# A Vector Space Approach to Models and Optimization 

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> To

Patricia Scott
Brett
Jonathan
Jennifer
Christopher

Every important idea is simple.
War and Peace Count Leo Tolstoy

## SYSTEMS ENGINEERING AND ANALYSIS SERIES

In a society which is producing more people, more materials, more things, and more information than ever before, systems engineering is indispensable in meeting the challenge of complexity. This series of books is an attempt to bring together in a complementary as well as unified fashion the many specialties of the subject, such as modeling and simulation, computing, control, probability and statistics, optimization, reliability, and economics, and to emphasize the interrelationship between them.

The aim is to make the series as comprehensive as possible without dwelling on the myriad details of each specialty and at the same time to provide a broad basic framework on which to build these details. The design of these books will be fundamental in nature to meet the needs of students and engineers and to insure they remain of lasting interest and importance.

## Preface

Models and optimization are fundamental to the design and operation of complex systems. This book is intended as an intuitive, probing, unified treatment of the mathematics of model analysis and optimization. It explores in a unifying framework the structure of deterministic linear system models and the optimization of both linear and nonlinear models. The unification is accomplished by means of the vector space language and a relatively small number of vector space concepts. The mathematical concepts and techniques, although not new, become more accessible when treated in an intuitive, unified manner.
This book is broader in coverage than most books on the subject, and is relatively low in its level of mathematical sophistication. I have deemphasized mathematical proofs; I have attempted instead to develop concepts by means of geometrical intuition and analogies to ideas familiar to engineering graduates. All concepts are illustrated with specific detailed examples. In addition, I have tried to relate the mathematical concepts to the real world by presenting practical applications and by discussing practical computer implementations of techniques for model analysis and optimization. Thus the development is less sterile than the treatments found in mathematics books.

I have attempted to build up the mathematical machinery in a way that demonstrates what can and cannot be accomplished with each tool. This methodical buildup helps to develop a fundamental feel for the mathematical concepts. For example, I withhold the definition of the inner product until late in the development in order that it be clear that perpendicular coordinate systems are not fundamental to the modeling process.

The background required of the reader is a familiarity with elementary matrix manipulations and elementary differential equation concepts. The selection of topics and the order of presentation reflect seven years of experience in presenting the material to full-time graduate students and to practicing engineers at the Moore School of Electrical Engineering, University of Pennsylvania. The book is designed for use as a text in a two-semester course sequence for first-year graduate students in engineering, operations research, and other disciplines which deal with systems. As
a consequence of the extensive cross-referencing and the numerous detailed examples, the book is also suitable for self-study. At the end of each chapter references that are good general references for much of the material in that chapter are indicated with asterisks (*). Answers to selected problems are included.

The symbols P \& C appear frequently throughout the text in reference to the Problems and Comments sections at the end of each chapter. These problems and comments form an important part of the book. Those problems that are in the form of statements are intended to be proved or verified by simple examples. The problems marked with asterisks (*) present concepts which are used later in the book, or which are significant extensions of the text material; these problems should at least be read and understood.

The reader will find that abstract symbols can be understood more easily if they are thought of in terms of simple examples. If possible, concepts should be illustrated geometrically with two- or three-dimensional arrow vectors.

In a two-semester course sequence, it would be appropriate to treat Sections 1.1-5.3 in the first semester (a vector space approach to models) and Sections 5.4-8.5 in the second (a vector space approach to optimization). There is not sufficient time in two semesters to include all the applications if all the mathematical concepts are covered. (I have usually omitted some of Section 4.4 and some of the applications.) By deleting Chapter 3 and by de-emphasizing differential systems and nondiagonalizable matrices in the remaining chapters, the two semesters can be reduced to two quarters. The accompanying diagram shows how the chapters depend on each other.


Because the concepts treated in this book find application in many fields, it is difficult to avoid conflicts between different standards in
notation. I have tried to be as consistent as possible with previous standards. Because instructors cannot use bold print at the blackboard, I have avoided the use of boldfaced type as a primary means of distinguishing vectors and transformations. However, I do use boldfaced type redundantly to emphasize the interpretation of an object as a vector or transformation of vectors.

