

# A General Approach for Stochastic Correlation using Hyperbolic Functions

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## 1 Introduction

## 2 Stochastic Model for Correlation

- Building-up the Model
- Stochastically correlated Brownian Motions
- Model calibration

## 3 Example: Pricing Quanto

## 4 Conclusions



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## The correlated Brownian Motions

Using the notation

$$dW_t^1 dW_t^2 = \rho dt,$$

the two BMs  $W_t^1$  and  $W_t^2$  are correlated with  $\rho \in [-1, 1]$ .

### Linear Correlation

Correlation  $\rho(X_1, X_2)$  between random variables  $X_1$  and  $X_2$  reads

$$\rho(X_1, X_2) := \frac{\text{cov}(X_1, X_2)}{\sqrt{\text{var}(X_1)\text{var}(X_2)}}$$

Example: Heston Model [Heston 1993]:

$$\begin{aligned} dS_t &= \mu S dt + \sqrt{\nu(t)} S dW_t^1 \\ d\nu(t) &= \kappa(\theta - \nu(t)) dt + \sigma \sqrt{\nu(t)} dW_t^2 \end{aligned}$$

$\mu, \kappa, \theta$  and  $\sigma$  are positive constants

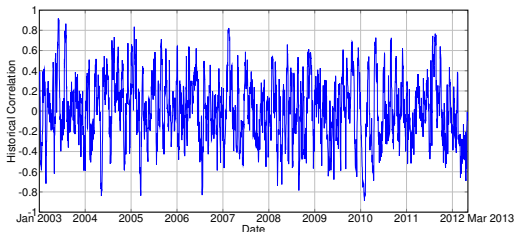


## The estimator of correlation

The correlation  $\rho_T(X_1, X_2)$  for the time region  $T$  with observed values  $\hat{X}_1(t)$  and  $\hat{X}_2(t)$ ,  $t \in T$  can be estimated as

$$\rho_T(X_1, X_2) \approx \hat{\rho}_T = \frac{\sum_{t \in T} (\hat{X}_1(t) - \frac{1}{n_T} \sum_{t \in T} \hat{X}_1(t)) (\hat{X}_2(t) - \frac{1}{n_T} \sum_{t \in T} \hat{X}_2(t))}{\sqrt{\sum_{t \in T} (\hat{X}_1(t) - \frac{1}{n_T} \sum_{t \in T} \hat{X}_1(t))^2 \sum_{t \in T} (\hat{X}_2(t) - \frac{1}{n_T} \sum_{t \in T} \hat{X}_2(t))^2}}$$

where  $n_T$  is the number of pairs  $(\hat{X}_1(t), \hat{X}_2(t))$  with  $t \in T$ .

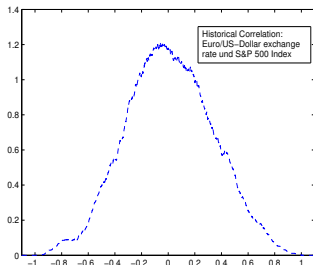


Historical Correlation between Euro/US-Dollar exchange Rate and S&P 500 [yahoo.com]

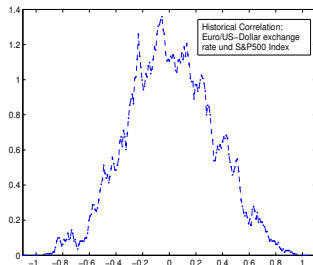
- Not constant over times
- As stochastic process?



Bandwidth:  $\frac{1}{20}$



Bandwidth:  $\frac{1}{80}$



## Empirical Density function

The correlation should

- only take values on  $[-1, 1]$
- vary around a mean value
- the probability mass tends to zero in the boundary values



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## Model

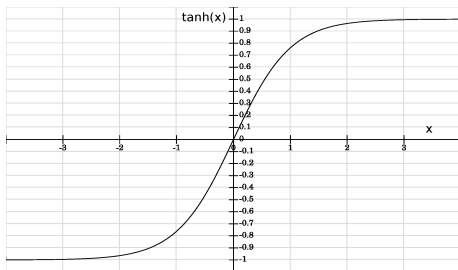
Based on the SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad t \geq 0, X_0 = x_0$$

we assume a **stochastic correlation**

$$\rho_t = \tanh(X_t)$$

where  $\rho_0 = \tanh(x_0) \in (-1, 1)$ .



