# QUARTERLY OF APPLIED MATHEMATICS

## A VARIATIONAL PROBLEM ARISING IN THE DESIGN OF COOLING FINS*

By

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**Abstract.** The efficiency of a cooling fin of given weight is measured by the amount of heat dissipated per unit time by the fin. It is known that the efficiency of a given fin can be altered by changing the shape of the fin. In this paper we determine the shape of the most efficient fin of given weight and length, and thickness $\leq H$ and $\geq h$.

**1. Introduction.** Cooling fins are used to conduct heat away from machines to an ambient medium. The question of the efficiency of a fin of given weight arises naturally. One wishes to determine how a fin of given weight should taper so as to maximize the amount of heat dissipated per unit time. It was conjectured by Schmidt in 1926 (cf. [1]) that optimum fins should taper, narrowing in the direction of heat flow, so as to make the temperature gradient constant along the fin. In 1959 Duffin (cf. [2]) gave a variational formulation of the problem of designing cooling fins. The solution of this problem proved the correctness of Schmidt's conjecture.

In this paper we consider Duffin's problem with the added constraint that the thickness of the fin must not be greater than some constant $H$, nor less than some constant $h$, at any point. Equivalent variational problems arise in several other contexts (see, for example, [3] and the references therein). [3] describes an iterative procedure for determining approximate solutions of these problems. Our purpose here is to derive explicit formulae for these solutions.

Other variants of the cooling fin problem are discussed in [4–8]. In particular, [8] treats problems where the thickness of the fin comes into consideration. For example, for straight fins, the optimum ratio of length to thickness is determined. [8] also provides a good discussion of the physics and geometry of cooling fins.

**2. The variational problem.** Let distance along the fin be measured by $x$. Let $y(x)$ denote the temperature in the fin at point $x$. Then, assuming that the temperature of the machine is in a steady state, $y$ satisfies the differential equation

$$(p(x)y')' - qy = 0, \qquad 0 \leq x \leq l, \tag{2.1}$$

with boundary conditions

$$y(0) = T, \qquad y'(l) = 0. \tag{2.2}$$

$p(x)$ denotes the thickness of the fin at $x$, $q$ is a constant denoting the cooling coefficient of the fin, $T$ is the steady-state temperature of the machine and $l$ is the length of the fin.

The amount of heat dissipated by the fin per unit time is given by

$$\int_0^l qy(x)\,dx. \tag{2.3}$$

The weight of the fin is proportional to

$$\int_0^l p(x)\,dx = W. \tag{2.4}$$

$W$ is a given constant, proportional to the weight of the fin. $p(x)$ is required to satisfy a constraint of the form

$$h \le p(x) \le H \tag{2.5}$$

where $0 < h < H$. We shall assume that $hl < W < Hl$.

Our problem is to determine $p$ satisfying the conditions (2.1), (2.2), (2.4), (2.5), and maximizing the integral (2.3). For convenience we reformulate the problem. From (2.1) we have that

$$\int_0^l [(p(x)y')' - qy]y\,dx = 0$$

which can be written as

$$p(l)y'(l)y(l) - p(0)y'(0)y(0) = \int_0^l [p(y')^2 + qy^2]\,dx, \tag{2.6}$$

or, in view of the boundary conditions (2.2),

$$-p(0)y'(0)T = \int_0^l [p(y')^2 + qy^2]\,dx. \tag{2.7}$$

On the other hand, by simply integrating Eq. (2.1) we obtain

$$\int_0^l qy\,dx = p(l)y'(l) - p(0)y'(0) = -p(0)y'(0).$$

Combining this with (2.7) we obtain

$$\int_0^l qy\,dx = \frac{1}{T}\int_0^l [p(y')^2 + qy^2]\,dx. \tag{2.8}$$

Now (2.1), together with (2.2), is just the Euler equation for the variational problem

$$\text{minimize } \int_0^l [p(y')^2 + qy^2]\,dx \text{ subject to } y(0) = T.$$

It therefore follows from (2.8) that the variational problem formulated above is equivalent to the max-min problem

$$\underset{\substack{h \le p(x) \le H \\ \int_0^l p(x)\,dx = W}}{\text{maximize}} \left\{ \text{minimum } \int_0^l [p(y')^2 + qy^2]\,dx \right\} \tag{2.9}$$

where the minimum is taken over all absolutely continuous functions $y$ having square integrable derivatives on $[0, l]$ and satisfying $y(0) = T$. We shall therefore solve problem (2.9). This max-min problem has meaning even if $p$ is only required to be measurable. And since it is easy to prove the existence of a solution in the class of measurable functions, we shall now expand our attention to this larger class of functions. The solution of (2.9) in this class will turn out to be continuous.

*Definition.* A measurable function $p$, satisfying the condition (2.5), will be called an admissible shape.

For convenience we introduce the function $g$ defined on the class of admissible shapes by

$$g(p) = \operatorname*{minimum}_{y(0)=T} \int_0^l [p(x)(y'(x))^2 + qy^2(x)] \, dx. \tag{2.10}$$

The transformed version of our problem is then to

maximize $g(p)$ subject to $\int_0^l p(x) \, dx = W$, $h \le p(x) \le H$, $p$ measurable. (2.11)

To show that this problem is well defined, and also for purposes of the development to follow, we shall verify that the minimum in (2.10) is actually attained.

LEMMA 2.1. The minimum in (2.10) is attained by an absolutely continuous function $y$, having a square integrable derivative on $[0, l]$.

