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A VARIATIONAL APPROACH TO THE CONSTRUCTION AND MALLIAVIN DIFFERENTIABILITY OF STRONG SOLUTIONS OF SDE'S

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Abstract. In this article we develop a new approach to construct strong solutions of stochastic equations with merely measurable coefficients. We aim at demonstrating the principles of our technique by analyzing stochastic differential equations driven by Brownian motion. An important and rather surprising consequence of our method which is based on Malliavin calculus is that the solutions derived by A. Y. Veretennikov [45] for Brownian motion with bounded and measurable drift in $\mathbb{R}^d$ are Malliavin differentiable. Moreover, it is conceivable that our approach which doesn't rely on a pathwise uniqueness argument is also applicable to the construction of strong solutions of stochastic equations in infinite dimensions.

Key words and phrases: strong solutions of SDE's, irregular drift coefficient, Malliavin calculus, relative L^2-compactness.
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1. Introduction

In this paper we are mainly interested to study the following stochastic differential equation (SDE) given by

$$dX_t = b(t, X_t)dt + dB_t, \ \ 0 \le t \le T, \quad X_0 = x \in \mathbb{R}^d, \tag{1.1}$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable function and B_t is a $d-$dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \pi)$. We denote by $\mathcal{F}_t$ the augmented filtration generated by B_t.

If b in (1.1) is of linear growth and (globally) fulfills a Lipschitz condition it is well known that there exists a unique global strong solution to the SDE (1.1). More precisely, there exists a continuous $\mathcal{F}_t-$adapted process X_t solving (1.1) such that

$$E\left[\int_0^T X_t^2 dt\right] < \infty.$$

Important applications, however, of SDE's of the type (1.1) to physics or stochastic control theory show that Lipschitz continuity imposed on the drift coefficient b is a rather severe restriction. For example, in statistical mechanics, where one is interested in solutions of (1.1) as functionals of the driving noise (i.e. strong solutions) to model interacting infinite particle systems, the drift b is typically discontinuous or singular. See e.g. [19] and the references therein.

Strong solutions of SDE's with non-Lipschitz coefficients have been investigated by many authors in the past decades. To begin with we mention the work of Zvonkin [47], where the author obtains unique strong solutions of (1.1) in the one-dimensional case, when b is merely bounded and measurable. The latter result can be regarded as a milestone in the theory of SDE's. Subsequently, this result was generalized by Veretennikov [45] to the multidimensional case. The tools used by these authors to derive strong solutions are based on estimates of solutions of parabolic partial differential equations and a pathwise uniqueness argument.

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Other important and more recent results in this direction based on a pathwise uniqueness argument (in connection with other techniques due to Portenko [32] or the Skorohod embedding) can be e.g. found in Krylov, Röckner [19], Gyöngy, Krylov [14] or Gyöngy, Martínez [15]. We also refer to [10], where the authors employ a modified version of Gronwall's Lemma. In this context we shall also point out the paper of Davie [7], who even establishes uniqueness of strong solutions of (1.1) for almost all Brownian paths in the case of bounded and measurable drift coefficients.

In this paper we further develop the new approach devised in [28] to construct strong solutions of SDE's with irregular drift coefficients which additionally yields the important insight that these solutions are Malliavin differentiable. See also [26] and [34]. More precisely, we derive the results in [28] without assuming a certain symmetry condition [27, Definition 3] on the drift b in (1.1), which severely restricts the class of SDE's to be studied. In particular, one of our main results is the extension of [27, Theorem 4] on the Malliavin differentiability of solutions of (1.1) for merely bounded Borel functions b from the one-dimensional to the multidimensional case.

Our approach is mainly based on Malliavin calculus. To be more precise, our technique relies on a compactness criterion based on Malliavin calculus and an approximation argument for certain generalized processes in the Hida distribution space which we directly verify to be strong solutions of (1.1). We remark that our construction method is different from the above mentioned authors' ones. The technique proposed in this paper is not based on a pathwise uniqueness argument (or the Yamada-Watanabe theorem). In fact we tackle the construction problem from the "opposite" direction and prove that strong existence in connection with uniqueness in law of solutions of SDE's enforces strong uniqueness.

The additional information that strong solutions of SDE's with merely measurable drift coefficients are Malliavin differentiable has important and interesting implications. For example, it entails that for all $0 \leq t \leq T$:

$$\left| X_t(\omega + \int_0^{\cdot} h(s)ds) - X_t(\omega) \right| \leq C \|h\|_{L^2([0,T])} , \tag{1.2}$$

for almost all $\omega \in \Omega = C_0([0,T])$ (Wiener space) and $h \in L^2([0,T])$, where C is a constant, see e.g. [30]. By considering the "initial condition" $y = x + B_t(\omega)$ in the ODE

$$\begin{aligned} \frac{d}{dt} X_t^y &= b(t, X_t^y) \\ X_0^y &= y \, , \end{aligned}$$

relation (1.2) in connection with (1.1) actually gives an interesting "link" to the flow property of solutions of ODE's with discontinuous coefficients. This may be of use in perturbation problems of discontinuous ordinary differential equations and other applications. See e.g. [24]. For recent advances on the existence of stochastic flows of Hölder homeomorphisms for solutions of SDEs with irregular drift coefficients see e.g. [11].

Finally, we mention that our technique may be applied to examine strong solutions of

$$dX_t = b(t, X_t)dt + dB_t^Q, \; X_0 = x \in H, \tag{1.3}$$

where B_t^Q is a Q-cylindrical Brownian motion on a Hilbert space H and Q a positive symmetric trace class operator. Applications to certain classes of SPDE's are also conceivable. See [25]. We point out that equations of the type (1.3) are not accessible within the framework of the above mentioned authors. For example, the construction method of the authors in [15] heavily rests on an estimate of Krylov [18], which has no extension to infinite dimensions.

The paper is organized as follows: In Section 2 we recall basic concepts of Malliavin calculus and Gaussian white noise theory. Section 3 is devoted to the study of the SDE (1.1). The main results of the paper are Theorem 3.3, Lemma 3.5, Corollary 3.6, and Theorem 3.17.

2. Framework

In this section we recall some facts from Gaussian white noise analysis and Malliavin calculus, which we aim at employing in Section 3 to construct strong solutions of SDE's. See [16, 31, 20] for more information on white noise theory. As for Malliavin calculus the reader is referred to [30, 22, 23, 8].

2.1. **Basic Facts of Gaussian White Noise Theory.** A building block of our proof for the constuction of strong solutions (see Section 3) is based on a generalized stochastic process in the Hida distribution space which we verify to be a SDE solution. In the following, we shall give the definition of this space which goes back to T. Hida (see [16]).

From now on we fix a time horizon $0 < T < \infty$. Consider a (positive) self-adjoint operator A on $L^2([0,T])$ with $Spec(A) > 1$. Let us require that A^{-r} is of Hilbert-Schmidt type for some $r > 0$. Denote by $\{e_j\}_{j\geq 0}$ a complete orthonormal basis of $L^2([0,T])$ in $Dom(A)$ and let $\lambda_j > 0$, $j \geq 0$ be the eigenvalues of A such that

$$1 < \lambda_0 \leq \lambda_1 \leq ... \longrightarrow \infty.$$

Let us assume that each basis element e_j is a continuous function on $[0,T]$. Further let $O_\lambda, \lambda \in \Gamma$, be an open covering of $[0,T]$ such that

$$\sup_{j\geq 0} \lambda_j^{-\alpha(\lambda)} \sup_{t\in O_\lambda} |e_j(t)| < \infty$$

for $\alpha(\lambda) \geq 0$.

