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A vector identity for the Dirichlet tessellation

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Summary. A vector identity associated with the Dirichlet tessellation is proved as a corollary of a more general result. The identity has applications in interpolation and smoothing problems in data analysis, and may be of interest in other areas.

The Dirichlet tessellation is a simple geometrical construct, long familiar to pure mathematicians as a consequence of its role in the proof of packing and covering theorems (Rogers (5)), and nowadays of interest in spatial data analysis (Sibson (6)) and interpolation (Lawson (2)). We give a formal definition as follows. Let E^K denote euclidean K-space, and let $P_1, \ldots, P_N$ be distinct points in E^K. Define

$$T_n = \{P \in E^K: d(P, P_n) < d(P, P_m) \quad \text{for} \quad m \neq n\}, \tag{1}$$

where d is euclidean distance. T_n is the *tile* (or *Voronoi* or *Thiessen polyhedron*) of P_n with respect to the configuration $P_1, \ldots, P_N$. Each T_n is clearly an intersection of finitely many open half-spaces, each being delimited by the perpendicular-bisector hyperplane of a line segment $P_n P_m$, and is thus open, convex and polyhedral; it is bounded if and only if P_n lies in the interior of the convex hull of $P_1, \ldots, P_N$. The T_n are disjoint, and cover E^K apart from a set of (Lebesgue) measure zero. T_n is simply the part of E^K nearer to P_n than to any other P_m, and this territorial interpretation is the basis of many of the applications of the construct, which is known as a whole as the *Dirichlet tessellation*.

A refinement of the Dirichlet tessellation is obtainable by subdividing E^K into regions, again open, convex and polyhedral, with specified first, second, third, ... nearest neighbours among $P_1, \ldots, P_N$. Some properties of such refined tessellations have been discussed by Miles (3), (4). We consider one step of this refinement, and define, for $m \neq n$,

$$T_{nm} = \{P \in E^K: d(P, P_n) < d(P, P_m) < d(P, P_l) \text{ for } l \neq m, n\}. \tag{2}$$

We call T_{nm} the *subtile* of the ordered pair (P_n, P_m) with respect to $P_1, \ldots, P_N$. T_{nm} may well be empty; it is non-empty if and only if T_n and T_m have a common facet ($(K-1)$-dimensional face), in which case we say that P_n, P_m are *contiguous*, or are *neighbours*. The contiguity relationship is symmetric, and can be shown to define a triangulation of $P_1, \ldots, P_N$ (the *Delaunay triangulation*, see Rogers (5)) when $P_1, \ldots, P_N$ are in general position in an appropriate sense. Clearly the T_{nm} are disjoint, and their union for fixed n is T_n apart from a set of measure zero. Let σ_K denote Lebesgue measure on E^K, and write κ_n for $\sigma_K(T_n)$, κ_{nm} for $\sigma_K(T_{nm})$. Because the T_{nm} almost partition T_n, we have $\Sigma\{\kappa_{nm}: m \neq n\} = \kappa_n$, in the sense that either both sides are infinite, or they are finite and equal.

Let u_n be the vector position of P_n. Then the result we prove is the following.

LOCAL COORDINATES PROPERTY (LCP). *With the notation above, when $\kappa_n < \infty$ we have*

$$\sum_{m \neq n} \kappa_{nm} u_m = \kappa_n u_n. \tag{3}$$

That is, if we place at P_m $(m \neq n)$ a mass equal to the measure of the subtile T_{nm}, then the centroid of these masses is P_n.

The reason why this result is of interest in applications is that, on division through by κ_n, (3) provides an expression for P_n as a convex combination of its neighbours; this expression is well-determined irrespective of the disposition of the remaining points (provided only that P_n remains in the interior of their convex hull), and, if the coordinate functions $\lambda_{nm} = \kappa_{nm}/\kappa_n$ are regarded as functions of the position u_n of P_n, then they are continuously differentiable except at the points P_m $(m \neq n)$, to which they may be extended by continuity. It is quite easy to establish this by explicit differentiation of the κ_{nm}, but we do not pursue the matter here. The author has exploited these local coordinates in the development of a new method of C^1 spatial interpolation and smoothing, and will be reporting this elsewhere in the literature. There would appear to be scope for applications in numerical analysis, for example in the approximation of partial differential operators by partial finite difference operators with respect to a (possibly irregular) discrete grid of points. In order to make practical use of local coordinates, it is necessary to be able to compute the κ_{nm}. In one dimension this is trivial. In two dimensions an efficient algorithm for computing the Dirichlet tessellation has been proposed and implemented by Green and Sibson(1); this can handle 10000 points easily, and has been used successfully on as many as 27000 points. It can be extended to compute the κ_{nm}. In three or more dimensions the computation of the Dirichlet tessellation in an efficient manner is still an open problem, although a new technique due to A. Bowyer (personal communication) shows great promise.

