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An $U_{qp}(\mathrm{u}_2)$ model for rotational bands of nuclei

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Abstract. A rotational model is developed from a new version of the two-parameter quantum algebra $U_{qp}(\mathrm{u}_2)$. This model is applied to the description of some recent experimental data for the rotating superdeformed nuclei $^{192-194-196-198}$Pb and $^{192-194}$Hg. A comparison between the $U_{qp}(\mathrm{u}_2)$ model presented here and the Raychev-Roussev-Smirnov model with $U_q(\mathrm{su}_2)$ symmetry shows the relevance of the introduction of a second parameter of a "quantum algebra" type.

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The concepts of quantum algebra (or Hopf algebra) and quantum group (or compact matrix pseudo-group), introduced in the eightees, continue to be objects of considerable interest from both a mathematical and a physical point of view. In nuclear physics, quantum algebras have been applied to rotational-vibrational spectroscopy of nuclei [1-6], to the interacting boson model [7] and to the U_3 shell model [8]. In particular, a q-rotor model, based on the use of the quantum algebra $U_q(\mathrm{su}_2)$, has been developed by Raychev *et al* [1] and, independently, by Iwao [2]. This model has been successfully applied to the description of rotational bands of deformed and superdeformed nuclei. The connection between this q-rotor model and the VMI (variable moment of inertia) model has been studied in great detail by the Demokritos-Moscow-Sofia collaboration [3-5].

Most of the applications of quantum algebras to (nuclear) physics have been restricted to the use of one-parameter algebras, as for instance $U_q(\mathrm{su}_2)$, although several theoretical works have been published in recent years on multi-parameter, at least two-parameter, quantum algebras. In this respect, we may quote among others the works of references [9-15] mainly devoted to the two-parameter algebra $U_{qp}(\mathrm{su}_2)$. However, it is well known (cf Drinfeld's theorem) that a two-parameter deformation of a semi-simple Lie algebra turns out to be essentially a one-parameter deformation. Therefore, in order to get a nontrivial two-parameter deformation, that reduces to $U_q(\mathrm{su}_2)$ in some limit, we have to deform the non semi-simple Lie algebra u_2 instead of su_2.

It is the aim of this letter to present a qp-rotor model based on the two-parameter quantum algebra $U_{qp}(\mathrm{u}_2)$ recently discussed in [16]. Such a model can be applied to nuclear physics and to molecular physics as well. We shall be concerned here with an application of the model to rotational bands of superdeformed nuclei. In nuclear physics, the motivation for constructing a qp-rotor model having the $U_{qp}(\mathrm{u}_2)$ symmetry is as follows. The $q-$rotor model, based on the one-parameter algebra $U_q(\mathrm{su}_2)$, is especially appropriate for describing rotational bands of deformed and superdeformed nuclei at weak and medium angular momenta ; however, the latter model is less convenient (as will be seen below) for describing those rotation energy levels, at high angular momenta, which come from recent experimental data in the $A \sim 190$ region. The objective of the present work is thus to develop a qp-rotor model and to test, on the recent experimental results for the superdeformed (SD) bands of $^{192-194}$Hg [17, 18] and $^{192-194-196-198}$Pb [19-21], the importance of introducing a second parameter when passing from the $U_q(\mathrm{su}_2)$ symmetry to the $U_{qp}(\mathrm{u}_2)$ symmetry.

The two-parameter deformation $U_{qp}(\mathrm{u}_2)$ of the Lie algebra u_2 is spanned by the four operators J_α $(\alpha = 0, 3, +, -)$ which satisfy the commutation relations [16]

$$[J_3, J_\pm] \;=\; \pm J_\pm \qquad [J_+, J_-] \;=\; (qp)^{J_0 - J_3}\,[[2J_3]]_{qp} \qquad [J_0, J_\alpha] \;=\; 0. \tag{1}$$