*Proof.* Let $d = \inf \int_0^l [p(x)(y'(x)^2 + qy^2(x)] \, dx$ where the infimum is taken over the class of absolutely continuous functions $y$, defined on $[0, l]$ and satisfying $y(0) = T$. We also assume that $y'$ is square integrable. Let $\{y_n\}$ be a sequence of such functions with the property that

$$\lim_{n \to \infty} \int_0^l [p(x)(y_n'(x))^2 + qy_n^2(x)] \, dx = d.$$

Let $\epsilon > 0$ be given and choose $N$ so large that

$$\int_0^l [p(x)(y_n'(x))^2 + qy_n^2(x)] \, dx < d + \epsilon$$

for $n \ge N$. Then for $n$ and $m \ge N$ we have

$$\int_0^l \left[ p\left(\frac{y_n' - y_m'}{2}\right)^2 + q\left(\frac{y_n - y_m}{2}\right)^2 \right] dx$$

$$= \frac{1}{2} \left\{ \int_0^l [p(y_n')^2 + qy_n^2] \, dx + \int_0^l [p(y_m')^2 + qy_m^2] \, dx \right\}$$

$$- \int_0^l \left[ p\left(\frac{y_n' + y_m'}{2}\right)^2 + q\left(\frac{y_n + y_m}{2}\right)^2 \right] dx$$

$$\le d + \epsilon - \int_0^l \left[ p\left(\frac{y_n' + y_m'}{2}\right)^2 + q\left(\frac{y_n + y_m}{2}\right)^2 \right] dx.$$

Now since the function $(y_n + y_m)/2$ is absolutely continuous on $[0, l]$ and has a square-integrable derivative there, and has value $T$ at $x = 0$, we have

$$\int_0^l \left[ p\left( \frac{y_n' + y_m'}{2} \right)^2 + q\left( \frac{y_n + y_m}{2} \right)^2 \right] dx \geq d.$$

It follows that

$$\int_0^l \left[ p\left( \frac{y_n' - y_m'}{2} \right)^2 + q\left( \frac{y_n - y_m}{2} \right)^2 \right] dx < \epsilon.$$

Since $p$ and $q$ are bounded from below by positive numbers, this shows that the sequence $\{y_n\}$, $\{y_n'\}$ are Cauchy in the space $L_2[0, l]$ of square-integrable functions on $[0, l]$. Therefore, there exist functions $y(x)$ and $\hat{y}'(x)$ in $L_2[0, l]$ with the property that

$$\lim_{n \to \infty} \int_0^l (y_n - y)^2 \, dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_0^l (y_n' - \hat{y}')^2 \, dx = 0.$$

(Here the prime on $\hat{y}$ is merely meant to be suggestive. It will turn out that $\hat{y}'$ is in fact the derivative of $y$.) Since each $y_n$ is absolutely continuous we have

$$y_n(x) = \int_0^x y_n'(t) \, dt + T, \qquad 0 \leq x \leq l.$$

Furthermore, since

$$\left| \int_0^x (y_n'(t) - \hat{y}'(t)) \, dt \right| \leq l^{1/2} \left[ \int_0^l (y_n' - \hat{y}')^2 \, dx \right]^{1/2},$$

it follows that

$$\lim_{n \to \infty} y_n(x) = \int_0^x \hat{y}'(t) \, dt + T$$

for each $x \in [0, l]$. And since $\{y_n\}$ converges in the mean square to $y$ we must have

$$y(x) = \int_0^x \hat{y}'(t) \, dt + T$$

for almost all $x \in [0, l]$. If necessary we redefine $y$ on a set of measure zero so that this equation holds for all $x \in [0, l]$. It then follows that $y(x)$ is absolutely continuous and that $y'(x) = \hat{y}'(x)$ for almost all $x \in [0, l]$. Finally, we have

$$\int_0^l [p(y')^2 + qy^2] \, dx = \int_0^l [p(\hat{y}')^2 + qy^2] \, dx$$

$$= \lim_{n \to \infty} \int_0^l [p(y_n')^2 + qy_n^2] \, dx = d.$$

Since $y$ is absolutely continuous and satisfies $y(0) = T$, we have established that the minimum in (2.10) is actually attained.

In (2.1) we have tacitly assumed $p$ to be continuous, whereas in (2.11) we have allowed $p$ to be simply measurable. The solution of (2.11) will turn out to be continuous, and therefore to be a solution of our original problem.

When $p$ is only measurable and $y$ is a solution of the minimization problem (2.10), we can only assert that $y$ satisfies the integrated version of the Euler equation (2.1) almost everywhere. That is

$$p(x)y'(x) = \int_0^x qy(t) \, dt - \int_0^l qy(t) \, dt \qquad (2.12)$$

for almost all $x$ in $[0, l]$. However, when $p$ is continuous it can be shown that this condition holds for all $x$ and that therefore, (2.1) holds.

**3. Analysis of the problem.** In this section we prove the existence of a solution of our problem and derive necessary and sufficient optimality conditions characterizing the solution.

To begin, note that the function $g$ is bounded from above. In fact if we take $y(x) \equiv T$ in (2.10) we obtain, for any measurable $p$,

$$g(p) \leq \int_0^l qT^2 \, dx = lqT^2. \tag{3.1}$$

THEOREM 3.1. For each continuous admissible shape $p$ the function $y$ which achieves the minimum in (2.10) satisfies

$$|y'(x)| \leq lqT/h \tag{3.2}$$

for each $x$ in $[0, l]$.

*Proof.* Since $p$ is continuous $y$ satisfies (2.1) and (2.2). Moreover, $y(x) \geq 0$ on $[0, l]$. To see this, note that $y(0) = T > 0$. Let $x_1$ be the first point in $[0, l]$ satisfying $y(x_1) = 0$. If there is no such point then $y(x) \geq 0$ on $[0, l]$ as we claimed. On replacing $0$ by $x_1$ in (2.6) we obtain

$$0 = -p(x_1)y'(x_1)y(x_1) = \int_{x_1}^l [p(x)(y'(x))^2 + qy^2(x)] \, dx$$

which shows that $y(x) \equiv 0$ on $[x_1, l]$. Thus $y(x) \geq 0$ on $[0, l]$.

Now (2.1) states that

$$\frac{d}{dx}[p(x)y'(x)] = qy(x) \geq 0$$

which shows that $p(x)y'(x)$ is monotone nondecreasing on $[0, l]$. It follows that

$$p(0)y'(0) \leq p(x)y'(x) \leq p(l)y'(l) = 0$$

for $0 \leq x \leq l$. And by (2.7) and (3.1) we have

$$p(0)y'(0) = -\frac{1}{T} g(p) \geq -lqT$$

so that $-lqT \leq p(x)y'(x) \leq 0$. Finally, we have, by (2.5),

$$-lqT/h \leq y'(x) \leq 0.$$

This completes the proof of the theorem.

Even if $p$ is only measurable the condition (3.2) holds for almost all $x$ in $[0, l]$. This result will come as a corollary in the next section.