Even in the first non-trivial case, $K = 2$, the LCP seems awkward to prove directly, although the author has a clumsy inductive proof for this case which does not appear to extend readily to higher values of K. It is, however, quite straightforward to prove for all K a more general result which has the LCP as a special case, and which may be of some further interest in itself. We now formulate and prove this more general result. Consider embedding E^K as a linear manifold in some higher-dimensional euclidean space E^L. Let $P'_1, \ldots, P'_N$ be distinct points in E^L, and $P_1, \ldots, P_N$ (which need not be distinct) their orthogonal projections on E^K. $P'_1, \ldots, P'_N$ generate a Dirichlet tessellation of E^L, with tiles $T'_1, \ldots, T'_N$ and we then define a generalized tessellation of E^K by considering their intersections with it, that is, $T_n = T'_n \cap E^K$. If all the P'_n lie in E^K, the usual Dirichlet tessellation is recovered as a special case; indeed, (1) may still serve as a definition for T_n in the generalized case, if we interpret d as distance in E^L and replace P_n, P_m by P'_n, P'_m. The refinement to subtiles generalizes in the same way; we define T'_{nm} to be a subtile in E^L, and then take $T_{nm} = T'_{nm} \cap E^K$; or alternatively make to (2) the parallel modification to that proposed for (1). Note that the definition of a generalized tessellation does not require that the P_n be distinct, merely that the P'_n be so; this on its own is not quite enough, because it leads to a defective construct if E^K happens to contain a facet of a T'_n or T'_{nm}. We accordingly confine our consideration to *proper* generalized tessellations, where this possibility is excluded; it would be excluded automatically if we required the P_n to be distinct, but we shall not wish to do this. The limitation to proper generalized tessellations ensures that the properties that the T_n almost partition E^K, and that the T_{nm} almost partition T_n, are retained. The T_n and T_{nm} continue to be open in E^K, convex and polyhedral, but in other ways they may look very different from the tiles and subtiles of the ordinary Dirichlet tessellation; in particular, T_n may be empty, and even if it is non-empty it may fail to contain P_n. The generalized Dirichlet tessellation has been discussed by Miles (4).

On carrying our earlier notation over to the generalized case, we are now able to formulate the result which we actually prove.

GENERALIZED LOCAL COORDINATES PROPERTY (GLCP). *With the notation above, and in the context of a proper generalized tessellation, if* $0 < \kappa_n < \infty$ *we have*

$$\sum_{m \neq n} \kappa_{nm} u_m = \kappa_n u_n. \tag{4}$$

COROLLARY. *The LCP is the special case arising when all* P'_n *lie in* E^K*; note that in this case properness is guaranteed.*

Proof of the GLCP. The proof consists of three steps. First an identity for $K = 1$ and for triples of points is proved, and this is then used to prove the GLCP for $K = 1$ by a summation argument; the final step is to deduce the GLCP for general K from this.

First consider E^1 embedded in E^L; without loss of generality choose coordinates (u, v) so that u is a coordinate in E^1 and v is the vector of the remaining $L-1$ coordinates. Write v^2 for the sum of the squares of the entries in v. Let R_1, R_2, R_3 be three points in E^L with $R_i = (u_i, v_i)$ $(i = 1, 2, 3)$ and u_1, u_2, u_3 all distinct. If $i \neq j$, the distinctness of u_i and u_j guarantees that the perpendicular-bisector hyperplane of $R_i R_j$ meets E^1; it does so in a unique point $(u_{ij}, 0)$ determined by the equation

$$(u_i - u_{ij})^2 + (v_i - 0)^2 = (u_j - u_{ij})^2 + (v_j - 0)^2.$$

On summing this equation for $(i, j) = (1, 2), (2, 3), (3, 1)$, simplifying, and rearranging, we obtain the identity

$$(u_{12} - u_{31})\, u_1 + (u_{23} - u_{12})\, u_2 + (u_{31} - u_{23})\, u_3 = 0 \tag{5}$$

in which v_1, v_2, v_3 do not appear.

