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A virtual element method for the Steklov eigenvalue problem.

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Abstract

The aim of this paper is to develop a virtual element method for the two-dimensional Steklov eigenvalue problem. We propose a discretization by means of the virtual elements presented in [L. Beirão da Veiga *et al.*, Math. Models Methods Appl. Sci., 23 (2013), pp. 199–214]. Under standard assumptions on the computational domain, we establish that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. We also prove higher order error estimates for the computation of the eigensolutions on the boundary, which in some Steklov problems (computing sloshing modes, for instance) provides the quantity of main interest (the free surface of the liquid). Finally, we report some numerical tests supporting the theoretical results.

Key words: Virtual Element Method, Steklov eigenvalue problem, error estimates
2000 MSC: 65N25, 65N30, 74S99.

1. Introduction

Very recently, a new evolution of the *Mimetic Finite Difference Method* was proposed in [7] under the name of *Virtual Element Method* (VEM). This approach takes the steps from the main ideas of modern mimetic schemes but follows from a Galerkin discretization of the problem and therefore can be fully interpreted as a generalization of the finite element method. Thus, VEM couples the flexibility of mimetic methods with the theoretical and applicative background of finite elements. Since VEM is very recent, the current published literature is still very limited [7, 8, 9, 10, 14].

The present paper deals with the solution of an eigenvalue problems by means of VEM. In particular, we have chosen the Steklov eigenvalue problem, which involves the Laplace operator but is characterized by the presence of the eigenvalue in the boundary condition. The reason of this choice is that the analysis turns out simpler, since the right-hand side involves only boundary terms whose approximation by virtual elements can be seen as a classical interpolation.

The numerical approximation of eigenvalue problems is object of great interest from both, the practical and theoretical points of view. We refer to [11] and the references therein for the state of the art in this subject area. In particular, the Steklov eigenvalue problem appears in many applications. For instance, we mention the study of the vibration modes of a structure in contact

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with an incompressible fluid (see [5]) and the analysis of the stability of mechanical oscillators immersed in a viscous media (see [26]). One of its main applications arises from the dynamics of liquids in moving containers, i.e., sloshing problems (see [6, 15, 16, 17, 20, 30]).

Among the existing techniques to solve this problem, various finite element methods have been introduced and analyzed. For instance, conforming finite element discretization have been considered in [1, 12], while [27, 25] deal with nonconforming finite elements. Other numerical treatment for the Steklov eigenvalue problem, including a posteriori error analysis can be found in [2, 3, 19, 21, 28] and the references cited therein. Traditionally, finite element methods rely on triangular (simplicial) or quadrilateral meshes. However, in complex simulations, it can be convenient to use more general polygonal meshes.

The aim of this paper is to introduce and analyze a virtual element method which applies to general polygonal (even non-convex) meshes for the solution of the two-dimensional Steklov eigenvalue problem. We begin with a variational formulation of the spectral problem. We propose a discretization based on the approach introduced in [7] for the Laplace equation. By using the abstract spectral approximation theory (see [4]), under rather mild assumptions on the polygonal meshes, we establish that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues.

The outline of this article is as follows: We introduce in Section 2 the variational formulation of the Steklov eigenvalue problem, define a solution operator and establish its spectral characterization. In Section 3, we introduce the virtual element discrete formulation and describe the spectrum of a discrete solution operator. In Section 4, we prove that the numerical scheme provides a correct spectral approximation and establish optimal order error estimates for the eigenvalues and eigenfunctions. We also prove an improved error estimate for the eigenfunctions on the free boundary, which allows computing a quantity of typical interest in sloshing problems. Finally, in Section 5, we report a couple of numerical tests that allow us to assess the convergence properties of the method, to confirm that it is not polluted with spurious modes and to check whether the experimental rates of convergence agree with the theoretical ones.

Throughout the article we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

2. The spectral problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\partial\Omega$. Let Γ_0 and Γ_1 be disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ and $|\Gamma_0| \neq 0$. We denote by n the outward unit normal vector to $\partial\Omega$ and by ∂_n the normal derivative.

We consider the following eigenvalue problem:

Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that

$$\left\{ \begin{array}{l} \Delta w = 0 \quad \text{in } \Omega, \\ \partial_n w = \left\{ \begin{array}{ll} \lambda w & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{array} \right. \end{array} \right.$$

By testing the first equation above with $v \in H^1(\Omega)$ and integrating by parts, we arrive at the following equivalent weak formulation:

Problem 1. *Find* $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, *such that*

$$\int_\Omega \nabla w \cdot \nabla v = \lambda \int_{\Gamma_0} wv \qquad \forall v \in H^1(\Omega).$$

Since the bilinear form on the left-hand side is not $H^1(\Omega)$-elliptic, it is convenient to use a shift argument to rewrite this eigenvalue problem in the following form:

Problem 2. *Find* $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, *such that*

$$\widehat{a}(w,v) = (\lambda + 1)\, b(w,v) \qquad \forall v \in H^1(\Omega),$$

where

$$\begin{aligned} \widehat{a}(w,v) &:= a(w,v) + b(w,v), \qquad w,v \in H^1(\Omega), \\ a(w,v) &:= \int_\Omega \nabla w \cdot \nabla v, \qquad w,v \in H^1(\Omega), \\ b(w,v) &:= \int_{\Gamma_0} wv, \qquad w,v \in H^1(\Omega) \end{aligned}$$

are bounded bilinear symmetric forms.