In what follows let $\mathcal{S}([0,T])$ denote the standard countably Hilbertian space constructed from $(L^2([0,T]), A)$. See [31]. Then $\mathcal{S}([0,T])$ is a nuclear subspace of $L^2([0,T])$. We denote by $\mathcal{S}'([0,T])$ the corresponding conuclear space, that is the topological dual of $\mathcal{S}([0,T])$. Then the Bochner-Minlos theorem provides the existence of a unique probability measure π on $\mathcal{B}(\mathcal{S}'([0,T]))$ (Borel $\sigma-$algebra of $\mathcal{S}'([0,T])$) such that

$$\int_{\mathcal{S}'([0,T])} e^{i\langle\omega,\phi\rangle}\pi(d\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2([0,T])}}$$

holds for all $\phi \in \mathcal{S}([0,T])$, where $\langle\omega,\phi\rangle$ is the action of $\omega \in \mathcal{S}'([0,T])$ on $\phi \in \mathcal{S}([0,T])$. Set

$$\Omega_i = \mathcal{S}'([0,T])\,, \quad \mathcal{F}_i = \mathcal{B}(\mathcal{S}'([0,T]))\,, \quad \mu_i = \pi\,,$$

for $i = 1, \ldots, d$. Then the product measure

$$\mu = \mathop{\times}_{i=1}^{d} \mu_i \tag{2.1}$$

on the measurable space

$$(\Omega, \mathcal{F}) := \left(\prod_{i=1}^{d} \Omega_i, \bigotimes_{i=1}^{d} \mathcal{F}_i\right) \tag{2.2}$$

is referred to as *d-dimensional white noise probability measure.*

Consider the Doleans-Dade exponential

$$\widetilde{e}(\phi,\omega) = \exp\left(\langle\omega,\phi\rangle - \frac{1}{2}\|\phi\|^2_{L^2([0,T];\mathbb{R}^d)}\right),$$

for $\omega = (\omega_1, \ldots, \omega_d) \in (\mathcal{S}'([0,T]))^d$ and $\phi = (\phi^{(1)}, \ldots, \phi^{(d)}) \in (\mathcal{S}([0,T]))^d$, where $\langle\omega,\phi\rangle := \sum_{i=1}^d \langle\omega_i,\phi_i\rangle$.

In the following let $\left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes}n}$ be the $n-$th completed symmetric tensor product of $(\mathcal{S}([0,T]))^d$ with itself. One verifies that $\widetilde{e}(\phi,\omega)$ is holomorphic in ϕ around zero. Hence there exist generalized Hermite polynomials $H_n(\omega) \in \left(\left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes}n}\right)'$ such that

$$\widetilde{e}(\phi,\omega) = \sum_{n\geq 0} \frac{1}{n!}\left\langle H_n(\omega), \phi^{\otimes n}\right\rangle \tag{2.3}$$

for ϕ in a certain neighbourhood of zero in $(\mathcal{S}([0,T]))^d$. It can be shown that

$$\left\{\left\langle H_n(\omega), \phi^{(n)}\right\rangle : \phi^{(n)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes}n},\ n \in \mathbb{N}_0\right\} \tag{2.4}$$

is a total set of $L^2(\mu)$. Further one finds that the orthogonality relation

$$\int_{\mathcal{S}'} \left\langle H_n(\omega), \phi^{(n)} \right\rangle \left\langle H_m(\omega), \psi^{(m)} \right\rangle \mu(d\omega) = \delta_{n,m} n! \left(\phi^{(n)}, \psi^{(n)} \right)_{L^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})} \tag{2.5}$$

is valid for all $n, m \in \mathbb{N}_0$, $\phi^{(n)} \in \left((\mathcal{S}([0,T]))^d \right)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \left((\mathcal{S}([0,T]))^d \right)^{\widehat{\otimes} m}$ where

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{else} \end{cases} .$$

Define $\widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ as the space of square integrable symmetric functions $f(x_1, \ldots, x_n)$ with values in $(\mathbb{R}^d)^{\otimes n}$. Then the orthogonality relation (2.5) implies that the mappings

$$\phi^{(n)} \longmapsto \left\langle H_n(\omega), \phi^{(n)} \right\rangle$$

from $\left(\mathcal{S}([0,T])^d \right)^{\widehat{\otimes} n}$ to $L^2(\mu)$ possess unique continuous extensions

$$I_n : \widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n}) \longrightarrow L^2(\mu)$$

for all $n \in \mathbb{N}$. We remark that $I_n(\phi^{(n)})$ can be viewed as an $n-$fold iterated Itô integral of $\phi^{(n)} \in \widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a $d-$dimensional Wiener process

$$B_t = \left(B_t^{(1)}, \ldots, B_t^{(d)} \right) \tag{2.6}$$

on the white noise space

$$(\Omega, \mathcal{F}, \mu) . \tag{2.7}$$

It turns out that square integrable functionals of B_t admit a Wiener-Itô chaos representation which can be regarded as an infinite-dimensional Taylor expansion, that is

$$L^2(\mu) = \bigoplus_{n \geq 0} I_n(\widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})). \tag{2.8}$$

We construct the Hida stochastic test function and distribution space by using the Wiener-Itô chaos decomposition (2.8). For this purpose let

$$A^d := (A, \ldots, A) , \tag{2.9}$$

where A was the operator introduced in the beginning of the section. We define the *Hida stochastic test function space* $(\mathcal{S})$ via a second quantization argument, that is we introduce $(\mathcal{S})$ as the space of all $f = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \in L^2(\mu)$ such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| \left((A^d)^{\otimes n} \right)^p \phi^{(n)} \right\|^2_{L^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})} < \infty \tag{2.10}$$

for all $p \geq 0$. It turns out that the space $(\mathcal{S})$ is a nuclear Fréchet algebra with respect to multiplication of functions and its topology is given by the seminorms $\|\cdot\|_{0,p}$, $p \geq 0$. Further one observes that

$$\widetilde{e}(\phi, \omega) \in (\mathcal{S}) \tag{2.11}$$

for all $\phi \in (\mathcal{S}([0,T]))^d$.

In the sequel we refer to the topological dual of $(\mathcal{S})$ as *Hida stochastic distribution space* $(\mathcal{S})^*$. Thus we have constructed the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*.$$

The Hida distribution space $(\mathcal{S})^*$ exhibits the crucial property that it contains the *white noise* of the coordinates of the $d-$dimensional Wiener process B_t, that is the time derivatives

$$W_t^i := \frac{d}{dt} B_t^i, \; i = 1, \ldots, d , \tag{2.12}$$

belong to $(\mathcal{S})^*$.

We shall also recall the definition of the S-transform which is an important tool to characterize elements of the Hida test function and distribution space. See [33]. The $S-$transform of a $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined by the dual pairing

$$S(\Phi)(\phi) = \langle \Phi, \widetilde{e}(\phi, \omega) \rangle \tag{2.13}$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$. Here $\mathcal{S}_{\mathbb{C}}([0,T])$ the complexification of $\mathcal{S}([0,T])$. We mention that the $S-$transform is a monomorphism from $(\mathcal{S})^*$ to $\mathbb{C}$. In particular, if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One checks that

$$S(W_t^i)(\phi) = \phi^i(t), \;\; i = 1, ..., d \tag{2.14}$$

for $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$.

Finally, we need the important concept of the *Wick* or *Wick-Grassmann product*, which we want to use in Section 3 to represent solutions of SDE's. The Wick product can be regarded as a tensor algebra multiplication on the Fock space and can be defined as follows: The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi) S(\Psi)(\phi) \tag{2.15}$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$. As an example we find that

$$\left\langle H_n(\omega), \phi^{(n)} \right\rangle \diamond \left\langle H_m(\omega), \psi^{(m)} \right\rangle = \left\langle H_{n+m}(\omega), \phi^{(n)} \widehat{\otimes} \psi^{(m)} \right\rangle \tag{2.16}$$

for $\phi^{(n)} \in \left((\mathcal{S}([0,T]))^d \right)^{\widehat{\otimes} n}$ and $\psi^{(m)} \in \left((\mathcal{S}([0,T]))^d \right)^{\widehat{\otimes} m}$. The latter in connection with (2.3) shows that

$$\widetilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \tag{2.17}$$

for $\phi \in (\mathcal{S}([0,T]))^d$. Here the Wick exponential $\exp^{\diamond}(X)$ of a $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \tag{2.18}$$

where $X^{\diamond n} = X \diamond \ldots \diamond X$, if the sum on the right hand side converges in $(\mathcal{S})^*$.

2.2. **Basic elements of Malliavin Calculus.** In this Section we briefly elaborate a framework for Malliavin calculus.

Without loss of generality we consider the case $d = 1$. Let $F \in L^2(\mu)$. Then it follows from (2.8) that

$$F = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \tag{2.19}$$

for unique $\phi^{(n)} \in \widehat{L}^2([0,T]^n)$. Assume that

$$\sum_{n \geq 1} n n! \left\| \phi^{(n)} \right\|^2_{L^2([0,T]^n)} < \infty \, . \tag{2.20}$$

Then the *Malliavin derivative* D_t of F in the direction of B_t is defined by

$$D_t F = \sum_{n \geq 1} n \left\langle H_{n-1}(\cdot), \phi^{(n)}(\cdot, t) \right\rangle . \tag{2.21}$$

We introduce the stochastic Sobolev space $\mathbb{D}_{1,2}$ as the space of all $F \in L^2(\mu)$ such that (2.20) is fulfilled. The Malliavin derivative $D.$ is a linear operator from $\mathbb{D}_{1,2}$ to $L^2(\lambda \times \mu)$, where λ denotes the Lebesgue measure. We mention that $\mathbb{D}_{1,2}$ is a Hilbert space with the norm $\left\| \cdot \right\|_{1,2}$ given by

$$\left\| F \right\|^2_{1,2} := \left\| F \right\|^2_{L^2(\mu)} + \left\| D.F \right\|^2_{L^2([0,T] \times \Omega, \lambda \times \mu)} . \tag{2.22}$$

We obtain the following chain of continuous inclusions:

$$(\mathcal{S}) \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathbb{D}_{-1,2} \hookrightarrow (\mathcal{S})^*, \tag{2.23}$$

where $\mathbb{D}_{-1,2}$ is the dual of $\mathbb{D}_{1,2}$.

