RR-75-22

A UTILITY FUNCTION FOR TIME STREAMS HAVING INTER PERIOD DEPENDENCIES

David E. Bell

July 1975

Research Reports are publications reporting on the work of the author. Any views or conclusions are those of the author, and do not necessarily reflect those of IIASA.

A UTILITY FUNCTION FOR TIME STREAMS HAVING INTER PERIOD DEPENDENCIES

David E. Bell

Abstract

Frequently a decision maker's preferences for consequences in a given period will depend on the particular outcome in the previous and/or following period. This paper gives a simple functional form which enables such preferences to be explicitly included in a utility function for time streams.

In evaluating alternative strategies which have impacts over time, many factors serve to complicate matters, and principal amongst them are the necessity of making tradeoffs between consequences in different periods and the uncertainty of the precise outcomes as to their magnitude and timing. It is precisely these issues that Von Neumann-Morgenstern utility functions [9] are able to resolve. Such a utility function serves as a preference (or value) function in that it provides an ordering over certain outcomes and its expected value provides a preference function over uncertain outcomes.

The difficulties associated with this approach arise when the number of time periods is large, for the dimension of the utility function is equal to the number of time periods; and whilst the assessment of a one-dimensional (or one-attribute) utility function is relatively easy and that of a two-dimensional utility function still practical, it soon becomes impossible without major simplifying assumptions.

The problem is to find reasonable assumptions which reduce the assessment of the utility function to a manageable level without losing the flexibility to reflect the decision maker's true preferences accurately, and without losing the property of the utility function as an evaluator of uncertain outcomes.

Both Meyer [8] and Fishburn [3] have given assumptions which do fulfil these requirements. Let $X = X_1 \times X_2 \times X_3 \times \cdots \times X_T$ be a set of time streams where X_i is the set of possible outcomes in period i whose elements are scalars or vectors. X_i will also be used to denote the attribute as well as its set of values.

Meyer assumes utility independence (see next section) to exist between the sets $X_1 \times \cdots \times X_j$ and $X_{j+1} \times \cdots \times X_T$ for each $j = 1, \ldots, T-1$ and shows that this implies that

$$u(x_1, \ldots, x_T) = \sum_{i=1}^{T} c_i \, u_i(x_i) \qquad c_i > 0$$

or (1)

$$\prod_{i=1}^{T} (a_i + b_i \, u_i(x_i)) \qquad b_i > 0 ,$$

where u_i is a utility function assessed over only the set X_i.

Fishburn assumes Markovian dependence which, briefly stated, says that the decision maker is indifferent between two uncertain time streams if and only if the probability distribution over set pairs $X_i \times X_{i+1}$ is the same for all i. This yields the form

$$u(x_1, \ldots, x_T) = \sum_{i=1}^{T-1} \lambda_i(x_i, x_{i+1}) , \tag{2}$$

where

$$\lambda_i(x_i, x_{i+1}) = u(x_1^0, x_2^0, \ldots, x_{i-1}^0, x_i, x_{i+1}, x_{i+2}^0, \ldots, x_T^0)$$
$$- u(x_1^0, \ldots, x_i^0, x_{i+1}, x_{i+2}^0, \ldots, x_T^0) .$$

In a recent study by the Ecology Project of a forest pest problem, the impacts on society were summarized by three attributes, profit from lumber (P), employment within the industry (E) and the recreational potential of the forest (R), each of these being time streams of attributes. The decision maker felt that the recreation streams and the profit/employment streams were mutually utility independent, which yielded a form

$$u(p,e,r) = u_R(r) + u_s(p,e) + ku_R(r)\ u_s(p,e) \quad ; \qquad (3)$$

and that the assumptions leading to the product from (1) were appropriate for the utility function u_R of recreation, but that preferences for (p_i,e_i) pairs depended heavily on the employment figures in neighbouring time periods. Analysis showed that for a given level of e_{i-1} he was risk averse for values of e_i which were greater than e_{i-1} but risk prone otherwise. The Markovian dependence assumptions leading to form (2) were not felt to be sufficiently intuitively meaningful to be used.

The results that follow give a functional form for a general utility function for time streams which is less restrictive than, but can be specialized to, the forms of Meyer and Fishburn.

A final section returns to the forest pest problem and gives a functional form for $u_s(\underline{p},\underline{e})$.

