Commun. Korean Math. Soc. **21** (2006), No. 4, pp. 665–673

A VARIANT OF THE GENERALIZED VECTOR VARIATIONAL INEQUALITY WITH OPERATOR SOLUTIONS

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ABSTRACT. In a recent paper, Domokos and Kolumbán [2] gave an interesting interpretation of variational inequalities (VI) and vector variational inequalities (VVI) in Banach space settings in terms of variational inequalities with operator solutions (in short, OVVI). Inspired by their work, in a former paper [4], we proposed the scheme of generalized vector variational inequality with operator solutions (in short, GOVVI) which extends (OVVI) into a multivalued case. In this note, we further develop the previous work [4]. A more general pseudomonotone operator is treated. We present a result on the existence of solutions of (GVVI) under the weak pseudomonotonicity introduced in Yu and Yao [8] within the framework of (GOVVI) by exploiting some techniques on (GOVVI) or (GVVI) in [4].

1. Introduction

In a recent paper, Domokos and Kolumbán [2] gave an interesting interpretation of variational inequalities (VI) and vector variational inequalities (VVI) in Banach space settings in terms of variational inequalities with operator solutions (in short, OVVI). They designed (OVVI) to provide a unified approach to several kinds of (VI) and (VVI) problems in Banach spaces, and successfully described those problems in a wider context of (OVVI). This (OVVI) is suitable for the general trend in the study of (VI) and (VVI) for abstract spaces, that is, to extend and unify earlier results by using a more general scheme. So the fruitfulness

Received July 24, 2006.

2000 Mathematics Subject Classification: 47J20, 49J40.

Key words and phrases: vector variational inequality, weakly C-pseudemonotone operator, generalized hemicontinuity, Fan-Browder fixed point theorem.

This work was supported by the research grant of the Chungbuk National University in 2005.

of (OVVI) can be served as a good motivation for further developments. Actually, motivated by the work of Domokos and Kolumbán [2], in a former paper [4], we proposed generalized vector variational inequality with operator solutions (in short, GOVVI) which extends (OVVI) into a multi-valued case.

In this paper, we add another development for (GOVVI) to the previous work [4]. In [4], only the standard pseudomonotonicity of operator was dealt with. However, it is necessary that a more general pseudomonotone operator should be treated. This direction to weaken the monotonicity of given operators is natural and reasonable. In fact, we present a result on the existence of solutions of (GVVI) under the weak pseudomonotonicity introduced in Yu and Yao [8] within the framework of (GOVVI). The notion of 'weak pseudomonotonicity' was also used in Lee and Kum [5] as well as in Konnov and Yao [3] . For the simplicity of argument, we restrict our concern to the basic case, where the domain X is compact and the underlying space Y is a normed space. This compact case can be a prototype for noncompact cases. To achieve our goal, we exploit some techniques on (GOVVI) or (GVVI) appeared in [4].

2. Preliminaries

Let E, F be Hausdorff t.v.s., and let X be a nonempty convex subset of E. Let $C_1 : X \rightrightarrows F$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in F with $\operatorname{int} C_1(x) \neq \emptyset$ and $C_1(x) \neq F$. Let $\mathcal{L}(E,F)$ be the space of all continuous linear operators from E to F and $T_1 : X \rightrightarrows \mathcal{L}(E,F)$ a multifunction. From now on, unless otherwise specified, we work under the following settings:

Let X' be a nonempty convex subset of $\mathcal{L}(E,F)$ and $T : X' \rightrightarrows E$ be a multifunction. Let $C : X' \rightrightarrows F$ be a multifunction such that for each $f \in X'$, $C(f)$ is a convex cone in F with $0 \notin C(f)$. Then the generalized variational inequalities with operator solutions (GOVVI) is defined as follows:

(GOVVI)

Find $f_0 \in X'$ such that $\forall f \in X', \exists x \in T(f_0)$ with $\langle f - f_0, x\rangle \notin C(f_0)$.

When T is single-valued, (GOVVI) reduces to (OVVI) due to Domokos and Kolumbán [2]. As pointed out in [2], the notation (GOVVI) is motivated by the fact that the solutions are sought in the space of continuous linear operators.

