J. Korean Math. Soc. **40** (2003), No. 4, pp. 747–756

A VANISHING THEOREM FOR L^2 COHOMOLOGY ON COMPLETE MANIFOLDS

JEFFERY D. MCNEAL

ABSTRACT. We establish a vanishing theorem on the square-integrable cohomology associated to the Cauchy-Riemann complex on some complete Kaehler manifolds. The hypothesis needed for this result is a growth condition on a primitive of the Kaehler form.

1. Introduction

Let (X^n, ω) be a complete, Kähler manifold, with $\dim_{\mathbb{C}} X = n$. A basic question, pertaining both to the function theory and topology on X, is: when are there non-trivial harmonic forms on X, in the various bi-degrees (p, q) determined by the complex structure? When X is not compact, a growth condition on the harmonic forms at infinity must be imposed, in order that the answer to this question be useful. A natural growth condition is square-integrability; if $\Omega^{p,q}_{(2)}(X)$ denotes the L^2-forms of type (p, q) on X and $\mathcal{H}^{p,q}_{(2)}(X)$ the harmonic forms in $\Omega^{p,q}_{(2)}(X)$, one version of the basic question is: what is the structure of $\mathcal{H}^{p,q}_{(2)}(X)$, $0 \leq p, q \leq n$?

Recall that the Hodge theorem for a compact manifold Y says that each (real) cohomology class of Y is represented by a unique harmonic form. This marvelous theorem creates a dictionary between the topology of Y (its real cohomology) and the analysis, or geometry, of Y (the space of solutions to $\triangle u = 0$), and allows results on either class of objects to be transformed back and forth. The study of $\mathcal{H}^{p,q}_{(2)}(X)$, a question of the so-called L^2-cohomology of X, is rooted in the attempt to extend Hodge theory to non-compact manifolds. This extension is not yet complete,

Received March 17, 20003.
2000 Mathematics Subject Classification: 32Q15, 53C55.
Key words and phrases: Kaehler manifold, cohomology, vanishing theorems.
Research supported by a grant from the National Science Foundation.

but there are numerous partial results about the L^2-cohomology of non-compact manifolds.

In this note, we establish a vanishing result on $\mathcal{H}^{p,q}_{(2)}(X)$ when $p+q \neq n$, under the hypothesis that a primitive of ω does not grow too fast at infinity. This result extends a vanishing theorem from [16] in two ways: 1. we remove the assumption from [16] that the fundamental form ω is given by a global potential, and 2. we allow faster growth on the primitive of ω than was stated in [16]. An application of the vanishing results in [16] is given by Hunsicker in [13]. We also mention that the results in [16] themselves were extensions of vanishing theorems obtained by Gromov [11], and that somewhat similar extensions of [11] were obtained by Cao-Xavier [4] and Jost-Zuo [14]. The papers by Anderson [1], Atiyah-Patodi-Singer [2], Dodziuk [5], Donnelly-Fefferman [8], Donnelly [6]-[7], Zucker [22] and papers cited in their references, are recommended as sources for further, significant results on analytic aspects of L^2-cohomology.

There actually are two separate vanishing theorems in [16], only one of which is extended in this paper. The other result, Theorem 2.1 in [16], gave solutions to the first-order operators $\bar\partial$ and $\bar\partial^*$, with estimates in the ω-metric, in addition to the vanishing of $\mathcal{H}^{p,q}_{(2)}(X)$. This theorem, of course, required stronger control at infinity on a primitive of ω. In the last section of this paper, we examine some examples which illustrate the borderline between manifolds where we have vanishing of $\mathcal{H}^{p,q}_{(2)}(X)$ without estimates on the first-order operators $\bar\partial$ and $\bar\partial^*$, i.e. manifolds where only Theorem 2.6 in [16] (or, more generally, Theorem 1 below) hold, and cases covered by Theorem 2.1 in [16].

2. Background and statement of result

Throughout, (X,ω) denotes a connected, complete, Kähler manifold of complex dimension n. The fundamental form ω, which in local coordinates $(z_1,\dots,z_n)$ can be written as $\omega = i\sum_{k,l} g_{kl}dz_k\wedge d\bar z_l$, gives rise to a metric g, which has the local expression

$$g=\sum_{k,l} g_{kl}dz_k\otimes d\bar z_l,$$

if we view g as a complex inner product on $T^{1,0}(X)$. The space of all measurable (p,q)-forms on X will be denoted $\Omega^{p,q}(X)$. The inner product g may be functorially lifted to all the spaces $\Omega^{p,q}(X)$, and we let $\langle \cdot, \cdot \rangle$ denote this pointwise inner product.

