

A variation on ward continuity

Hüseyin Çakallı^a

^aMaltepe University, Faculty of Arts and Science, Department of Mathematics, Istanbul, Turkey

Abstract. In this paper, we prove that any ideal ward continuous function is uniformly continuous either on an interval or on an ideal ward compact subset of \mathbf{R} . A characterization of uniform continuity is also given via ideal quasi-Cauchy sequences.

1. Introduction

A real function is continuous on the set of real numbers if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated. Not all but some of them we recall in the following: slowly oscillating continuity ([10]), quasi-slowly oscillating continuity, Δ -quasi-slowly oscillating continuity ([15], and [5]), ward continuity ([6]), δ -ward continuity ([7]), statistical ward continuity, lacunary statistical ward continuity ([8, 9, 11]), and λ -statistically ward continuity ([14]). Investigation of some of these kinds of continuities lead some authors to find certain characterizations of uniform continuity of a real function in terms of sequences in the above manner ([21, Theorem 8], [8, Theorem 6], [2, Theorem 1], [3, Theorem 3.8]).

The concept of ideal convergence, which is a generalization of statistical convergence, was introduced by Kostyrko, Šalát and Wilczyński [18] by using the ideal I of subsets of the set of positive integers. Using the main idea in defining new continuities, the concept of I -ward continuity of a real function is recently introduced and investigated.

The purpose of this paper is to continue the investigation given in [4], and obtain further interesting results on I -ward continuity.

2. Preliminaries

Some definitions and notation will be given in the following. Throughout this paper, \mathbf{N} , and \mathbf{R} will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\alpha, \beta, x, y, z, \dots$ for sequences $(\alpha_n), (\beta_n), (x_n), (y_n), (z_n), \dots$ of terms in \mathbf{R} .

Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set, here in our study it suffices to take I as a family of subsets of \mathbf{N} such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I . An ideal I is called non-trivial if $I \neq \emptyset$ and $\mathbf{N} \notin I$. A non-trivial ideal I is called *admissible* if $\{\{n\} : n \in \mathbf{N}\} \subset I$. Further details on ideals can be found in Kostyrko, et.al (see

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Email address: hcakalli@maltepe.edu.tr (Hüseyin Çakallı)

[18]). Throughout this paper we assume I is a non-trivial admissible ideal in \mathbf{N} , and an ideal will mean a non-trivial admissible ideal I . Recall that a sequence (α_n) of points in \mathbf{R} is I -convergent to a real number ℓ if $\{n \in \mathbf{N} : |\alpha_n - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$. In this case we write $I - \lim \alpha_n = \ell$. A sequence (α_n) of points in \mathbf{R} is said to be I -quasi-Cauchy if $I - \lim(\alpha_{n+1} - \alpha_n) = 0$. We note that the definition of a quasi-Cauchy sequence is a special case of an ideal quasi-Cauchy sequence where I is taken as the finite subsets of the set of positive integers. The notion is useful not only in mathematical analysis but also in other branches for example in the ergodic theory and computer science theory. Now we give the following interesting examples which show emphasis the interest in different research areas.

Example 2.1. Let n be a positive integer. In a group of n people, each person selects at random and simultaneously another person of the group. All of the selected persons are then removed from the group, leaving a random number $n_1 < n$ of people which form a new group. The new group then repeats independently the selection and removal thus described, leaving $n_2 < n_1$ persons, and so forth until either one person remains, or no persons remain. Denote by p_n the probability that, at the end of this iteration initiated with a group of n persons, one person remains. Then the sequence $\mathbf{p} = (p_1, p_2, p_n, \dots)$ is a quasi-Cauchy sequence, and $\lim p_n$ does not exist (see [22]).

Example 2.2. Let n be a positive integer. In a group of n people, each person selects independently and at random one of three subgroups to which to belong, resulting in three groups with random numbers n_1, n_2, n_3 of members; $n_1 + n_2 + n_3 = n$. Each of the subgroups is then partitioned independently in the same manner to form three sub subgroups, and so forth. Subgroups having no members or having only one member are removed from the process. Denote by t_n the expected value of the number of iterations up to complete removal, starting initially with a group of n people. Then the sequence $(t_1, \frac{t_2}{2}, \frac{t_3}{3}, \dots, \frac{t_k}{k}, \dots)$ is a bounded nonconvergent quasi-Cauchy sequence (see [16]).

Example 2.3. Let $x := (x_n)$ be a sequence such that for each nonnegative integer n , x_n is either 0 or 1. For each positive integer n set $a_n = \frac{x_1 + x_2 + \dots + x_n}{n}$. Then a_n is the arithmetic mean average of the sequence up to time or position n . Clearly for each n , $0 \leq a_n \leq 1$. (a_n) is a quasi-Cauchy sequence. That is, the sequence of averages of 0 s and 1 s is always a quasi-Cauchy sequence (see [16]).