Conditional Utility Independence

Consider a situation involving two attributes X_1 and X_2, each of which may be vector valued. Then X_1 is said to be utility independent of X_2 if, in cases where only consequences for X_1 are uncertain and the value of X_2 is known with certainty, the decision maker's attitude towards risk taking is independent of the particular value at which X_2 is fixed.
More precisely,

$$u(x_1,x_2) = f(x_2) + g(x_2)\ u(x_1,x_2^0) \qquad (4)$$

for some fixed $x_2^0 \varepsilon X_2$ where the only restrictions on f and g are that $g(x_2) > 0$. This form comes from the fact that two utility functions are strategically equivalent under uncertainty if and only if they are positive linear transformations of each other. If, in addition, X_2 is utility independent of X_1, then (see Keeney [5])

$$u(x_1,x_2) = u(x_1,x_2^0) + u(x_1^0,x_2) + ku(x_1,x_2^0)\, u(x_1^0,x_2) \qquad (5)$$

for some constant $k > 0$, where $u(x_1^0,x_2^0) = 0$.

For a comprehensive discussion of utility functions and related independence assumptions see Keeney and Raiffa [6].

A generalization of utility independence is <u>conditional utility independence</u> (see Chapter 6 of [6]). For a situation having three attributes X_1, X_2, X_3 (each possibly vector valued), X_1 is conditionally utility independent of X_2, given X_3, if, whenever X_3 is fixed, X_1 is utility independent of X_2. In this case

$$u(x_1,x_2,x_3) = f(x_2,x_3) + g(x_2,x_3)\, u(x_1,x_2^0,x_3) \qquad (6)$$

for some fixed value x_2^0.

If in addition X_2 is conditionally utility independent of X_1, given X_3, then we will say that X_1 and X_2 are <u>mutually conditionally utility independent</u> and write X_1 mcui X_2. The conditional "given X_3" will be omitted since it will always be assumed here that the conditioning is on all the attributes not explicitly mentioned.

<u>Lemma 1</u>: If X_1 mcui X_2, then

$$u(x_1,x_2,x_3) = [1-k(x_3)\, u(x_1^0,x_2^0,x_3)][u(x_1,x_2^0,x_3) + u(x_1^0,x_2,x_3) - u(x_1^0,x_2^0,x_3^0] + k(x_3)\, u(x_1,x_2^0,x_3)\, u(x_1^0,x_2,x_3) ,$$

where x_1^0, x_2^0 and x_3^0 are arbitrary values of X_1, X_2, X_3; $k(x_3)$ is an arbitrary function, and X_3 represents all attributes other than X_1 and X_2.

<u>Proof</u>: With X_3 fixed at an arbitrary level $\bar{x}_3$ and defining $\bar{u}(x_1,x_2,x_3) = u(x_1,x_2,x_3) - u(x_1^0,x_2^0,x_3)$, it can be seen from (5)

that

$$\bar{u}(x_1,x_2,\bar{x}_3) = \bar{u}(x_1,x_2^0,\bar{x}_3) + \bar{u}(x_1^0,x_2,\bar{x}_3) + k\bar{u}(x_1,x_2^0,\bar{x}_3)\ \bar{u}(x_1^0,x_2,\bar{x}_3) \quad , \qquad (7)$$

where k now may depend upon the value $\bar{x}_3$. Since (7) will hold for all choices $\bar{x}_3$, resubstitution in (7) for u, explicitly recognizing $k \equiv k(x_3)$ and rearranging terms, gives the result. ||

It will be most convenient to abuse normal notation slightly in order to make a number of subsequent formulas look neater and more meaningful. For each attribute X_i a value x_i^0 is selected and the utility function scaled so that $u(\underline{x}^0) = 0$. Then whenever an attribute is at its fixed level it will be omitted from explicit mention in the vector array of attribute values. Hence $u(x_1,x_2^0,x_3)$ will be written $u(x_1,x_3)$, $u(x_1^0,x_2,x_3^0)$ as $u(x_2)$ and so on. Thus when X_1 mcui X_2 we may write

$$u(x_1,x_2,x_3) = [1-k(x_3)\ u(x_3)][u(x_1,x_3) + u(x_2,x_3) - u(x_3)]$$

$$+ k(x_3)\ u(x_1,x_3)\ u(x_2,x_3) \quad .$$

The following lemma will enable us to use induction to prove the main result.

<u>Lemma 2</u>: If X_1 mcui $\{X_2,X_3\}$ then X_1 mcui X_2 and X_1 mcui X_3.