Next, we define the solution operator associated with Problem 2:

$$\begin{aligned} T:\ H^1(\Omega) &\longrightarrow H^1(\Omega), \\ f &\longmapsto Tf := u, \end{aligned}$$

where $u \in H^1(\Omega)$ is the solution of the corresponding source problem:

$$\widehat{a}(u,v) = b(f,v) \qquad \forall v \in H^1(\Omega). \tag{2.1}$$

The following lemma allows us to establish the well-posedness of this source problem.

Lemma 2.1. *There exists a constant* $\alpha > 0$, *depending on* Ω, *such that*

$$\widehat{a}(v,v) \geq \alpha \left\| v \right\|_{1,\Omega}^2 \qquad \forall v \in H^1(\Omega).$$

Proof. The result follows immediately from the generalized Poincaré inequality. □

We deduce from Lemma 2.1 that the linear operator T is well defined and bounded. Notice that $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$ solves Problem 2 (and hence Problem 1) if and only if $Tw = \mu w$ with $\mu \neq 0$ and $w \neq 0$, in which case $\mu := \frac{1}{1+\lambda}$. Moreover, it is easy to check that T is self-adjoint with respect to the inner product $\widehat{a}(\cdot,\cdot)$ in $H^1(\Omega)$. Indeed, given $f, g \in H^1(\Omega)$,

$$\widehat{a}(Tf, g) = b(f,g) = b(g,f) = \widehat{a}(Tg, f) = \widehat{a}(f, Tg).$$

The following is an additional regularity result for the solution of problem (2.1) and consequently, for the eigenfunctions of T.

Lemma 2.2. *There exists* $r_\Omega > \frac{1}{2}$ *such that the following results hold:*

i) for all $f \in H^1(\Omega)$ *and for all* $r \in [\frac{1}{2}, r_\Omega)$, *the solution* u *of problem* (2.1) *satisfies* $u \in H^{1+r_1}(\Omega)$ *with* $r_1 := \min\{r, 1\}$ *and there exists* $C > 0$ *such that*

$$\left\| u \right\|_{1+r_1,\Omega} \leq C \left\| f \right\|_{1,\Omega};$$

ii) if w is an eigenfunction of Problem 1 with eigenvalue λ, for all $r \in [\frac{1}{2}, r_\Omega)$, $w \in H^{1+r}(\Omega)$ and there exists $C > 0$ (depending on λ) such that

$$\|w\|_{1+r,\Omega} \leq C \|w\|_{1,\Omega}.$$

Proof. The proof of (i) follows from the classical regularity result for the Laplace equation with Neumann boundary conditions (cf. [23]). The proof of (ii) follows from the same arguments and the fact that w is the solution of problem (2.1) with $f = \lambda w$, combined with a bootstrap trick. $\square$

The constant $r_\Omega > \frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. If Ω is convex, then $r_\Omega > 1$, whereas, otherwise, $r_\Omega := \frac{\pi}{\omega}$ with ω being the largest reentrant angle of Ω (see [23])). Hence, because of the compact inclusion $H^{1+r}(\Omega) \hookrightarrow H^1(\Omega)$, T is a compact operator. Therefore, we have the following spectral characterization result.

Theorem 2.1. *The spectrum of T decomposes as follows:* $\operatorname{sp}(T) = \{0, 1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, *where:*

i) $\mu = 1$ is an eigenvalue of T and its associated eigenspace is the space of constant functions in Ω;

ii) $\mu = 0$ is an infinite-multiplicity eigenvalue of T with associated eigenspace is $H^1_{\Gamma_0}(\Omega) := \left\{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_0\right\}$;

iii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0, 1)$ is a sequence of finite-multiplicity eigenvalues of T which converge to 0 and their corresponding eigenspaces lie in $H^{1+r}(\Omega)$.

Proof. Properties (i) and (ii) are easy to check. Properties (iii) follows from the classical spectral characterization of compact operators and Lemma 2.2(ii). $\square$

3. The discrete problem

In this section, first we recall the mesh construction and the assumptions considered in [7] for the virtual element method. Then, we will introduce a virtual element discretization of Problems 1 and 2 and provide a spectral characterization of the resulting discrete eigenvalue problems.

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K\in\Omega} h_K$.

For the analysis, we will make as in [7] the following assumptions.

- **A0.1.** Every mesh $\mathcal{T}_h$ consists of a finite number of *simple* polygons (i.e. open simply connected sets with non self intersecting polygonal boundaries).
- **A0.2.** There exists $\gamma > 0$ such that, for all meshes $\mathcal{T}_h$, each polygon $K \in \mathcal{T}_h$ is star-shaped with respect to a ball of radius greater than or equal to γh_K.
- **A0.3.** There exists $\widehat{\gamma} > 0$ such that, for all meshes $\mathcal{T}_h$, for each polygon $K \in \mathcal{T}_h$, the distance between any two of its vertices is greater than or equal to $\widehat{\gamma} h_K$.

We consider now a simple polygon K and, for $k \in \mathbb{N}$, we define

$$\mathbb{B}_k(\partial K) := \left\{v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e) \text{ for all edges } e \subset \partial K\right\}.$$

We then consider the finite-dimensional space defined as follows:

$$V_k^K := \left\{v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v|_K \in \mathbb{P}_{k-2}(K)\right\},$$

where, for $k = 1$, we have used the convention that $\mathbb{P}_{-1}(K) := \{0\}$. We choose in this space the degrees of freedom introduced in [7, Section 4.1]. Finally, for every decomposition $\mathcal{T}_h$ of Ω into simple polygons K and for a fixed $k \in \mathbb{N}$, we define

$$V_h := \left\{ v \in H^1(\Omega) : \ v|_K \in V_k^K \right\}.$$

In what follows, we will also use the broken H^1-seminorm

$$|v|_{1,h}^2 := \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{0,K}^2 ,$$

which is well defined for every $v \in L^2(\Omega)$ such that $v|_K \in H^1(K)$ for all polygon $K \in \mathcal{T}_h$.