The global inner product is defined

$$(u, v) = \int_X \langle u, v \rangle \, dV_\omega,$$

where $dV_\omega = c_n \overbrace{\omega \wedge \cdots \wedge \omega}^{n \text{ times}}$ is the volume form determined by ω. We also write $|u|^2 = \langle u, u \rangle$, $||u||^2 = \int_X |u|^2 \, dV_\omega$, and let

$$\Omega^{p,q}_{(2)}(X, \omega) = \{u \in \Omega^{p,q}(X) : ||u|| < \infty\}.$$

Let ϑ denote the formal adjoint of the Cauchy-Riemann operator $\bar{\partial}$ relative to the inner product above. The complex Laplacian, $\Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}$, is defined on Dom($\Box$), where Dom($\Box$) is domain of the unbounded operator $\Box$. The L^2 harmonic forms are denoted

$$\mathcal{H}^{p,q}_{(2)}(X) = \{\phi \in \Omega^{p,q}_{(2)}(X) : \Box\phi = 0\}.$$

Here $\Box\phi = 0$ is meant in the sense of distributions.

Recall that a function $E : X \longrightarrow \mathbb{R}$ is called an *exhaustion function* if, for any $k \in \mathbb{R}$,

$$X_k = \{p \in X : E(p) < k\} \subset X.$$

We shall only seriously consider C^1 exhaustion functions below.

Definition 1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be continuous and E be a C^1 exhaustion function. We say that a 1-form α on X *is $f(E)$-bounded*, if

$$|\alpha(p)| \leq f\left(E(p)\right), \qquad \text{for all } p \in X.$$

Note that the (regularized) distance function, ρ, associated to the metric g has the property that its differential $d\rho$ is $f(E)$-bounded for $f \equiv$ constant and for any exhaustion function E.

Definition 2. The Kähler manifold (X^n, ω) is *vanishingly exhaustible* if there exists a C^1 exhaustion function E on X, continuous, non-decreasing functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}^+$, and a C^1 1-form α on X such that

(i) $\bar{\partial}\alpha = \omega$,

(ii) α is $f(E)$-bounded, dE is $g(E)$-bounded,
(iii) the series

$$\sum_{k=N}^{\infty} \frac{1}{g(k)f(k)}$$

diverges.

Note that since $d\omega = 0$ (the Kähler condition), a solution to $\bar{\partial} u = \omega$ exists locally. However, for more topological purposes than we are concerned with here, Definition 2 should probably be formulated on the universal cover of X. That is, if $\tilde{X}$ is a simply-connected space which covers X, ω lifts to a Kähler form $\tilde{\omega}$ on $\tilde{X}$ and one can always find a global solution to $\bar{\partial}\tilde{u} = \tilde{\omega}$; thus it is not necessary to hypothesize (i) on $\tilde{X}$. (This was what Gromov did in [11] with his condition of Kähler hyperbolicity). Conditions (ii) and (iii) are not automatic and, taken together, are the growth hypotheses mentioned in the introduction.

The vanishing result that follows from these notions is

THEOREM 1. *Let (X^n, ω) be a complete, Kähler manifold which is vanishingly exhaustible. Then*

$$\mathcal{H}^{p,q}_{(2)}(X) = 0 \qquad \text{if } p+q \neq n.$$

Let us explicitly connect this to previous results. Theorem 2.6 of [16] follows from Theorem 1: there we assumed ω had a global potential, $\omega = i\partial\bar{\partial}\lambda$ for some smooth function λ, with certain estimates, and it is easy to check that these estimates imply ω satisfies the hypotheses of Theorem 1 with $f(x) = g(x) = \sqrt{A + Bx}$ and $E(p) = \lambda(p)$. Theorem 2 of [4] similarly follows from our Theorem 1, by taking $f(x) = c(1 + x)$, $g(x) = 1$, and $E(p) = \rho(p, p_0)$ where $\rho(p, p_0)$ denotes the Riemannian distance between p and a fixed base point $p_0 \in X$.