(In this letter, we use the notations $[[X]]_{qp} = (q^X - p^X)/(q-p)$ and $[X]_q \equiv [[X]]_{qq^{-1}} = (q^X - q^{-X})/(q - q^{-1})$, where X may stand for an operator or a number.) The co-product Δ_{qp} of the quantum algebra $U_{qp}(\mathrm{u}_2)$ is defined by [16]

$$\begin{aligned} \Delta_{qp}(J_\pm) &= J_\pm \otimes (qp)^{\frac{1}{2}J_0}(qp^{-1})^{\frac{1}{2}J_3} + (qp)^{\frac{1}{2}J_0}(qp^{-1})^{-\frac{1}{2}J_3} \otimes J_\pm \\ \Delta_{qp}(J_3) &= J_3 \otimes I + I \otimes J_3 \qquad\quad \Delta_{qp}(J_0) \;=\; J_0 \otimes I + I \otimes J_0. \end{aligned} \tag{2}$$

The universal $R-$matrix, denoted here as R_{qp} and (partially) defined via $\Delta_{qp}(J_\alpha) = R_{qp}\Delta_{pq}(J_\alpha)(R_{qp})^{-1}$, depends on the two parameters q and p, as does the co-product Δ_{qp}. For instance, the R-matrix corresponding to the coupling of two angular momenta $j = \frac{1}{2}$ reads (in terms of the unit matrices E_{ab})

$$R_{qp} \;=\; q\,(E_{11} + E_{44}) + \sqrt{qp}\,(E_{22} + E_{33}) + (q-p)\,E_{32}. \tag{3}$$

It can be checked that the operator

$$C_2(U_{qp}(\mathrm{u}_2)) \;=\; \frac{1}{2}\,(J_+J_- + J_-J_+) + \frac{1}{2}\,[[2]]_{qp}\,(qp)^{J_0-J_3}\,[[J_3]]^2_{qp} \tag{4}$$

is an invariant of the quantum algebra $U_{qp}(\mathrm{u}_2)$. This invariant is the main mathematical ingredient for the qp-rotor Hamiltonian model to be developed (see (10)). For generic q and p, each irreducible representation of $U_{qp}(\mathrm{u}_2)$ is characterized by a Young pattern $[\varphi_1, \varphi_2]$ with $\varphi_1 - \varphi_2 = 2j$, where $2j$ is a nonnegative integer (j will represent a spin angular momentum in what follows) ; we note $|[\varphi_1, \varphi_2]m\rangle$ (with $m = -j, -j+1, \cdots, j$) the basis vectors for the representation $[\varphi_1, \varphi_2]$. For physical reasons, we shall see in the following that $q = p^*$. In this connection, it should be observed that the constraint $q = p^*$ ensures that $(\Delta_{qp}(J_\pm))^\dagger = \Delta_{pq}(J_\mp)$ and is compatible with the commutation relations for the operators $\Delta_{qp}(J_\alpha)$ ($\alpha = 0, 3, +, -$).

Two particular cases are worth noticing. First, in the limiting situation where $q = p^{-1} \to 1$, it is clear that equations (1)-(4) reduce to relations characterizing the Lie algebra u_2. Second, when $p = q^{-1}$ (q not being a root of unity), the three generators J_3, J_+ and J_- of the algebra $U_{qq^{-1}}(\mathrm{u}_2)$ generate the one-parameter quantum algebra $U_q(\mathrm{su}_2)$, ie the usual deformation of the Lie algebra su_2 introduced in the pioneer works by Kulish, Reshetikhin, Sklyanin and other authors (see references in [16]). At this stage, we may wonder whether we really gain something when passing from the "classical" quantum algebra $U_q(\mathrm{su}_2)$ to the quantum algebra $U_{qp}(\mathrm{u}_2)$. In this respect, let us define the operators A_α ($\alpha = 0, 3, +, -$) through

$$J_\pm \;=\; (qp)^{\frac{1}{2}(J_0-\frac{1}{2})}\,A_\pm \qquad J_0 \;=\; A_0 \qquad J_3 \;=\; A_3 \tag{5}$$

and let us introduce

$$Q = (qp^{-1})^{\frac{1}{2}} \qquad P = (qp)^{\frac{1}{2}}. \tag{6}$$

Then, we can verify that the set $\{A_3, A_+, A_-\}$ spans $U_Q(\mathrm{su}_2)$, which commutes with A_0, so that we have the decomposition