In order to construct the discrete scheme, we need some preliminary definitions. First, we split the bilinear form $\widehat{a}(\cdot,\cdot)$ as follows:

$$\widehat{a}(u,v) = \sum_{K \in \mathcal{T}_h} a^K(u,v) + b(u,v), \qquad u, v \in H^1(\Omega),$$

where

$$a^K(u,v) := \int_K \nabla u \cdot \nabla v, \qquad u, v \in H^1(\Omega).$$

To compute the local matrix a^K for $u, v \in V_h$, we must have into account that due to the implicit space definition, we would not know how to compute the bilinear form exactly. Nevertheless, the final output will be a local matrix on each element K whose associated bilinear form is exact whenever one of the two entries is a polynomial of degree k. This will allow us to retain the optimal approximation properties of the space V_h.

With this end, for any $K \in \mathcal{T}_h$ and for any sufficiently regular function φ, we define first

$$\bar{\varphi} := \frac{1}{N_K} \sum_{i=1}^{N_K} \varphi(P_i),$$

where P_i, $1 \le i \le N_K$, are the vertices of K. Now, we define the projector $\Pi_k^K : \ V_k^K \longrightarrow \mathbb{P}_k(K) \subseteq V_k^K$ for each $v \in V_k^K$ as the solution of

$$a^K\left(\Pi_k^K v, q\right) = a^K(v,q) \qquad \forall q \in \mathbb{P}_k(K), \tag{3.1a}$$

$$\overline{\Pi_k^K v} = \overline{v}. \tag{3.1b}$$

On the other hand, let $S^K(\cdot,\cdot)$ be any symmetric positive definite bilinear form to be chosen as to satisfy

$$c_0\, a^K(v,v) \le S^K(v,v) \le c_1\, a^K(v,v) \qquad \forall v \in V_k^K \text{ with } \Pi_k^K v = 0, \tag{3.2}$$

for some positive constants c_0 and c_1 independent of K. Then, set

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h), \qquad u_h, v_h \in V_h,$$

where $a_h^K(\cdot,\cdot)$ is the bilinear form defined on $V_k^K \times V_k^K$ by

$$a_h^K(u,v) := a^K\left(\Pi_k^K u, \Pi_k^K v\right) + S^K\left(u - \Pi_k^K u, v - \Pi_k^K v\right), \qquad u, v \in V_k^K. \tag{3.3}$$

The following properties of the bilinear form $a_h^K(\cdot,\cdot)$ have been established in [7, Theorem 4.1].

- *k-Consistency*:

$$a_h^K(p, v_h) = a^K(p, v_h) \qquad \forall p \in \mathbb{P}_k(K), \quad \forall v_h \in V_k^K.$$

- *Stability*: There exist two positive constants α_* and α^*, independent of K, such that:

$$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \qquad \forall v_h \in V_k^K. \tag{3.4}$$

Now, we are in a position to write the virtual element discretization of Problem 1.

Problem 3. *Find $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, such that*

$$a_h(w_h, v_h) = \lambda_h b(w_h, v_h) \qquad \forall v_h \in V_h.$$

We use again a shift argument to rewrite this discrete eigenvalue problem in the following convenient equivalent form.

Problem 4. *Find $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, such that*

$$\widehat{a}_h(w_h, v_h) = (\lambda_h + 1)\, b(w_h, v_h) \qquad \forall v_h \in V_h,$$

where

$$\widehat{a}_h(w_h, v_h) := a_h(w_h, v_h) + b(w_h, v_h), \qquad w_h, v_h \in V_h.$$

We observe that by virtue of (3.4) and the trace theorem, the bilinear form $\widehat{a}_h(\cdot, \cdot)$ is bounded. Moreover, as shown in the following lemma, it is also uniformly elliptic.

Lemma 3.1. *There exists a constant $\beta > 0$, independent of h, such that*

$$\widehat{a}_h(v_h, v_h) \geq \beta \, \|v_h\|_{1,\Omega}^2 \qquad \forall v_h \in V_h.$$

Proof. Thanks to (3.4) and Lemma 2.1, it is easy to check that the above inequality holds with $\beta := \alpha \min\{\alpha_*, 1\}$. $\square$

The discrete version of the operator T is then given by

$$\begin{aligned} T_h : \ H^1(\Omega) &\longrightarrow H^1(\Omega), \\ f &\longmapsto T_h f := u_h, \end{aligned}$$

where $u_h \in V_h$ is the solution of the corresponding discrete source problem

$$\widehat{a}_h(u_h, v_h) = b(f, v_h) \qquad \forall v_h \in V_h.$$

Because of Lemma 3.1, the linear operator T_h is well defined and bounded uniformly with respect to h. Once more, as in the continuous case, $(\lambda_h, w_h) \in \mathbb{R} \times V_h$ solves Problem 4 (and hence Problem 3) if and only if $T_h w_h = \mu_h w_h$ with $\mu_h \neq 0$ and $w_h \neq 0$, in which case $\mu_h := \frac{1}{1+\lambda_h}$. Moreover, $T_h|_{V_h} : \ V_h \longrightarrow V_h$ is self-adjoint with respect to $\widehat{a}_h(\cdot, \cdot)$. Indeed, given $f, g \in V_h$,

$$\widehat{a}_h(T_h f, g) = b(f, g) = b(g, f) = \widehat{a}_h(T_h g, f) = \widehat{a}_h(f, T_h g).$$

As a consequence, we have the following spectral characterization.

