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A VARIATIONAL APPROACH TO SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS WITH TURNING POINTS*

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Abstract. In studying singularity perturbed boundary value problems for second order linear differential equations with a simple turning point, R. C. Ackerberg and R. E. O'Malley [2] pointed out a number of interesting anomalies. In particular they observed that standard application of the method of matched asymptotic expansions did not suffice to uniquely determine the asymptotic expansion of the solution. They further noted that the standard construction in that method led to boundary layers at both ends of the interval, even for problems where in fact there is only one boundary layer located at one or other of the endpoints. In this paper we employ a variational formulation of the problem to resolve the question of the number and location of the boundary layers as well as to uniquely determine the asymptotic expansion of the solution. The results are then extended to analogous problems for partial differential equations, and new results are obtained for a class of singularly perturbed elliptic boundary value problems with turning points.

1. Introduction. We consider singularly perturbed boundary value problems of the form

$$(1)\qquad \begin{aligned} &\varepsilon y''+f(x;\varepsilon)y'+g(x;\varepsilon)y=0, && -a<x<b,\\ &y(-a;\varepsilon)=\alpha(\varepsilon), && y(b;\varepsilon)=\beta(\varepsilon),\end{aligned}$$

where $a, b>0$, $0<\varepsilon \ll 1$, and α and β are prescribed. As in [2], [3], the functions f and g are assumed to be analytic in x and ε. In § 4, we discuss analogous problems for partial differential equations.

When f has one sign throughout the interval $[-a, b]$, the unique asymptotic expansion of the solution can be constructed by the well-known method of matched asymptotic expansions [1], which we now briefly describe.

In attempting to approximate the solution of (1) for small but nonzero ε by a solution of the reduced equation

$$(2)\qquad f_0(x)w'+g_0(x)w=0$$

with

$$f_0(x)=f(x;0); \qquad g_0(x)=g(x,0),$$

we note that $w(x)$, a solution of a first order differential equation, cannot in general satisfy both of the prescribed boundary conditions. While the function $w(x)$ which satisfies (2) and one of the boundary conditions may provide a good approximation to y elsewhere in the interval, it cannot, in general, provide a good approximation in a neighborhood of the excluded boundary point. In this neighborhood, referred to as a boundary layer, another approximation to y is necessary. It is well known that the boundary layer occurs at the left (right) end of the interval depending on whether the function f is positive (negative).

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We define the outer expansion by

$$w(x;\varepsilon)=\sum_{j=0}^{\infty} w^j(x)\varepsilon^j, \tag{3}$$

where $w^0(x)$ satisfies the reduced equation and the boundary condition at $x=b$ ($x=-a$) if $f(x;\varepsilon)$ is positive (negative), and the functions $w^j(x)$ ($j\geqq 1$) are determined by inserting (3) into (1) and equating the coefficient of each power of ε separately to zero.

It is then possible to prove that $w(x;\varepsilon)$ is an asymptotic expansion of $y(x;\varepsilon)$ in the region $x_0\leqq x\leqq b$ ($-a\leqq x\leqq x_1$) when $f(x;\varepsilon)$ is positive (negative). Here x_0 and x_1 are close to but bounded away from $-a$ and b, respectively.

We then construct one of the boundary layer expansions:

$$Z_1=\sum_{j=0}^{\infty} Z_1^j(\eta_1)\varepsilon^j, \tag{4a}$$

with η_1 given by the stretching transformation

$$\eta_1=(x+a)/\varepsilon \tag{4b}$$

if $f(x;\varepsilon)$ is positive, or

$$Z_2=\sum_{j=0}^{\infty} Z_2^j(\eta_2)\varepsilon^j, \tag{5a}$$

with η_2 given by the stretching transformation

$$\eta_2=(b-x)/\varepsilon \tag{5b}$$

if $f(x;\varepsilon)$ is negative. Equations for $Z_1^j(Z_2^j)$ are obtained by inserting (5) into (1) and equating the coefficient of each power of ε separately to zero. In addition, $Z_1(Z_2)$ satisfies the boundary condition at $x=-a(x=b)$ as well as a boundary condition as $\eta_1\to\infty$ ($\eta_2\to\infty$) obtained by matching the boundary layer expansion to the outer expansion. The matching procedure involves the assumption that there exists a domain of overlap in which the outer and boundary layer expansions are both valid. This overlap domain consists of points which are close to but bounded away from $x=-a(x=b)$. As $\varepsilon\to 0$ at such values of x, $\eta_1\to\infty$ ($\eta_2\to\infty$). Thus, the matching condition involves a comparison of the boundary layer expansion as $\eta_1\to\infty$ ($\eta_2\to\infty$) and the outer expansion as $x\to -a$ ($x\to b$). The boundary condition and matching condition, together with the differential equations mentioned above, serve to uniquely determine the functions Z_1^j (Z_2^j) in the boundary layer expansion.

Finally, it is possible to construct a composite expansion and prove that it is valid uniformly throughout $-a\leqq x\leqq b$, by adding together the outer and boundary layer expansions and then subtracting certain terms which the two expansions have in common so that they are not counted twice. These terms are precisely those terms in the boundary layer expansion which do not vanish as $\eta_1\to\infty$ ($\eta_2\to\infty$). For the problem (1), this construction yields

$$y_0\sim\begin{cases} w^0(x)+(\alpha-w^0(-a))\,e^{-f_0(-a)(x+a)/\varepsilon} & \text{if } f>0,\\ w^0(x)+(\beta-w^0(b))\,e^{f_0(b)(b-x)/\varepsilon} & \text{if } f<0,\end{cases} \tag{6}$$

as the leading term of the asymptotic expansion.

However, when f changes sign in the interval, the situation is not as clear. Points where f changes sign, i.e., singular points of the lower order equation (2), are often referred to as turning points. The case when f has a single simple zero at $x=0$ and $f'(x;\varepsilon)<0$ throughout the interval, with f and g analytic in x and ε, was considered in [2]. In that paper the authors pointed out that the method of matched asymptotic expansions led to certain paradoxical results. In particular, for problems which exhibit what they termed the resonance phenomenon, they observed that boundary layers of the form $c_1 \exp[-f_0(-a)(x+a)/\varepsilon]$ and $c_2 \exp[f_0(b)(b-x)/\varepsilon]$ with $c_{1,2}$ constant, are possible at the endpoints $x=-a$ and $x=b$ respectively. In fact there is only one boundary layer, located at left (right) end point if $I \equiv \int_{-a}^{b} f(x,0)\,dx$ is positive (negative). The term resonance was applied to problems for which the outer expansion of the solution was nonzero. In addition, they noted that for such problems, a unique asymptotic expansion of the solution cannot be obtained by standard application of matched asymptotic expansions, since one constant in the expansion remains undetermined after all the conditions are applied. Thus the method yields a one parameter family of possible expansions, with no condition for determining which member of the family is in fact the asymptotic expansion of the solution.

In subsequent papers, various authors considered the question of determining which problems exhibit resonance, and also proposed alternatives to, or modifications of, the method of matched asymptotic expansions, in order to determine the unique expansion of the solution. A discussion of these approaches, as well as references, is given in [3].

In this paper, we adopt a different approach. We consider problems which are known to exhibit resonance, and which therefore exhibit the above mentioned paradoxical results. We propose to employ the standard method of matched asymptotic expansions rather than modifications or alternatives of that method, and to augment the standard method in a natural way, by an additional condition that will determine the number and location of the boundary layers as well as the unique asymptotic expansion of the solution. One of the advantages of this approach is that it can readily be generalized to more complicated problems including boundary value problems for partial differential equations. In so doing we obtain some new and interesting results for singularly perturbed boundary value problems for partial differential equations with turning points.