$$U_{qp}(u_2) = u_1 \oplus U_Q(su_2). \tag{7}$$

On the other hand, the invariant $C_2(U_{qp}(\mathrm{u}_2))$ can be developed as

$$C_2(U_{qp}(\mathrm{u}_2)) \;=\; P^{2A_0-1}\, C_2(U_Q(\mathrm{su}_2)) \tag{8}$$

where

$$C_2(U_Q(\mathrm{su}_2)) \;=\; \frac{1}{2}\,(A_+A_- + A_-A_+) + \frac{1}{2}\,[2]_Q\,[A_3]_Q^2 \tag{9}$$

is an invariant of $U_Q(\mathrm{su}_2)$. Therefore, in spite of the fact that the transformation (5)-(6) allows us to generate the one-parameter algebra $U_Q(\mathrm{su}_2)$ from the two-parameter algebra $U_{qp}(\mathrm{u}_2)$, see (7), the invariant $C_2(U_{qp}(\mathrm{u}_2))$, as given by (8), still exhibits two independent parameters (Q and P instead of q and p).

We are now in a position to develop an $U_{qp}(\mathrm{u}_2)$ model for describing rotational bands of a deformed or superdeformed nucleus. The first input of this model lies on the use of the *qp*-rotor Hamiltonian

$$H \;=\; \frac{1}{2\mathcal{I}}\, C_2(U_{qp}(\mathrm{u}_2)) + E_0 \tag{10}$$

where $\mathcal{I}$ denotes the moment of inertia of the nucleus and E_0 the bandhead energy. The second input consists of choosing to diagonalise H on the subspace $\{|jm\rangle = |[2j,0]m\rangle\ :\ m = -j, -j+1, \cdots, +j\}$ of constant spin j corresponding to the irreducible representation for which $\varphi_1 = 2j$ and $\varphi_2 = 0$. Then, the eigenvalues of H take the form

$$E \;=\; \frac{1}{2\mathcal{I}}\,[[j]]_{qp}\,[[j+1]]_{qp} + E_0 \tag{11a}$$

or equivalently

$$E = \frac{1}{2\mathcal{I}} \, e^{(2j-1)\frac{s+r}{2}} \, \frac{\sinh(j\frac{s-r}{2}) \, \sinh[(j+1)\frac{s-r}{2}]}{\sinh^2(\frac{s-r}{2})} + E_0 \tag{11b}$$

where $s = \ln q$ and $r = \ln p$. Two important constraints can be imposed on the parameters s and r. Indeed, *(i)* in the particular case $p = q^{-1}$ (ie, $r = -s$), we want that our model reduces to the $U_q(\mathrm{su}_2)$ model developed by Raychev *et al* [1] and *(ii)* for obvious reasons, E should be real. Points *(i)* and *(ii)* lead to the constraints $(s-r) \in \mathrm{i}\mathbb{R}$ and $(s+r) \in \mathbb{R}$. Therefore, we shall assume

$$\begin{array}{lcl} \frac{s+r}{2} = \beta\cos\gamma & & q = e^{\beta\cos\gamma} \, e^{+\mathrm{i}\beta\sin\gamma} \\ & \Longleftrightarrow & \\ \frac{s-r}{2\mathrm{i}} = \beta\sin\gamma & & p = e^{\beta\cos\gamma} \, e^{-\mathrm{i}\beta\sin\gamma} \end{array} \tag{12}$$

where β and γ are two independent real parameters. (Note that the parameters q and p defined by (12) satisfy $q = p^*$.) By introducing (12) into (11b), we obtain the rotational energy

$$E = \frac{1}{2\mathcal{I}} \, e^{(2j-1)\beta\cos\gamma} \, \frac{\sin(j\beta\sin\gamma) \, \sin[(j+1)\beta\sin\gamma]}{\sin^2(\beta\sin\gamma)} + E_0. \tag{13}$$

This expression for E constitutes our basic result for applications. In the particular case where $\gamma = \frac{\pi}{2}$ (ie, $q = p^{-1} = e^{\mathrm{i}\beta}$), (13) coincides with the corresponding expression derived in [1].