Theorem 3.1. *The spectrum of $T_h|_{V_h}$ consists of $M_h := \dim(V_h)$ eigenvalues, repeated according to their respective multiplicities. It decomposes as follows: $\mathrm{sp}(T_h|_{V_h}) = \{0, 1\} \cup \{\mu_{hk}\}_{k=1}^{N_h}$, where:*

i) the eigenspace associated with $\mu_h = 1$ is the space of constant functions in Ω;

ii) the eigenspace associated with $\mu_h = 0$ is $Z_h := V_h \cap H^1_{\Gamma_0}(\Omega) = \{q_h \in V_h : \ q_h = 0 \text{ on } \Gamma_0\}$;

iii) $\mu_{hk} \subset (0, 1)$, $k = 1, \ldots, N_h := M_h - \dim(Z_h) - 1$, are non-defective eigenvalues repeated according to their respective multiplicities.

4. Spectral approximation

To prove that T_h provides a correct spectral approximation of T, we will resort to the classical theory for compact operators (see [4]), which is based on the convergence in norm of T_h to T as $h \to 0$. With the aim of proving this, the first step is to establish the following result.

Lemma 4.1. *There exists $C > 0$ such that, for all $f \in H^1(\Omega)$, if $u = Tf$ and $u_h = T_h f$, then*

$$\|(T - T_h) f\|_{1,\Omega} = \|u - u_h\|_{1,\Omega} \le C \left(\|u - u_I\|_{1,\Omega} + |u - u_\pi|_{1,h} \right),$$

for all $u_I \in V_h$ and for all $u_\pi \in L^2(\Omega)$ such that $u_\pi|_K \in \mathbb{P}_k(K)$ $\forall K \in \mathcal{T}_h$.

Proof. Let $f \in H^1(\Omega)$. For $u_I \in V_h$, we set $v_h := u_h - u_I$ and thanks to Lemma 3.1, the definitions (3.3) of a_h^K and those of T and T_h, we have

$$\begin{aligned}
\beta \|v_h\|_{1,\Omega}^2 &\le \widehat{a}_h(v_h, v_h) = \widehat{a}_h(u_h, v_h) - \widehat{a}_h(u_I, v_h) \\
&= b(f, v_h) - \sum_{K \in \mathcal{T}_h} a_h^K(u_I, v_h) - b(u_I, v_h) \\
&= b(f, v_h) - b(u_I, v_h) - \sum_{K \in \mathcal{T}_h} \left(a_h^K(u_I - u_\pi, v_h) + a^K(u_\pi - u, v_h) + a^K(u, v_h) \right) \\
&= b(u - u_I, v_h) - \sum_{K \in \mathcal{T}_h} \left(a_h^K(u_I - u_\pi, v_h) + a^K(u_\pi - u, v_h) \right).
\end{aligned}$$

Therefore, from the trace theorem, (3.4) and the boundedness of $a_h^K(\cdot,\cdot)$ and $a^K(\cdot,\cdot)$,

$$\begin{aligned}
\beta \|v_h\|_{1,\Omega}^2 &\le \|u - u_I\|_{0,\Gamma_0} \|v_h\|_{0,\Gamma_0} + \sum_{K \in \mathcal{T}_h} \left(\alpha^* |u_I - u_\pi|_{1,K} |v_h|_{1,K} + |u_\pi - u|_{1,K} |v_h|_{1,K} \right) \\
&\le \|u - u_I\|_{1,\Omega} \|v_h\|_{1,\Omega} + \sum_{K \in \mathcal{T}_h} \left(\alpha^* |u_I - u|_{1,K} |v_h|_{1,K} + (\alpha^* + 1) |u - u_\pi|_{1,K} |v_h|_{1,K} \right) \\
&\le C \left(\|u - u_I\|_{1,\Omega} + |u - u_\pi|_{1,h} \right) \|v_h\|_{1,\Omega}.
\end{aligned}$$

Hence, the proof follows from the triangular inequality. □

The next step is to find appropriate terms u_I and u_π that can be used in the above lemma to prove the claimed convergence. For the latter we have the following proposition, which is derived by interpolation between Sobolev spaces (see for instance [22, Theorem I.1.4]) from the analogous result for integer values of s. In its turn, the result for integer values is stated in [7, Proposition 4.2] and follows from the classical Scott-Dupont theory (see [13]).

Proposition 4.1. *If the assumption* **A0.2** *is satisfied, then there exists a constant C, depending only on k and γ, such that for every s with $0 \le s \le k$ and for every $v \in H^{1+s}(K)$, there exists $v_\pi \in \mathbb{P}_k(K)$ such that*

$$\|v - v_\pi\|_{0,K} + h_K |v - v_\pi|_{1,K} \le C h_K^{1+s} \|v\|_{1+s,K}.$$

For the term $u_I \in V_h$ in Lemma 4.1, we have the following result which is an extension of [7, Proposition 4.3] to less regular functions.

Proposition 4.2. *If the assumptions* **A0.2** *and* **A0.3** *are satisfied, then, for each s with $0 \leq s \leq k$, there exists a constant C, depending only on k, γ and $\widehat{\gamma}$, such that for every $v \in H^{1+s}(\Omega)$, there exists $v_I \in V_h$ that satisfies*

$$\|v - v_I\|_{0,\Omega} + h\,|v - v_I|_{1,\Omega} \leq Ch^{1+s}\,\|v\|_{1+s,\Omega}\,.$$

Proof. Let $v \in H^{1+s}(\Omega)$, $0 \leq s \leq k$. Since we are assuming **A0.2**, let $v_\pi \in L^2(\Omega)$ be defined on each $K \in \mathcal{T}_h$ so that $v_\pi|_K \in \mathbb{P}_k(K)$ and the estimate of Proposition 4.1 holds true.