Tangens Hyperbolicus





Applying *Itô's Lemma*

$$d\rho_t = d \tanh(X_t) = \frac{\partial \tanh(X_t)}{\partial t} dt + \frac{\partial \tanh(X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 \tanh(X_t)}{\partial x^2} (dX_t)^2$$

we obtain

$$d\rho_t = (1 - \rho_t^2) \left( (\tilde{a} - \rho_t \tilde{b}^2) dt + \tilde{b} dW_t \right), \quad t \geq 0$$

where  $\rho_0 \in (-1, 1)$ ,  $\tilde{a} = a(t, \operatorname{artanh}(\rho_t))$  and  $\tilde{b} = b(t, \operatorname{artanh}(\rho_t))$

The generated correlation

- only takes value on  $(-1, 1)$  agreed with market observation
- can vary around a mean value
- approaches zero in the boundary values



## Example

We choose the *Ornstein-Uhlenbeck process*

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t \quad \kappa, \sigma > 0 \text{ and } X_0, \mu \in \mathbb{R}$$

Applying *Itô's Lemma* with  $\rho_t = \tanh(X_t)$

$$d\rho_t = \frac{\partial \tanh(X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 \tanh(X_t)}{\partial x^2} \sigma^2 dt$$

gives **stochastic correlation process** as

$$d\rho_t = (1 - \rho_t^2) (\kappa(\mu - \operatorname{artanh}(\rho_t)) - \rho_t \sigma^2) dt + (1 - \rho_t^2) \sigma dW_t,$$

where  $t \geq 0$ ,  $\rho_0 \in (-1, 1)$ ,  $\kappa, \sigma > 0$  and  $\mu \in \mathbb{R}$ .

For  $t \rightarrow \infty$  the probability density function  $f(\tilde{\rho})$  can be derived as

$$f(\tilde{\rho}) = \frac{1}{1 - \tilde{\rho}^2} \cdot \frac{\sqrt{\kappa}}{\sigma \sqrt{\pi}} \cdot e^{-\frac{\kappa(\operatorname{artanh}(\tilde{\rho}) - \mu)^2}{\sigma^2}}$$



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Based on two independent Brownian motions  $W_t^2$  and  $W_t^3$  we define

$$W_t^1 = \int_0^t \rho_s dW_s^2 + \int_0^t \sqrt{1 - \rho_s^2} dW_s^3$$

which satisfies  $W_0^1 = 0$ ,  $\mathbb{E}[(W_t^1)^2] = t$  and  $\mathbb{E}[W_t^1 | \mathcal{F}_s] = W_s^1, t \geq s$ .

By the stochastic correlation process  $\rho_t$  correlated this two Brownian motions  $W_t^1$  and  $W_t^2$  holds

$$\mathbb{E}[W_t^1 \cdot W_t^2] = \mathbb{E}\left[\int_0^t \rho_s ds\right]$$

which agrees for constant correlation  $\rho$  with

$$dW_t^1 \cdot dW_t^2 = \rho dt.$$



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# The Fokker-Planck equation

Recall the *stochastic correlation process*

$$d\rho_t = \underbrace{(1 - \rho_t^2)(\tilde{a} - \rho_t \tilde{b}^2)}_{:=\hat{a}(t, \rho_t)} dt + \underbrace{(1 - \rho_t^2)\tilde{b}}_{:=\hat{b}(t, \rho_t)} dW_t, \quad t \geq 0$$

where  $\rho_0 \in (-1, 1)$ . Assuming it possesses a transition density  $p(t, \tilde{\rho}|\rho_0)$  which satisfies the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} p(t, \tilde{\rho}) + \frac{\partial}{\partial \tilde{\rho}} (\hat{a}(t, \tilde{\rho}) p(t, \tilde{\rho})) - \frac{1}{2} \frac{\partial^2}{\partial \tilde{\rho}^2} (\hat{b}(t, \tilde{\rho})^2 p(t, \tilde{\rho})) = 0$$

