Bull. Aust. Math. Soc. **97** (2018), 367–373
doi:10.1017/S0004972717001411

A VARIATION ON THE THEME OF NICOMACHUS

FLORIAN LUCA, GEREMÍAS POLANCO and WADIM ZUDILIN✉

(Received 28 September 2017; accepted 14 December 2017; first published online 28 March 2018)

Abstract

In this paper, we prove some conjectures of K. Stolarsky concerning the first and third moments of the Beatty sequences with the golden section and its square.

2010 *Mathematics subject classification*: primary 11G20; secondary 11T23, 11Y16.

Keywords and phrases: elliptic curves, generators, finite fields.

1. Introduction

Nicomachus' theorem asserts that the sum of the first m cubes is the square of the mth triangular number,

$$1^3 + 2^3 + \cdots + m^3 = (1 + 2 + \cdots + m)^2. \tag{1.1}$$

(See [2].) With the notation

$$Q(\alpha, m) := \frac{\sum_{n=1}^{m} \lfloor \alpha n \rfloor^3}{(\sum_{n=1}^{m} \lfloor \alpha n \rfloor)^2}, \tag{1.2}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, it implies that

$$\lim_{m\to\infty} Q(\alpha, m) = \alpha. \tag{1.3}$$

Here, $\lfloor x \rfloor$ is the integer part of the real number x. The limit in (1.3) follows from $\lfloor \alpha n \rfloor = \alpha n + O(1)$ and Nicomachus' theorem (1.1).

Recall that the Fibonacci and Lucas sequences, $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$, are given by $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ and the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad L_{n+2} = L_{n+1} + L_n$$

The first author was supported in part by NRF (South Africa) Grant CPRR160325161141 and an A-rated researcher award, and by CGA (Czech Republic) Grant 17-02804S. The second author was supported in part by the Institute of Mathematics of Universidad Autonoma de Santo Domingo, Grant FONDOCyT 2015-1D2-186, Ministerio de Educación Superior Ciencia y Tecnología (Dominican Republic).

for $n \geq 0$. In a personal communication to the second author, K. Stolarsky observed that the limit relation (1.3) can be 'quantified' for $\alpha = \phi$ and ϕ^2, where $\phi := \frac{1}{2}(1 + \sqrt{5})$ is the golden mean, and a specific choice of m along the Fibonacci sequence. The corresponding result is Theorem 2.1 below. We complement it by a general analysis of moments of the Beatty sequences and give a solution to a related arithmetic question in Theorem 2.5.

2. Principal results

THEOREM 2.1. *For $k \geq 1$ an integer, define $m_k := F_k - 1$. Then*

$$Q(\phi^2, m_{2k}) - Q(\phi, m_{2k}) = \begin{cases} 1 - \dfrac{1}{(F_{k+1})^2 L_{k+2} L_{k-1}} & \text{if } k \text{ is even,} \\ 1 - \dfrac{1}{(L_{k+1})^2 F_{k+2} F_{k-1}} & \text{if } k \text{ is odd} \end{cases}$$

and

$$Q(\phi^2, m_{2k-1}) - Q(\phi, m_{2k-1}) = \begin{cases} 1 - \dfrac{F_{k-2}}{F_{k+1}(F_k)^2 (L_{k-1})^2} & \text{if } k \text{ is even,} \\ 1 - \dfrac{L_{k-2}}{L_{k+1}(L_k)^2 (F_{k-1})^2} & \text{if } k \text{ is odd.} \end{cases}$$

The theorem motivates our interest in the numerators and denominators of $Q(\phi, F_k - 1)$ and $Q(\phi^2, F_k - 1)$, which can be thought of as expressions of the form

$$A(k, s) := \sum_{n=1}^{F_k - 1} \lfloor \phi n \rfloor^s \quad \text{and} \quad A'(k, s) := \sum_{n=1}^{F_k - 1} \lfloor \phi^2 n \rfloor^s$$

for $k = 1, 2, \ldots$ and $s = 1, 3$. More generally, our analysis in Section 3 covers the sums

$$A(k, s, j) := \sum_{n=1}^{F_k - 1} n^j \lfloor \phi n \rfloor^s \quad \text{where } k = 1, 2, \ldots \text{ and } s, j = 0, 1, 2, \ldots. \tag{2.1}$$

Namely, we find a recurrence relation for $A(k, j, s)$ and deduce recursions from it for $A(k, s) = A(k, s, 0)$ and $A'(k, s)$. The strategy leads to the following expressions for the numerators and denominators in Theorem 2.1, which are given in Lemmas 2.2–2.4.