THEOREM 3.2. If $p_1$ and $p_2$ are continuous admissible shapes then

$$\int_0^l y_1'^2(p_2 - p_1) \, dx - \frac{2}{h}\left(\frac{lqT}{h}\right)^2 \int_0^l (p_2 - p_1)^2 \, dx$$

$$\leq g(p_2) - g(p_1) \leq \int_0^l y_1'^2(p_2 - p_1) \, dx, \tag{3.3}$$

where $y_1$ is the solution of (2.10) corresponding to $p_1$.

*Proof.* Let $y_2$ be the solution of (2.10) for $p = p_2$. Then

$$g(p_2) - g(p_1) = \int_0^l [p_2 y_2'^2 + q y_2^2]\, dx - \int_0^l [p_1 y_1'^2 + q y_1^2]\, dx$$

$$= \int_0^l y_1'^2 (p_2 - p_1)\, dx + \int_0^l [p_2 y_2'^2 + q y_2^2]\, dx - \int_0^l [p_2 y_1'^2 + q y_1^2]\, dx. \qquad (3.4)$$

Now

$$\int_0^l [p_2 y_2'^2 + q y_2^2]\, dx - \int_0^l [p_2 y_1'^2 + q y_1^2]\, dx \le 0$$

by definition. It therefore follows that

$$g(p_2) - g(p_1) \le \int_0^l y_1'^2 (p_2 - p_1)\, dx,$$

which is the second inequality in (3.3).

To obtain the first inequality note that

$$\int_0^l (y_2'^2 - y_1'^2)(p_1 - p_2)\, dx = \int_0^l \{[p_1 y_2'^2 + q y_2^2] - [p_1 y_1'^2 + q y_1^2]\}\, dx$$

$$+ \int_0^l \{[p_2 y_1'^2 + q y_1^2] - [p_2 y_2'^2 + q y_2^2]\}\, dx. \qquad (3.5)$$

These last two integrals are $\ge 0$ by the definitions of $y_1$ and $y_2$. Consider the first of these. It is equal to

$$\int_0^l [2 p_1 y_1'(y_2' - y_1') + 2 q y_1(y_2 - y_1)]\, dx + \int_0^l [p_1(y_2' - y_1')^2 + q(y_2 - y_1)^2]\, dx.$$

Now

$$\int_0^l [2 p_1 y_1'(y_2' - y_1') + 2 q y_1(y_2 - y_1)]\, dx = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \left[ \int_0^l [p_1(y_1' + \epsilon(y_2' - y_1'))^2 \right.$$

$$\left. + q(y_1 + \epsilon(y_2 - y_1))^2]\, dx - \int_0^l [p_1 y_1'^2 + q y_1^2]\, dx \right]$$

and consequently

$$\int_0^l [2 p_1 y_1'(y_2' - y_1') + 2 q y_1(y_2 - y_1)]\, dx \ge 0.$$

It therefore follows that

$$\int_0^l \{[p_1 y_2'^2 + q y_2^2] - [p_1 y_1'^2 + q y_1^2]\}\, dx \ge \int_0^l [p_1(y_2' - y_1')^2 + q(y_2 - y_1)^2]\, dx$$

$$\ge \int_0^l h(y_2' - y_1')^2\, dx.$$

Similarly

$$\int_0^l \{[p_2 y_1'^2 + q y_1^2] - [p_2 y_2'^2 + q y_2]\}\, dx \ge \int_0^l h(y_2' - y_1')^2\, dx. \qquad (3.6)$$

From (3.5) we now deduce that

$$\int_0^l (y_2'^2 - y_1'^2)(p_1 - p_2)\, dx \geq \int_0^l 2h(y_2' - y_1')^2\, dx. \tag{3.7}$$

On the other hand,

$$\int_0^l (y_2'^2 - y_1'^2)(p_1 - p_2)\, dx = \int_0^l (y_2' + y_1')(y_2' - y_1')(p_1 - p_2)\, dx$$

$$\leq 2\frac{lqT}{h}\left(\int_0^l (y_2' - y_1')^2\, dx\right)^{1/2}\left(\int_0^l (p_1 - p_2)^2\, dx\right)^{1/2} \tag{3.8}$$

by Theorem 3.1. Together, (3.7) and (3.8) imply that

$$\int_0^l (y_1' - y_2')^2\, dx \leq \left(\frac{lqT}{h^2}\right)^2 \int_0^l (p_1 - p_2)^2\, dx. \tag{3.9}$$

Substituting this into (3.8), we obtain

$$\int_0^l (y_1'^2 - y_2'^2)(p_1 - p_2)\, dx \leq \frac{2}{h}\left(\frac{lqT}{h}\right)^2 \int_0^l (p_1 - p_2)^2\, dx.$$

Finally we have

$$g(p_2) = \int_0^l [p_2 y_2'^2 + q y_2^2]\, dx$$

$$= \int_0^l y_1'^2(p_2 - p_1)\, dx - \int_0^l (y_2'^2 - y_1'^2)(p_1 - p_2)\, dx + \int_0^l [p_1 y_2'^2 + q y_2^2]\, dx$$

$$\geq \int_0^l y_1'^2(p_2 - p_1)\, dx - \frac{2}{h}\left(\frac{lqT}{h}\right)^2 \int_0^l (p_1 - p_2)^2\, dx + g(p_1).$$

This proves the first inequality in (3.3) and completes the proof of the theorem.

COROLLARY. Theorem (3.2) remains valid if $p_1$ and $p_2$ are assumed to be only measurable. Also, Theorem 3.1 holds for measurable shapes $p$. In this case (3.2) is an almost-everywhere condition.

*Proof.* Since the only use made of the continuity properties of $p_1$ and $p_2$ in Theorem 3.2 was the inequality (3.2) from Theorem 3.1, it suffices to prove that Theorem 3.1 is valid almost everywhere for measurable $p$.

Let $\{p_n\}$ be a sequence of continuous functions converging in the mean square to $p$. Let $\{y_n\}$ be the corresponding solutions of (2.10). Then by (3.9) we have

$$\int_0^l (y_i'(x) - y_j'(x))^2\, dx \leq \left(\frac{lqT}{h^2}\right)^2 \int_0^l (p_i(x) - p_j(x))^2\, dx$$

for any pair $p_i$, $p_j$ from the sequence $\{p_n\}$. This shows that the sequence $\{y_n'(x)\}$ is Cauchy in the space $L_2[0, l]$ of square-integrable functions on $[0, l]$, and hence converges in the mean square to a function $\hat{y}'(x) \in L_2[0, l]$. We define

$$y(x) = \int_0^x \hat{y}'(x)\, dx + T.$$

Then $y'(x) = \hat{y}'(x)$ for almost all $x$ in $[a, b]$, which implies that

$$\lim_{n \to \infty} \int_0^l (y_n'(x) - y'(x))^2 \, dx = 0 \tag{3.10}$$

and since $|y_n'(x)| \leq lqT/h$ for each $x$ in $[0, l]$ we must have $|y'(x)| \leq lqT/h$ for almost all $x$ in $[0, l]$.