2. Variational characterization. We augment the method of matched asymptotic expansions as follows. We construct the functional

$$J=\frac{1}{2}\int_{-a}^{b}\left\{\varepsilon(y')^2+fyy'+\left(\frac{f^2}{2\varepsilon}+\frac{f'}{2}-g\right)y^2\right\}\exp\left(\frac{1}{\varepsilon}\int_0^x f(t;\varepsilon)\,dt\right)dx \tag{7}$$

whose Euler–Lagrange differential equation is the given equation (1). We employ as the class of admissable functions the one parameter family determined by the method of matched asymptotic expansions, and choose that member of the family which makes the functional stationary.

3. Application of the variational condition. We consider the problem (1) with $f(0;\varepsilon)=0$, $f'(x;\varepsilon)<0$ throughout $[-a,b]$. We further assume that

$g_0(0)/f'_0(0) = -n$ $(n = 0, 1, 2, \cdots)$, which is the first of the conditions necessary for resonance to occur. We also assume that all the other conditions are satisfied and that resonance does occur. The leading term of the expansion of the solution is constructed as

$$y_0 \sim c_0 k(x) + A \exp[-f(-a)(x+a)/\varepsilon] + B \exp[f(b)(b-x)/\varepsilon], \tag{8}$$

where

$$k(x) = x^n \exp\left[-\int_0^x \left(\frac{g_0(s)}{f_0(s)} + \frac{n}{s}\right) ds\right] \tag{9}$$

is a solution of the reduced equation (2), and

$$A = \alpha_0 - c_0 k_1, \tag{10}$$

$$B = \beta_0 - c_0 k_2. \tag{11}$$

Here $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$ and

$$k_1 = k(-a), \tag{12}$$

$$k_2 = k(b). \tag{13}$$

Thus equation (8) represents a one parameter family of possible asymptotic expansions, where the constant c_0, as yet undetermined, labels the members of the family. We observe that the expression (8) seems to exhibit boundary layers at both endpoints. To determine c_0, as well as the number and location of the boundary layers, we employ the family (8) as the class of admissable functions in (7). Then, retaining only the highest order terms, we obtain

$$J \sim \frac{c_0^2}{4\varepsilon} J_1 + \frac{Ac_0}{2\varepsilon} J_2 + \frac{Bc_0}{2\varepsilon} J_3 + \frac{A^2}{2\varepsilon} J_4 + \frac{B^2}{2\varepsilon} J_5, \tag{14}$$

where

$$J_1 = \int_{-a}^{b} (f_0(x)k(x))^2 \exp\left\{\frac{1}{\varepsilon}\int_0^x f_0(t)\,dt\right\} dx, \tag{15}$$

$$J_2 = \int_{-a}^{b} (f_0(x)^2 - f_0(x)f_0(-a))k(x) \exp\left\{-\frac{1}{\varepsilon}\left(f_0(-a)(x+a) - \int_0^x f_0(t)\,dt\right)\right\} dx, \tag{16}$$

$$J_3 = \int_{-a}^{b} (f_0^2(x) - f_0(x)f_0(b))k(x) \exp\left\{\frac{1}{\varepsilon}\left(f_0(b)(b-x) + \int_0^x f_0(t)\,dt\right)\right\} dx, \tag{17}$$

$$J_4 = \int_{-a}^{b} \left(f_0^2(-a) - f_0(x)f_0(-a) + \frac{f_0^2(x)}{2}\right) \exp\left\{-\frac{1}{\varepsilon}\left(2f_0(-a)(x+a) - \int_0^x f_0(t)\,dt\right)\right\} dx \tag{18}$$

and

$$J_5=\int_{-a}^{b}\left(f_0^2(b)-f_0(x)f_0(b)+\frac{f_0^2(x)}{2}\right)\exp\left\{\frac{1}{\varepsilon}\left(2f_0(b)(b-x)+\int_0^x f_0(t)\,dt\right)\right\}dx. \tag{19}$$