Now consider points $P'_1, \ldots, P'_N$ in E^L generating a proper generalized tessellation of E^1; let P'_n be such that $0 < \kappa_n < \infty$. Call this point Q_0. The generalized tile T_n ($= U_0$, say) of this point is almost-partitioned (because the generalized tessellation is proper) into generalized subtiles; label those P'_m which have non-empty subtiles T_{nm} as $Q_1, \ldots, Q_M$ from negative to positive along E^1, and correspondingly label the subtiles as $U_{01}, \ldots, U_{0M}$. Let $Q_m = (u_m, v_m)$ $(m = 0, \ldots, M)$. The endpoints of U_0, and also of U_{0m} $(m = 1, \ldots, M)$ are points where perpendicular-bisector hyperplanes meet E^1 and, because such intersections occur, we can deduce that

$$u_1 < u_2 < \ldots < u_{M-1} < u_M \quad \text{and} \quad u_1 < u_0 < u_M.$$

This allows the three-point identity to be applied to the triples

$$Q_1 Q_2 Q_M, \ldots, \quad Q_{M-2} Q_{M-1} Q_M \quad \text{and} \quad Q_0 Q_1 Q_M.$$

Summing the identity (5) over such triples gives

$$(u_{01} - u_{M0}) u_0 + (u_{12} - u_{01}) u_1 + \ldots + (u_{M0} - u_{M-1,M}) u_M = 0, \tag{6}$$

where $(u_{ij}, 0)$ is the point of intersection of the perpendicular-bisector hyperplane of $Q_i Q_j$ with E^1. But $u_{M0} - u_{01}$ is $\sigma_1(U_0)$, and $u_{12} - u_{01}$ is $\sigma_1(U_{01})$, etc., and on reverting to the original labelling of points we see that (6) coincides with (4) except that in (6) those P_m for which T_{nm} is empty are excluded from the summation. For such P_m, κ_{nm} is zero, and thus (6) does indeed give the GLCP for $K = 1$.

The third and final step is an integration argument. It is enough for general K to prove that (4) holds for components in an arbitrary direction; without loss of generality we may choose coordinates so that this direction is that of the first coordinate x, and let y be the vector of the remaining coordinates in E^K, so that $u = (x, y)$. Suppose then that $P'_1, \ldots, P'_N$ in E^L define a proper generalized tessellation of E^K, and let P'_n be such that $0 < \kappa_n < \infty$. Write $E^1(y)$ for $\{(x,y): x \in E^1\}$, $T_n(y)$ for $T_n \cap E^1(y)$, $T_{nm}(y)$ for $T_{nm} \cap E^1(y)$, $\kappa_n(y)$ for $\sigma_1(T_n(y))$, $\kappa_{nm}(y)$ for $\sigma_1(T_{nm}(y))$. We thereby construct the generalized tessellation defined on $E^1(y)$ by $P'_1, \ldots, P'_N$. This is proper for σ_{K-1} - almost - all values of y, and when it is proper we have, from the GLCP for $K = 1$,

$$\sum_{m \neq n} \kappa_{nm}(y) x_m = \kappa_n(y) x_n. \tag{7}$$

Integration of (7) over $y \in E^{K-1}$ with respect to σ_{K-1} then gives

$$\sum_{m \neq n} \kappa_{nm} x_m = \kappa_n x_n, \tag{8}$$

which is simply (4) for components in the direction under consideration; the proof is complete.

It should be apparent from the first part of the above proof that there is an element of spuriousness about the definition of a generalized tessellation, in that it does not depend on the positions of the P'_n in E^L except via the positions P_n of their projections on E^K and the values of $d(P'_n, P_n)$. Thus we can in fact construct any generalized tessellation by taking $L = K + 1$, and even within that one extra dimension each P'_n can independently be chosen to be on either side of E^K. However, no simplification of the proof of the GLCP seems to follow from this observation. As yet, no applications for the full strength of the GLCP have arisen, although generalized tessellations themselves can be useful.

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