To complete the proof of the corollary we need only show that $y$ is the solution of (2.10) corresponding to $p$. To do this note that

$$\lim_{n \to \infty} \int_0^l (y_n(x) - y(x))^2 \, dx = 0.$$

This follows from (3.10). Now let $\bar{y}$ be any function having a square-integrable derivative on $[0, l]$ and satisfying $\bar{y}(0) = T$. Then

$$\int_0^l [p_n \bar{y}'^2 + q\bar{y}^2] \, dx \geq \int_0^l [p_n y_n'^2 + qy_n^2] \, dx$$

by the definition of $y_n$. Letting $n \to \infty$, we see that $y$ solves (2.10).

THEOREM 3.3.   The problem (2.11) has a solution.

*Proof.*   Let $\{p_n\}$ be a sequence of shapes satisfying the conditions in (2.11) and having the property that $\lim_{n \to \infty} g(p_n) = \sup g(p) \equiv g^*$ where the supremum is taken over all shapes satisfying the conditions in (2.11). This class of functions is weakly compact in $L_2[0, l]$. Therefore we can select an infinite subsequence of the $p_n$, which we again denote by $\{p_n\}$, converging weakly to an element $p^*$ in the same class. By (3.3) we have

$$g(p_n) - g(p^*) \leq \int_0^l (y^{*\prime})^2 (p_n - p^*) \, dx$$

where $y^*$ is the solution of (2.10) corresponding to $p^*$. Letting $n \to \infty$, we obtain

$$\sup g(p) \leq g(p^*)$$

which proves the optimality of $p^*$.

THEOREM 3.4.   An admissible shape $p^*$, satisfying $\int_0^l p^*(x) \, dx = W$, is optimal if and only if there exists a constant $\eta$ such that

$$\int_0^l p^*(x)[(y^{*\prime}(x))^2 - \eta] \, dx \geq \int_0^l p(x)[(y^{*\prime}(x))^2 - \eta] \, dx \tag{3.11}$$

for all admissible shapes $p$. $y^*$ is the solution of (2.10) corresponding to $p^*$.

*Proof.*   Assume that the condition (3.11) is satisfied for some $\eta$ and $p^*$. Then for any admissible $p$ satisfying

$$\int_0^l p(x) \, dx = \int_0^l p^*(x) \, dx = W$$

we have

$$g(p) - g(p^*) = g(p) - g(p^*) + \eta \int_0^l [p^*(x) - p(x)] \, dx$$

$$\leq \int_0^l p(x)[(y^{*\prime}(x))^2 - \eta] \, dx - \int_0^l p^*(x)[(y^{*\prime}(x))^2 - \eta] \, dx$$

$$\leq 0.$$

The inequalities are due to (3.3) and (3.11) respectively. They prove that $p^*$ is optimal.

Now assume that $p^*$ is optimal. We shall prove the existence of a constant $\eta$ satisfying (3.11). First we assert that $g$ is concave. To see this, let admissible shapes $p_1$ and $p_2$, and a constant $0 \leq \lambda \leq 1$ be given. Then

$$g(\lambda p_1 + (1 - \lambda)p_2) = \min_{\nu(0) = T} \int_0^l [(\lambda p_1 + (1 - \lambda)p)y'^2 + qy^2] \, dx$$

$$\geq \min_{\nu(0) = T} \int_0^l \lambda[p_1 y'^2 + qy^2] \, dx$$

$$+ \min_{\nu(0) = T} \int_0^l (1 - \lambda)[p_2 y'^2 + qy^2] \, dx$$

$$= \lambda g(p_1) + (1 - \lambda)g(p_2), \tag{3.12}$$

which proves that $g$ is concave. Now consider the set $S$ consisting of all pairs $(\alpha, \beta)$ with the property that

$$\alpha = \int_0^l p(x) \, dx \quad \text{and} \quad \beta \leq g(p)$$

for some admissible shape $p$. $S$ is clearly a convex set, bounded from above. The point

$$(\alpha^*, \beta^*) = \left( \int_0^l p^*(x) \, dx, \, g(p^*) \right)$$

is the highest point in $S$ along the ray $\alpha = W$. It is therefore a boundary point of $S$. Consequently there exists an outward normal $(\nu, \nu_0)$ to $S$ at the point $(\alpha^*, \beta^*)$. We therefore have

$$(\nu, \nu_0) \cdot ((\alpha, \beta) - (\alpha^*, \beta^*)) \leq 0 \tag{3.13}$$

for every point $(\alpha, \beta) \in S$. The condition $hl < W < Hl$, assumed at the outset, implies that the ray $\alpha = W$ has points in the interior of $S$. If $(\alpha, \beta)$ is such a point, then strict inequality holds in (3.13), which now reduces to $\nu_0(\beta - \beta^*) < 0$. This implies that $\nu_0 > 0$ since $\beta < \beta^*$. Without loss of generality we may assume that $\nu_0 = 1$.

Now consider the points in $S$ given by

$$(\alpha, \beta) = \left( \int_0^l p(x) \, dx, \, g(p) \right)$$

for $p$ admissible. (3.13) then implies that

$$\nu \int_0^l (p(x) - p^*(x)) \, dx + g(p) - g(p^*) \leq 0.$$

In particular, if we replace $p$ by the admissible shape $p^* + \lambda(p - p^*)$ for $0 < \lambda < 1$ we obtain

$$\nu \int_0^l (p(x) - p^*(x)) \, dx + \frac{g(p^* + \lambda(p - p^*)) - g(p^*)}{\lambda} \leq 0.$$

Allowing $\lambda$ to approach zero, we deduce from this, together with the first inequality in (3.3), that

$$\nu \int_0^l (p(x) - p^*(x))\, dx + \int_0^l (y^{*\prime}(x))^2 (p(x) - p^*(x))\, dx \leq 0.$$

This inequality gives the conclusion of the theorem with $\eta = -\nu$.