<u>Proof</u>: Let X_4 be a vector attribute combining all attributes not including X_1, X_2, X_3, if any.

If X_1 mcui $\{X_2,X_3\}$ then by Lemma 1

$$u(x_1,x_2,x_3,x_4) = [1-k(x_4)\ u(x_4)][u(x_1,x_4) + u(x_2,x_3,x_4) - u(x_4)]$$

$$+ k(x_4)\ u(x_1,x_4)\ u(x_2,x_3,x_4) \quad . \qquad (8)$$

for some function $k(x_4)$. If X_3 and X_4 are fixed at any values $\bar{x}_3$, $\bar{x}_4$, say, and denoting $u(\bar{x}_4)$ by c_1, $k(\bar{x}_4)$ by c_2, $u(x_1,\bar{x}_4)$ by $u_1(x_1)$, $u(x_2,\bar{x}_3,\bar{x}_4)$ by $u_2(x_2)$, then from (8) we have

$$u(x_1,x_2,x_3,x_4) = -c_1(1-c_1c_2) + (1-c_1c_2)(u_1(x_1) + u_2(x_2)) + c_2u_1(x_1)\,u_2(x_2) ,$$

which shows that for any fixed values of X_3 and X_4, X_1 and X_2 are mutually utility independent (compare (5)), and thus X_1 mcui X_2. Similarly X_1 mcui X_3. $||$

The converse of the lemma is not always true although it is easily shown that X_1 cui X_2, X_1 cui X_3 does imply X_1 cui $\{X_2,X_3\}$.

Theorem: Assuming that

(i) $\{X_1,X_2,\ldots,X_{i-1}\}$ is mutually conditionally utility independent with $\{X_{i+1},\ldots,X_T\}$ for all $i = 2,\ldots,T-1$,

(ii) For each value x_i of X_i there exist values x^*_{i-1} of X_{i-1} and x^*_{i+1} of X_{i+1} such that

$$u(x^*_{i-1},x_i) \neq u(x^0_{i-1},x_i)$$

$$u(x_i,x^*_{i+1}) \neq u(x_i,x^0_{i+1}) ,$$

then for $T \geq 4$ either

a) $$u(x_1,x_2,\ldots,x_T) = \sum_{i=1}^{T-1} u(x_i,x_{i+1}) - \sum_{i=2}^{T-1} u(x_i) ,$$

or

b) $$u(x_1,x_2,\ldots,x_T) = \left[\prod_{i=2}^{T-1} (\lambda + u(x_i))\right]^{-1} \left[\prod_{i=1}^{T-1} (\lambda + u(x_i,x_{i+1}))\right] -\lambda ,$$

where λ is a constant which may be taken as ± 1.

Proof: First, fix X_i at level x_i^0 for all $i = 5,6,\ldots,T$; then by assumption (i) and Lemma 2 we have that $\{X_1,X_2\}$ is mutually conditionally utility independent with X_4, and X_1 is mutually conditionally utility independent with $\{X_3,X_4\}$. Regarding X_1, X_2 as one vector attribute we may use Lemma 1 to give that

$$u(x_1,x_2,x_3,x_4) = [1-s(x_3)\ u(x_3)][u(x_1,x_2,x_3) + u(x_3,x_4) - u(x_3)]$$
$$+ s(x_3)\ u(x_1,x_2,x_3)\ u(x_3,x_4)\ ; \qquad (9)$$

and regarding X_3, X_4 as a single attribute Lemma 1 gives

$$u(x_1,x_2,x_3,x_4) = [1-k(x_2)\ u(x_2)][u(x_1,x_2) + u(x_2,x_3,x_4) - u(x_2)]$$
$$+ k(x_2)\ u(x_1,x_2)\ u(x_2,x_3,x_4) \qquad (10)$$

for some functions $s(x_3)$ and $k(x_2)$.
Substitution of $X_1 = x_1^0$ in (9) gives

$$u(x_2,x_3,x_4) = [1-s(x_3)\ u(x_3)][u(x_2,x_3) + u(x_3,x_4) - u(x_3)]$$
$$+ s(x_3)\ u(x_2,x_3)\ u(x_3,x_4)\ , \qquad (11)$$

and $X_4 = x_4^0$ in (10) gives

$$u(x_1,x_2,x_3) = [1-k(x_2)\ u(x_2)][u(x_1,x_2) + u(x_2,x_3) - u(x_2)]$$
$$+ k(x_2)\ u(x_1,x_2)\ u(x_2,x_3)\ . \qquad (12)$$