Note that the hypothesis (ii) in Theorem 1 simplifies when ω is given by a global potential which is also an exhaustion function on X, e.g. the Bergman metric for X a pseudoconvex domain in $\mathbb{C}^n$. To illustrate, suppose that $X \subset\subset \mathbb{C}^n$ and $\omega = i\partial\bar{\partial}\lambda$ for a function $\lambda \in C^2(X)$ which satisfies $\lambda(z) \longrightarrow \infty$ as $z \to bX$. Suppose that there is no $C < \infty$ such that

$$\left|\sum_{j=1}^{n} \frac{\partial\lambda}{\partial z_j}(p)\xi_j\right|^2 \leq C\lambda(p) \cdot \sum_{j,k=1}^{n} \frac{\partial^2\lambda}{\partial z_j \partial\bar{z}_k}(p)\xi_j\bar{\xi}_k, \qquad \xi \in \mathbb{C}^n,$$

(so Theorem 2.6 in [16] does not apply), but that there is some $K < \infty$ such that

$$|\sum_{j=1}^{n} \frac{\partial \lambda}{\partial z_j}(p)\xi_j|^2 \leq K\lambda(p)\log\lambda(p) \cdot \sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}(p)\xi_j\bar{\xi}_k, \qquad \xi \in \mathbb{C}^n.$$

Then Theorem 1 implies that $\mathcal{H}^{p,q}_{(2)}(X) = 0$ if $p+q \neq n$, by letting $f(x) = g(x) = \sqrt{Kx \cdot \log x}$.

3. Proof of Theorem 1

The proof of Theorem 1 follows very closely the method used to prove Theorem 2 in [4] and Theorem 2.6 in [16].

We first recall some basic facts from Kähler geometry. For a proof of the first 2 results, see [21] or [10]; for a proof of the third result, see [9].

LEMMA 3.1. *On a Kähler manifold (X^n, ω), let*

$$L(\beta) = \omega \wedge \beta$$

be the operator of multiplication by the Kähler form (the Lefschetz map). Then

$$[\Box, L] = 0.$$

Thus, if $\beta \in \mathcal{H}^{p,q}_{(2)}(X)$, then $L(\beta) \in \mathcal{H}^{p+1,q+1}_{(2)}(X)$.

LEMMA 3.2. *The operator*

$$L : \mathcal{H}^{p,q}_{(2)}(X) \longrightarrow \mathcal{H}^{p+1,q+1}_{(2)}(X)$$

is injective, if $p+q < n$.

LEMMA 3.3. *On a complete, Kähler manifold (X^n, ω), if $u \in \mathcal{H}^{p,q}_{(2)}(X)$, then u is in the domain of the operators $\bar{\partial}$ and $\bar{\partial}^*$, and $\bar{\partial}u = \vartheta u = 0$.*

Proof of Theorem 1. Let $\chi : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \chi \leq 1$, with

$$\chi(x) = \begin{cases} 1 \text{ if } x \geq 1 \\ 0 \text{ if } x \leq 0, \end{cases}$$

and define, for $k \in \mathbb{Z}^+$,

$$\chi_k(p) = \chi\left(k - E(p)\right).$$

Note that supp $\chi_k \subset X_k$ and $\chi_k \equiv 1$ on X_{k-1}.

Suppose that $p+q<n$ and let $h \in \mathcal{H}^{p,q}_{(2)}(X)$. By Lemma 3.1, $\omega \wedge h \in \mathcal{H}^{p+1,q+1}_{(2)}(X)$ and, so, Lemma 3.3 implies that $\omega \wedge h$ is ϑ-closed. Let $\mathbf{h} = \alpha \wedge h$. Since $\chi_k \cdot \mathbf{h}$ has compact support, an integration by parts gives

$$(3.1)\qquad \begin{aligned}\left(\omega \wedge h, \bar\partial(\chi_k \cdot \mathbf{h})\right) &= (\vartheta(\omega \wedge h), \chi_k \cdot \mathbf{h}) \\ &= 0.\end{aligned}$$

Again by Lemma 3.3, h is $\bar\partial$-closed. Thus

$$(3.2)\qquad \begin{aligned}\bar\partial\left(\chi_k \cdot \mathbf{h}\right) = \chi'\left(k-E\right) \cdot \bar\partial E \wedge \alpha \wedge h \\ + \chi_k \cdot \omega \wedge h.\end{aligned}$$

We now substitute (3.2) into (3.1) and consider the two terms coming from the right-hand side of (3.2) separately. For the first term, the fact that supp $\chi_k' \subset X_k \setminus X_{k-1}$ and the fact that ω is bounded in the $<,>$ inner product imply

(3.3)

$$\begin{aligned}\left|\left(\omega \wedge h, \chi_k' \cdot \bar\partial E \wedge \alpha \wedge h\right)\right| &\le \int_{X_k \setminus X_{k-1}} \left|\bar\partial E \wedge \alpha\right| |h|^2\, dV \\ &\le \int_{X_k \setminus X_{k-1}} (f(E)g(E))\, |h|^2\, dV \\ &\le (f(k)\cdot g(k)) \int_{X_k \setminus X_{k-1}} |h|^2\, dV,\end{aligned}$$

for the functions f, g in Definition 2. The second inequality follows from our hypotheses on E and α. The assumption that $h \in \mathcal{H}^{p,q}_{(2)}(X)$ implies that there exists a subsequence $\{k_l\}$ such that

$$(3.4)\qquad f(k_l)\cdot g(k_l) \int_{X_{k_l} \setminus X_{k_{l-1}}} |h|^2 \longrightarrow 0 \text{ as } l \to \infty.$$