A more useful version of Theorem 3.4 is provided by the following corollary.

COROLLARY.  An admissible shape $p^*$, satisfying $\int_0^l p^*(x)\, dx = W$ is optimal if and only if there exists a constant $\eta$ such that

$$\max_{h \leq p \leq H} p[(y^{*\prime}(x))^2 - \eta] = p^*(x)[(y^{*\prime}(x))^2 - \eta] \qquad (3.14)$$

or almost all $x$ in $[0, l]$.

*Proof.*  It suffices to show that (3.14) is equivalent to (3.11). We begin by observing that (3.14) trivially implies (3.11).

Now assume that (3.11) holds for each admissible shape. Recall that if $f(x)$ is an integrable function on $[0, l]$, the indefinite integral $\int_0^x f(t)\, dt$ is differentiable almost everywhere (cf. [9], Theorem 6.3, p. 118) and at points $x$ where the derivative exists it is given by $f(x)$. (We have already made uses of this fact in Secs. 1 and 2.) Such a point is called a Lebesgue point. Thus if $x$ is a Lebesgue point for $f$, we have, by definition,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_x^{x+\epsilon} f(t)\, dt = f(x).$$

Let $0 < \hat{x} < l$ be a Lebesgue point for each of the functions

$$f_1(x) = p^*(x)[y^{*\prime}(x))^2 - \eta], \qquad f_2(x) = (y^{*\prime}(x))^2 - \eta.$$

This is possible since almost all points in $[0, l]$ are Lebesgue points for each of these functions. Let $\hat{p}$ be any number satisfying $h \leq \hat{p} \leq H$. Choose $\epsilon > 0$ so small that $0 < \hat{x} + \epsilon < l$. Then define the admissible shape $p$ as follows:

$$p(x) = \hat{p} \quad \text{if} \quad x\epsilon[\hat{x}, \hat{x} + \epsilon]$$
$$= p^*(x) \quad \text{otherwise.}$$

Substituting $p$ into (3.11) and simplifying a bit, we obtain

$$\int_{\hat{x}}^{\hat{x}+\epsilon} p^*(x)[(y^{*\prime}(x))^2 - \eta]\, dx \geq \int_{\hat{x}}^{\hat{x}+\epsilon} \hat{p}[(y^{*\prime}(x))^2 - \eta]\, dx.$$

Multiplying both sides of this inequality by $1/\epsilon$ and allowing $\epsilon$ to tend to zero, we obtain

$$p^*(\hat{x})[(y^{*\prime}(\hat{x}))^2 - \eta] \geq \hat{p}[y^{*\prime}(\hat{x}))^2 - \eta].$$

Now since $\hat{p}$ was chosen arbitrarily in $[h, H]$ we have

$$p^*(\hat{x})[(y^{*\prime}(\hat{x}))^2 - \eta] \geq \max_{h \leq p \leq H} p[(y^{*\prime}(\hat{x}))^2 - \eta]$$

and equality holds since $p^*(\hat{x}) \in [h, H]$. Since the choice of $\hat{x}$ only excluded a set of measure zero, we have established (3.14) for almost all $x$.

LEMMA 3.1.  The constant $\eta$ in (3.14) is positive.

*Proof.*  Assume that $\eta < 0$. Then for each $x$ in $[0, l]$ we have $(y^{*\prime}(x))^2 - \eta > 0$. It follows that

$$\max_{h \leq p \leq H} p[(y^{*\prime}(x))^2 - \eta] = H[(y^{*\prime}(x))^2 - \eta]$$

for each $x$ in $[0, l]$. But this implies that $p^*(x) \equiv H$ which is impossible since $\int_0^l p^*(x)\, dx = W$ and we assumed at the outset that $W < Hl$.

Now assume that $\eta = 0$. Since (3.14) and (3.11) are equivalent, we can deduce from (3.11) that

$$\int_0^l (y^{*\prime}(x))^2 (p(x) - p^*(x))\, dx \leq 0$$

for each admissible shape $p$. This inequality implies that $g(p) \leq g(p^*)$ for each admissible shape $p$. To see this, take $p_2 = p$ and $p_1 = p^*$ in the second inequality in (3.3). In particular, if we denote by $H$ the admissible shape defined by $p(x) \equiv H$, then

$$g(H) \leq g(p^*). \tag{3.15}$$

But this inequality is impossible. To see this let, $y_H$ denote the solution of the problem

$$\min_{y(0) = T} \int_0^l [Hy^{\prime 2} + qy^2]\, dx = g(H).$$

We then have

$$g(p^*) = \min_{y(0) = T} \int_0^l [p^* y^{\prime 2} + qy^2]\, dx \leq \int_0^l [p^* y_H^{\prime 2} + qy_H^2]\, dx$$

$$\leq \int_0^l [Hy_H^{\prime 2} + qy_H^2]\, dx = g(H). \tag{3.16}$$

The last inequality follows from the fact that $p^*(x) \leq H$. But we can say even more. Since $\int_0^l p^*(x) = W < Hl$ we must have $p^*(x) < H$ on a set of positive measure. I claim that $y_H^{\prime}(x)$ is not zero on this set and so the last inequality in (3.16) holds with strict inequality. To see that $y_H^{\prime}(x)$ is never zero on $[0, l]$ notice that $y_H(x)$ satisfies

$$Hy^{\prime\prime} + qy = 0, \qquad y(0) = T, \qquad y^{\prime}(l) = 0.$$

Solving this gives

$$y_H(x) = T \cosh \sqrt{\frac{q}{H}}\, (l - x) \Big/ \cosh \sqrt{\frac{q}{H}}\, l$$

and $y_H^{\prime}(x) \neq 0$ for $0 \leq x < l$.

(3.16) now implies that $g(p^*) < g(H)$, which contradicts (3.15). It now follows that we cannot have $\eta = 0$, and so $\eta > 0$ as claimed.

**4. Computing the optimal shape.** For the purpose of computing $p^*$ it is convenient to introduce the variables $y_1(x) = y^*(x)$ and $y_2(x) = p^*(x)y^{*\prime}(x)$. On intervals where $p^*$ is continuous we deduce from (2.1) that

$$y_1^{\prime} = \frac{1}{p^*(x)}\, y_2, \qquad y_2^{\prime} = qy_1. \tag{4.1}$$

We shall follow the optimal path traced out by this system starting at $x = l$ and working backwards. $y_2(l) = 0$, but $y_1(l)$ is an unknown value which we denote by $K$.