Before testing formula (13) on some recent experimental data, we would like to show that our $U_{qp}(\mathrm{u}_2)$ model is not phenomenologically equivalent (in the sense discussed in [3-5]) to the VMI model. As a matter of fact, if we try to obtain an *à la* Dunham expansion of E, we find

$$E = \frac{1}{2\mathcal{I}_{\beta\gamma}} \left(\sum_{n=0}^{\infty} d_n(\beta,\gamma) \, [C_2(\mathrm{su}_2)]^n + [2C_1(\mathrm{u}_1)+1] \sum_{n=0}^{\infty} c_n(\beta,\gamma) \, [C_2(\mathrm{su}_2)]^n \right) + E_0 \tag{14}$$

where

$$\mathcal{I}_{\beta\gamma} = \mathcal{I} \, e^{2\beta\cos\gamma} \qquad C_2(\mathrm{su}_2) = j(j+1) \qquad C_1(\mathrm{u}_1) = j \tag{15a}$$

while the expansion coefficients $c_n(\beta,\gamma)$ and $d_n(\beta,\gamma)$ are given by the series

$$c_n(\beta,\gamma) = \frac{2^{2n}}{2\sin^2(\beta\sin\gamma)}$$

$$\sum_{k=0}^{\infty} \{(\cos\gamma)^{2k+1+2n}\cos(\beta\sin\gamma) - \cos[(2k+1+2n)\gamma]\} \frac{\beta^{2k+1+2n}}{(2k+1+2n)!} \frac{(k+n)!}{k!\,n!}$$

$$d_n(\beta,\gamma) = \frac{2^{2n}}{2\sin^2(\beta\sin\gamma)}$$

$$\sum_{k=0}^{\infty} \{(\cos\gamma)^{2k+2n}\cos(\beta\sin\gamma) - \cos[(2k+2n)\gamma]\} \frac{\beta^{2k+2n}}{(2k+2n)!} \frac{(k+n)!}{k!\,n!}. \tag{15b}$$

Note that for $\gamma = \frac{\pi}{2}$, we have $c_n(\beta, \frac{\pi}{2}) = 0$ and the expression (14) simplifies to

$$E = \frac{1}{2\mathcal{I}} \frac{\beta^2}{\sin^2\beta} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n-1}}{n!} \beta^{n-1} j_{n-1}(\beta) [j(j+1)]^n + E_0 \tag{16}$$

in terms of the spherical Bessel functions of the first kind j_n. The particular expansion (16) is in agreement with the result of Bonatsos *et al* [3] (see also [4, 5]). The more general expansion (14) provides us with a development of E in terms of the second-order invariant $C_2(\mathrm{su}_2)$ of the Lie algebra su_2, which invariant characterizes the energy of a rigid rotor, and of the first-order invariant $C_1(\mathrm{u}_1)$ of the Lie algebra u_1. The expansion (14) differs from the ones for the VMI model and for the $U_q(\mathrm{su}_2)$ model by the occurrence of $C_1(\mathrm{u}_1)$. We thus expect that our $U_{qp}(\mathrm{u}_2)$ model leads to results different from the ones obtained by using the VMI model or the $U_q(\mathrm{su}_2)$ model.

We have applied the energy formula (13) to the SD bands which appear for some nuclei in the $A \sim 190$ region. We report here on the analysis of seven SD bands in six even-even nuclei, namely $^{192-194}$Hg [17, 18] and $^{192-194-196-198}$Pb [19-21]. In the framework of our $U_{qp}(\mathrm{u}_2)$ model, the two parameters $\beta\sin\gamma$ and $\beta\cos\gamma$ occurring in (13) are taken as free independent parameters. Furthermore, the moment of inertia $\mathcal{I}$ is kept constant and chosen as the static moment extrapolated at zero spin from the experimental data of [17-21]. From equation (13), we can compute the transition energies $E_\gamma(j) = E(j) - E(j-2)$ and we choose to minimize

$$\chi^2 = \frac{1}{N} \sum_j \left[\frac{E_\gamma^{\mathrm{th}}(j) - E_\gamma^{\mathrm{ex}}(j)}{\Delta E_\gamma(j)} \right]^2 \tag{17}$$

where $\Delta E_\gamma(j)$ are the experimental errors and N is the number of experimental points entering the fitting procedure. For the purpose of comparison, the same SD bands have also been analysed in the framework of the $U_q(\mathrm{su}_2)$ model by using a similar fitting procedure (same χ^2 ; same moment of inertia ; the parameter γ is kept to the value 90°and only the parameter β', with $q = \mathrm{e}^{\mathrm{i}\beta'}$, is freely varied).