Otherwise, for some $c>0$,

$$\begin{aligned}\int_X |h|^2 &= \sum_{k=1}^{\infty} \int_{X_k \setminus X_{k-1}} |h|^2 \\ &\ge c \sum_{k=1}^{\infty} \frac{1}{f(k)g(k)} \\ &= \infty.\end{aligned}$$

a contradiction.

So, for the sequence given by (3.4), it follows from (3.3) that

$$\lim_{l\to\infty}\left(\omega\wedge h, \chi'_{k_l}\cdot\bar\partial E\wedge\alpha\wedge h\right)=0. \tag{3.5}$$

For the term coming from the second term on the right-hand side of (3.2), the dominated convergence theorem implies

$$\lim_{k\to\infty}(\omega\wedge h, \chi_k\cdot\omega\wedge h)=||\omega\wedge h||^2. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.1), it follows that $\omega\wedge h=0$. Lemma 3.2 implies that $h=0$. Finally, Poincaré duality extends the argument just given to the case when $p+q>n$. This completes the proof. □

4. Strictly versus weakly Kähler convex

In this section we assume that $\omega=i\partial\bar\partial\lambda$, for some $\lambda\in C^2(X)$ with $\lambda\geq 1$ on X; many standard Kähler manifolds are of this type. Suppose that for all $p\in X$, there exists constants $A,B<\infty$ such that

$$i\partial\lambda(p)\wedge\bar\partial\lambda(p)\leq[A+B\,\lambda(p)]\,i\partial\bar\partial\lambda(p), \tag{4.1}$$

where the inequality is meant in the sense of currents. In [16], (X,ω) was then called *Kähler convex* and, if B could be chosen strictly less than 1 in (4.1), (X,ω) was called *strictly Kähler convex.* The distinction was made because if (X,ω) was Kähler convex than we could only conclude that $\mathcal{H}^{p,q}_{(2)}(X)=0$ if $p+q\neq n$ (Theorem 2.6 [16]) but, if (X,ω) was strictly Kähler convex, then we could also show there exist constants m,M depending only on universal constants and the constants A,B in (4.1), such that

$$m\,|p+q-n|\int_X\frac{1}{\lambda+M}|u|^2\leq||\bar\partial u||^2+||\bar\partial^* u||^2,\qquad u\in\Lambda_0^{p,q}(X). \tag{4.2}$$

(Theorem 2.1 [16]). Inequality (4.2) is highly desirable, as it gives information about the first-order $\bar\partial$-cohomology, in addition to the second-order $\Box$-cohomology; see, for example, Proposition 2.4 and Corollary 2.5 in [16].

There are a number of natural situations where (X, ω) just fails to be strictly Kähler convex, i.e. (4.1) holds with $B = 1$. A family of geometrically interesting examples, which interpolate between cases where (4.1) holds for $B < 1$ and where it does not, is the following. Let $h : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a holormorphic and non-constant. Let $V = \{z : h(z) = 0\}$, and consider $X = \mathbb{C}^n \setminus V$ with various metrics constructed from $|h|^2$. First, for $s > 0$, consider $\omega^{-s} = i\partial\bar{\partial}\lambda^{-s}$ with $\lambda^{-s} = (|h|^2)^{-s}$. Computing derivatives gives

$$\lambda^{-s}_{z_k} = -s\left(|h|^2\right)^{-s-1} h_{z_k}\bar{h}, \tag{4.3}$$

and

(4.4)
$$\begin{aligned}\lambda^{-s}_{\bar{z}_l z_k} &= -s\left(|h|^2\right)^{-s-1} h_{z_k}\bar{h}_{\bar{z}_l} + s(s+1)\left(|h|^2\right)^{-s-2}|h|^2 h_{z_k}\bar{h}_{\bar{z}_l} \\ &= s^2\left(|h|^2\right)^{-s-1} h_{z_k}\bar{h}_{\bar{z}_l}.\end{aligned}$$

Clearly, (4.3) and (4.4) show that (X, ω^{-s}) is Kähler convex, and they also show that the smallest B that satisfies (4.1) is $B = 1$. So, in these cases, while $\mathcal{H}^{p,q}_{(2)}(X, \omega^{-s}) = 0$ if $p + q \neq n$, we get no estimates on $\bar{\partial}$ in the metric ω^{-s}.