Consider the multifunction $T_1 : X \rightrightarrows \mathcal{L}(E,F)$. Then T_1 is said to be

(1) *weakly C_1-pseudomonotone* if for any $x, y \in X$ and for any $s \in T_1(x)$, we have

$$\langle s, y-x\rangle \notin -\mathrm{int}C_1(x) \ \text{ implies } \ \langle t, y-x\rangle \notin -\mathrm{int}C_1(x) \text{ for some } t \in T_1(y);$$

and
(2) *C_1-pseudomonotone* if for any $x, y \in X$ and for any $s \in T_1(x)$, we have

$$\langle s, y-x\rangle \notin -\mathrm{int}C_1(x) \ \text{ implies } \ \langle t, y-x\rangle \notin -\mathrm{int}C_1(x) \ \text{ for all } \ t \in T_1(y);$$

and
(3) *generalized hemicontinuous* if for any $x, y \in X$, the multifunction

$$\alpha \mapsto \langle T_1(x+\alpha(y-x)), y-x\rangle, \quad \forall \alpha \in [0,1]$$

is upper semicontinuous at 0^+, where

$$\langle T_1(x+\alpha(y-x)), y-x\rangle = \{\langle s, y-x\rangle \mid s \in T_1(x+\alpha(y-x))\}.$$

It is obvious that the weak C_1-pseudomonotonicity implies C_1-pseudomonotonicity, but not vice versa. In regard to monotonicity and continuity of T, two analogous definitions to those of T_1 in the above are necessary; $T : X' \rightrightarrows E$ is said to be
(1)$'$ *weakly C-pseudomonotone* if for any $f, g \in X'$ and for any $s \in T(f)$, we have

$$\langle g-f, s\rangle \notin C(f) \ \text{ implies } \ \langle g-f, t\rangle \notin C(f) \ \text{ for some } \ t \in T(g); \text{ and}$$

(2)$'$ *C-pseudomonotone* if for any $f, g \in X'$ and for any $s \in T(f)$, we have

$$\langle g-f, s\rangle \notin C(f) \ \text{ implies } \ \langle g-f, t\rangle \notin C(f) \ \text{ for all } \ t \in T(g); \text{ and}$$

(3)$'$ *generalized hemicontinuous* if for any $f, g \in X'$, the multifunction

$$\alpha \mapsto \langle g-f, T(f+\alpha(g-f))\rangle, \quad \forall \alpha \in [0,1]$$

is upper semicontinuous at 0^+, where

$$\langle g-f, T(f+\alpha(g-f))\rangle = \{\langle g-f, s\rangle \mid s \in T(f+\alpha(g-f))\}.$$

Let us recall two topologies on $\mathcal{L}(E,F)$: *The topology of pointwise convergence* and *the topology of bounded convergence.* The former is the topology generated by the 0-neighborhood base $\{M(S,V) \mid S$ is a finite subset of $E,\ V \in \mathcal{B}\}$, where $M(S,V) = \{f \in \mathcal{L}(E,F) \mid f(S) \subset V\}$ and $\mathcal{B}$ is a 0-neighborhood base in F. The latter is the one which has $\{M(S,V) \mid S$ is a bounded subset of $E,\ V \in \mathcal{B}\}$ as a 0-neighborhood base. The following Fan-Browder fixed point theorem [1, Theorem 1] which is a particular case of Park [6, Theorem 5], is a basic machinery of deriving main results of this paper.

LEMMA 2.1. *Let X be a nonempty compact convex subset of a real (not necessarily) Hausdorff topological vector space E. Let $A,\ B : X \rightrightarrows X$ be two multifunctions. Suppose that*

(i) *for each $x \in X,\ Ax \subset Bx$;*
(ii) *for each $x \in X,\ Bx$ is convex;*
(iii) *for each $x \in X,\ Ax$ is nonempty ;*
(iv) *for each $y \in X,\ A^{-1}y$ is open in X.*

Then B has a fixed point x_0; that is, $x_0 \in Bx_0$.

3. Main result

We begin with the following lemma. For the sake of completeness, we provide a detailed proof.