3. Main results

In this section, we want to further develop the ideas introduced in [28] to derive Malliavin differentiable strong solutions of stochastic differential equations with discontinuous coefficients. More precisely, we aim at analyzing the SDE's of the form

$$dX_t = b(t, X_t)dt + dB_t,\ 0 \le t \le 1,\ \ X_0 = x \in \mathbb{R}^d\,, \tag{3.1}$$

where the drift coefficient $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable function and B_t is a d-dimensional Brownian motion with respect to the stochastic basis

$$(\Omega, \mathcal{F}, \mu)\,, \{\mathcal{F}_t\}_{0\le t\le T} \tag{3.2}$$

for the $\mu-$augmented filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ generated by B_t. At the end of this section we shall also apply our technique to equations with more general diffusions coefficients (Theorem 3.17).

Our method to construct strong solution is actually motivated by the following observation in [21] and [26] (see also [27]).

Proposition 3.1. *Suppose that the drift coefficient $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ in (3.1) is bounded and Lipschitz continuous. Then the unique strong solution $X_t = (X_t^1, ..., X_t^d)$ of (3.1) allows for the explicit representation*

$$\varphi\left(t, X_t^i(\omega)\right) = E_{\widetilde{\mu}}\left[\varphi\left(t, \widetilde{B}_t^i(\widetilde{\omega})\right)\mathcal{E}_T^{\diamond}(b)\right] \tag{3.3}$$

for all $\varphi : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\varphi\left(t, B_t^i\right) \in L^2(\mu)$ for all $0 \le t \le T$, $i = 1, \ldots, d,$. The object $\mathcal{E}_T^{\diamond}(b)$ is given by

$$\begin{aligned}\mathcal{E}_T^{\diamond}(b)(\omega, \widetilde{\omega}) := \quad & \exp^{\diamond}\Big(\textstyle\sum_{j=1}^d \int_0^T \left(W_s^j(\omega) + b^j(s, \widetilde{B}_s(\widetilde{\omega}))\right) d\widetilde{B}_s^j(\widetilde{\omega}) \\ & -\tfrac{1}{2}\int_0^T \left(W_s^j(\omega) + b^j(s, \widetilde{B}_s(\widetilde{\omega}))\right)^{\diamond 2} ds\Big).\end{aligned} \tag{3.4}$$

Here $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mu}\right), \left(\widetilde{B}_t\right)_{t\ge 0}$ is a copy of the quadruple $(\Omega, \mathcal{F}, \mu)\,, (B_t)_{t\ge 0}$ in (3.2). Further $E_{\widetilde{\mu}}$ denotes a Pettis integral of random elements $\Phi : \widetilde{\Omega} \longrightarrow (\mathcal{S})^*$ with respect to the measure $\widetilde{\mu}$. The Wick product $\diamond$ in the Wick exponential of (3.4) is taken with respect to μ and W_t^j is the white noise of B_t^j in the Hida space $(\mathcal{S})^*$ (see (2.12)). The stochastic integrals $\int_0^T \phi(t, \widetilde{\omega})d\widetilde{B}_s^j(\widetilde{\omega})$ in (3.4) are defined for predictable integrands ϕ with values in the conuclear space $(\mathcal{S})^*$. See [17] *for definitions. The other integral type in (3.4) is to be understood in the sense of Pettis.*

Remark 3.2. *Let $0 = t_1^n < t_2^n < \ldots < t_{m_n}^n = T$ be a sequence of partitions of the interval $[0,T]$ with $\max_{i=1}^{m_n-1}\left|t_{i+1}^n - t_i^n\right| \longrightarrow 0$. Then the stochastic integral of the white noise W^j can be approximated as follows:*

$$\int_0^T W_s^j(\omega)d\widetilde{B}_s^j(\widetilde{\omega}) = \lim_{n\longrightarrow\infty}\sum_{i=1}^{m_n}(\widetilde{B}_{t_{i+1}^n}^j(\widetilde{\omega}) - \widetilde{B}_{t_i^n}^j(\widetilde{\omega}))W_{t_i^n}^j(\omega)$$

in $L^2(\lambda \times \widetilde{\mu}; (\mathcal{S})^*)$. For more information about stochastic integration on conuclear spaces the reader may consult [17].

In the sequel we shall use the notation $Y_t^{i,b}$ for the expectation on the right hand side of (3.3) for $\varphi(t,x) = x$, that is

$$Y_t^{i,b} := E_{\widetilde{\mu}}\left[\widetilde{B}_t^{(i)}\mathcal{E}_T^{\diamond}(b)\right]$$

for $i = 1, \ldots, d$. We set

$$Y_t^b = \left(Y_t^{1,b}, \ldots, Y_t^{d,b}\right). \tag{3.5}$$

The form of Formula (3.3) in Proposition 3.1 actually suggests that the expectation on the right hand side or Y_t^b in (3.5) may also represent solutions of (3.1) for merely measurable drift coefficients b. The latter naturally leads to the following question: Can one specify conditions on b under which one succeeds to directly verify the generalized process Y_t^b to be a (strong) solution of (3.1)? This question was successfully treated for the one-dimensional case using a comparison argument in [26] and for the multidimensional case under a rather strong symmetry condition on the drift b using Malliavin calculus in [28]. In this paper we considerably improve the results given in [28] by removing the symmetry condition on b. Our main result in this paper is the following theorem:

Theorem 3.3. *Suppose that the drift coefficient $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ in (3.1) is a bounded Borel-measurable function. Then there exists a unique global strong solution X to Equation (3.1) such that X_t is Malliavin differentiable for all $0 \le t \le 1$.*

Remark 3.4. *In the one-dimensional case the existence and uniqueness of strong solutions to (3.1) for bounded and measurable drift coefficients was first obtained by Zvonkin in his celebrated paper [47]. The extension to the multi-dimensional case was given by [45]. We point out that our solution technique grants the important additional insight that such solutions are Malliavin differentiable. We remark that Theorem 3.3 is a generalization of [27, Theorem 5] from the one-dimensional to the multi-dimensional case. Let us also mention that we considerably improve the technique initiated in [28] (see also [26] and [34]) by removing a certain symmetry condition on the drift coefficients in (3.1) (see [27, Definition 3]), which severely limits the class of SDE's to be analyzed. The removal of the latter condition, however, may actually pave the way for the construction of strong solutions of discontinuous infinite dimensional stochastic equations of the type (1.3) or SPDE's. See [25]. We point out that the methods of the authors mentioned in the introduction fail in this case.*

To prove Theorem 3.3 we follow a procedure consisting of two steps (compare [28]). In the **first step**, we show for a sequence of uniformly bounded, smooth coefficients $b_n : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $n \ge 1$, with compact support that for each $0 \le t \le 1$ the sequence of corresponding strong solutions $X_{n,t} = Y_t^{b_n}$, $n \ge 1$, is relatively compact in $L^2(\mu; \mathbb{R}^d)$ (Corollary 3.6). The main tool to prove compactness is the bound in Lemma 3.5 in connection with a compactness criteria in terms of Malliavin derivatives obtained in [6] (see Appendix A). This step is one of the main contribution of this paper.

Given a merely measurable and bounded drift coefficient b, we then show in the **second step** that Y_t^b, $0 \le t \le 1$ is a generalized process in the Hida distribution space, and we apply the S-transform 2.13 to prove that for a given sequence of a.e. approximating, uniformly bounded, smooth coefficients b_n with compact support a subsequence of the corresponding strong solutions $X_{n_j,t} = Y_t^{b_{n_j}}$ fulfills

$$Y_t^{b_{n_j}} \to Y_t^b$$

in $L^2(\mu; \mathbb{R}^d)$ for $0 \le t \le 1$ (Lemma 3.14). Using a certain transformation property for Y_t^b (Lemma 3.16) we directly verify Y_t^b as a solution to (3.1) which in addition is Malliavin differentiable.