I wish to express my appreciation to H. R. Howland, W. A. Gruver, and C. N. Campopiano, who read the full manuscript and suggested helpful improvements. Thanks are also due to Renate Schulz for her help in proofreading and drawing, to Pam Dorny and Nancy Maguire who did most of the typing, and to the Moore School of Electrical Engineering of the University of Pennsylvania which provided support for much of the effort. Most of all, I wish to express my gratitude to my wife and children who waited patiently for the long nights, weekends, and summers to end.

C. NELSON DORNY

May 1975
Philadelphia, Pennsylvania

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## Symbols

## Scalars

| $a, b, c, d, s, t$, |  |  |
| :--- | :--- | ---: |
| $\alpha, \beta, \gamma, \epsilon, \sigma, \tau$ | general scalars | 36 |
| $\bar{a}$ | complex conjugate of $a$ | 239 |
| $i, j, k, r$ | integer subscripts | 37 |
| $l, m, n, p, q$ | positive integers | 184 |
| $\xi_{i}, \eta_{i}$ | elements of vectors | 37 |
| $\lambda_{i}$ | eigenvalues | 151 |
| $\lambda_{i}, \mu_{i}, \nu_{i}$ | Lagrange multipliers | 413 |

## V ectors

$\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \quad$ general vectors 36
$\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \boldsymbol{\phi}$ general functions ..... 40
$\theta$ zero vector or zero function ..... 36
1 unit function ..... 455
standard basis vectors for $\Re^{n}$ or $\mathfrak{R}^{n \times 1}$ ..... 48$\boldsymbol{\varepsilon}_{i}$
$\lambda, \mu, \nu$Lagrange multiplier vectors415
Vector Spaces
$\mathfrak{V}, \mathscr{Q}, \mathscr{Q}$ general spaces or subspaces ..... 36
$\mathscr{H}$ Hilbert space ..... 276
$\Re^{n}$ real $n$-tuple space ..... 39
$\mathscr{P}^{n}$ polynomial functions of order less than n ..... 40
$\mathcal{C}(a, b)$ functions continuous on $[a, b]$ ..... 42
$e^{n}(a, b)$ functions with continuous $n$th derivatives ..... 130, 95
$\ell_{2}$ ..... $C_{2}(a, b)$
square-summable infinite sequences ..... 38, 276
functions square-integrable on $[a, b]$ ..... 42
$\mathscr{N}^{n \times 1}$ $n \times 1$ matrices ..... 39

## O rdered Bases

$\mathscr{X}=\left\{\mathbf{x}_{i}\right\}, \mathscr{Y}=\left\{\mathbf{y}_{i}\right\}$,
$\mathscr{Z}=\left\{\mathbf{z}_{\boldsymbol{i}}\right\} \quad$ general bases 48
$\mathscr{F}=\left\{\mathbf{f}_{i}\right\}, \mathcal{G}=\left\{\mathbf{g}_{i}\right\}$,
$\mathscr{F}=\left\{\mathbf{h}_{\boldsymbol{i}}\right\} \quad$ function space bases $\quad 50$
$\mathcal{E}=\left\{\boldsymbol{\varepsilon}_{i}\right\} \quad$ standard basis for $\Re^{n}$ or $\mathscr{R}^{n \times 1} 48$
$\mathfrak{K}$ natural basis 49
Transformations
$\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ general transformations 56
F,G usually nonlinear transformations 400
F usually a functional 403
B bounded linear functional 286
$\boldsymbol{\Theta}$ zero transformation 59
I identity transformation 58
D differentiation operator 64
$\mathbf{L} \quad$ general differential operator $\quad 103,64$
$\boldsymbol{\nabla}^{2} \quad$ Laplacian operator 85
$\mathcal{L}$ Lagrangian functional 420
$\mathbf{E} \quad$ expected value operator 189
$\boldsymbol{\Phi}_{\boldsymbol{i}} \quad$ penalty function 538
$\boldsymbol{\psi}_{c} \quad$ penalized objective function 538
$\boldsymbol{\beta}_{\boldsymbol{i}} \quad$ boundary condition 104
$\mathcal{L} \quad$ Laplace transform $\quad 65$