Example 2.4. Limit of the sequence of the ratios of Fibonacci numbers converge to the golden mean. This is a lacunary sequential method obtained via the sequence of Fibonacci numbers, method, i.e. $\theta = (k_r)$ is the lacunary sequence defined by writing $k_0 = 0$ and $k_r = F_{r+2}$ where (F_r) is the Fibonacci sequence, i.e. $F_1 = 1, F_2 = 1, F_r = F_{r-1} + F_{r-2}$ for $r \geq 3$ (see [3]). For this lacunary sequence $\theta = (k_r)$, any quasi-Cauchy sequence is also lacunary statistically quasi-Cauchy (see [11]).

A function f is called I -ward continuous on a subset E of \mathbf{R} if f preserves I -quasi-Cauchy sequences of points in E , i.e. $(f(\alpha_n))$ is I -quasi-Cauchy whenever (α_n) is an I -quasi-Cauchy sequences of numbers in E ([4]).

3. Results

$C(\mathbf{R})$ be the set of continuous functions of \mathbf{R} . Noticing the observation in [1], we have the following: if $f \in C(\mathbf{R})$ and E is a compact subset of \mathbf{R} , then $f|_E$ is uniformly continuous. But the standard proof shows something more is true: $\forall \varepsilon > 0 \exists \delta > 0$ such that if $|\alpha - \beta| < \delta$ and $\alpha, \beta \in E$, then $|f(\alpha) - f(\beta)| < \varepsilon$. To see this, suppose to the contrary that $\forall n \in \mathbf{N} \exists \alpha_n, \beta_n$ such that if $|\alpha_n - \beta_n| < \frac{1}{n}$ and $\alpha_n, \beta_n \in E$, then $|f(\alpha_n) - f(\beta_n)| \geq \varepsilon$. By compactness the sequence (α_n) has a cluster point $\alpha \in E$. But α is also a cluster point of the sequence (β_n) and so continuity of f at α fails. We replace the word "compactness" by ideal ward-compactness in this strong argument in proofs of our theorems in the following, which makes the proofs easy.

Theorem 3.1. If a function f is uniformly continuous on a subset E of \mathbf{R} , then $(f(x_n))$ is I -quasi Cauchy whenever (x_n) is a quasi-Cauchy sequence of points in E .

Proof. Let f be uniformly continuous on E . Suppose that there exists a quasi-Cauchy sequence (x_n) so that $(f(x_n))$ is not I -quasi-Cauchy. Therefore there is an $\varepsilon_0 > 0$ such that $\{n : |f(x_{n+1}) - f(x_n)| \geq \varepsilon_0\} \notin I$. As f is uniformly continuous on E , for this ε_0 , there exists a $\delta > 0$ such that $|f(\alpha) - f(\beta)| < \varepsilon_0$ whenever $|\alpha - \beta| < \delta$ and $\alpha, \beta \in E$. Since (x_n) is quasi-Cauchy, for this δ_0 there is an $n_0 \in \mathbf{N}$ such that $|x_{n+1} - x_n| < \delta$ for $n \geq n_0$. Admissibility of I implies that the set $\{n : |f(x_{n+1}) - f(x_n)| \geq \varepsilon_0\}$ is an infinite subset of \mathbf{N} . Hence there exists an $n_1 \in \mathbf{N}$, which is greater than n_0 , such that $|f(x_{n_1+1}) - f(x_{n_1})| \geq \varepsilon_0$. This contradiction completes the proof of the theorem. \square

It is well known that any continuous function on a compact subset E of \mathbf{R} is uniformly continuous on E . It is also true for the ideal case, i.e. any I -ward continuous function on an I -ward compact subset E of \mathbf{R} is also uniformly continuous on E .

Theorem 3.2. *Let E be an I -ward compact subset E of \mathbf{R} and $f : E \rightarrow \mathbf{R}$. f is I -ward continuous on E , then it is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous on E so that there exist an $\varepsilon_0 > 0$ and sequences (α_n) and (β_n) of points in E such that

$$|\alpha_n - \beta_n| < 1/n$$

and

$$|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0$$

for all $n \in \mathbf{N}$. Since E is I -ward compact, there is an I -quasi-Cauchy subsequence of (α_n) of (α_n) . On the other hand, there is an I -quasi-Cauchy subsequence of (β_n) of (β_n) . It is clear that the corresponding sequence $(\alpha_{n_{k_j}})$ is also an I -quasi-Cauchy, since $(\beta_{n_{k_j}})$ is an I -quasi-Cauchy sequence and

$$\alpha_{n_{k_j}} - \alpha_{n_{k_{j+1}}} = (\alpha_{n_{k_j}} - \beta_{n_{k_j}}) + (\beta_{n_{k_j}} - \beta_{n_{k_{j+1}}}) + (\beta_{n_{k_{j+1}}} - \alpha_{n_{k_{j+1}}}).$$