Figure 1 displays the results obtained from the $U_{qp}(\mathrm{u}_2)$ and $U_q(\mathrm{su}_2)$ models. Table 1 shows the values of the parameters β and γ for the $U_{qp}(\mathrm{u}_2)$ model and of the parameter β' for the $U_q(\mathrm{su}_2)$ model. For $\gamma = 90°$, the two models coincide and our results reflect this fact since for γ close to 90°, as for ^{198}Pb, we have β' close to β and the two models give similar results. However, when γ increases, a better agreement with experiment is globally obtained for the $U_{qp}(\mathrm{u}_2)$ model ; the largest discrepancy between the two models is reached in the case of ^{192}Hg for which $\gamma = 127°$and the values of β and β' are quite different. Finally, it is to be noted that the better agreement for the $U_{qp}(\mathrm{u}_2)$ model is also depicted by the values obtained for χ which range from 1 to 7 for the $U_{qp}(\mathrm{u}_2)$ model and from 2 to 1000 for the $U_q(\mathrm{su}_2)$ model. (In both cases, the large values of χ are due to the small values of the experimental errors, which are of the order of 0.1% of the transition energies.)

As a further test of our model, we have used the parameters β and γ obtained for ^{192}Hg in order to predict the transition energies $E_\gamma(44)$ and $E_\gamma(46)$. The so calculated energies are in good agreement with the new experimental data obtained with the EUROGAM detector [22].

A few words should be said about the physical interpretation of the parameters of the $U_{qp}(\mathrm{u}_2)$ model. Like the parameter β' in the $U_q(\mathrm{su}_2)$ model, the parameter $\beta \sin \gamma$ corresponds to the softness parameter of the VMI model. The parameter $\beta \cos \gamma$ strongly differentiates the two models at high spins: a much better agreement between theory and experiment is generally obtained at high spins with the $U_{qp}(\mathrm{u}_2)$ model ; this is especially true for ^{192}Hg and ^{194}Hg(1b) for which the values of γ are far from 90°. This suggests that the parameter $\beta \cos \gamma$ manifests itself at high spins by a weakening in the increasing (versus spin) of the moment of inertia.

In conclusion, we have presented an $U_{qp}(\mathrm{u}_2)$ model for rotational spectra of nuclei which presents some advantages over the $U_q(\mathrm{su}_2)$ model as far as energy levels are concerned. Our test has been concerned with seven SD bands. On the other hand, we have also tested our model on the ground state band of ^{238}U ; the results obtained for this case study are in good accordance with classical results [23]. Further applications of our model could be the analysis, for classification purposes, of identical SD bands: keeping their moment of inertia as constant, one could extract their dependency on the (q, p) parameters. Furthermore, it would be interesting to investigate $B(E2)$ transition probabilities in the qp-rotor model with $U_{qp}(\mathrm{u}_2)$ symmetry. These matters shall be the object of the thesis by the junior author (R B).