On the other hand, considering $\lambda^s = (|h|^2)^s$, s still positive and, to get a complete metric, $s < 1$, computations analogous to (4.3) and (4.4) show that

$$\begin{aligned}|\partial\lambda|^2 &= s^2\left(|h|^2\right)^{2s-1}|\partial h|^2 \\ i\partial\bar{\partial}\lambda &= s^2\left(|h|^2\right)^{s-1} i\partial h \wedge \bar{\partial}h.\end{aligned}$$

Since $2s-1 > s-1$, we have that (4.1) holds with an arbitrarily small B, or even $B = 0$; thus (4.2) is satisfied and we obtain good estimates on the Cauchy-Riemann operator in the norms determined by $\omega = i\partial\bar{\partial}(|h|^2)^s$. For more information about related weighted estimates for $\bar{\partial}$, we refer to the paper by Berndtsson [3].

Finally, for (regularized) Poincaré type metrics, $\omega = i\partial\bar{\partial}(-\log(|h|^2 + \epsilon))$, it is straightforward to compute as above that (4.1) is satisfied, but the smallest B is $B = 1$; we are therefore again in the situation of "vanishing without estimates".

References

[1] M. Anderson, *L^2-harmonic forms on complete Riemannian manifolds*, Lecture Notes in Math. **1339**, Springer-Verlag (1989), 1–19.

[2] M. Atiyah, V. Patodi, and I. Singer, *Spectral asymmetry and Riemannian geometry*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[3] B. Berndtsson, *Weighted estimates for the $\bar{\partial}$ equation*, Complex Analysis and Geometry, Proceedings of a a conference at Ohio State University, Walter De Gruyter **9** (2001), 43–58.

[4] J. Cao and F. Xavier, *Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature*, Math. Ann. **319** (2001), 483–491.

[5] J. Dodziuk, *L^2 harmonic forms on complete manifolds*, Seminar on differential geometry, Ann. of Math. Stud., Vol. **102** (1982).

[6] H. Donnelly, *L^2 cohomology of pseudoconvex domains with complete Kähler metric*, Mich. Math. J. **41** (1994), 433-442.

[7] ———, *L^2 cohomology of the Bergman metric for weakly pseudoconvex domains*, Ill. Math. J. **41** (1997), 151-160.

[8] H. Donnelly & C. Fefferman, *L^2 cohomology and index theorem for the Bergman metric*, Ann. Math. **118** (1983), 593–618.

[9] M. Gaffney, *A special Stoke's theorem for complete Riemannian manifolds*, Ann. Math **60** (1954), 140–145.

[10] P. Griffths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, 1978.

[11] M. Gromov, *Kähler hyperbolicity and L^2-Hodge theory*, J. Diff. Geo. **33** (1991), 263–292.

[12] G. Herbort, *Logarithmic growth of the Bergman kernel for weakly pseudoconvex domains in $\mathbb{C}^3$ of finite type*, Manuscripta Math. **45** (1983), 69–76.

[13] E. Hunsicker, *L^2 harmonic forms for a class of complete Kähler metrics*, Mich. Math. J. **50** (2002), 339–349.

[14] J. Jost and Z. Zuo, *Vanishing theorems for L^2-cohomology on infinite coverings pf compact Kähler manifolds and applications in algebraic geometry*, Comm. Anal. Geo. **8** (2000), 1–30.

[15] J. Lott, *The zero-in-the-spectrum question*, L'Enseign. Math. **42** (1996), 341–376.

[16] J. D. McNeal, *L^2 harmonic forms on some complete Kähler manifolds*, Math. Ann. **323** (2002), 319–349.

[17] A. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. Math. **132** (1990), 549–596.

[18] T. Ohsawa, *A remark on the completeness of the Bergman metric*, Proc. Japan Acad. Sci. **57** (1981), 238–240.

[19] ———, *On the L^2 cohomology groups of isolated singularities*, Adv. Stud. Pure. Math., Math. Soc. Japan **22** (1993), 247–263.

[20] ———, *Applications of L^2 estimates and some complex geometry*, Geometric Complex Analysis, Hayama. World Scientific (1996), 505–523.

[21] R. O. Wells, *Differential Analysis on Complex Manifolds*, Springer–Verlag, 1980.

[22] S. Zucker, *Hodge theory with degenerating coefficients. L^2 cohomology in the Poincaré metric*, Ann. Math. **109** (1979), 415–476.

Department of Mathematics
Ohio State University
Columbus, Ohio, 43210
E-mail: mcneal@math.ohio-state.edu