Now substitute (11) into (10) and (12) into (9), then subtraction of (10) from (9) gives that

$$A(x_2,x_3)[-u(x_2)\ u(x_3) + u(x_3)\ u(x_1,x_2) + u(x_2)\ u(x_3,x_4) - u(x_1,x_2)\ u(x_3,x_4)] \equiv 0$$

where

$$A(x_2,x_3) = s(x_3) - k(x_2) - s(x_3)\ k(x_2)[u(x_2) - u(x_3)]\ .$$

Suppose that there exist values of X_2, X_3, say x_2^*, x_3^*, such that

$$A(x_2^*,x_3^*) \neq 0\ ,$$

then it must be that

$$- u(x_2^*)\ u(x_3^*) + u(x_3^*)\ u(x_1,x_2^*) + u(x_2^*)\ u(x_3^*,x_4)$$

$$- u(x_1,x_2^*)\ u(x_3^*,x_4) \equiv 0 \quad \text{for all } x_1,x_4 \ . \tag{13}$$

By assumption we may choose a value of X_4, x_4^* such that $u(x_3^*,x_4^*) \neq u(x_3^*,x_4^0)$; hence from (13)

$$[u(x_1,x_2^*) - u(x_2^*)][u(x_3^*,x_4^*) - u(x_3^*,x_4^0)] = 0 \ ,$$

which implies that

$$u(x_1,x_2^*) = u(x_1^0,x_2^*)$$

for all x_1 -- a contradiction to assumption (ii).
Hence $A(x_2,x_3) \equiv 0$.

Thus

$$s(x_3) = k(x_2)/[1-k(x_2)\ u(x_2) + k(x_2)\ u(x_3)] \ ,$$

so that if $k(x_2^*) = 0$ for some x_2^*, then $s(x_3) \equiv 0$ (similarly $s(x_3^*) = 0$ implies $k(x_2) \equiv 0$); otherwise

$$s(x_3) = 1/[k(x_2)^{-1} - u(x_2) + u(x_3)] \ ,$$

implying that

$$k(x_2)^{-1} - u(x_2) = \text{constant} = \lambda, \text{ say,}$$

or $$k(x_2) = (\lambda + u(x_2))^{-1}, \tag{14}$$

and $$s(x_3) = (\lambda + u(x_3))^{-1} \ . \tag{15}$$

Substituting (12), (14) and (15) into (9) gives

$$u(x_1,x_2,x_3,x_4) =$$

$$[\lambda + u(x_1,x_2)][\lambda + u(x_2,x_3)][\lambda + u(x_3,x_4)]/[\lambda + u(x_2)][\lambda + u(x_3)] - \lambda \ . \quad (16)$$

If $k(x_2) \equiv s(x_3) \equiv 0$ then

$$u(x_1,x_2,x_3,x_4) = u(x_1,x_2) + u(x_2,x_3) + u(x_3,x_4) - u(x_2) - u(x_3) \ . \quad (17)$$

Equations (16) and (17) show that the Theorem is true when X_i is fixed at x_i^0 for $i \geq 5$. Now we proceed by induction, assume the theorem is true whenever X_i is fixed at x_i^0 for $i \geq n+1$ and consider the case when X_i is fixed at x_i^0 for $i \geq n+2$.

First of all, regarding $X_n' = X_n \times X_{n+1}$ as a single attribute, by induction we know that either

$$u(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n-2} u(x_i,x_{i+1}) + u(x_{n-1},x_n,x_{n+1}) - \sum_{i=2}^{n-1} u(x_i) \quad (18)$$

or

$$u(x_1,\ldots,x_{n+1}) = \frac{\prod_{i=1}^{n=2} (\lambda + u(x_i,x_{i+1})(\lambda + u(x_{n-1},x_n,x_{n+1}))}{\prod_{i=2}^{n-1} (\lambda + u(x_i))} - \lambda \ . \quad (19)$$

Similarly, regarding $X_1 \times X_2$ as a single attribute we know that either

$$u(x_1,\ldots,x_{n+1}) = u(x_1,x_2,x_3) + \sum_{i=3}^{n} u(x_i,x_{i+1}) - \sum_{i=3}^{n} u(x_i) \quad (20)$$

or

$$u(x_1,\ldots,x_{n+1}) = \frac{(\Theta + u(x_1,x_2,x_3)) \prod_{i=3}^{n} (\Theta + u(x_i,x_{i+1}))}{\prod_{i=3}^{n} (\Theta + u(x_i))} - \Theta \ . \quad (21)$$

If (18) and (20) are true simultaneously then simple comparison gives

$$u(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n} u(x_i,x_{i+1}) - \sum_{i=2}^{n} u(x_i)$$

as required.