LEMMA 2.2. *Let $k \geq 1$ be an integer. Then*

$$\begin{aligned} A(k, 1) &= \tfrac{1}{2}(F_{k+1} - 1)(F_k - 1), \\ A'(k, 1) &= \tfrac{1}{2}(F_{k+2} - 1)(F_k - 1). \end{aligned} \tag{2.2}$$

LEMMA 2.3. *Let $k \geq 1$ be an integer. Then*

$$\begin{aligned} A(2k, 3) &= \tfrac{1}{4}(F_{2k-1} - 1)(F_{2k+1} - 1)^2 (F_{2k+2} - 1), \\ A(2k - 1, 3) &= \tfrac{1}{4}(F_{2k-1} - 1)(F_{2k} - 1) \times \tfrac{1}{5}(L_{4k} - 3L_{2k+1} - L_{2k} + 3). \end{aligned}$$

LEMMA 2.4. *Let $k \geq 1$ be an integer. Then*

$$A'(2k,3) = \tfrac{1}{4}(F_{2k}-1)(F_{2k+2}-1) \times \tfrac{1}{5}(L_{4k+4} - 5L_{2k+3} + 13),$$
$$A'(2k-1,3) = \tfrac{1}{4}(F_{2k-1}-1)(F_{2k+1}-1) \times \tfrac{1}{5}(L_{4k+2} - 5L_{2k+2} + 7).$$

Finally, we present an arithmetic formula inspired by Stolarsky's original question.

THEOREM 2.5. *For $k \geq 1$,*

$$\mathrm{LCM}(A(2k,1), A'(2k,1)) = \begin{cases} \frac{1}{2}F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1} & \text{if } 2 \mid k, \\ \frac{1}{2}F_{k+2}F_{k+1}F_{k-1}L_{k+1}L_k & \text{if } 2 \nmid k. \end{cases}$$

REMARK 2.6. Lemmas 2.2–2.4 indicate that the expression

$$Q(\phi^2, F_k - 1) - Q(\phi, F_k - 1)$$

is expressible as a fraction whose numerator and denominator are polynomials in Fibonacci and Lucas numbers with indices depending linearly on k according to the parity of k, yet the statement of Theorem 2.1 presents formulas for these quantities according to the congruence class of k modulo 4 rather than modulo 2. The discrepancy is related to different factorisations of the factors $F_n - 1$ that occur in the formulas for $A(k, j)$ and $A'(k, j)$ for $j \in \{1, 3\}$, since each of the factors $F_n - 1$ happens to be a product of a Fibonacci and a Lucas number according to the congruence class of n modulo 4 (see formulas (6.1)).

3. Recurrence relations for auxiliary sums

Here, we show how to compute the integer-part sums (2.1). This clearly covers the cases $A(k, s) = A(k, s, 0)$. On using

$$\phi^2 = 1 + \phi,$$

which upon multiplication by the integer n and taking integer parts becomes

$$\lfloor \phi^2 n \rfloor = n + \lfloor \phi n \rfloor,$$

one also gets the explicit formulas

$$A'(k, s) = \sum_{i=0}^{s} \binom{s}{i} A(k, s-i, i).$$

Using the Binet formula

$$F_k = \frac{\phi^k - (-\phi^{-1})^k}{\sqrt{5}} \quad \text{for all } k \geq 0,$$

one easily proves that

$$\lfloor \phi F_k \rfloor = F_{k+1} - \epsilon_k \quad \text{where } \epsilon_k = \frac{1 + (-1)^k}{2}$$

and

$$\lfloor\phi(F_k+n)\rfloor = F_{k+1} + \lfloor\phi n\rfloor \quad \text{for } 1 \le n \le F_{k-1}-1$$