We now turn to the first step of our procedure. The successful completion of the first step relies on the following essential lemma:

Lemma 3.5. *Let $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function with compact support. Then the corresponding strong solution X in (3.1) fulfills*

$$E\left[\|D_t X_s - D_{t'} X_s\|^2\right] \le C_d(\|b\|_\infty)|t - t'|^\alpha$$

for $0 \le t' \le t \le 1$, $\alpha = \alpha(s) > 0$ and

$$\sup_{0 \le t \le 1} E\left[\|D_t X_s\|^2\right] \le C_d(\|b\|_\infty)$$

where $C_d : [0,\infty) \to [0,\infty)$ is an increasing, continuous function, $\|\cdot\|$ a matrix-norm on $\mathbb{R}^{d\times d}$ and $\|\cdot\|_\infty$ the supremum norm.

From Lemma 3.5 together with Corollary A.3 we immediately obtain the main result of step one of our procedure:

Corollary 3.6. *Let $b_n : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $n \geq 1$, be a sequence of uniformly bounded, smooth coefficients with compact support. Then for each $0 \leq t \leq 1$ the sequence of corresponding strong solutions $X_{n,t} = Y_t^{b_n}$, $n \geq 1$, is relatively compact in $L^2(\mu; \mathbb{R}^d)$.*

In order to prove Lemma 3.5 we need the following estimate, which can be considered a generalization of a bound given in [7, Proposition 2.2]:

Proposition 3.7. *Let B be a d-dimensional Brownian Motion starting from the origin and $b_1, \ldots, b_n$ be compactly supported continuously differentiable functions $b_i : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ for $i = 1, 2, \ldots n$. Let $\alpha_i \in \{0,1\}^d$ be a multiindex such that $|\alpha_i| = 1$ for $i = 1, 2, \ldots, n$. Then there exists a universal constant C (independent of $\{b_i\}_i$, n, and $\{\alpha_i\}_i$) such that*

$$\left| E\left[\int_{t_0 < t_1 < \cdots < t_n < t} \left(\prod_{i=1}^n D^{\alpha_i} b_i(t_i, B(t_i)) \right) dt_1 \ldots dt_n \right] \right| \leq \frac{C^n \prod_{i=1}^n \|b_i\|_\infty (t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)} \tag{3.6}$$

where Γ is the Gamma-function. Here D^{α_i} denotes the partial derivative with respect to the j'th space variable, where j is the position of the 1 in α_i.

Proof. Without loss of generality, assume that $\|b_i\|_\infty \leq 1$ for $i = 1, 2 \ldots, n$. Denote by $z = (z^{(1)}, \ldots z^{(d)})$ a generic element of $\mathbb{R}^d$ and by $\| \cdot \|$ the usual Euclidian norm. With $P(t,z) = (2\pi t)^{-d/2} e^{-\|z\|^2/2t}$, write the left hand side in (3.6) as

$$\left| \int_{t_0 < t_1 < \cdots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \ldots dz_n dt_1 \ldots dt_n \right| .$$

Introduce the notation

$$J_n^\alpha(t_0, t, z_0) = \int_{t_0 < t_1 < \cdots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \ldots dz_n dt_1 \ldots dt_n$$

where $\alpha = (\alpha_1, \ldots \alpha_n) \in \{0,1\}^{nd}$. We shall show that $|J_n^\alpha(t_0, t, 0)| \leq C^n (t - t_0)^{n/2} / \Gamma(n/2 + 1)$, thus proving the proposition.

To do this, we will use integration by parts to shift the derivatives onto the Gaussian kernel. This will be done by introducing the alphabet $\mathcal{A}(\alpha) = \{P, D^{\alpha_1} P, \ldots, D^{\alpha_n} P, D^{\alpha_1} D^{\alpha_2} P, \ldots D^{\alpha_{n-1}} D^{\alpha_n} P\}$ where D^{α_i}, $D^{\alpha_i} D^{\alpha_{i+1}}$ denotes the derivatives in z on $P(t,z)$.

Take a string $S = S_1 \cdots S_n$ in $\mathcal{A}(\alpha)$ and define

$$I_S^\alpha(t_0, t, z_0) = \int_{t_0 < \cdots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n b_i(t_i, z_i) S_i(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \ldots dz_n dt_1 \ldots dt_n .$$

We will only need a special type of strings, and we say that a string is *allowed* if, when all the $D^{\alpha_i} P$'s are removed from the string, a string of the form $P \cdot D^{\alpha_s} D^{\alpha_{s+1}} P \cdot P \cdot D^{\alpha_{s+1}} D^{\alpha_{s+2}} P \cdots P \cdot D^{\alpha_r} D^{\alpha_{r+1}} P$ for $s \geq 1$, $r \leq n-1$ remains. Also, we will require that the first derivatives $D^{\alpha_i} P$ are written in an increasing order with respect to i.

Before we proceed with the proof of Proposition 3.7 we will need some intermediate results.

Lemma 3.8. *We can write*

$$J_n^\alpha(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} \epsilon_j I_{S^j}^\alpha(t_0, t, z_0)$$

where each ϵ_j is either -1 or 1 and each S^j is an allowed string in $\mathcal{A}(\alpha)$.

Proof. The equation obviously holds for $n=1$. Assume the equation holds for $n \ge 1$, and let b_0 be another function satisfying the requirements of the proposition. Likewise with α_0. Then

$$\begin{aligned} J_{n+1}^{(\alpha_0,\alpha)}(t_0,t,z_0) &= \int_{t_0}^t \int_{\mathbb{R}^d} D^{\alpha_0} b_0(t_1,z_1) P(t_1-t_0, z_1-z_0) J_n^\alpha(t_1,t,z_1) dz_1 dt_1 \\ &= -\int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1,z_1) D^{\alpha_0} P(t_1-t_0,z_1-z_0) J_n^\alpha(t_1,t,z_1) dz_1 dt_1 \\ &\quad - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1,z_1) P(t_1-t_0,z_1-z_0) D^{\alpha_0} J_n^\alpha(t_1,t,z_1) dz_1 dt_1 \,. \end{aligned}$$

Notice that

$$D^{\alpha_0} I_S^\alpha(t_1,t,z_1) = -I_{\tilde{S}}^{(\alpha_0,\alpha)}(t_1,t,z_1)$$

where

$$\tilde{S} = \begin{cases} D^{\alpha_0} P \cdot S_2 \cdots S_n & \text{if } S = P \cdot S_2 \cdots S_n \\ D^{\alpha_0} D^{\alpha_1} P \cdot S_2 \cdots S_n & \text{if } S = D^{\alpha_1} P \cdot S_2 \cdots S_n \,. \end{cases}$$

Here, $\tilde{S}$ is not an allowed string in $\mathcal{A}(\alpha)$. So from the induction hyptothesis $D^{\alpha_0} J_n^\alpha(t_0,t,z_0) = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{\tilde{S}}^{(\alpha_0,\alpha)}(t_0,t,z_0)$ this gives

$$J_{n+1}^{(\alpha_0,\alpha)} = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{D^{\alpha_0} P \cdot S^j}^{(\alpha_0,\alpha)} + \sum_{j=1}^{2^{n-1}} \epsilon_j I_{P \cdot \tilde{S}^j} \,.$$

It is easily checked that when S^j is an allowed string in $\mathcal{A}(\alpha)$, both $D^{\alpha_0} P \cdot S^j$ and $P \cdot \tilde{S}^j$ are allowed strings in $\mathcal{A}(\alpha_0,\alpha)$.

□

For the rest of the proof of Proposition 3.7 we will bound I_S^α when S is an allowed string, and the result will follow from the above representation.

Lemma 3.9. *Let $\phi, h : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that $|\phi(s,z)| \le e^{-\|z\|^2/3s}$ and $\|h\|_\infty \le 1$. Also let $\alpha, \beta \in \{0,1\}^d$ be multiindices such that $|\alpha| = |\beta| = 1$. Then there exists a universal constant C (independent of ϕ, h, α and β) such that*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s,z) h(t,y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \right| \le C \,.$$

Proof. Let $l, m \in \mathbb{Z}^d$ and denote $[l, l+1) := [l^{(1)}, l^{(1)}+1) \times \cdots \times [l^{(d)}, l^{(d)}+1)$ and similarly for $[m, m+1)$. Define $\phi_l(s,z) = \phi(s,z) 1_{[l,l+1)}(z)$ and $h_m(t,y) = h(t,y) 1_{[m,m+1)}$.

Denote the above integral by I, and $I_{l,m}$ the integral when ϕ, h is replaced by ϕ_l, h_m. Then we can write $I = \sum_{l,m \in \mathbb{Z}^d} I_{l,m}$. Below we let C be a generic constant that may vary from line to line.