If (19) and (20) are true simultaneously then fixing X_i at x_i^* except for i = 1, n+1 and equating gives

$$u(x_1) + u(x_{n+1}) = -\lambda + \frac{(\lambda + u(x_1))(\lambda + u(x_{n+1})}{\lambda} \quad ,$$

implying that

$$\frac{1}{\lambda}\, u(x_1)\; u(x_{n+1}) = 0 \quad ,$$

which if true would contradict assumption (ii). The same applies if (18) and (21) are true simultaneously.

If the remaining combination of (19) and (20) are true together then again fixing X_i at x_i^0 for all i except 1, n+1 and equating we have that

$$-\lambda + \frac{(\lambda + u(x_1))(\lambda + u(x_{n+1})}{\lambda} = -\Theta + \frac{(\Theta + u(x_1))(\Theta + u(x_{n+1})}{\Theta} \quad ,$$

implying that

$$\Theta u(x_1)\; u(x_{n+1}) = \lambda u(x_1)\; u(x_{n+1}) \quad .$$

This can be achieved without contradicting assumption (ii) only if $\Theta = \lambda$.

As the method of induction may be continued up to and including n+1 = T the theorem is proved. ||

Note that because utility functions are unique only up to positive linear transformations it is possible to scale $u(x_1,\ldots,x_T)$ so that $\lambda = \pm 1$. The transformation $u = \lambda u'$ if $\lambda > 0$ or $u = -\lambda u'$ if $\lambda < 0$ will serve to cancel the λ's.

The functional form of the utility function still requires the assessment of T-1 two-attribute utility functions

$$u_i(x_i,x_{i+1}) = u(x_1^0,x_2^0,\ldots,x_{i-1}^0,x_i,x_{i+1},x_{i+2}^0,\ldots,x_T^0) \quad ,$$

with the restriction that

$$u_i(x_i^0,x_{i+1}) = u_{i+1}(x_{i+1},x_{i+2}^0) \quad i = 1,\ldots,T-2 \quad . \tag{22}$$

This can be simplified still further if the decision maker's preferences are stationary (Koopmans [7]), for then for each i

$$u_i(x,y) = \alpha_i \; u^*(x,y)$$

for some constant α_i, where normally $0 \le \alpha_i \le 1$, and some function u^*. But because of (22)

$$\alpha_i \; u^*(x_i^0,x_{i+1}) = \alpha_{i+1} \; u^*(x_{i+1},x_{i+2}^0) \quad ,$$

so if $x_1^0 = x_2^0 =,\cdots,x_T^0$ then

$$\frac{\alpha_{i+1}}{\alpha_i} = \alpha, \text{ a constant.}$$

Corollary: The conditions of the theorem, plus an assumption of stationarity and $x_i^0 = x^0$ for all i, implies that

$$u(x_1,\ldots,x_T) = u^*(x_1,x_2) + \sum_{i=2}^{T-1} \alpha^{i-1}(u^*(x_1,x_{i+1}) - u^*(x_i,x^0))$$

or

$$u(x_1,\ldots,x_T) = \left[\prod_{i=2}^{T-1} (\lambda + \alpha^{i-1}u^*(x_i,x^0))\right]^{-1}\left[\prod_{i=1}^{T-1} (\lambda + \alpha^{i-1}u^*(x_i,x_{i+1}))\right] -\lambda$$

where α is constant and $u^*(x^0,y) = \alpha u^*(y,x^0)$.

The Application

Returning to the forest pest study which led to the foregoing theory, the decision maker felt that as long as he knew the levels of profit and employment in neighbouring years, his attitude to risk in any given year was independent of the profit/employment levels in other years. Hence regarding $X_i = P_i \times E_i$ as a single attribute the conditions of the theorem seemed to be appropriate. This meant that the four-dimensional functions

$$u_i(p_i, e_i, p_{i+1}, e_{i+1})$$

were required for all i. But also, he felt that given everything else was fixed, (P_i, E_i) pairs were mutually utility independent with P_{i-1} and P_{i+1}.