Since $y^{*\prime}(l) = 0$, we have $(y^{*\prime}(x))^2 < \eta$ for almost all $x$ in some left neighborhood of $l$. For $x$ in this neighborhood we deduce from (3.14) that $p^*(x) = h$. It therefore

follows from (4.1) that

$$\frac{d}{dx}\left[qy_1{}^2(x) - \frac{1}{h}\,y_2{}^2(x)\right] = 0$$

for $x$ in this neighborhood. Thus initially the point $(y_1(x), y_2(x))$ travels along a branch of the hyperbola

$$qy_1{}^2 - \frac{1}{h}\,y_2{}^2 = qK^2 \tag{4.2}$$

where $K$ denotes the value $y(l)$ to be determined (see Fig. 1). During this time $(y^{*\prime}(x))^2$ is increasing as $x$ decreases and $y^{*\prime}(x)$ is negative. $p^*(x) \equiv h$ is not a solution to our problem since we have required that $hl < \int_0^l p^*(x)\,dx < Hl$. It therefore follows that there is a point $0 < x_2 < l$ satisfying $(y^{*\prime}(x_2))^2 = \eta$.

I claim that $(y^{*\prime}(x))^2$ remains constant with value $\eta$ for $x$ in some positive interval of the form $[x_1, x_2]$, where $x_1 \geq 0$. To see this note that if $(y^{*\prime}(x))^2 > \eta$ for $x$ in some left neighborhood of $x_2$, we have, by (3.14), $p^*(x) = H$ in this neighborhood. It follows that

$$(p^*(x)y^{*\prime}(x))^2 > H^2\eta = h^2\eta + (H^2 - h^2)\eta$$

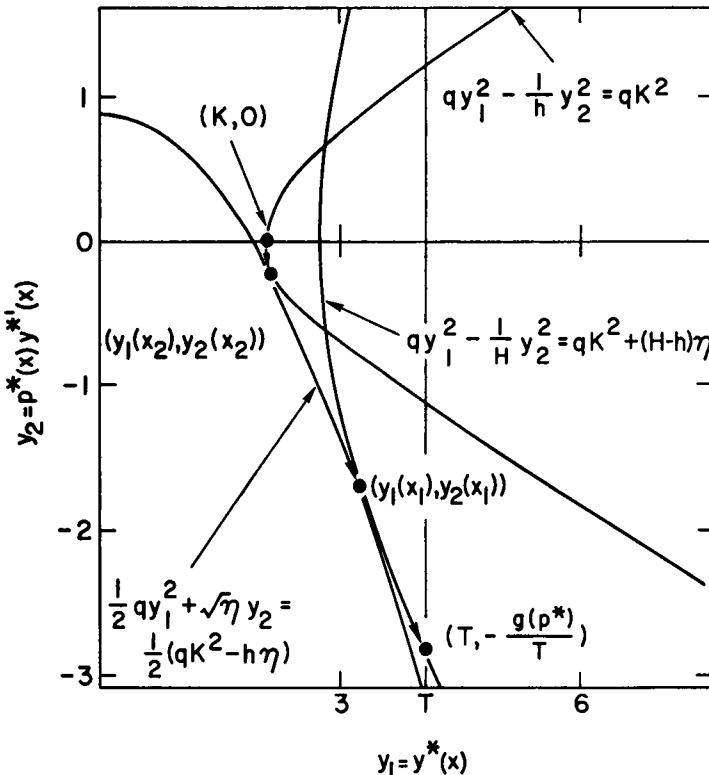$$= (p^*(x_2)y^{*\prime}(x_2))^2 + (H^2 - h^2)\eta$$



FIG. 1.   The optimal path is indicated by arrows.

for $x$ in this neighborhood. This contradicts the fact, which is evident from (2.12), that

$$\lim_{x \to x_2} (p^*(x)y^{*\prime}(x))^2 = (p^*(x_2)y^{*\prime}(x_2))^2.$$

It follows that for $x$ sufficiently close to $x_2$ we must have $(y^{*\prime}(x))^2 \le \eta$.

Similarly, we cannot have $(y^{*\prime}(x))^2 < \eta = (y^{*\prime}(x_2))^2$ in some left neighborhood of $x_2$ since this implies that $p^*(x) = h$ in an interval containing $x_2$ in its interior. And, as we have seen, on intervals where $p(x) = h$, $y^{*\prime}(x)$ is negative and decreasing with $x$.

Thus to continue the construction of the optimal path in Fig. 1 beyond the point $(y_1(x_2), y_2(x_2))$ we must move so that the condition $(y^{*\prime}(x))^2 = \eta$ is satisfied. Substituting this into (4.1), we deduce that

$$\frac{d}{dx} \left(\tfrac{1}{2}qy_1^{\,2}(x) + \sqrt{\eta} y_2(x)\right) = 0.$$

This shows that the point $(y_1(x), y_2(x))$ is traveling along the parabola

$$\tfrac{1}{2}qy_1^{\,2} + \sqrt{\eta}\, y_2 = \tfrac{1}{2}qy_1^{\,2}(x_2) + \sqrt{\eta}\, y_2(x_2) \qquad (4.3)$$

on some interval of the form $x_1 \le x \le x_2$, as in Fig. 1. On this interval, we deduce from the first equation in (4.1) that

$$p^*(x) = -(1/\sqrt{\eta})y_2(x). \qquad (4.4)$$

There are now two cases to consider. The first is where $x$ decreases from $x_2$ to zero without $p^*(x)$, given by (4.4), violating the inequality $p^*(x) < H$. Let us consider writing down the formula for $p^*$ in this case.