The integrals J_i $(i=1,\cdots,5)$ are now evaluated asymptotically in the following manner. J_1 is integrated by parts to give

$$J_1\sim\varepsilon[f_0(b)k_2^2\exp(I_2/\varepsilon)-f_0(-a)k_1^2\exp(I_1/\varepsilon)], \tag{20}$$

where

$$I_1=\int_0^{-a}f_0(x)\,dx \tag{21}$$

and

$$I_2=\int_0^{b}f_0(x)\,dx. \tag{22}$$

The major contributions to J_2 and J_3 come from the endpoints $x=-a$ and $x=b$ respectively. We introduce the transformations $x=-a+\varepsilon^{1/2}\eta$ and $x=b-\varepsilon^{1/2}\eta$ in J_2 and J_3 respectively to obtain

$$J_2\sim-\varepsilon f_0(-a)k_1\exp(I_1/\varepsilon) \tag{23}$$

and

$$J_3\sim\varepsilon f_0(b)k_2\exp(I_2/\varepsilon). \tag{24}$$

The major contributions to J_4 and J_5 also come from the endpoints $x=-a$ and $x=b$ respectively. Here we introduce the transformations $x=-a+\varepsilon\eta$ and $x=b-\varepsilon\eta$ in J_4 and J_5 respectively to obtain

$$J_4\sim\frac{\varepsilon f_0(-a)}{2}\exp(I_1/\varepsilon) \tag{25}$$

and

$$J_5\sim\frac{-\varepsilon f_0(b)}{2}\exp(I_2/\varepsilon). \tag{26}$$

Therefore to leading order

$$J\sim\left[-\frac{c_0^2}{4}k_1^2-\frac{Ac_0}{2}k_1+\frac{A^2}{4}\right]f_0(-a)\exp(I_1/\varepsilon)+\left[\frac{c_0^2}{4}k_2^2+\frac{Bc_0}{2}k_2-\frac{B^2}{4}\right]f_0(b)\exp(I_2/\varepsilon). \tag{27}$$

The constant c_0 is then determined by the condition that $dJ/dc_0=0$. We distinguish three cases according as $I_2 \gtreqless I_1$ $(I \gtreqless 0)$.

Case 1. $I_2 > I_1$ $(I > 0)$. In this case the second term in J is transcendentally smaller in ε than the first and is therefore asymptotically negligible. Then the condition $dJ/dc_0 = 0$ implies that

(28) $$c_0 = \alpha_0 / k_1$$

or

(29) $$A = 0$$

so that there is a boundary layer at $x = b$ only.

Case 2. $I_2 < I_1$ $(I < 0)$. In this case the first term in J is transcendentally smaller in ε than the second so that it is negligible. Then c_0 is determined as

(30) $$c_0 = \beta_0 / k_2$$

or

(31) $$B = 0$$

so that there is a boundary layer at $x = -a$ only.

Case 3. $I_1 = I_2$ $(I = 0)$. In this case each term contributes equally and C_0 is determined as

(32) $$c_0 = \frac{\alpha_0 k_1 f_0(-a) - \beta_0 k_2 f_0(b)}{f_0(-a) k_1^2 - f_0(b) k_2^2}$$

and there are boundary layers at both endpoints.

Our expressions for C_0 are in agreement with the result obtained in [3] for the problems considered there. They also agree with the results of Kreiss and Parter [4] and Zauderer [5], and appear to be in simpler form than those results.

4. Partial differential equations. Our method can be extended to singularly perturbed elliptic boundary value problems. As an illustration we consider a number of problems of the form[1]

(33) $$\begin{aligned} &\varepsilon \Delta u - x u_x - y u_y + n u = 0, && (x, y) \in D, \\ &u = f, && (x, y) \in \partial D. \end{aligned}$$

Here Δ is the Laplace operator, D is a bounded domain in the x, y plane, and n is an integer. The problems considered differ only in the specification of the domain D and the values of n. Problem (33) may be written in polar coordinates as

(34) $$\begin{aligned} &\varepsilon\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) - r u_r + n u = 0, && r < \bar{r}(\theta), \\ &u(\bar{r}(\theta), \theta) = f(\theta) && 0 \leqq \theta \leqq 2\pi. \end{aligned}$$

[1] The problems considered can easily be generalized though we do not do so here.