with the conditions

$$\int_{-1}^1 p(t, \tilde{\rho}) d\tilde{\rho} = 1 \quad \text{and} \quad \int_{-1}^1 \tilde{\rho} \cdot p(t, \tilde{\rho}) d\tilde{\rho} \rightarrow_{t \rightarrow \infty} \text{mean value}$$

and the stationary density can be computed as

$$p(\tilde{\rho}) := \lim_{t \rightarrow \infty} p(t, \tilde{\rho}|\rho_0).$$



## Example

Correlation process based on the Ornstein-Uhlenbeck process

$$d\rho_t = \underbrace{(1 - \rho_t^2)(\kappa(\mu - \operatorname{artanh}(\rho_t)) - \rho_t\sigma^2)}_{:=\hat{a}(t,\rho_t)} dt + \underbrace{(1 - \rho_t^2)\sigma}_{:=\hat{b}(t,\rho_t)} dW_t$$

where  $t \geq 0$  and  $\rho_0 \in (-1, 1)$ .

The stationary density can be computed as

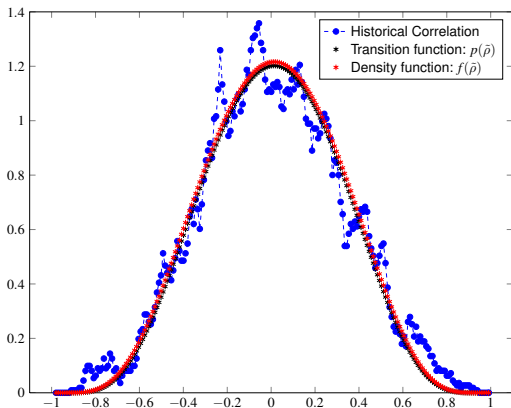
$$p(\tilde{\rho}) = \frac{c}{\tilde{\rho}^2 - 1} \cdot e^{-\frac{\kappa \operatorname{artanh}(\tilde{\rho})}{\sigma^2} (\operatorname{artanh}(\tilde{\rho}) - 2\mu)}$$

with  $c$  such that  $\int_{-1}^1 p(\tilde{\rho}) = 1$ .

Compare to function  $f(\tilde{\rho})$

$$f(\tilde{\rho}) = \frac{1}{1 - \tilde{\rho}^2} \cdot \frac{\sqrt{\kappa}}{\sigma\sqrt{\pi}} \cdot e^{-\frac{\kappa(\operatorname{artanh}(\tilde{\rho}) - \mu)^2}{\sigma^2}}$$





The estimated parameters for:

$p(\tilde{\rho}) : \kappa = 25.66, \mu = 0.01, \sigma = 2.31$  and  $c = -1.2$  ( $MSE = 0.0011$ )

$f(\tilde{\rho}) : \kappa = 23.33, \mu = 0.01$  and  $\sigma = 2.24$  ( $MSE = 0.0014$ )





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## Quanto (Quantity Adjusting Option)

A call on the S&P500 with payoff in Euro

$$\underbrace{\text{exchangeRate}}_{:=R_t} \cdot \underbrace{(S\&P500)}_{:=S_t} - \text{Strike})^+$$

is modeled by

$$\begin{cases} dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S \\ dR_t = \mu_R R_t dt + \sigma_R R_t dW_t^R \end{cases}$$

where  $W_t^S$  and  $W_t^R$  are correlated with

$$d\rho_t = (1 - \rho_t^2)(\kappa(\mu - \text{artanh}(\rho_t)) - \rho_t \sigma^2) dt + (1 - \rho_t^2) \sigma dW_t$$