We have seen that $y^{*\prime}(x_2) = -\sqrt{\eta}$, which implies that $y_2(x_2) = -h\sqrt{\eta}$. Substituting this into (4.2) gives

$$y_1(x_2) = \left(\frac{h}{q}\eta + K^2\right)^{1/2} = y^*(x_2).$$

Now, by assumption $y^{*\prime} = -\sqrt{\eta}$ on $[0, x_2]$ and so

$$y^*(x) = -\sqrt{\eta}\,(x - x_2) + \left(\frac{h}{q}\eta + K^2\right)^{1/2} \qquad (4.5)$$

on this interval. Also on this interval, we deduce from (4.1) that $y_2'(x) = qy_1(x) = qy^*(x)$ which implies that

$$y_2(x) = -\frac{q\sqrt{\eta}}{2}(x - x_2)^2 + q\left(\frac{h}{q}\eta + K^2\right)^{1/2}(x - x_2) - h\sqrt{\eta}.$$

Finally we have, by (4.4),

$$p^*(x) = \frac{q}{2}(x - x_2)^2 - q\left(\frac{h}{8} + \frac{K^2}{\eta}\right)^{1/2}(x - x_2) + h, \qquad 0 \le x \le x_2$$

$$= h, \qquad x_2 \le x \le 1. \qquad (4.6)$$

This expression for $p^*$ contains the three unknowns $\eta$, $K$ and $x_2$. These quantities can be determined from the conditions

$$\int_0^l p^*(x)\, dx = W,$$

$$y^*(0) = T, \tag{4.7}$$

$$y^*(x_2) = \left(\frac{h}{q}\,\eta + K^2\right)^{1/2}.$$

The first of these conditions can be written as

$$\frac{1}{6}\,qx_2{}^3 + \frac{q}{2}\left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2} x_2{}^2 + hl = W. \tag{4.8a}$$

Using (4.5), the second condition can be written as

$$\sqrt{\eta}x_2 + \left(\frac{h}{q}\,\eta + K^2\right)^{1/2} = T. \tag{4.8b}$$

To state the third condition explicitly, we substitute $y_1 = y^*(x)$, $y_2 = hy^{*\prime}(x)$ into (4.2) to obtain

$$qy^{*2}(x) - h(y^{*\prime}(x))^2 = qK^2$$

from which we deduce that

$$\sqrt{\frac{h}{q}} \int_{y^*(x)}^K \frac{dy}{(y^2 - K^2)^{1/2}} = -\int_x^l dx$$

or

$$y^*(x) = K \cosh \sqrt{\frac{q}{h}}\,(l - x).$$

(4.5) shows that

$$\frac{1}{\sqrt{\eta}}\,y^*(x_2) = \left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2};$$

that is, that

$$\frac{K}{\sqrt{\eta}} \cosh \sqrt{\frac{q}{h}}\,(l - x_2) = \left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2}. \tag{4.8c}$$

This equation can be solved for $K^2/\eta$ in terms of $x_2$ as

$$\frac{K^2}{\eta} = \frac{h}{q}\left(\sinh \sqrt{\frac{q}{h}}\,(l - x_2)\right)^{-2}. \tag{4.9}$$

Substituting this into (4.8a), we obtain

$$\frac{1}{6}\,qx_2{}^3 + \frac{q}{2}\left\{\frac{h}{q} + \frac{h}{q}\left(\sinh \sqrt{\frac{q}{h}}\,(l - x_2)\right)^{-2}\right\}^{1/2} x_2{}^2 + hl = W. \tag{4.10}$$

At $x_2 = 0$ the left side of this equation has the value $hl < W$. As $x_2 \to l$, the left side approaches $+\infty$. It follows that there is a value $0 < x_2 < l$ for which (4.10) holds. This value can be determined, at least approximately, by numerical methods. Given $x_2$, $K^2/\eta$ can be determined from (4.9). Given these values $p^*$ is completely determined

by (4.6). If we divide (4.8b) by $\sqrt{\eta}$ we obtain a simple expression for $\eta$. We can then determine $K$ since $K^2/\eta$ is known.

Suppose now that the shape $p^*$, computed by the procedure just described, satisfies $p^*(x_1) = H$ for some point $0 \leq x_1 < x_2$. We are then in the second case alluded to above. In this case it is easily seen that $p^*(x)$ remains fixed at the value $H$ throughout the interval $[0, x_1]$. We therefore have

$$
\begin{aligned}
p^*(x) &= H \quad \text{if} \quad 0 \leq x \leq x_1 \\
&= \frac{q}{2}(x - x_2)^2 - q\left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2}(x - x_2) + h \quad \text{if} \quad x_1 \leq x \leq x_2 \qquad (4.11) \\
&= h \quad \text{if} \quad x_2 \leq x \leq l.
\end{aligned}
$$

As before, the conditions (4.7) are to be used to determine the unknowns $x_1$, $x_2$, $\eta$, and $K$. The first of these conditions states that

$$
Hx_1 + h(l - x_2) + \frac{q}{6}(x_2 - x_1)^3 + \frac{q}{2}\left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2}(x_2 - x_1)^2 + h(x_2 - x_1) = W. \qquad (4.12)
$$

we have

$$
p^*(x_1) = \frac{q}{2}(x_1 - x_2)^2 - q\left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2}(x_1 - x_2) + h = H
$$

which implies that

$$
x_1 - x_2 = \left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2} - \left(\frac{K^2}{\eta} + \frac{2H - h}{q}\right)^{1/2}. \qquad (4.13)
$$

From (4.8c) we deduce that

$$
l - x_2 = \sqrt{\frac{h}{q}}\, \cosh^{-1}\left(1 + \frac{h}{q}\frac{\eta}{K^2}\right)^{1/2}. \qquad (4.14)
$$

Combining these last two equations we obtain

$$
x_1 = l + \left(\frac{K^2}{\eta} + \frac{h}{q}\right)^{1/2} - \left(\frac{K^2}{\eta} + \frac{2H - h}{q}\right)^{1/2} - \sqrt{\frac{h}{q}}\, \cosh^{-1}\left(1 + \frac{h}{q}\frac{\eta}{K^2}\right)^{1/2}. \qquad (4.15)
$$

Substituting (4.13), (4.14), and (4.15) into Eq. (4.12), we obtain a nonlinear equation for the unknown $r \equiv K^2/\eta$. If this equation is solved for $r$, $x_1$ and $x_2$ are given by (4.15) and (4.14) respectively. Given $x_1$, $x_2$, and $r$, $p^*(x)$ is completely determined by (4.11).