Assume $\|l-m\|_\infty := \max_i |l^{(i)} - m^{(i)}| \ge 2$. For $z \in [l, l+1)$ and $y \in [m, m+1)$ we have $\|z-y\| \ge \|l-m\|_\infty - 1$. If $\alpha \ne \beta$ we have that

$$D^\alpha D^\beta P(t-s, z-y) = \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t-s)^2} P(t-s, y-z)$$

for a suitable choice of i, j. Then we can find C such that

$$|D^\alpha D^\beta P(t-s, z-y)| \le C e^{-(\|l-m\|_\infty - 2)^2/4}.$$

If $\alpha = \beta$, we have

$$(D^\alpha)^2 P(t-s, y-z) = \left(\frac{(y^{(i)} - z^{(i)})^2}{t-s} - 1 \right) \frac{P(t-s, y-z)}{t-s}$$

and similarly we find C such that

$$|(D^\alpha)^2 P(t-s, y-z)| \le C e^{-(\|l-m\|_\infty - 2)^2/4} \,.$$

In both cases we have $|I_{l,m}| \leq Ce^{-\|l\|^2/8}e^{-(\|l-m\|_\infty-2)^2/4}$ and it follows that

$$\sum_{\|l-m\|_\infty \geq 2} |I_{l,m}| \leq C.$$

Assume $\|l-m\|_\infty \leq 1$ and let $\hat{\phi}_l(s,u)$ and $\hat{h}_m(t,u)$ be the Fourier transform in the second variable. By the Plancherel theorem we have that

$$\int_{\mathbb{R}^d} \hat{\phi}_l(s,u)^2 du = \int_{\mathbb{R}^d} \phi_l(s,z)^2 dz \leq Ce^{-\|l\|^2/6}$$

for all $s \in [0,1]$ and

$$\int_{\mathbb{R}^d} \hat{h}_m(t,u)^2 du = \int_{\mathbb{R}^d} h_m(t,y)^2 dy \leq 1.$$

We can write

$$I_{l,m} = \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s,u)\hat{h}_m(t,-u)u^{(i)}u^{(j)}e^{-(t-s)\|u\|^2/2}dudsdt$$

for a suitable choice of i and j. To see this, notice that with $p(u) = u^{(i)}u^{(j)}$ and $f(u) = e^{-(t-s)\|u\|^2/2}$ we have $\widehat{(p\cdot f)}(y-z) = D^\alpha D^\beta \hat{f}(y-z)$. Also, note that $\hat{P}(1,\cdot) = P(1,\cdot)$. The result follows by substituting $v = \sqrt{t-s}u$ in the integral.

Applying $ab \leq \frac{1}{2}a^2c + \frac{1}{2}b^2c^{-1}$ with $a = \hat{\phi}_l(s,u)u^{(i)}$, $b = \hat{h}_m(t,-u)u^{(j)}$ and $c = e^{\|l\|^2/12}$ we get

$$\begin{aligned}
|I_{l,m}| &\leq \frac{1}{2}\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s,u)^2(u^{(i)})^2 e^{\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \\
&\quad + \frac{1}{2}\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t,-u)^2(u^{(j)})^2 e^{-\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \\
&\leq \frac{1}{2}\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s,u)^2\|u\|^2 e^{\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \\
&\quad + \frac{1}{2}\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t,-u)^2\|u\|^2 e^{-\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt.
\end{aligned}$$

For the first term, integrate first with respect to t in order to get

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s,u)^2\|u\|^2 e^{\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \leq Ce^{-\|l\|^2/12}$$

and for the second term, integrate with respect to s first to get

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t,-u)^2\|u\|^2 e^{-\|l\|^2/12}e^{-(t-s)\|u\|^2/2}dudsdt \leq Ce^{-\|l\|^2/12}$$

which gives $|I_{l,m}| \leq Ce^{-\|l\|^2/12}$ and hence

$$\sum_{\|l-m\|_\infty \leq 1} |I_{l,m}| \leq C.$$

□

Corollary 3.10. *There exists an absolute constant C such that for measurable functions g and h bounded by 1*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)dydzdsdt \right| \leq C$$

and

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,z)D^\gamma P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)dydzdsdt \right| \leq C\,.$$

Notice that we have $\int_{\mathbb{R}^d} P(t,z)dz = 1$ and that

$$\int_{\mathbb{R}^d} |D^\alpha P(t,z)|dz \le Ct^{-1/2}\,, \tag{3.7}$$

$$\int_{\mathbb{R}^d} |D^\alpha D^\beta P(t,z)|dz \le Ct^{-1}\,. \tag{3.8}$$

Lemma 3.11. *There is an absolute constant C such that for every Borel-measurable functions g and h bounded by 1, and $r \ge 0$*

$$\left|\int_{t_0}^{t}\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t_2,z)P(t_2-t_0,z)h(t_1,y)D^\alpha D^\beta P(t_1-t_2,y-z)(t-t_1)^r dydzdt_2dt_1\right| \le C(1+r)^{-1}(t-t_0)^{r+1}$$

and

$$\left|\int_{t_0}^{t}\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t_2,z)D^\gamma E(t_2-t_0,z)h(t_1,y)D^\alpha D^\beta P(t_1-t_2,y-z)(t-t_1)^r dydzdt_2dt_1\right| \le C(1+r)^{-1/2}(t-t_0)^{r+1/2}\,.$$

Proof. We begin by proving the estimate for $t = t_0 = 0$. From Corollary 3.10 we have that for each $k \ge 0$

$$\left|\int_{2^{-k-1}}^{2^{-k}}\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)(1-t)^r dydzdsdt\right| \le C(1-2^{-k-1})^r 2^{-k}\,.$$

To see this, make the substitutions $t' = 2^k t$ and $s' = 2^k s$. Use the easily verified fact that $P(at,z) = a^{-d/2}P(t,a^{-1/2}z)$ and substitute $z' = 2^{k/2}z$ and $y' = 2^{k/2}y$. Using $\tilde{h}(t,y) := \frac{(1-t)^r}{(1-2^{-k-1})^r}h(t,y)$ in Corollary (3.10), the result follows.

Summing this equation over k gives

$$\left|\int_{0}^{1}\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)(1-t)^r dydzdsdt\right| \le C(1+r)^{-1}$$

Moreover from the bound (3.8)

$$\left|\int_{0}^{1}\int_{0}^{t/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,z)P(s,z)h(t,y)D^\alpha D^\beta P(t-s,y-z)(1-t)^r dydzdsdt\right| \le C\int_0^1\int_0^{t/2}(t-s)^{-1}(1-t)^r dsdt \le C(1+r)^{-1}$$

and combining these bounds gives the first assertion for $t = t_0 = 0$. For general t and t_0 use the change of variables $t_1' = \frac{t_1-t_0}{t-t_0}$, $t_2 = \frac{t_2-t_0}{t-t_0}$, $y' = (t-t_0)^{-1/2}y$ and $z' = (t-t_0)^{-1/2}z$.

The second assertion is proved similary. □

We turn to the completion of the proof of Proposition 3.7 by showing that there exists a constant M such that for each allowed string S in the alphabet $\mathcal{A}(\alpha)$ we have

$$I_S^\alpha(t_0,t,z_0) \le \frac{M^n(t-t_0)^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

We will prove this by induction on n. The case $n = 0$ is immediate, so assume $n > 0$ and that this holds for all allowed strings of length less than n. There are three cases

1. $S = D^{\alpha_1}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_2,\dots,\alpha_n)$
2. $S = P \cdot D^{\alpha_1}D^{\alpha_2}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_3,\dots,\alpha_n)$
3. $S = P \cdot D^{\alpha_1}P \cdots D^{\alpha_m}P \cdot D^{\alpha_{m+1}}D^{\alpha_{m+2}}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_{m+3},\dots,\alpha_n)$.

In each case, S' is an allowed string in the given alphabet.

(1) We use the inductive hypothesis to bound $I^{\alpha'}_{S'}(t_1, t, z_1)$ and the bound (3.7) to get

$$\begin{aligned}
|I^{\alpha}_S(t_0, t, z_0)| &= \left| \int_{t_0}^t \int_{\mathbb{R}^d} b_1(t_1, z_1) D^{\alpha_1} P(t_1 - t_0, z_1 - z_0) I^{\alpha'}_{S'}(t_1, t, z_1) dz_1 dt_1 \right| \\
&\leq \frac{M^{n-1}}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} \int_{\mathbb{R}^d} |D^{\alpha_1} P(t_1 - t_0, z_1 - z_0)| dz_1 dt_1 \\
&\leq \frac{M^{n-1} C}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} (t_1 - t_0)^{-1/2} dt_1 \\
&= \frac{M^{n-1} C \sqrt{\pi} (t - t_0)^{k/2}}{\Gamma(\frac{n}{2} + 1)}.
\end{aligned}$$

The result follows if M is large enough.