The variational condition that we employ is to make stationary the functional

$$K=\frac{1}{2}\int_0^{2\pi}\int_0^{\bar r(\theta)}\left\{\varepsilon\left(\frac{\partial u}{\partial r}\right)^2+\frac{\varepsilon}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2+\left(\frac{\varepsilon}{r}-r\right)u\frac{\partial u}{\partial r}+\left(\frac{r^2}{2\varepsilon}-\frac{3}{2}-n\right)u^2\right\}\exp\left(\frac{-r^2}{2\varepsilon}\right)r^2\,dr\,d\theta. \tag{35}$$

Problem I. Let D be the unit circle and $n=0$. We compute the leading term of the expansion of u as

$$u_0\sim c_0+(f(\theta)-c_0)\exp\left\{-\frac{(1-r)}{\varepsilon}\right\} \tag{36}$$

with the constant c_0 to be determined by the variational condition. Thus inserting (36) into (35) and retaining only the highest order terms we obtain

$$K\sim\frac{K_1c_0^2}{2\varepsilon}\int_0^{2\pi}d\theta+\frac{K_2C_0}{\varepsilon}\int_0^{2\pi}(f(\theta)-c_0)\,d\theta+\frac{K_3}{\varepsilon}\int_0^{2\pi}(f(\theta)-c_0)^2\,d\theta, \tag{37}$$

where

$$K_1=\int_0^1\exp(-r^2/(2\varepsilon))r^4\,dr, \tag{38}$$

$$K_2=\int_0^1 r^3(r-1)\exp\left\{-\frac{1}{\varepsilon}\left(1-r+\frac{r^2}{2}\right)\right\}dr \tag{39}$$

and

$$K_3=\int_0^1 r^2\left(1-r+\frac{r^2}{2}\right)\exp\left\{-\frac{1}{\varepsilon}\left(2(1-r)+\frac{r^2}{2}\right)\right\}dr. \tag{40}$$

The integrals K_i $(i=1,2,3)$ are evaluated asymptotically as in the previous section. K_1 is integrated by parts and the transformation $r=1-\varepsilon\eta$ is employed in K_2 and K_3 to yield

$$K\sim\left[\pi c_0^2-c_0\int_0^{2\pi}f(\theta)\,d\theta+\frac{1}{4}\int_0^{2\pi}f^2(\theta)\,d\theta\right]\exp(-1/(2\varepsilon)). \tag{41}$$

Then the condition that $dK/dc_0=0$ implies that

$$c_0=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta. \tag{42}$$

This result was obtained by De Groen [6] who expanded the exact solution of the problem which was expressed as an infinite sum of confluent hypergeometric functions and exponentials.

Problem II. Let D be the unit circle and let $\eta=1$. Then

$$u_0\sim c_0(\theta)r+(f(\theta)-c_0(\theta))\exp\left\{\frac{-(1-r)}{\varepsilon}\right\}, \tag{43}$$

where

(44) $$c_0(\theta) = \alpha \cos \theta + \beta \sin \theta.$$

Proceeding as above, we find that the constants α and β are determined by the variational condition as

(45) $$\alpha = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos \theta \, d\theta$$

and

(46) $$\beta = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin \theta \, d\theta.$$

Problem III. Let $n = 0$ and let the boundary $r = \bar{r}(\theta)$ of D have a unique point P nearest the origin. With no loss of generality we take P to be the point $(r, \theta) = (1, 0)$. We assume that near P, $\bar{r}(\theta) \sim 1 + \alpha\theta^2$ with $\alpha > 0$ (i.e., that $\bar{r}(\theta)$ has first order contact with the unit circle at P). The leading term of the expansion of u is constructed as

(47) $$u \sim c_0 + (f(\theta) - c_0) \exp\left\{\frac{-\bar{r}(\theta)(\bar{r}(\theta) - r)}{\varepsilon}\right\}.$$

After asymptotically evaluating the integrals that result from the variational condition, we find that c_0 is given by

(48) $$c_0 = f(0).$$

This result agrees with that in [10].