To determine $\eta$ and $K$, first note that by (4.1) we have

$$
\frac{d}{dx}\left(qy_1{}^2(x) - \frac{1}{H}y_2{}^2(x)\right) = 0 \qquad (4.16)
$$

for $0 \leq x \leq x_1$. Thus to determine the motion of the point $(y_1(x), y_2(x))$ for $0 \leq x \leq x_1$ we need only determine the values $y_1(x_1)$ and $y_2(x_1)$. Since $p^*(x_1) = H$ we deduce from (4.4) that $y_2(x_1) = -H\sqrt{\eta}$. To determine $y_1(x_1)$ we substitute the values

$$
y_1(x_2) = \left(\frac{h}{q}\eta + K^2\right)^{1/2}, \qquad y_2(x_2) = -h\sqrt{\eta},
$$

into (4.3), obtaining

$$\tfrac{1}{2}qy_1^2(x) + \sqrt{\eta}y_2(x) = \tfrac{1}{2}(qK^2 - h\eta) \quad \text{for} \quad x_1 \le x \le x_2 .$$

In particular, for $x = x_1$ we have

$$y_1(x_1) = \left(K^2 + (2H - h)\frac{\eta}{q}\right)^{1/2}. \tag{4.17}$$

Thus for $0 \le x \le x_1$ we have, by (4.16),

$$qy_1^2(x) - \frac{1}{H} y_2^2(x) = qK^2 + (H - h)\eta. \tag{4.18}$$

The point $(y_1(x), y_2(x))$ is therefore moving along a branch of the hyperbola

$$qy_1^2 - \frac{1}{H} y_2^2 = qK^2 + (H - h)\eta$$

as in Fig. 1.

Substituting $y_1(x) = y^*(x)$, $y_2(x) = p^*(x)y^{*\prime}(x)$ into (4.18), we obtain

$$qy^{*2}(x) - H(y^{*\prime}(x))^2 = qK^2 + (H - h)\eta.$$

Solving this equation gives

$$y^*(x) = \left(K^2 + \frac{H - h}{q}\eta\right)^{1/2} \cosh\left[\cosh^{-1} \frac{T}{\left(K^2 + \dfrac{H - h}{q}\eta\right)^{1/2}} - \sqrt{\frac{q}{H}}\, x\right].$$

If we take $x = x_1$ and equate this expression to (4.17) we obtain, by simple manipulations

$$\frac{T}{\left(\eta\left(r + \dfrac{H - h}{q}\right)\right)^{1/2}} = \cosh\left[\cosh^{-1} \left(\frac{qr + 2H - h}{qr + H - h}\right)^{1/2} + \sqrt{\frac{q}{H}}\, x_1\right].$$

If $r$ and $x_1$ are known, this provides a simple expression for $\sqrt{\eta}$. Given $\sqrt{\eta}$ we have $K = (r\eta)^{1/2}$.

**5. Determining the ratio $K^2/\eta$.** We have reduced the problem of finding the optimal shape $p^*$ to the task of solving one or on two nonlinear equations for the unknown $r \equiv K^2/\eta$. In the case where $p^* < H$ for $0 \le x \le l$ we must first solve Eq. (4.14) for $x_2$ . We have already shown that this equation has a solution in $[0, l]$. This solution can be obtained to any desired degree of accuracy by the secant method. Given $x_2$ , we determine $r = K^2/\eta$ from (4.9). Now consider the function $p^*$ specified by (4.6). If $p^*(x) < H$ for $0 < x \le l$, then $p^*$ is the optimal shape we seek.

On the other hand, if $p^*(x_1) = H$ for some point $0 < x_1 < x_2$ , then the shape (4.6) is not admissible and hence not optimal. Let $r_1$ denote the value of the ratio $K^2/\eta$ used in determining this shape. Then, as we have seen, the optimal shape is given by (4.11) where $r = K^2/\eta$ satisfies a certain nonlinear equation. We wish to determine an interval containing $r$. A precise estimate of $r$ can then be determined by the secant method.

I claim that $r > r_1$ . To see this, recall that $r_1$ is given by (4.9) where $x_2$ satisfies (4.10). Now if $p^*$ is defined by (4.6), we have seen that there is an $x_1$ between 0 and $x_2$

satisfying $p^*(x_1) = H$. $x_1$ and $x_2$ therefore satisfy (4.13), (4.14), and (4.15). The left side of (4.12) is therefore given by

$$Hx_1 + \int_{x_1}^{l} p^*(x)\,dx < \int_{0}^{l} p^*(x)\,dx = W. \tag{5.1}$$

On the other hand, as $r = K^2/\eta \to \infty$, $x_2 \to l$. This follows from (4.14). Also, by (4.13) $x_2 - x_1 \to 0$ as $K^2/\eta \to \infty$. Consider now the left side of (4.12) for large $r$. We have

$$Hx_1 + h(l - x_2) + \frac{q}{6}(x_2 - x_1)^3 + \frac{q}{2}\left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2}(x_2 - x_1)^2 + h(x_2 - x_1)$$

$$> Hx_1 + h(l - x_2) + (x_2 - x_1)\left\{\frac{q}{6}(x_2 - x_1)^2 + \frac{q}{3}\left(\frac{h}{q} + \frac{K^2}{\eta}\right)^{1/2}(x_2 - x_1) + \tfrac{1}{3}h\right\}$$

$$= Hx_1 + h(l - x_2) + \frac{(x_2 - x_1)}{3}p^*(x_1)$$

$$= H(\tfrac{2}{3}x_1 + \tfrac{1}{3}x_2) + h(l - x_2). \tag{5.2}$$

Since $x_1$ and $x_2 \to l$ as $K^2/\eta \to \infty$, and since $W < Hl$ by assumption, (5.2) shows that the left side of (4.12) is $> W$ for $K^2/\eta$ sufficiently large. Therefore, in order to find an interval containing a value of the ratio $K^2/\eta$ which solves the system (4.12)–(4.15), we successively substitute the values $2r_1$, $3r_1$, $4r_1$, $\cdots$, of $K^2/\eta$ into this system. Let $nr_1$ be the first of these values for which the left side of (4.12) is $\geq W$. Then the value of $K^2/\eta$ we seek is in the interval $[(n - 1)r_1, nr_1]$, and can be found by the secant method.

*Remark.* We have shown that the optimal shape is given by (4.6) or by (4.11), and have given equations for determining the unknowns $x_1$, $x_2$, $\eta$, and $K$. We have demonstrated that these equations have solutions, and have described how these solutions can be obtained. The question of the uniqueness of these solutions has been ignored. This is permissible, since, as Theorem 3.4 shows, any $p^*$, computed by the procedure we have outlined, is optimal.

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