LEMMA 3.1. *Let $T : X' \rightrightarrows E$ be a weakly C-pseudomonotone and generalized hemicontinuous multifunction with $T(f) \neq \emptyset$ for all $f \in X'$. Let $W : X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$, where $\mathcal{L}(E,F)$ is endowed with either the topology of pointwise convergence or the topology of bounded convergence. Then the following two problems are equivalent:*

(i) *Find $f \in X'$ such that $\forall g \in X',\ \exists x \in T(f)$ with $\langle g-f,x\rangle \notin C(f)$.*

(ii) *Find $f \in X'$ such that $\forall g \in X',\ \exists x \in T(g)$ with $\langle g-f,x\rangle \notin C(f)$.*

PROOF. (i) $\Rightarrow$ (ii) This is immediate from the weak C-pseudomonotonicity of T.

(ii) $\Rightarrow$ (i) Let $f \in X'$ be a solution of (ii). Suppose to the contrary that f is not a solution of (i). Then there exists $g_0 \in X'$ such that

$$\forall x \in T(f),\ \langle g_0 - f, x\rangle \in C(f). \tag{3.1}$$

Since f is a solution of (ii), we have, for each $t \in (0,1)$,

$$\langle tg_0 + (1-t)f - f, x_t \rangle \notin C(f) \quad \text{for some} \quad x_t \in T(f + t(g_0 - f)).$$

Hence

$$\langle g_0 - f, x_t \rangle \notin C(f) \quad \text{for some} \quad x_t \in T(f + t(g_0 - f)). \tag{3.2}$$

As T is generalized hemicontinuous, the multifunction $H : [0,1] \to 2^F$ defined by $H(t) = \langle g_0 - f, T(f + t(g_0 - f)) \rangle$ is upper semicontinuous at 0^+. It follows from (3.1) that

$$H(0) = \langle g_0 - f, T(f) \rangle \subset C(f).$$

Observe that the closedness of $Gr(W)$ implies that of $W(f)$ for every $f \in X'$. Thus $C(f)$ is open in F for every $f \in X'$. Hence there exists $\bar{t} \in (0,1)$ such that

$$H(t) = \langle g_0 - f, T(f + t(g_0 - f)) \rangle \subset C(f) \text{ for all } t \in (0, \bar{t}),$$

which contradicts (3.2). This completes the proof. □

In what follows, the sets X and X' are always assumed to be compact.

THEOREM 3.1. *Let X' be a nonempty compact convex subset of $\mathcal{L}(E,F)$ endowed with the topology of bounded convergence. Let $T : X' \rightrightarrows E$ be a weakly C-pseudomonotone and generalized hemicontinuous multifunction such that $T(f)$ is nonempty and compact for all $f \in X'$. Let $W : X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$. Then (GOVVI) is solvable.*

PROOF. First note that $\mathcal{L}(E,F)$ equipped with the topology of bounded convergence is a locally convex space. We define two multifunctions $A,\ B : X' \rightrightarrows X'$ to be

$$A(f) := \{g \in X' \mid \forall x \in T(g),\ \langle g - f, x \rangle \in C(f)\},$$
$$B(f) := \{g \in X' \mid \forall x \in T(f),\ \langle g - f, x \rangle \in C(f)\}.$$

The proof is organized in the following parts.

(i) Since T is weakly C-pseudomonotone, we have $A(f) \subset B(f)$ for all $f \in X'$.

(ii) For each $f \in X'$, $B(f)$ is convex. Indeed, let g_1 and g_2 be in $B(f)$. Fix $t \in [0,1]$ and $x \in T(f)$. Then we have

$$\langle tg_1 + (1-t)g_2 - f, x\rangle = t\langle g_1 - f, x\rangle + (1-t)\langle g_2 - f, x\rangle \in C(f),$$

which implies that $tg_1 + (1-t)g_2 \in B(f)$. Hence $B(f)$ is convex.

(iii) Clearly B has no fixed point because $0 \notin C(f)$ for all $f \in X'$.