For each polygon $K \in \mathcal{T}_h$, consider the triangulation $\mathcal{T}_h^K$ obtained by joining each vertex of K with the midpoint of the ball with respect to which K is starred. Let $\widehat{\mathcal{T}}_h := \bigcup_{K\in\mathcal{T}_h} \mathcal{T}_h^K$. Since we are also assuming **A0.3**, $\{\widehat{\mathcal{T}}_h\}_h$ is a shape-regular family of triangulations of Ω.

Let v_{c} be the Clément interpolant of degree k of v over $\widehat{\mathcal{T}}_h$ (cf. [18]). Then, $v_{\mathrm{c}} \in H^1(\Omega)$ and the following error estimate follows by interpolation between Sobolev spaces from the analogous result for integer values of s (which in turn has been proved in [18]):

$$\|v - v_{\mathrm{c}}\|_{0,\Omega} + h\,|v - v_{\mathrm{c}}|_{1,\Omega} \leq Ch^{1+s}\,\|v\|_{1+s,\Omega}\,. \tag{4.1}$$

Now, for each $K \in \mathcal{T}_h$, we define $v_I|_K \in H^1(K)$ as the solution of the following problem:

$$\begin{cases} -\Delta v_I = -\Delta v_\pi & \text{in } K, \\ v_I = v_{\mathrm{c}} & \text{on } \partial K. \end{cases}$$

Note that $v_I|_K \in V_k^K$. Moreover, although v_I is defined locally, since on the boundary of each element it coincides with v_{c} which belongs to $H^1(\Omega)$, we have that also v_I belongs to $H^1(\Omega)$ and, hence, $v_I \in V_h$.

According to the above definition we have that

$$\begin{cases} -\Delta(v_\pi - v_I) = 0 & \text{in } K, \\ v_\pi - v_I = v_\pi - v_{\mathrm{c}} & \text{on } \partial K, \end{cases}$$

and, hence, it is easy to check that

$$|v_\pi - v_I|_{1,K} = \inf\left\{|z|_{1,K}\,,\ z \in H^1(K):\ z = v_\pi - v_{\mathrm{c}} \ \text{ on } \partial K\right\} \leq |v_\pi - v_{\mathrm{c}}|_{1,K}\,.$$

Therefore,

$$|v - v_I|_{1,K} \leq |v - v_\pi|_{1,K} + |v_\pi - v_I|_{1,K} \leq |v - v_\pi|_{1,K} + |v_\pi - v_{\mathrm{c}}|_{1,K} \leq 2\,|v - v_\pi|_{1,K} + |v - v_{\mathrm{c}}|_{1,K}\,,$$

which together with Proposition 4.1 and (4.1) lead to

$$|v - v_I|_{1,\Omega} \leq Ch^s\,\|v\|_{1+s,\Omega}\,. \tag{4.2}$$

On the other hand, for all $K \in \mathcal{T}_h$, each triangle $T \in \mathcal{T}_h^K$ has one edge on ∂K. Hence, since $v_I = v_{\mathrm{c}}$ on ∂K, a scaling argument and the classical Poincaré inequality yield

$$\|v_{\mathrm{c}} - v_I\|_{0,T} \leq Ch_K\,|v_{\mathrm{c}} - v_I|_{1,T}\,.$$

Thus, from the above inequality, (4.1) and (4.2), we have

$$\begin{aligned} \|v - v_I\|_{0,\Omega} &\leq \|v - v_{\mathrm{c}}\|_{0,\Omega} + \|v_{\mathrm{c}} - v_I\|_{0,\Omega} \leq \|v - v_{\mathrm{c}}\|_{0,\Omega} + Ch\,|v_{\mathrm{c}} - v_I|_{1,\Omega} \\ &\leq \|v - v_{\mathrm{c}}\|_{0,\Omega} + Ch\,|v - v_{\mathrm{c}}|_{1,\Omega} + Ch\,|v - v_I|_{1,\Omega} \\ &\leq Ch^{1+s}\,\|v\|_{1+s,\Omega}\,, \end{aligned}$$

which together with (4.2) allow us to conclude the proof. □

The following result yields the convergence in norm of T_h to T as $h \to 0$.

Lemma 4.2. *For all $r \in [\frac{1}{2}, r_\Omega)$, let $r_1 := \min\{r, 1\}$ as defined in Lemma 2.2(i). Then, there exists $C > 0$ such that*

$$\|(T - T_h) f\|_{1,\Omega} \leq C h^{r_1} \|f\|_{1,\Omega} \qquad \forall f \in H^1(\Omega).$$

Proof. The result follows from Lemma 4.1, Propositions 4.1 and 4.2, and Lemma 2.2(i). $\square$

4.1. Error estimates

As a direct consequence of Lemma 4.2, standard results about spectral approximation (see [24], for instance) show that isolated parts of $\mathrm{sp}(T)$ are approximated by isolated parts of $\mathrm{sp}(T_h)$. More precisely, let $\mu \in (0, 1)$ be an isolated eigenvalue of T with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. Then, there exist m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which converge to μ. Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces.

We recall the definition of the *gap* $\widehat{\delta}$ between two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $H^1(\Omega)$:

$$\widehat{\delta}(\mathcal{X}, \mathcal{Y}) := \max\{\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\}, \qquad \text{where} \quad \delta(\mathcal{X}, \mathcal{Y}) := \sup_{x \in \mathcal{X}:\ \|x\|_{1,\Omega} = 1} \left(\inf_{y \in \mathcal{Y}} \|x - y\|_{1,\Omega} \right).$$

The following error estimates for the approximation of eigenvalues and eigenfunctions hold true.