(2) For this case we can write

$$\begin{aligned}
I^{\alpha}_S(t_0, t, z_0) = \int_{t_0}^t \int_{t_1}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & b_1(t_1, z_1) b_2(t_2, z_2) \\
& \times P(t_1 - t_0, z_1 - z_0) D^{\alpha_1} D^{\alpha_2} P(t_2 - t_1, z_2 - z_1) I^{\alpha'}_{S'}(t_2, t, z_2) dz_1 dz_2 dt_2 dt_1.
\end{aligned}$$

We set $h(t_2, z_2) := b_2(t_2, z_2) I^{\alpha'}_{S'}(t_2, z_2)(t - t_2)^{1-n/2}$ so that by the inductive hypothesis we have

$$\|h\|_\infty \leq M^{n-2}/\Gamma(n/2).$$

Use this in the first part of Lemma 3.11 with $g = b_1$ and integrate with respect to t_2 first, to get

$$|I^{\alpha}_S(t_0, t, z_0)| \leq \frac{C M^{n-2} (t - t_0)^{n/2}}{n \Gamma(n/2)},$$

and the result follows if M is large enough.

(3) We have

$$\begin{aligned}
I^{\alpha}_S(t_0, t, z_0) = & \int_{t_0 < \dots t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^{m+2} b_j(t_j, z_j) \\
& \times \prod_{j=2}^{m} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) \\
& \times I^{\alpha'}_{S'}(t_{m+2}, t, z_{m+2}) dz_1 \dots dz_{m+2} dt_1 \dots dt_{m+2}.
\end{aligned}$$

Let $h(t_{m+2}, z_{m+2}) = b_{m+2}(t_{m+2}, z_{m+2}) I^{\alpha'}_{S'}(t_{m+2}, t, z)(t - t_{m+2})^{(2+m-n)/2}$, so that from the inductive hypothesis we have $\|h\|_\infty \leq M^{n-m-2}/\Gamma((n-m)/2)$. Write

$$\begin{aligned}
\Omega(t_m, z_m) := & \int_{t_m}^t \int_{t_{m+1}}^t \int_{\mathbb{R}^{2d}} b_{m+1}(t_{m+1}, z_{m+1}) h(t_{m+2}, z_{m+2}) \\
& \times (t - t_{m+2})^{(n-m-2)/2} D^{\alpha_m} P(t_{m+1} - t_m, z_{m+1} - z) \\
& \times D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) dz_{m+1} dz_{m+2} dt_{m+1} dt_{m+2},
\end{aligned}$$

so that from Lemma (3.11) we have that

$$|\Omega(t_m, z_m)| \leq \frac{C(n-m)^{-1/2} M^{n-m-2} (t - t_m)^{(n-m-1)/2}}{\Gamma(\frac{n-m}{2})}.$$

Using this in

$$I_S^\alpha(t_0,t,z_0) = \int_{t_0<\ldots t_{m+2}<t}\int_{\mathbb{R}^{(m+2)d}} P(t_1-t_0,z_1-z_0)\prod_{j=1}^m b_j(t_j,z_j)$$
$$\times \prod_{j=1}^{m-1} D^{\alpha_j}P(t_j-t_{j-1},z_j-z_{j-1})\Omega(t_m,z_m)dz_1\ldots dz_m dt_1\ldots dt_m\,,$$

and using the bound (3.7) several times gives

$$\begin{aligned}|I_S^\alpha(t_0,t,z_0)| &\le C^{m+1}(n-m)^{-1/2}\frac{M^{n-m-2}}{\Gamma((n-m)/2)}\\ &\quad\times\int_{t_0<\ldots t_m<t}(t_2-t_1)^{-1/2}\ldots(t_m-t_{m-1})^{-1/2}(t-t_m)^{(n-m-1)/2}dt_1\ldots dt_m\\ &= C^{m+1}(n-m)^{-1/2}\frac{M^{n-m-2}\pi^{(m-1)/2}\Gamma(\frac{n-m+1}{2})}{\Gamma(\frac{n-m}{2})\Gamma(\frac{n}{2}+1)}(t-t_0)^{n/2}\,,\end{aligned}$$

and the result follows when M is large enough, thus proving the induction step.

□

We are now ready to complete the proof of Lemma 3.5.

Proof of Lemma 3.5. Using the chain-rule of the Malliavin derivative D_t (see [30]) we find that

$$D_tX_s = \mathcal{I}_d + \int_t^s b'(u,X_u)D_tX_u du \tag{3.9}$$

μ-a.e. for all $1\ge t\ge s$, where $\mathcal{I}_d$ is the $d\times d$ identity matrix and $b' = \left(\frac{\partial}{\partial x_i}b^{(j)}(t,x)\right)_{1\le i,j\le d}$ is the (bounded) space derivative of b.

Fix $0\le t'\le t<1$. Then, for $1\ge s\ge t$ we have

$$\begin{aligned}D_{t'}X_s - D_tX_s &= \int_{t'}^s b'(u,X_u)D_{t'}X_u du - \int_t^s b'(u,X_u)D_tX_u du\\ &= \int_{t'}^t b'(u,X_u)D_{t'}X_u du + \int_t^s b'(u,X_u)\left(D_{t'}X_u - D_tX_u\right)du\\ &= D_{t'}X_t - \mathcal{I}_d + \int_t^s b'(u,X_u)\left(D_{t'}X_u - D_tX_u\right)du.\end{aligned}$$

Applying Picard iteration to the above equation we find that

$$\begin{aligned}&D_{t'}X_s - D_tX_s\\ &= \left(\mathcal{I}_d + \sum_{n=1}^\infty\int_{t<s_1<\cdots<s_n<s} b'(s_1,X_{s_1}):\cdots:b'(s_n,X_{s_n})ds_1\ldots ds_n\right)(D_{t'}X_t - \mathcal{I}_d)\end{aligned} \tag{3.10}$$

in $L^2(\mu)$, uniformly in s, where : denotes (non-commutative) matrix multiplication. On the other hand we also observe that

$$D_{t'}X_t - \mathcal{I}_d = \sum_{n=1}^\infty\int_{t'<s_1<\cdots<s_n<t} b'(s_1,X_{s_1}):\cdots:b'(s_n,X_{s_n})ds_1\ldots ds_n\,. \tag{3.11}$$

Denote by $\|\cdot\|$ the maximum norm on $\mathbb{R}^{d\times d}$. Then Girsanov's theorem, Hölder's inequality and the Novikov condition in connection with (3.10) and (3.11) yield

$$\begin{aligned}
E\left[\|D_{t'}X_s - D_tX_s\|^2\right] =& E\Bigg[\Bigg\|\left(\mathcal{I}_d + \sum_{n=1}^{\infty}\int_{t<s_1<\cdots<s_n<s} b'(s_1,B_{s_1}):\cdots:b'(s_n,B_{s_n})ds_1\ldots ds_n\right) \\
&\times\left(\sum_{n=1}^{\infty}\int_{t'<s_1<\cdots<s_n<t} b'(s_1,B_{s_1}):\cdots:b'(s_n,B_{s_n})ds_1\ldots ds_n\right)\Bigg\|^2 \\
&\times\mathcal{E}\left(\sum_{j=1}^{d}\int_0^1 b^{(j)}(u,B_u)dB_u^{(j)}\right)\Bigg] \\
\leq& C_1\left\|\mathcal{I}_d + \sum_{n=1}^{\infty}\int_{t<s_1<\cdots<s_n<s} b'(s_1,B_{s_1}):\cdots:b'(s_n,B_{s_n})ds_1\ldots ds_n\right\|^2_{L^8(\mu;\mathbb{R}^{d\times d})} \\
&\times\left\|\sum_{n=1}^{\infty}\int_{t'<s_1<\cdots<s_n<t} b'(s_1,B_{s_1}):\cdots:b'(s_n,B_{s_n})ds_1\ldots ds_n\right\|^2_{L^8(\mu;\mathbb{R}^{d\times d})}
\end{aligned}$$

where C_1 is a constant and $\mathcal{E}(M_t)$ denotes the Doleans-Dade exponential of a martingale M_t.

