nabla\psi \tag{16}$$

via the calculation:

$$\begin{aligned}
\{f_\ell,g_\ell\}_\ell &= df_\ell\cdot X_{g_\ell} = \sum_j \frac{\partial f_\ell}{\partial\sigma_j} d\sigma_j\cdot X_{g_\ell} \\
&= \sum_j \frac{\partial f_\ell}{\partial\sigma_j}\{\sigma_j,g_\ell\}_\ell = -\sum_j \frac{\partial f_\ell}{\partial\sigma_j} dg_\ell\cdot X_{\sigma_j} \\
&= \sum_{j,k,i} \frac{\partial f_\ell}{\partial\sigma_j}\frac{\partial g_\ell}{\partial\sigma_k}\epsilon_{jki}\frac{\partial\psi}{\partial\sigma_i} \\
&= (\nabla f_\ell\times\nabla g_\ell)\cdot\nabla\psi.
\end{aligned}$$

Therefore Hamilton's equations for $h_\ell \in W^\infty(M_\ell)$ are

$$\dot\sigma = \nabla h_\ell \times \nabla\psi$$

where $\times$ is the vector product on $\mathbf{R}^3$, because

$$\begin{aligned} \dot{\sigma}_i &= \{\sigma_i, h_\ell\}_\ell \\ &= (\nabla\sigma_i \times \nabla h_\ell) \cdot \nabla\psi \\ &= (\nabla h_\ell \times \nabla\psi) \cdot \nabla\sigma_i \\ &= (\nabla h_\ell \times \nabla\psi)_i. \end{aligned}$$

So far we have only treated the S^1 symmetry of the spherical pendulum. To treat the dynamics, we consider the Hamiltonian

$$H : T\mathbf{R}^3 \longrightarrow \mathbf{R} : (x, y) \longrightarrow \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + x_3.$$

When restricted to TS^2, H is the Hamiltonian of the spherical pendulum. Since H is S^1-invariant, it is a function of the invariants, namely

$$H = \frac{1}{2}\sigma_3 + \sigma_1. \tag{17}$$

The reduced Hamiltonian on M_ℓ is h_ℓ, given also by (17). A short calculation shows that the critical points of h_ℓ occur only when $\sigma_2 = 0$. Geometrically they occur in the σ_1, σ_3-plane when the line defined by $h_\ell = e$ is tangent to the fold curve of the projection of M_ℓ on the σ_1, σ_3-plane (see figure 3).

Figure 3. The darkened curve is the fold curve of the projection of M_ℓ on the $\{\sigma_2 = 0\}$ plane. The dots are the critical points of h_ℓ on $M_\ell \cap \{\sigma_2 = 0\}$.

Over every point on a line in figure 4 lie two points of M_ℓ except when (i) the line intersects the darkened fold curve where there is only one point of M_ℓ or (ii) outside the darkened curve where there are no points of M_ℓ. Hence P_ℓ, P_0 are stable equilibrium points of X_{h_ℓ}, and correspond to stable periodic orbits of X_H. On the other hand, P_0^* is an unstable equilibrium point of X_H.

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6 Appendix: Notation and terminology

We deal with varieties in a generalized sense. For us a variety consists of a topological space X together with a choice of a set of "smooth" functions $C^\infty(X) \subseteq C^0(X)$. This choice defines a differential structure on X. If $\pi : X \longrightarrow Y$ is a surjection of a variety X onto a space Y, we make Y a quotient variety of X by putting the quotient topology on Y and taking

$$C^\infty(Y) = \{f \in C^0(Y) | f \circ \pi = F \text{ for some } F \in C^\infty(X)\}.$$

Similarly if Y is a subset of a variety X, we make Y into a subvariety of X by putting the relative topology on Y and taking $C^\infty(Y) = W^\infty(Y)$, where

$$W^\infty(Y) = \{f \in C^0(Y) | f = F|Y \text{ for some } F \in C^\infty(X)\}$$

are the Whitney smooth functions on Y. A map $f : X \longrightarrow Y$ between varieties is smooth provided it is continuous and $f^* C^\infty(Y) \subseteq C^\infty(X)$. If $Y \subseteq X$ is a subvariety, then the ideal $\mathcal{I}(Y) \subseteq C^\infty(X)$ of Y consists of all smooth functions on X which vanish when restricted to Y.

Let G be a Lie group, which acts smoothly on a variety X. Let $Y = X/G$ be the corresponding orbit space. From the definitions it follows that $C^\infty(X/G) = C^\infty(X)^G$, the space of G-invariant smooth functions on X.

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