Theorem 4.1. *There exists a strictly positive constant C such that*

$$\widehat{\delta}(\mathcal{E}, \mathcal{E}_h) \leq C \gamma_h,$$
$$\left| \mu - \mu_h^{(i)} \right| \leq C \gamma_h, \qquad i = 1, \ldots, m,$$

where

$$\gamma_h := \sup_{f \in \mathcal{E}:\ \|f\|_{1,\Omega} = 1} \|(T - T_h) f\|_{1,\Omega}.$$

Proof. As a consequence of Lemma 4.2, T_h converges in norm to T as h goes to zero. Then, the proof follows as a direct consequence of Theorems 7.1 and 7.3 from [4]. $\square$

The theorem above yields error estimates depending on γ_h. The next step is to show an optimal order estimate for this term.

Theorem 4.2. *For all $r \in [\frac{1}{2}, r_\Omega)$ there exists a positive constant C such that*

$$\|(T - T_h) f\|_{1,\Omega} \leq C h^{\min\{r,k\}} \|f\|_{1,\Omega} \qquad \forall f \in \mathcal{E} \tag{4.3}$$

and, consequently,

$$\gamma_h \leq C h^{\min\{r,k\}}. \tag{4.4}$$

Proof. The proof is identical to that of Lemma 4.2, but using now the additional regularity from Lemma 2.2(ii). $\square$

The error estimate for the eigenvalue $\mu \in (0, 1)$ of T leads to an analogous estimate for the approximation of the eigenvalue $\lambda = \frac{1}{\mu} - 1$ of Problem 1 by means of the discrete eigenvalues $\lambda_h^{(i)} := \frac{1}{\mu_h^{(i)}} - 1$, $1 \leq i \leq m$, of Problem 3. However, the order of convergence in Theorem 4.1 is not optimal for μ and, hence, not optimal for λ either. Our next goal is to improve this order.

Theorem 4.3. *For all $r \in [\frac{1}{2}, r_\Omega)$, there exists a strictly positive constant C such that*

$$\left|\lambda - \lambda_h^{(i)}\right| \leq C h^{2\min\{r,k\}}.$$

Proof. Let w_h be such that $(\lambda_h^{(i)}, w_h)$ is a solution of Problem 3 with $\|w_h\|_{1,\Omega} = 1$. According to Theorem 4.1, there exists a solution (λ, w) of Problem 1 such that

$$\|w - w_h\|_{1,\Omega} \leq C\gamma_h. \tag{4.5}$$

From the symmetry of the bilinear forms and the facts that $a(w,v) = \lambda b(w,v)$ for all $v \in H^1(\Omega)$ (cf. Problem 1) and $a_h(w_h, v_h) = \lambda_h^{(i)} b(w_h, v_h)$ for all $v_h \in V_h$ (cf. Problem 3), we have

$$\begin{aligned} a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) &= a(w_h, w_h) - \lambda b(w_h, w_h) \\ &= [a(w_h, w_h) - a_h(w_h, w_h)] - \left(\lambda - \lambda_h^{(i)}\right) b(w_h, w_h), \end{aligned}$$

from which we obtain the following identity:

$$\left(\lambda_h^{(i)} - \lambda\right) b(w_h, w_h) = a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) + [a_h(w_h, w_h) - a(w_h, w_h)]. \tag{4.6}$$

The next step is to bound each term on the right hand side above. The first and the second ones are easily bounded from the continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the trace theorem, (4.5) and (4.4):

$$|a(w - w_h, w - w_h)| + \lambda\,|b(w - w_h, w - w_h)| \leq C h^{2\min\{r,k\}}. \tag{4.7}$$

For the third term, we use (3.2) and (3.1a) to write:

$$\begin{aligned} &|a_h(w_h, w_h) - a(w_h, w_h)| \\ &\quad = \left| \sum_{K \in \mathcal{T}_h} \left[a^K(\Pi_k^K w_h, \Pi_k^K w_h) + S^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right)\right] - \sum_{K \in \mathcal{T}_h} a^K(w_h, w_h) \right| \\ &\quad \leq \left| \sum_{K \in \mathcal{T}_h} \left[a^K\left(\Pi_k^K w_h, \Pi_k^K w_h\right) - a^K(w_h, w_h)\right] \right| + \sum_{K \in \mathcal{T}_h} c_1\, a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right) \\ &\quad = \sum_{K \in \mathcal{T}_h} \left[a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right)\right] + \sum_{K \in \mathcal{T}_h} c_1\, a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) \\ &\quad = \sum_{K \in \mathcal{T}_h} (1 + c_1)\, a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right). \end{aligned}$$

Therefore, from the boundedness of $a^K(\cdot,\cdot)$ and the stability of Π_k^K (see (3.1a)), we obtain

$$\begin{aligned} |a_h(w_h, w_h) - a(w_h, w_h)| &\leq C \sum_{K \in \mathcal{T}_h} \left|w_h - \Pi_k^K w_h\right|_{1,K}^2 \\ &\leq C \sum_{K \in \mathcal{T}_h} \left(|w_h - w|_{1,K} + \left|w - \Pi_k^K w\right|_{1,K} + \left|\Pi_k^K (w - w_h)\right|_{1,K} \right)^2 \\ &\leq C \sum_{K \in \mathcal{T}_h} \left(|w_h - w|_{1,K} + \left|w - \Pi_k^K w\right|_{1,K} \right)^2. \end{aligned}$$