So we obtain that

$$\begin{aligned}
&E\left[\|D_{t'}X_s - D_tX_s\|^2\right] \\
&\leq C_1\left\|\mathcal{I}_d + \sum_{n=1}^{\infty}\int_{t<s_1<\cdots<s_n<s} b'(s_1,B_{s_1}):\cdots:b'(s_n,B_{s_n})ds_1\ldots ds_n\right\|^2_{L^8(\mu;\mathbb{R}^{d\times d})} \\
&\times\left\|\sum_{n=1}^{\infty}\int_{t'<s_1<\cdots<s_n<t} b'(s_1,B_{s_1}):\cdots:b'(s_n,B_{s_n})ds_1\ldots ds_n\right\|^2_{L^8(\mu;\mathbb{R}^{d\times d})} \\
&\leq C_1\Bigg(1+\sum_{n=1}^{\infty}\sum_{i,j=1}^{d}\sum_{l_1,\ldots l_{n-1}=1}^{d}\Bigg\|\int_{t<s_1<\cdots<s_n<s}\frac{\partial}{\partial x_{l_1}}b^{(i)}(s_1,B_{s_1})\frac{\partial}{\partial x_{l_2}}b^{(l_1)}(s_2,B_{s_2})\ldots \\
&\qquad\ldots\frac{\partial}{\partial x_j}b^{(l_{n-1})}(s_n,B_{s_n})ds_1\ldots ds_n\Bigg\|_{L^8(\mu;\mathbb{R})}\Bigg)^2 \\
&\times\Bigg(\sum_{n=1}^{\infty}\sum_{i,j=1}^{d}\sum_{l_1,\ldots l_{n-1}=1}^{d}\Bigg\|\int_{t'<s_1<\cdots<s_n<t}\frac{\partial}{\partial x_{l_1}}b^{(i)}(s_1,B_{s_1})\frac{\partial}{\partial x_{l_2}}b^{(l_1)}(s_2,B_{s_2})\ldots \\
&\qquad\ldots\frac{\partial}{\partial x_j}b^{(l_{n-1})}(s_n,B_{s_n})ds_1\ldots ds_n\Bigg\|_{L^8(\mu;\mathbb{R})}\Bigg)^2. \qquad (3.12)
\end{aligned}$$

Now, look at the expression

$$A := \int_{t'<s_1<\cdots<s_n<t}\frac{\partial}{\partial x_{l_1}}b^{(i)}(s_1,B_{s_1})\frac{\partial}{\partial x_{l_2}}b^{(l_1)}(s_2,B_{s_2})\ldots\frac{\partial}{\partial x_{l_n}}b^{(l_n)}(s_n,B_{s_n})ds_1\ldots ds_n. \qquad (3.13)$$

Then, using (deterministic) integration by parts, repeatedly, one finds that A^2 can be written as a sum of at most 2^{2n} summands of the form

$$\int_{t'<s_1<\cdots<s_{2n}<t} g_1(s_1)\ldots g_{2n}(s_{2n})ds_1\ldots ds_{2n}\,, \qquad (3.14)$$

where $g_l \in \left\{\frac{\partial}{\partial x_j}b^{(i)}(\cdot,B_\cdot) : 1\leq i,j\leq d\right\}$, $l=1,2\ldots 2n$. Since $A^4 = A^2A^2$, we can argue similarly and conclude that there are *at most* 2^{8n} such summands (of length $4n$). Using this principle once more we see that A^8 can be represented as a sum of at most 2^{32n} summands of the form (3.14) now with lenght $8n$.

Combining this with Proposition 3.7 we get that

$$\begin{aligned}\left\| \int_{t'<s_1<\cdots<s_n<t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \ldots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \ldots ds_n \right\|_{L^8(\mu;\mathbb{R})} \\ \leq \left(\frac{2^{32n} C^{8n} \|b\|_\infty^{8n} |t-t'|^{4n}}{\Gamma(4n+1)} \right)^{1/8} \\ \leq \frac{2^{4n} C^n \|b\|_\infty^n |t-t'|^{n/2}}{(4n!)^{1/8}}. \qquad (3.15)\end{aligned}$$

Then it follows from (3.12) that

$$\begin{aligned} E\left[\|D_t X_s - D_{t'} X_s\|^2\right] &\leq C_1 \left(1 + \sum_{n=1}^{\infty} \frac{d^{n+2} 2^{4n} C^n \|b\|_\infty^n |t-s|^{n/2}}{(4n!)^{1/8}}\right)^2 \\ &\times \left(\sum_{n=1}^{\infty} \frac{d^{n+2} 2^{4n} C^n \|b\|_\infty^n |t-t'|^{(n-1)/2}}{(4n!)^{1/8}}\right)^2 |t-t'| \\ &\leq C_d(\|b\|_\infty)|t-t'| \end{aligned}$$

for a function C_d as claimed in the theorem.

Similarly, we deduce the estimate for $\sup_{0\leq t\leq s} E[\|D_t X_s\|^2]$.

□

This concludes step one in our program and we are now coming to the second step. For a Borel-measurable, bounded coefficient b we gradually show the following:

- Y_t^b in (3.5) is a well-defined object in the Hida distribution space $(\mathcal{S})^*$, $0 \leq t \leq 1$, (Lemma 3.12).
- For any a.e. approximating sequence of uniformly bounded, smooth coefficients b_n with compact support a subsequence of the corresponding strong solutions $X_{n_j,t} = Y_t^{b_{n_j}}$, fulfills $Y_t^{b_{n_j}} \to Y_t^b$ in $L^2(\mu)$ for $0 \leq t \leq 1$ (in particular $Y_t^b \in L^2(\mu)$, $0 \leq t \leq 1$), (Lemma 3.14).
- We apply a transformation property for Y_t^b (Lemma 3.16) and identify Y_t^b as a Malliavin differential strong solution to (3.1).

The first lemma gives a criterion under which the process Y_t^b belongs to the Hida distribution space.

Lemma 3.12. *Suppose that*

$$E_\mu \left[\exp\left(36 \int_0^1 \|b(s, B_s)\|^2 \, ds\right)\right] < \infty, \qquad (3.16)$$

where the drift $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is measurable (in particular, (3.16) is valid for b bounded). Then the coordinates of the process Y_t^b, defined in (3.5), that is

$$Y_t^{i,b} = E_{\widetilde{\mu}} \left[\widetilde{B}_t^{(i)} \mathcal{E}_T^\diamond(b)\right], \qquad (3.17)$$

are elements of the Hida distribution space.

Proof. See [28] □

Lemma 3.13. *Let $b_n : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a sequence of Borel measurable functions with $b_0 = b$ such that*

$$\sup_{n\geq 0} E\left[\exp\left(512 \int_0^1 \|b_n(s, B_s)\|^2 ds\right)\right] < \infty \qquad (3.18)$$

holds. Then

$$\left| S(Y_t^{i,b_n} - Y_t^{i,b})(\phi) \right| \leq const \cdot E[J_n]^{\frac{1}{2}} \cdot \exp(34 \int_0^1 \|\phi(s)\|^2 \, ds)$$

for all $\phi \in (S_{\mathbb{C}}([0,1]))^d$, $i = 1, \ldots, d$, *where the factor* J_n *is defined by*

$$J_n = \sum_{j=1}^{d} \left(2 \int_0^1 \left(b_n^{(j)}(u, B_u) - b^{(j)}(u, B_u) \right)^2 du + \left(\int_0^1 \left| (b_n^{(j)}(u, B_u))^2 - (b^{(j)}(u, B_u))^2 \right| du \right)^2 \right). \tag{3.19}$$

In particular, if b_n *approximates* b *in the following sense*

$$E[J_n] \to 0 \tag{3.20}$$

as $n \to \infty$, *it follows that*

$$Y_t^{b_n} \to Y_t^b \text{ in } (\mathcal{S})^*$$

as $n \to \infty$ *for all* $0 \leq t \leq 1$, $i = 1, \ldots, d$.

Proof. See [28] □

Lemma 3.14. *Let* $b_n : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be a sequence of Borel-measurable, uniformly bounded, smooth functions with compact support which approximates a Borel-measurable, bounded coefficient* $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *a.e. with respect to the Lebesgue measure. Then for any* $0 \leq t \leq 1$ *there exists a subsequence of the corresponding strong solutions* $X_{n_j,t} = Y_t^{b_{n_j}}$, $j = 1, 2...$, *such that*

$$Y_t^{b_{n_j}} \longrightarrow Y_t^b$$

for $j \to \infty$ *in* $L^2(\mu)$. *In particular this implies* $Y_t^b \in L^2(\mu)$, $0 \leq t \leq 1$.

Proof. By Corollary 3.6 we know that there exists a subsequence $Y_t^{b_{n_j}}$, $j = 1, 2...$, converging in $L^2(\mu)$. Further, by boundedness obviously $E[J_{n_j}] \to 0$ in (3.20), and thus $Y_t^{b_{n_j}} \to Y_t^b$ in $(\mathcal{S})^*$. But then, by uniqueness of the limit, also $Y_t^{b_{n_j}} \to Y_t^b$ in $L^2(\mu)$. □

Remark 3.15. *Note that by well known approximation results there always exists a sequence of functions* b_n, $n \geq 1$, *fulfilling the assumptions in Lemma 3.14. Then Lemma 3.14 guarantees that we are now ready to state the following "transformation property" for* Y_t^b.