Now define a sequence $\mathbf{z} = (z_j)$ by setting $z_1 = \alpha_{n_{k_1}}, z_2 = \beta_{n_{k_1}}, z_3 = \alpha_{n_{k_2}}, z_4 = \beta_{n_{k_2}}, z_5 = \alpha_{n_{k_3}}, z_6 = \beta_{n_{k_3}}$, and so on. Thus the sequence $\mathbf{z} = (z_j)$ defined in this way is an I -quasi-Cauchy while $f(\mathbf{z}) = (f(z_j))$ is not I -quasi-Cauchy. Hence this establishes a contradiction so this completes the proof of the theorem. \square

Corollary 3.3. *If a real valued function defined on a bounded subset of \mathbf{R} is I -ward continuous, then it is uniformly continuous on E .*

Proof. The proof follows from the preceding theorem and [4, Theorem 8]. \square

We give below that any real ideal-ward continuous function defined on an interval is uniformly continuous. First we give the following lemma.

Lemma 3.4. *If (ξ_n, η_n) is a sequence of ordered pairs of points in an interval such that $\lim_{n \rightarrow \infty} |\xi_n - \eta_n| = 0$, then there exists an I -quasi-Cauchy sequence (α_n) with the property that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (\alpha_{j-1}, \alpha_j)$.*

Proof. Although the following proof is similar to that of [9], we give it for completeness. For each positive integer k , we can fix $z_0^k, z_1^k, \dots, z_{n_k}^k$ in E with $z_0^k = \eta_k, z_{n_k}^k = \xi_{k+1}$, and $|z_i^k - z_{i-1}^k| < \frac{1}{k}$ for $1 \leq i \leq n_k$. Now write

$$(\xi_1, \eta_1, z_1^1, \dots, z_{n_1-1}^1, \xi_2, \eta_2, z_1^2, \dots, z_{n_2-1}^2, \xi_3, \eta_3, \dots, \xi_k, \eta_k, z_1^k, \dots, z_{n_k-1}^k, \xi_{k+1}, \eta_{k+1}, \dots)$$

Then denoting this sequence by (α_n) we obtain that for any $i \in \mathbf{N}$ there exists a $j \in \mathbf{N}$ such that $(\xi_i, \eta_i) = (\alpha_{j-1}, \alpha_j)$. This completes the proof of the lemma. \square

Theorem 3.5. *If a function defined on an interval E is I -ward continuous, then it is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous on E . Then there is an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exist $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For every $n \in \mathbf{N}$ fix $\xi_n, \eta_n \in E$ with $|\xi_n - \eta_n| < \frac{1}{n}$ and $|f(\xi_n) - f(\eta_n)| \geq \varepsilon_0$. By Lemma 3.4, there exists an I -quasi-Cauchy sequence (α_i) such that for any integer $i \geq 1$ there exists a j with $\xi_i = \alpha_j$ and $\eta_i = \alpha_{j+1}$. This implies that $|f(\alpha_{j+1}) - f(\alpha_j)| \geq \varepsilon_0$; hence $(f(\alpha_i))$ is not I -quasi-Cauchy. Thus f does not preserve I -quasi-Cauchy sequences. This completes the proof of the theorem. \square

Since the sequence constructed in Lemma 3.4 is also quasi-Cauchy, we see that the statement $(f(x_n))$ is I -quasi Cauchy whenever (x_n) is quasi-Cauchy sequence of points in E implies the uniform continuity of f on E . Now combining Theorem 3.1 with this observation we have the following result.

Corollary 3.6. *Let f be a function defined on an interval E . Then f is uniformly continuous on E if and only if $(f(x_n))$ is I -quasi Cauchy whenever (x_n) is quasi-Cauchy sequence of points in E .*

Corollary 3.7. *If a function defined on an interval is I -ward continuous, then it is ward continuous.*

Proof. The proof follows from Theorem 3.5, and [9, Theorem 5] so it is omitted. \square

Corollary 3.8. *If a function defined on an interval is I -ward continuous, then it is slowly oscillating continuous.*

Proof. The proof follows from Theorem 3.5, and [9, Theorem 5] so it is omitted. \square

Now we give a result related to N_θ -quasi-Cauchy sequences (see [13] for concepts related to N_θ -quasi sequences).

Corollary 3.9. *If f is I -ward continuous on an interval, then $(f(x_n))$ is N_θ -quasi Cauchy whenever (x_n) is a quasi-Cauchy sequence of points in E .*

Corollary 3.10. *If f is I -ward continuous on an interval, then $(f(x_n))$ is lacunary statistically quasi-Cauchy whenever (x_n) is a quasi-Cauchy sequence of points in E .*

Corollary 3.11. *If f is I -ward continuous on an interval, then $(f(x_n))$ is λ -statistically quasi-Cauchy whenever (x_n) is quasi-Cauchy sequence of points in E .*

Finally we note the following further investigation problem arises.

Remark 3.12. For further study, we suggest to investigate I -quasi-Cauchy sequences of fuzzy points and I -ward continuity for the fuzzy functions. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (for example see [12, 17, 19, 20]).

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