Now, also from (3.1a) it is immediate to check that

$$\left|w - \Pi_k^K w\right|_{1,K} \leq |w - w_\pi|_{1,K} \qquad \forall w_\pi \in \mathbb{P}_k(K).$$

Then, from the last two inequalities, Proposition 4.1, (4.5) and (4.4), we obtain

$$|a_h(w_h, w_h) - a(w_h, w_h)| \leq Ch^{2\min\{r,k\}}.$$

On the other hand, by virtue of Lemma 3.1 and the fact that $\lambda_h^{(i)} \to \lambda$ as h goes to zero, we know that there exists $C > 0$ such that

$$b(w_h, w_h) = \frac{\widehat{a}_h(w_h, w_h)}{\lambda_h^{(i)} + 1} \geq \frac{\beta \|w_h\|_{1,\Omega}^2}{\lambda_h^{(i)} + 1} \geq \frac{\beta}{C} > 0.$$

By using this estimate to bound the left-hand side of (4.6) from below, together with the previous one and (4.7) for an upper bound of the right-hand side, we conclude that

$$\left|\lambda - \lambda_h^{(i)}\right| \leq Ch^{2\min\{r,k\}}$$

and we end the proof. □

4.2. Error estimates for the eigenfunctions on Γ_0.

Our next goal is to improve the error estimate for the trace of the eigenfunctions in the $L^2(\Gamma_0)$-norm. With this end, we will resort to a duality technique. Given $u \in H^1(\Omega)$ and $u_h \in V_h$, let $v \in H^1(\Omega)$ be the solution of the following problem:

$$\begin{cases} \Delta v = 0 \quad \text{in } \Omega, \\ \partial_n v + v = \begin{cases} u - u_h & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{cases} \end{cases}$$

By testing the first equation above with functions in $H^1(\Omega)$ and integrating by parts, we obtain

$$\widehat{a}(v, z) := \int_\Omega \nabla v \cdot \nabla z + \int_{\Gamma_0} vz = \int_{\Gamma_0} (u - u_h)z =: b(u - u_h, z) \qquad \forall z \in H^1(\Omega). \tag{4.8}$$

Therefore, $v = T(u - u_h)$, so that according to Lemma 2.2(i), for all $r \in [\frac{1}{2}, r_\Omega)$, $v \in H^{1+r_1}(\Omega)$ (recall that $r_1 := \min\{r, 1\}$) and

$$\|v\|_{1+r_1,\Omega} \leq C \|u - u_h\|_{1,\Omega}. \tag{4.9}$$

The improved error estimate will be a consequence of the following result.

Lemma 4.3. *Let $f \in \mathcal{E}$ be an eigenfunction of the operator T. If $u = Tf$ and $u_h = T_h f$, then, for all $r \in [\frac{1}{2}, r_\Omega)$, there exists $C > 0$ such that*

$$\|(T - T_h)f\|_{0,\Gamma_0} = \|u - u_h\|_{0,\Gamma_0} \leq Ch^{r_1/2 + \min\{r,k\}} \|f\|_{1,\Omega}.$$

Proof. Let v be as defined above and $v_I \in V_h$ so that the estimate of Proposition 4.2 holds true. Testing (4.8) with $z = (u - u_h) \in H^1(\Omega)$, we obtain

$$\|u - u_h\|_{0,\Gamma_0}^2 = \widehat{a}(u - u_h, v) = \widehat{a}(u - u_h, v - v_I) + \widehat{a}(u - u_h, v_I). \tag{4.10}$$

To bound the first term on the right-hand side above, we use the continuity of the bilinear form $\widehat{a}(\cdot,\cdot)$, Proposition 4.2 and (4.9):

$$\begin{aligned} \widehat{a}(u - u_h, v - v_I) &\leq C \|u - u_h\|_{1,\Omega} \|v - v_I\|_{1,\Omega} \\ &\leq Ch^{r_1} \|u - u_h\|_{1,\Omega} \|v\|_{1+r_1,\Omega} \leq Ch^{r_1} \|u - u_h\|_{1,\Omega}^2. \end{aligned} \tag{4.11}$$

For the second term, we use that $\widehat{a}(u, v_h) = b(f, v_h) = \widehat{a}_h(u_h, v_h)$ for all $v_h \in V_h$ to write

$$
\begin{aligned}
\widehat{a}(u - u_h, v_I) = \widehat{a}_h(u_h, v_I) - \widehat{a}(u_h, v_I) &= \sum_{K \in \mathcal{T}_h} \left(a_h^K(u_h, v_I) - a^K(u_h, v_I) \right) \\
&= \sum_{K \in \mathcal{T}_h} \left(a^K(\Pi_k^K u_h, \Pi_k^K v_I) + S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) - a^K(u_h, v_I) \right) \\
&= \sum_{K \in \mathcal{T}_h} \left(a^K(\Pi_k^K u_h - u_h, v_I - \Pi_k^K v_I) + S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) \right), \qquad (4.12)
\end{aligned}
$$

where we have used (3.1a) to derive the last equality.