## M atrices

| $\mathbf{A}, \mathbf{B}, \mathbf{C}$ | general matrices | 559,62 |
| :--- | :--- | ---: |
| $\mathbf{Q}, \mathbf{R}$ | positive definite matrices | 319 |
| $\mathbf{S}$ | change-of-coordinates matrix | 77 |
| $\mathbf{E}_{i j}$ | constituent matrices | 212 |
| $\boldsymbol{\Theta}$ | zero matrix | 559 |
| $\mathbf{I}$ | identity matrix | 561 |
| $\mathbf{\Lambda}$ | diagonal or Jordan form | 156,188 |

## M iscellaneous Symbols

| $\mathcal{\delta}$ | general set | 56 |
| :--- | :--- | ---: |
| $\mathcal{S}^{\perp}$ | orthogonal complement of $\mathcal{S}$ | 250 |
| $\mathscr{\Re}_{g}, \Re_{g}$ | generalized nullspace and range | 182 |
| $\stackrel{\Delta}{=}$ | "defined as" | 18,37 |
| $\Rightarrow$ | "implies" | 47 |


| [ $a, b$ ] | real numbers between and including points $a$ and $b$ | end $42$ |
| :---: | :---: | :---: |
| $(a, b)$ | real numbers between but excluding points $a$ and $b$ | e end 115 |
| $\langle\cdot, \cdot\rangle$ | inner product | 239 |
| $\cdot\rangle\langle\cdot$ | outer product | 326 |
| $\\|\cdot\\|$ | norm | 240, 464, 325 |
| $\times$ | Cartesian product | 39 |
| $\left\{\mathbf{x}_{1}\right\}$ | a set of vectors denoted $\mathbf{x}_{1}, \mathbf{x}_{2}$, and so on | 47 |
| $\mathrm{A}^{\mathbf{T}}$ | transpose of matrix $\mathbf{A}$ | 560 |
| $\mathbf{A}^{-1}, \mathbf{T}^{-1}$ | inverse of $\mathbf{A}$ or $\mathbf{T}$ | 58, 564, 22 |
| $\overline{\mathbf{A}}$ | complex conjugate of $\mathbf{A}$ | 245 |
| $\mathbf{A}^{\dagger}, \mathbf{T}^{\dagger}$ | pseudoinverse of $\mathbf{A}$ or $\mathbf{T}$ | 368 |
| T* | adjoint of $\mathbf{T}$ | 288 |
| (A $\vdots$ B) | A augmented with $\mathbf{B}$ | 20 |
| $\operatorname{det}(\mathbf{A})$ | determinant of $\mathbf{A}$ | 561 |
| $\operatorname{dim}(\mathscr{V})$ | dimension of $\mathscr{V}$ | 53 |
| $\oplus$ | direct sum | 145 |
| $\stackrel{+}{\oplus}$ | orthogonal direct sum | 294 |
| VF | gradient of $\mathbf{F}$ | 404 |
| dG(x,h) | Fréchet differential of $\mathbf{G}$ at $\mathbf{x}$ | 400 |
| $\mathbf{G}^{\prime}(\mathbf{x})$ | Fréchet derivative of $\mathbf{G}$ at $\mathbf{x}$ | 400 |
| $\mathrm{F}^{\prime \prime}$ | Second Fréchet derivative of $\mathbf{F}$ | 465 |
| $\frac{\partial \mathbf{G}}{\partial \mathbf{x}}$ | Jacobian matrix of $\mathbf{G}$ | 405 |
| $\underset{[\mathbf{x}]_{\chi}}{\substack{\text { ¢ }}}$ | coordinate matrix of x relative to $\mathcal{X}$ | 49 |
| $[\mathrm{T}]_{X \mathscr{Y}}$ | matrix of $\mathbf{T}$ relative to $\mathscr{X}$ and $\mathscr{Y}$ | 72 |
| $\mathbf{Q}_{\chi}$ | matrix of an inner product relative to $\mathfrak{X}$ | 245 |
| $k(t, s)$ | Green's function | 110 |
| $\rho_{j}(t)$ | boundary kernel | 110 |
| $\delta$ | delta function | 568, 575 |