(see, for example, [1]). Thus,

$$\begin{aligned}
A(k+1,s,j) &= \sum_{n=1}^{F_k-1} n^j\lfloor\phi n\rfloor^s + F_k^j\lfloor\phi F_k\rfloor^s + \sum_{n=F_k+1}^{F_{k+1}-1} n^j\lfloor\phi n\rfloor^s \\
&= A(k,s,j) + F_k^j(F_{k+1}-\epsilon_k)^s + \sum_{n=1}^{F_{k+1}-F_k-1} (F_k+n)^j\lfloor\phi(F_k+n)\rfloor^s \\
&= A(k,s,j) + F_k^j(F_{k+1}-\epsilon_k)^s + \sum_{n=1}^{F_{k-1}-1} (F_k+n)^j(F_{k+1}+\lfloor\phi n\rfloor)^s \\
&= A(k,s,j) + F_k^j\sum_{i=0}^{s}\binom{s}{i}F_{k+1}^i(-\epsilon_k)^{s-i} \\
&\quad + \sum_{n=1}^{F_{k-1}-1}\sum_{\ell=0}^{j}\binom{j}{\ell}F_k^\ell n^{j-\ell}\sum_{i=0}^{s}\binom{s}{i}F_{k+1}^i\lfloor\phi n\rfloor^{s-i} \\
&= A(k,s,j) + \sum_{i=0}^{s}\binom{s}{i}(-\epsilon_k)^{s-i}F_k^jF_{k+1}^i \\
&\quad + \sum_{\ell=0}^{j}\sum_{i=0}^{s}\binom{j}{\ell}\binom{s}{i}F_k^\ell F_{k+1}^i A(k-1,s-i,j-\ell).
\end{aligned}$$

The above reduction, the identity $A(k,0,0) = F_k - 1$ and induction on $k+j+s$ imply that

$$A(k,s,j) \in \operatorname{span}\{(\phi^i)^k,\ (-\phi^i)^k : |i| \le j+s+1\};$$

in particular, for a fixed choice of s and j, the sequence $\{A(k,s,j)\}_{k\ge1}$ is linearly recurrent of order at most $4(s+j)+6$. In the following section, we will use this observation about linear recurrency together with the following facts.

- If $\boldsymbol{u} = \{u_n\}_{n\ge0}$ is a linearly recurrent sequence whose roots are all simple in some set U, then, for fixed integers p and q, the sequence $\{u_{pn+q}\}_{n\ge0}$ is linearly recurrent with simple roots in $\{\alpha^p : \alpha \in U\}$.
- If $\boldsymbol{u} = \{u_n\}_{n\ge0}$ and $\boldsymbol{v} = \{v_n\}_{n\ge0}$ are linearly recurrent and their roots are all simple in some sets U and V, respectively, then $\boldsymbol{uv} = \{u_nv_n\}_{n\ge0}$ is linearly recurrent and its roots are all simple in $UV = \{\alpha\beta : \alpha \in U,\ \beta \in V\}$.

In this context, the roots of a linearly recurrent sequence are defined as the zeros of its characteristic polynomial, counted with their multiplicities. It then follows that, for a fixed s, each of the sequences $\{A(k,s)\}_{k\ge1}$ and $\{A'(k,s)\}_{k\ge1}$ is linearly recurrent of order at most $4s+6$.

4. The proofs of the lemmas

We first establish Lemma 2.2. By the argument in Section 3, both $A(k,1)$ and $A'(k,1)$ are linearly recurrent with simple roots in the set $\{\pm\phi^l : |l| \le 2\}$. The same is true for the right-hand sides in (2.2). Since the set of roots is contained in a set with 10 elements, it follows that the validity of (2.2) for $k = 1, \dots, 10$ implies that the relations hold for all $k \ge 1$.

Lemmas 2.3 and 2.4 are similar. The argument in Section 3 shows that the left-hand sides, $\{A(k,3)\}_{k\ge1}$ and $\{A'(k,3)\}_{k\ge1}$, are linearly recurrent with simple roots in $\{\pm\phi^l : |l| \le 4\}$. Splitting according to the parity of k, we deduce that $\{A(2k,3)\}_{k\ge1}$, $\{A(2k-1,3)\}_{k\ge1}$, $\{A'(2k,3)\}_{k\ge1}$ and $\{A'(2k-1,3)\}_{k\ge1}$ are linearly recurrent with simple roots in $\{\phi^{2l} : |l| \le 4\}$, a set with nine elements. The same is true about the right-hand sides in the lemmas. Thus, if the relations hold for $k = 1, \dots, 9$, then they hold for all $k \ge 1$.