Thus with $X_1, \ldots, X_{i-2}$, $X_{i+1}, \ldots, X_T$ fixed his assumptions were

$$(P_i, E_i) \quad \text{mcui} \quad P_{i-1}$$

and

$$(P_{i-1}, E_{i-1}) \quad \text{mcui} \quad P_i \ .$$

But these are exactly the assumptions used in the first part of the proof of the theorem (associate P_{i-1} with X_1, E_{i-1} with X_2, E_i with X_3 and P_i with X_4); hence

$$u(p_{i-1}, e_{i-1}, p_i, e_i) = u(p_{i-1}, e_{i-1}) + u(p_i, e_i) + u(e_{i-1}, e_i) - u(e_{i-1}) - u(e_i)$$

or

$$u(p_{i-1}, e_{i-1}, p_i, e_i) = \frac{(\lambda_i + u(p_{i-1}, e_{i-1}))(\lambda_i + u(p_i, e_i))(\lambda_i + u(e_{i-1}, e_i))}{(\lambda_i + u(e_{i-1}))(\lambda_i + u(e_i))} - \lambda_i \ .$$

If we assume stationarity of preferences in addition, it may be shown that $\lambda_i = \lambda$ for all i, and the assessment of the utility function $u_s(\underline{p}, \underline{e})$ of (3) requires only two-attribute utility functions $u_A(p, e)$ and $u_B(e_1, e_2)$ together with a number of constants, giving a final functional form of

$$\frac{\prod_{i=1}^{T} (\lambda + \alpha^{i} u_A(p_i, e_i)) \prod_{i=1}^{T-1} (\lambda + \alpha^{i} u_B(e_i, e_{i+1}))}{\prod_{i=1}^{T} (\lambda + \alpha^{i} u_B(e_i, e^{0})) \prod_{i=2}^{T-1} (\lambda + \alpha^{i} u_B(e_i, e^{0}))}$$

where $u_A(p^0, e) = u_B(e, e^0)$, $u_B(e^0, e) = \alpha u_B(e, e^0)$ and λ and α are constants.

A description of the forest pest problem and the preliminary analysis of the decision maker's preferences are reported in Holling et al. [4] and Bell [2] respectively. A future paper will discuss the assessment of the utility function in more detail.

Summary

It has been shown that it is not necessary to require utility independence amongst all attributes in a time stream in order to have a manageable assessment task, and that explicit account can be taken of preferences which depend on the values assumed by attributes in other periods.

Acknowledgement

I would like to thank Ralph L. Keeney and Craig W. Kirkwood for their helpful comments and suggestions on this paper.

References

[1] Bell, D.E. "Evaluating Time Streams of Income." IIASA RR-73-7 Laxenburg, Austria, International Institute for Applied Systems Analysis, 1974. Also in Omega 2 (1974), 691-699.

[2] Bell, D.E. "Defining and Quantifying Goals for Measuring the State of an Ecosystem." IIASA RR-74-26. Laxenburg, Austria, International Institute for Applied Systems Analysis, 1974.

[3] Fishburn, P.C. "Markovian Dependence in Utility Theory with Whole Product Sets." Operations Research, 13 (1965), 238-257.

[4] Holling, C.S., et al. "A Case Study of Ecosystem Management." Status Report 74-2-EC. Laxenburg, Austria, International Institute for Applied Systems Analysis, 1974.

[5] Keeney, R.L. "Utility Functions for Multiattributed Consequences." Management Science, 18 (1972), 276-287.

[6] Keeney, R.L. and Raiffa, H. "Decision Analysis with Multiple Conflicting Objectives: Preferences and Value Tradeoffs." Manuscript to be published by John Wiley, New York.

[7] Koopmans, T.C. "Stationary Ordinal Utility and Impatience." Econometrica, 28 (1960), 297-309.

[8] Meyer, R.F. "On the Relationship among the Utility of Assets, the Utility of Consumption and Investment Strategy in an Uncertain but Time Invariant World." J. Lawrence (ed.), OR69: Proceedings of the Fifth International Conference on Operational Research, Tavistock Publications, London (1970), 627-48.

[9] Von Neumann, J. and Morgenstern, O. "Theory of Games and Economic Behaviour." Princeton University Press, Princeton, New Jersey (1953).