(iv) For each $g \in X'$, $A^{-1}(g)$ is open in X'. In fact, let $\{f_\lambda\}$ be a net in $(A^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin A(f_\lambda)$ and hence for some $x_\lambda \in T(g)$,

$$\langle g - f_\lambda, x_\lambda\rangle \notin C(f_\lambda).$$

Thus $\langle g - f_\lambda, x_\lambda\rangle \in W(f_\lambda)$. As $T(g)$ is compact, we may assume without loss of generality that $x_\lambda \to x$ for some $x \in T(g)$. Since $\mathcal{L}(E,F)$ is endowed with the topology of bounded convergence and $T(g)$ is compact, $\langle g - f_\lambda, x_\lambda\rangle \to \langle g - f, x\rangle$. By virtue of the closedness of $Gr(W)$, we have $(f, \langle g - f, x\rangle) \in Gr(W)$, that is, $\langle g - f, x\rangle \notin C(f)$ for the particular $x \in T(g)$. Hence $g \notin A(f)$, so $f \in (A^{-1}(g))^c$. This shows that $(A^{-1}(g))^c$ is closed, therefore $A^{-1}(g)$ is open in X'.

(v) From (i)-(iv), we see, by Lemma 2.1, there must be an $f_0 \in K'$ such that $A(f_0) = \emptyset$, namely,

$$\forall g \in X',\ \exists x \in T(g) \text{ such that } \langle g - f_0, x\rangle \notin C(f_0).$$

It follows from Lemma 3.1 that f_0 is a solution of (GOVVI). This completes the proof. □

As an application of Theorem 3.1, we prove the following generalized VVI in a normed space.

THEOREM 3.2. *Let Y and Z be two normed spaces. Let X be a nonempty compact convex subset of Y and $C_1 : X \rightrightarrows Z$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in Z with $intC_1(x) \neq \emptyset$ and $C_1(x) \neq Z$. Let $T_1 : X \rightrightarrows \mathcal{L}(Y,Z)$ be a weakly C_1-pseudomonotone and generalized hemicontinuous multifunction with nonempty compact values, where $\mathcal{L}(Y,Z)$ is the normed space of the continuous linear operators between Y and Z with the usual norm. Let $W_1 : X \rightrightarrows Z$ be defined by $W_1(x) = Z \setminus -intC_1(x)$ such that the graph $Gr(W_1)$ of W_1 is closed in $X \times Z$. Then there exists $x_0 \in X$ such that*

$$\forall x \in X,\ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0\rangle \notin -intC_1(x_0).$$

PROOF. We consider $E = \mathcal{L}(Y, Z)$ as the normed space of the continuous linear operators between Y and Z with the usual norm, and $F = (Z, \|\cdot\|)$. Define a mapping $\phi : Y \to \mathcal{L}(E, F)$ by $\phi(x) = f_x$, where $f_x(l) = \langle l, x\rangle$ for all $l \in E$.

Claim 1: ϕ is well-defined, linear and injective.

Indeed, assume that $l_i \to l$ in E. This implies that $\forall x \in Y$, $\langle l_i, x\rangle \to \langle l, x\rangle$ in $F = Z$. Thus $f_x(l_i) \to f_x(l)$ in F, so $f_x \in \mathcal{L}(E, F)$. The linearity of ϕ is obvious. To show the injectivity of ϕ, it suffices to check that for each nonzero $x \in Y$, there exists an $l \in E$ such that $\langle l, x\rangle \neq 0$. By the separation theorem, we can find a $g \in Y^*$ with $g(x) = 1$. Define a linear operator $l : Y \to Z$ by

$$\langle l, y\rangle = g(y)z_0 \text{ for some } z_0 \neq 0 \text{ in } Z.$$

Clearly $l \in \mathcal{L}(Y, Z)$ and $\langle l, x\rangle = g(x)z_0 = z_0 \neq 0$.

Claim 2. ϕ is an isometry from X onto $\phi(X) = X'$.