Lemma 3.16. *Assume that* $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *is Borel-measurable and bounded. Then*

$$\varphi^{(i)}\left(t, Y_t^b\right) = E_{\widetilde{\mu}}\left[\varphi^{(i)}\left(t, \widetilde{B}_t\right) \mathcal{E}_T^{\diamond}(b)\right] \tag{3.21}$$

a.e. for all $0 \leq t \leq 1, i = 1, \ldots, d$ *and* $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(d)})$ *such that* $\varphi(B_t) \in L^2(\mu; \mathbb{R}^d)$.

Proof. See [34, Lemma 16] or [26]. □

Using the above auxiliary results we can finally give the proof of Theorem 3.3.

Proof of Theorem 3.3. We aim at employing the transformation property (3.21) of Lemma 3.16 to verify that Y_t^b is a unique strong solution of the SDE (3.1). To shorten notation we set $\int_0^t \varphi(s, \omega) dB_s := \sum_{j=1}^d \int_0^t \varphi^{(j)}(s, \omega) dB_s^{(j)}$ and $x = 0$. Also, let b_n, $n = 1, 2, ...$, be a sequence of functions as required in Lemma 3.14 (see Remark 3.15).

We first remark that $Y_\cdot^b$ has a continuous modification. The latter can be checked as follows: Since each $Y_t^{b_n}$ is a strong solution of the SDE (3.1) with respect to the drift b_n we obtain from Girsanov's theorem and our assumptions that

$$\begin{aligned} E_\mu\left[\left(Y_t^{i,b_n} - Y_u^{i,b_n}\right)^2\right] &= E_{\widetilde{\mu}}\left[\left(\widetilde{B}_t^{(i)} - \widetilde{B}_u^{(i)}\right)^2 \mathcal{E}\left(\int_0^1 b_n(s, \widetilde{B}_s) d\widetilde{B}_s\right)\right] \\ &\leq const \cdot |t - u| \end{aligned}$$

for all $0 \le u, t \le 1$, $n \ge 1$, $i = 1, ..., d$. By Lemma 3.14 we know that

$$Y_t^{b_{n_j}} \longrightarrow Y_t^b \text{ in } L^2(\mu; \mathbb{R}^d)$$

for a subsequence, $0 \le t \le 1$. So we get that

$$E_\mu \left[\left(Y_t^{i,b} - Y_u^{i,b} \right)^2 \right] \le const \cdot |t - u| \tag{3.22}$$

for all $0 \le u, t \le 1$, $i = 1, ..., d$. Then Kolmogorov's Lemma provides a continuous modification of Y_t^b.

Since $\widetilde{B}_t$ is a weak solution of (3.1) for the drift $b(s,x) + \phi(s)$ with respect to the measure $d\mu^* = \mathcal{E}\left(\int_0^1 \left(b(s, \widetilde{B}_s) + \phi(s)\right) d\widetilde{B}_s\right) d\mu$ we obtain that

$$\begin{aligned} S(Y_t^{i,b})(\phi) &= E_{\widetilde{\mu}}\left[\widetilde{B}_t^{(i)} \mathcal{E}\left(\int_0^1 \left(b(s, \widetilde{B}_s) + \phi(s)\right) d\widetilde{B}_s\right)\right] \\ &= E_{\mu^*}\left[\widetilde{B}_t^{(i)}\right] \\ &= E_{\mu^*}\left[\int_0^1 \left(b^{(i)}(s, \widetilde{B}_s) + \phi^{(i)}(s)\right) ds\right] \\ &= \int_0^t E_{\widetilde{\mu}}\left[b^{(i)}(s, \widetilde{B}_s) \mathcal{E}\left(\int_0^1 \left(b(u, \widetilde{B}_u) + \phi(u)\right) d\widetilde{B}_u\right)\right] ds + S\left(B_t^{(i)}\right)(\phi). \end{aligned}$$

Hence the transformation property (3.21) applied to b gives

$$S(Y_t^{i,b})(\phi) = S(\int_0^t b^{(i)}(u, Y_u^{i,b}) du)(\phi) + S(B_t^{(i)})(\phi).$$

Then the injectivity of S implies that

$$Y_t^b = \int_0^t b(s, Y_s^b) ds + B_t \, .$$

The Malliavin differentiability of Y_t^b follows from the fact that

$$\sup_{n \ge 1} \left\| Y_t^{i,b_n} \right\|_{1,2} \le M < \infty$$

for all $i = 1, \ldots, d$ and $0 \le t \le 1$. See e.g. [30].

On the other hand our conditions allow the application of Girsanov's theorem to any other strong solution. Then the proof of Proposition 3.1 (see e.g. [34, Proposition 1]) shows that any other solution necessarily takes the form Y_t^b.

□

Finally, we give an extension of Theorem 3.3 to a class of non-degenerate $d-$dimensional Itô-diffusions.

Theorem 3.17. *Consider the time-homogeneous $\mathbb{R}^d-$valued SDE*

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t, \quad X_0 = x \in \mathbb{R}^d, \quad 0 \le t \le T, \tag{3.23}$$

where the coefficients $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Require that there exists a bijection $\Lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, which is twice continuously differentiable. Let $\Lambda_x : \mathbb{R}^d \longrightarrow L\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and $\Lambda_{xx} : \mathbb{R}^d \longrightarrow L\left(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d\right)$ be the corresponding derivatives of Λ and assume that

$$\Lambda_x(y)\sigma(y) = id_{\mathbb{R}^d} \text{ for } y \text{ a.e.}$$

as well as

$$\Lambda^{-1} \text{ is Lipschitz continuous.}$$

Suppose that the function $b_* : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *given by*

$$b_*(x) := \Lambda_x\left(\Lambda^{-1}(x)\right)\left[b(\Lambda^{-1}(x))\right] + \frac{1}{2}\Lambda_{xx}\left(\Lambda^{-1}(x)\right)\left[\sum_{i=1}^{d}\sigma(\Lambda^{-1}(x))\,[e_i]\,,\sum_{i=1}^{d}\sigma(\Lambda^{-1}(x))\,[e_i]\right]$$

satisfies the conditions of Theorem 3.3, where e_i, $i = 1, \ldots, d$, *is a basis of* $\mathbb{R}^d$. *Then there exists a Malliavin differentiable solution* X_t *to (3.23).*

Proof. The proof can be directly obtained from Itô's Lemma. See [28]. □

Appendix A.

The following result which is due to [6, Theorem 1] provides a compactness criterion for subsets of $L^2(\mu; \mathbb{R}^d)$ using Malliavin calculus.

Theorem A.1. *Let* $\{(\Omega, \mathcal{A}, P)\,; H\}$ *be a Gaussian probability space, that is* $(\Omega, \mathcal{A}, P)$ *is a probability space and* H *a separable closed subspace of Gaussian random variables of* $L^2(\Omega)$, *which generate the* σ*-field* $\mathcal{A}$. *Denote by* $\mathbf{D}$ *the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \ldots, h_n)) = \sum_{i=1}^{n} \partial_i f(h_1, \ldots, h_n) h_i, \;\; h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ *be the closure of the family of elementary smooth random variables with respect to the norm*

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega;H)}\,.$$

Assume that C *is a self-adjoint compact operator on* H *with dense image. Then for any* $c > 0$ *the set*

$$\mathcal{G} = \left\{G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \left\|C^{-1}\mathbf{D}\,G\right\|_{L^2(\Omega;H)} \leq c\right\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [6].

Lemma A.2. *Let* $v_s, s \geq 0$ *be the Haar basis of* $L^2([0,1])$. *For any* $0 < \alpha < 1/2$ *define the operator* A_α *on* $L^2([0,1])$ *by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ *and*

$$A_\alpha 1 = 1.$$

Then for all β *with* $\alpha < \beta < (1/2)$, *there exists a constant* c_1 *such that*

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt\, dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem A.1 and Lemma A.2 is now the following compactness criteria which is essential for the proof of Corollar 3.6:

Corollary A.3. *Let a sequence of* $\mathcal{F}_1$*-measurable random variables* $X_n \in \mathbb{D}_{1,2}$, $n = 1, 2...$, *be such that there exist constants* $\alpha > 0$ *and* $C > 0$ *with*

$$\sup_n E\left[\|D_t X_n - D_{t'} X_n\|^2\right] \leq C|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$ *and*

$$\sup_n \sup_{0 \leq t \leq 1} E\left[\|D_t X_n\|^2\right] \leq C\,.$$

Then the sequence X_n, $n = 1, 2...$, *is relatively compact in* $L^2(\Omega)$.

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