No-Arbitrage Condition requires

- $\frac{1}{R_0} \exp(r_e T) \mathbb{E}[R_T] = \exp(r_d T) \Rightarrow \mu_R = r_d - r_e$
- $\frac{1}{R_0} \frac{1}{S_0} \mathbb{E}[S_T R_T] = \exp(r_d T) \Rightarrow \mu_S = r_d - \mu_R - \sigma_S \sigma_R \frac{1}{T} \int_0^T \rho_t dt$

$r_e$  and  $r_d$  denote the risk-free interest rates in Euro/US-Dollar



# Conditional Monte Carlo Approach

Price of the Quanto in the Black-Scholes formula with continuous dividend yield

$$C_{\text{Quanto}}(S_0, K, r_d, D(\rho_t), \sigma_S, T) = R_0 \left( S_0 e^{-D(\rho_t)T} \mathcal{N}(d_1) - K e^{-r_d T} \mathcal{N}(d_2) \right)$$

with

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + ((r_d - D(\rho_t)) + \frac{\sigma_S^2}{2})/T}{\sigma_S \sqrt{T}}, \quad d_2 = d_1 - \sigma_S \sqrt{T}$$

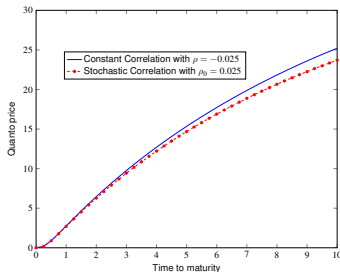
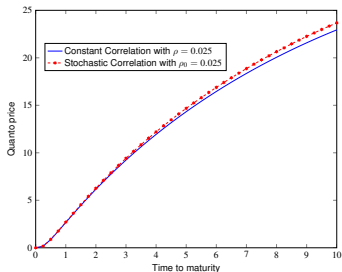
and

$$D(\rho_t) = r_d - r_e + \sigma_S \sigma_R \frac{1}{T} \int_0^T \rho_t dt.$$

The fair price  $P_0$  is given by

$$P_0 = \mathbb{E} \left[ \mathbb{E} [C_{\text{Quanto}}(S_0, K, r_d, D(\rho_t), \sigma_S, T) | \mathcal{F}(\rho_{\{0 \leq s \leq t\}})] \right]$$





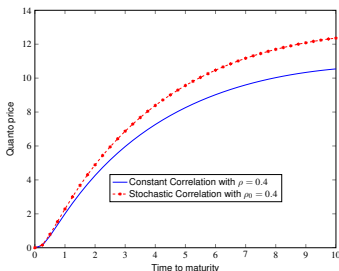
BS parameters:

$$K = 120, S_0 = 100, R_0 = 1, r_d = 0.05, r_e = 0.03, \sigma_S = 0.2, \sigma_R = 0.4$$

Correlation process parameters:

$$\kappa = 23.33, \mu = 0.01 \text{ and } \sigma = 2.24$$





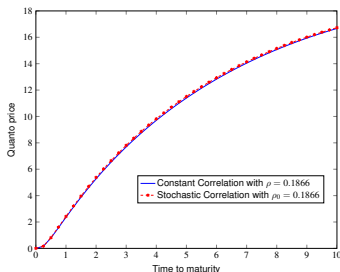
BS parameters:

$$K = 120, S_0 = 100, R_0 = 1, r_d = 0.05, r_e = 0.03, \sigma_S = 0.2, \sigma_R = 0.4$$

Correlation process parameters:

$$\kappa = 23.33, \mu = 0.4 \text{ and } \sigma = 2.24$$





BS parameters:

$$K = 120, S_0 = 100, R_0 = 1, r_d = 0.05, r_e = 0.03, \sigma_S = 0.2, \sigma_R = 0.4$$

Correlation process parameters:

$$\kappa = 43.33, \mu = 0.2, \text{ and } \sigma = 2.24$$

$$\Rightarrow \text{mean value} = 0.1866$$



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- Correlation modeled as fixed number may lead to correlation risk
- Correlation can be modeled as the Hyperbolic Function  $\tanh$  of a stochastic process
- Effect of considering stochastic correlation on pricing the quanto as Example





Thank you for your attention!



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