In fact, $\forall l \in L(Y, Z)$, $\|f_x(l)\| = \|\langle l, x\rangle\| \leq \|l\|\|x\|$, hence $\|f_x\| \leq \|x\|$. For the converse inequality, it suffices to show that for a nonzero $x \in Y$, there is an $l \in L(Y, Z)$ such that $\|f_x(l)\| = \|\langle l, x\rangle\| \geq \|l\|\|x\|$. By the Hahn-Banach theorem (see Rudin [7, 5.20 Theorem]), we have an $h \in Y^*$ such that $h(x) = \|x\|$ and $\|h\| = 1$. Fix a $z_0 \in Z$ with $\|z_0\| = 1$. Define a continuous linear mapping $J : \mathbb{R} \to Z$ by $J(\alpha) = \alpha z_0$, $\forall \alpha \in \mathbf{R}$. Clearly $\|J\| = 1$. Consider the continuous linear operator $l = J \circ h$. Then $l \in \mathcal{L}(Y, Z)$ and $\|l\| \leq \|J\|\|h\| = 1$. On the other hand, we have

$$\|\langle l, x\rangle\| = \|J(h(x))\| = \|x\|\|z_0\| = \|x\|.$$

This implies $\|l\| \geq 1$, hence $\|l\| = 1$. Therefore $\|f_x(l)\| = \|\langle l, x\rangle\| = \|x\| \geq \|l\|\|x\|$, as desired.

Now we define $T : X' \rightrightarrows E$, $C : X' \rightrightarrows F$ and $W : X' \rightrightarrows F$ as follows:

$$T(f_x) = T_1(x),\ C(f_x) = -\text{int}C_1(x),\ W(f_x) = W_1(x),$$

where $\text{int}C_1(x)$ is the interior of $C_1(x)$ in the normed space Z. Then $0 \notin C(f_x)$ because $\text{int}C_1(x)$ is a proper convex cone of Z. The proof is organized in the following parts.

(i) The weak C_1-pseudomonotonicity of T_1 implies the weak C-pseudomonotonicity of T. In fact, for any $f_x,\ f_y \in X'$ and $s \in T(f_x) = T_1(x)$,

$$
\begin{aligned}
&\langle f_y - f_x, s\rangle \notin C(f_x) \\
\Rightarrow\ &\langle s, y - x\rangle \notin -\mathrm{int}C_1(x) \\
\Rightarrow\ &\langle t, y - x\rangle \notin -\mathrm{int}C_1(x) \quad \text{for some } t \in T_1(y) = T(f_y) \\
\Rightarrow\ &\langle f_y - f_x, t\rangle \notin C(f_x) \quad \text{for some } t \in T(f_y).
\end{aligned}
$$

(ii) The generalized hemicontinuity of T_1 amounts to that of T. Actually, for any $f_x,\ f_y \in X'$ and $\alpha \in [0,1]$,

$$
\begin{aligned}
\alpha \mapsto\ &\langle f_y - f_x, T(f_x + \alpha(f_y - f_x))\rangle \\
=\ &\langle T_1(x + \alpha(y - x)), y - x\rangle
\end{aligned}
$$

is upper semicontinuous at 0^+.

(iii) By the hypothesis, $T(f_x) = T_1(x)$ is nonempty and compact.

(iv) The graph $Gr(W)$ of W is closed in $X' \times F$. Indeed, let $\{f_{x_i}\}$ be a sequence in X' convergent to $f_x \in X'$ with respect to the usual norm on $\mathcal{L}(E,F)$. Let $w_i \in W(f_{x_i}) = W_1(x_i)$ such that $w_i \to w$ in F. Since ϕ is a homeomorphism, $\phi^{-1}(f_{x_i}) = x_i \to x = \phi^{-1}(f_x)$. Because the graph $Gr(W_1)$ of W_1 is closed in $X \times Z$, we have $w \in W_1(x) = W(f_x)$. This implies that $Gr(W)$ is closed in $X' \times F$.

It follows directly from Theorem 3.1 that there exists $f_{x_0} \in X'$ such that for each $f_x \in X'$, there is $t \in T(f_{x_0})$ with $\langle f_x - f_{x_0}, t\rangle \notin C(f_{x_0})$. Therefore, there exists $x_0 \in X$ such that

$$\forall x \in X,\ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0\rangle \notin -\mathrm{int}C_1(x_0).$$

This completes the proof. □

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