Problem IV. Let $n = 0$ and let the boundary $r = \bar{r}(\theta)$ of D have two distinct points, P_1 and P_2, nearest the origin, say at $(r, \theta) = (1, 0)$ and at $(1, \pi)$. Further, near P let $\bar{r}(\theta) \sim 1 + \alpha\theta^{2p}$ with $\alpha > 0$, and near P_2 let $\bar{r}(\theta) \sim 1 + \beta(\theta - \pi)^{2q}$ with $\beta > 0$. The leading term of the expansion of u is again given by (47) with the constant c_0 determined by the variational condition. Its value depends on the relative values of p and q and of α and β. Specifically,

(49) $$c_0 = \begin{cases} f(0), & \text{if } p > q, \\ f(\pi), & \text{if } p < q. \end{cases}$$

If $p = q$, then c_0 is a weighted average of $f(0)$ and $f(\pi)$, where the weights depend on α and β. In particular if $p = q = 1$.

(50) $$c_0 = \frac{\sqrt{\beta} f(0) + \sqrt{\alpha} f(\pi)}{\sqrt{\beta} + \sqrt{\alpha}}.$$

Using these results, we can derive additional results for other interesting domains without much additional calculation. For example, if there are N nearest points, then c_0 equals the boundary value at the point of highest order contact. If there are R points of highest contact ($R \leqq N$), then c_0 is an appropriate weighted average of the R boundary values. As an example, let the domain be the square

$-1<x, y<1$. We observe that there are four points nearest the origin, and each has the same contact with the unit circle. Thus c_0 is given by

$$c_0=\frac{1}{4}\left[f(0)+f\left(\frac{\pi}{2}\right)+f(\pi)+f\left(\frac{3\pi}{2}\right)\right]. \tag{51}$$

5. Remarks. In treating the problems of § 4, it was necessary to consider the order of contact of the boundary with a circle, at the points of the boundary nearest the origin. This is due to the fact that circles are that family of curves, orthogonal to the characteristics of the lower order operator in (33) ((33) with $\varepsilon=0$) which enter the origin (the unique singular point of the operator). In general it is necessary to consider the contact of the boundary with the family of curves (not necessarily circles), orthogonal to the characteristics curves.

The fact that the study of singularly perturbed second order elliptic equations is intimately connected with the characteristics of the lower order operator was pointed out by Levinson [7]. When the characteristics cross the domain, results were already given by Levinson. However, the methods and results of [7] did not by themselves suffice to determine the asymptotic expansion, e.g., when the characteristics were closed orbits inside D or when all the characteristics entered D and met at the singular point. By introducing averages around the orbits, Khasminskii [8] showed how to obtain the expansion when the characteristics were closed. By introducing the variational characterization, we have shown how to obtain the asymptotic representation of the solution when the characteristics enter the domain. General results for problems of this type will be given in [9], [11].

Finally, we remark that the problem considered here has implications for the problem of random perturbations of dynamical systems. Specifically, the effect of even very small random perturbations may be considerable after sufficiently long times, so that even if the deterministic dynamical system has an asymptotically stable equilibrium point the trajectories of the system will leave any compact domain with probability one. It is of interest to compute the probability distribution of the points on the boundary where trajectories exit, at the first time of their exit from the domain. It can be shown that the distribution can be computed in terms of the solution of the problem considered here. Results of this problem will be reported elsewhere [9], [11].

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