A few words about the computation. For the identities presented in Lemmas 2.2–2.4, one can use brute force to compute $A(k,s)$ and $A'(k,s)$ for $s = 1, 3$ and $k = 2\ell + i$, where $i \in \{0,1\}$ and ℓ is reasonably small (we dealt with $\ell \le 9$), with any computer algebra system. To check them up to larger values of k (around 100, say), the brute force strategy no longer works since the summation range up to $F_k - 1$ becomes too large. Instead one can use the recursion from Section 3 together with $A(k,0,0) = F_k - 1$ to find $A(k,1,0)$, $A(k,2,0)$ and $A(k,3,0)$ for all desired k and, similarly, $A(k,s,j)$ for small j, to evaluate $A'(k,s)$.

5. The proof of Theorem 2.1

Let us now address Theorem 2.1. When $k = 4\ell$, this can be rewritten as

$$\begin{aligned} &F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} (A'(4\ell,3)A(4\ell,1)^2 - A(4\ell,3)A'(4\ell,1)^2) \\ &\quad = A(4\ell,1)^2 A'(4\ell,1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1). \end{aligned} \tag{5.1}$$

Since $A(4\ell,s)$ and $A'(4\ell,s)$ are linearly recurrent (in ℓ) with roots contained in $\{\phi^{4l} : |l| \le s+1\}$, and both the left-most factor in the left-hand side and the right-most factor in the right-hand side each have simple roots in $\{\phi^{4l} : |l| \le 2\}$, it follows that both the left-hand side and the right-hand side are linearly recurrent with simple roots contained in $\{\phi^{4l} : |l| \le 10\}$, a set with 21 elements. Thus, if the above formula holds for $\ell = 1, \dots, 21$, then it holds for all $\ell \ge 1$. A similar argument applies to the case when $k = 4\ell + i$ for $i \in \{1,2,3\}$. Hence, all claimed formulas hold provided they hold for all $k \le 100$, say.

Now we use the lemmas. For $k = 4\ell$, Lemmas 2.2–2.4 tell us that (5.1), after eliminating the common factor $(F_{4\ell}-1)^2(F_{4\ell+1}-1)^2(F_{4\ell+2}-1)/16$, is equivalent to

$$\begin{aligned} &F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} \times (\tfrac{1}{5}(F_{4\ell}-1)(L_{8\ell+4} - 5L_{4\ell+3} + 13) \\ &\qquad - (F_{4\ell+2}-1)(F_{4\ell-1}-1)(F_{4\ell+2}-1)) \\ &\quad = (F_{4\ell}-1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1) \end{aligned}$$

(and one can perform further reduction using (6.1)). It is sufficient to verify the resulting equality for $\ell = 1, \ldots, 15$ and we have checked it for all $\ell = 1, \ldots, 100$. The remaining cases for k modulo 4 are similar. We do not give further details here.

6. The proof of Theorem 2.5

This follows from Lemma 2.2, the classical formulas

$$\begin{aligned} F_{4\ell} - 1 &= F_{2\ell+1}L_{2\ell-1}, & F_{4\ell+1} - 1 &= F_{2\ell}L_{2\ell+1}, \\ F_{4\ell+2} - 1 &= F_{2\ell}L_{2\ell+2}, & F_{4\ell+3} - 1 &= F_{2\ell+2}L_{2\ell+1} \end{aligned} \tag{6.1}$$

as well as known facts about the greatest common divisor of Fibonacci and Lucas numbers with close arguments. For example, for $k = 2\ell$,

$$\begin{aligned} \mathrm{LCM}(2A(4\ell, 1), 2A'(4\ell, 1)) &= \mathrm{LCM}((F_{4\ell+1} - 1)(F_{4\ell} - 1), (F_{4\ell+2} - 1)(F_{4\ell} - 1)) \\ &= \mathrm{LCM}(F_{2\ell}L_{2\ell+1}, F_{2\ell}L_{2\ell+2})F_{2\ell+1}L_{2\ell-1} \\ &= F_{2\ell}L_{2\ell+1}L_{2\ell+2}F_{2\ell+1}L_{2\ell-1} \\ &= F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1}, \end{aligned}$$

where we used the fact that $\gcd(L_{2\ell+1}, L_{2\ell+2}) = 1$. The case $k = 2\ell + 1$ is similar.