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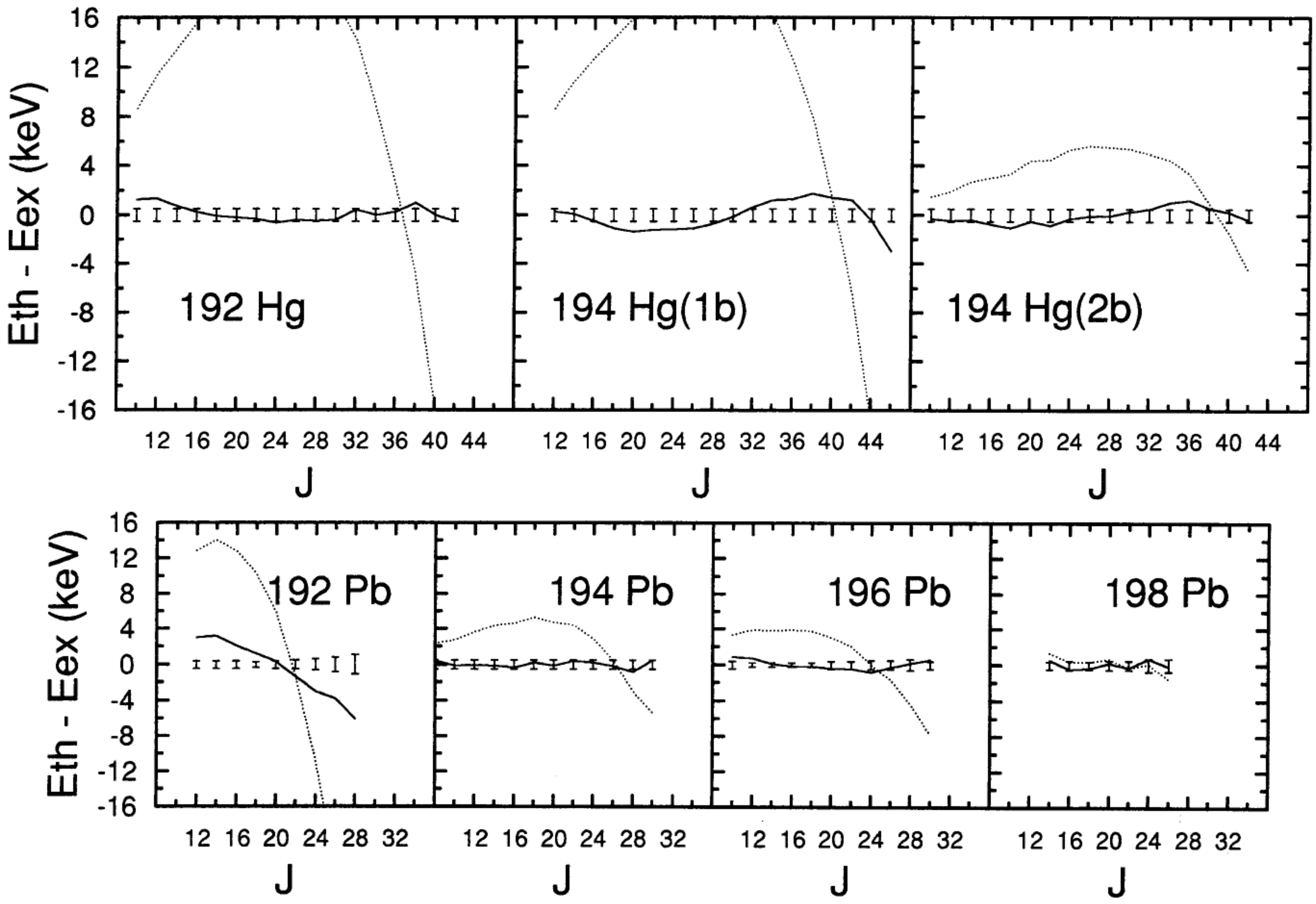
TABLE

TABLE I. Parameters from the $U_q(\mathrm{su}_2)$ and $U_{qp}(\mathrm{u}_2)$ models.

SD bands	$U_q(\mathrm{su}_2)$	$U_{qp}(\mathrm{u}_2)$		$\frac{1}{2\mathcal{I}}$ (keV)
	β'	β	γ (degrees)	
^{192}Hg	0.0146	0.0029	127	5.95
^{194}Hg(1b)	0.0134	0.0054	105	5.81
^{194}Hg(2b)	0.0110	0.0084	93	5.37
^{192}Pb	0.0210	0.0400	94	6.25
^{194}Pb	0.0147	0.0091	95	5.75
^{196}Pb	0.0139	0.0080	95	5.81
^{198}Pb	0.0117	0.0100	91	5.74

FIGURE

FIG. 1. Comparison between the theoretical and experimental transition energies (in keV). Solid lines and dot lines display the results for the $U_{qp}(\mathrm{u}_2)$ model and the $U_q(\mathrm{su}_2)$ model, respectively. Experimental error bars are taken from references [17-21].


192 Hg
194 Hg(1b)
194 Hg(2b)
192 Pb
194 Pb
196 Pb
198 Pb
Eth - Eex (keV)
J