7. Further variations

First, we give an informal account of a more general result lurking, perhaps, behind the formulas in Theorem 2.1. Consider a homogeneous (rational) function $r(\boldsymbol{x}) = r(x_1, \ldots, x_m)$ of degree 1, that is, satisfying

$$r(t\boldsymbol{x}) = tr(\boldsymbol{x}) \quad \text{for } t \in \mathbb{Q},$$

and an algebraic number α solving the equation

$$\sum_{k=0}^{m} c_k\alpha^k = 0, \tag{7.1}$$

where the c_k are integers. If $r(\boldsymbol{x})$ vanishes at a vector $\boldsymbol{x}^* = (x_1^*, \ldots, x_m^*)$, then automatically

$$\sum_{k=0}^{m} c_k r(\alpha^k\boldsymbol{x}^*) = 0 \tag{7.2}$$

in view of the homogeneity of the function. We can then enquire whether equation (7.2) is 'approximately' true if $r(\boldsymbol{x}^*) = 0$ is 'approximately' true. In this note, we merely examined the golden ratio case in which (7.1) is $\alpha^2 - \alpha - 1 = 0$, while the choice

$$r(x_1, \ldots, x_m) = \frac{\sum_{n=1}^{m} x_n^3}{(\sum_{n=1}^{m} x_n)^2}$$

for the rational function and $\boldsymbol{x}^* = (1, 2, \ldots, m)$ for its exact solution originated from the Nicomachus identity.

Notice that Nicomachus' theorem (1.1) is the first entry in the chain of identities

$$1^{2r-1} + 2^{2r-1} + \cdots + m^{2r-1} = P_r(1 + 2 + \cdots + m) \quad \text{for } r = 2, 3, \ldots,$$

where $P_r(x)$ are known as the Faulhaber polynomials. Our approach in this note gives a clear strategy to deal with the quantities that replace (1.2) in these settings.

Some further variations on the topic can be investigated in the q-direction, based on q-analogues of (1.1) (see [3]).

Acknowledgements

The first and third authors thank the Max Planck Institute for Mathematics (Bonn) for providing excellent working conditions. The second author thanks his former PhD advisor Kenneth Stolarsky for communicating to him the conjectures that became a theorem in this note. The authors thank the anonymous referee for careful and thoughtful reading of the note and for several helpful suggestions.

References

[1] C. Kimberling, 'The Zeckendorf array equals the Wythoff array', *Fibonacci Quart.* **33** (1995), 3–8.
[2] J. Stopple, *A Primer of Analytic Number Theory. From Pythagoras to Riemann* (Cambridge University Press, Cambridge, 2003).
[3] S. O. Warnaar, 'On the q-analogue of the sum of cubes', *Electron. J. Combin.* **11**(1) (2004), Note 13, 2 pages.

FLORIAN LUCA, School of Mathematics, University of the Witwatersrand,
Private Bag X3, Wits 2050, Johannesburg, South Africa
and
Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
and
Department of Mathematics, Faculty of Sciences, University of Ostrava,
30 dubna 22, 701 03 Ostrava 1, Czech Republic
e-mail: Florian.Luca@wits.ac.za

GEREMÍAS POLANCO, School of Natural Science, Hampshire College,
893 West St, Amherst, MA 01002, USA
e-mail: gpeNS@hampshire.edu

WADIM ZUDILIN, IMAPP, Radboud Universiteit, PO Box 9010,
6500 GL Nijmegen, The Netherlands
and
School of Mathematical and Physical Sciences, The University of Newcastle,
Callaghan, NSW 2308, Australia
e-mail: w.zudilin@math.ru.nl, wadim.zudilin@newcastle.edu.au