Now, from the symmetry of $S^K(\cdot, \cdot)$, inequality (3.2) and the definition of $a^K(\cdot, \cdot)$, we have that $S^K(v_h, z_h) \leq c_1 \left|v_h\right|_{1,K} \left|z_h\right|_{1,K}$ for all $v_h, z_h \in V_k^K$ such that $\Pi_k^K v_h = \Pi_k^K z_h = 0$. We use this inequality to bound the second term on the right-hand side of (4.12):

$$
\sum_{K \in \mathcal{T}_h} S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) \leq c_1 \sum_{K \in \mathcal{T}_h} \left|u_h - \Pi_k^K u_h\right|_{1,K} \left|v_I - \Pi_k^K v_I\right|_{1,K}. \qquad (4.13)
$$

By virtue of (3.1a) it is easy to check that

$$
\begin{aligned}
\left|u_h - \Pi_k^K u_h\right|_{1,K} &\leq \left|u_h - u\right|_{1,K} + \left|u - \Pi_k^K u\right|_{1,K} + \left|\Pi_k^K (u - u_h)\right|_{1,K} \\
&\leq 2 \left|u_h - u\right|_{1,K} + \left|u - u_\pi\right|_{1,K} \qquad \forall u_\pi \in P_k(K)
\end{aligned}
$$

and, analogously,

$$
\left|v_I - \Pi_k^K v_I\right|_{1,K} \leq 2 \left|v_I - v\right|_{1,K} + \left|v - v_\pi\right|_{1,K} \qquad \forall v_\pi \in P_k(K).
$$

Substituting these inequalities into (4.13) and using (4.3), Proposition 4.1 and Lemma 2.2(ii) (since $f \in \mathcal{E}$) for the former and Propositions 4.1 and 4.2 and (4.9) for the latter, we obtain

$$
\sum_{K \in \mathcal{T}_h} S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) \leq C h^{r_1 + \min\{r,k\}} \left\|f\right\|_{1,\Omega} \left\|u - u_h\right\|_{1,\Omega}.
$$

By repeating the same steps as above, we obtain a similar bound for the first term on the right hand side of (4.12):

$$
\sum_{K \in \mathcal{T}_h} a^K(u_h - \Pi_k^K u_h, v_h - \Pi_k^K v_h) \leq C h^{r_1 + \min\{r,k\}} \left\|f\right\|_{1,\Omega} \left\|u - u_h\right\|_{1,\Omega}.
$$

Hence,

$$
\widehat{a}(u - u_h, v_I) \leq C h^{r_1 + \min\{r,k\}} \left\|f\right\|_{1,\Omega} \left\|u - u_h\right\|_{1,\Omega}.
$$

The proof follows by substituting this inequality and (4.11) into (4.10) and using (4.3). □

The next step is to define a solution operator on the space $L^2(\Gamma_0)$:

$$
\begin{aligned}
\widetilde{T} :\ L^2(\Gamma_0) &\longrightarrow L^2(\Gamma_0), \\
\widetilde{f} &\longmapsto \widetilde{T}\widetilde{f} := u|_{\Gamma_0},
\end{aligned}
$$

where $u \in H^1(\Omega)$ is the solution of the following problem:

$$
\widehat{a}(u, v) = \int_{\Gamma_0} \widetilde{f} v \qquad \forall v \in H^1(\Omega). \qquad (4.14)
$$

It is easy to check that the operator $\widetilde{T}$ is compact and self-adjoint. We also define the corresponding discrete solution operator:

$$\begin{aligned}\widetilde{T}_h :\ L^2(\Gamma_0) &\longrightarrow L^2(\Gamma_0),\\ \widetilde{f} &\longmapsto \widetilde{T}_h\widetilde{f} := u_h|_{\Gamma_0},\end{aligned}$$

where $u_h \in V_h$ is the solution of the discrete problem

$$\widehat{a}_h(u_h, v_h) = \int_{\Gamma_0} \widetilde{f} v_h \qquad \forall v_h \in V_h. \tag{4.15}$$

The spectra of T and $\widetilde{T}$ coincide. In fact, it is immediate to check that if $Tw = \mu w$, with $w \neq 0$ and $\mu \neq 0$, then $w|_{\Gamma_0} \neq 0$ and $\widetilde{T}(w|_{\Gamma_0}) = \mu w|_{\Gamma_0}$. Conversely, if $\widetilde{T}\widetilde{w} = \mu\widetilde{w}$, with $\widetilde{w} \neq 0$ and $\mu \neq 0$, then there exists $w \in H^1(\Omega)$, such that $Tw = \mu w$ and $w|_{\Gamma_0} = \widetilde{w}$. The same arguments allow us to show that the spectra of T_h and $\widetilde{T}_h$ also coincide and their respective eigenfunction are related in the same way as those of T and $\widetilde{T}$.

To prove that the operators $\widetilde{T}_h$ converge in norm to $\widetilde{T}$, we will use the following additional regularity estimate analogous to that in Lemma 2.2 but that only involves $\|f\|_{0,\Gamma_0}$.

Lemma 4.4. *For all $s \in (0, \frac{1}{2})$, there exists $C > 0$ such that, for all $f \in L^2(\Gamma_0)$, the solution u of problem (4.14) satisfies $u \in H^{1+s}(\Omega)$ and*

$$\|u\|_{1+s,\Omega} \leq C \|f\|_{0,\Gamma_0}.$$

Proof. The proof is a consequence of [29, Theorem 4]). □

Now, we are able to conclude the convergence in norm of $\widetilde{T}_h$ to $\widetilde{T}$.

Lemma 4.5. *For all $s \in (0, \frac{1}{2})$, there exists $C > 0$ such that*

$$\left\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\right\|_{0,\Gamma_0} \leq Ch^s \left\|\widetilde{f}\right\|_{0,\Gamma_0}.$$

Proof. Given $\widetilde{f} \in L^2(\Gamma_0)$, let $u \in H^1(\Omega)$ and $u_h \in V_h$ be the solutions of problems (4.14) and (4.15), respectively, so that $\widetilde{T}\widetilde{f} = u|_{\Gamma_0}$ and $\widetilde{T}_h\widetilde{f} = u_h|_{\Gamma_0}$. The arguments used in the proof of Lemma 4.1 can be repeated in this case yielding

$$\|u - u_h\|_{1,\Omega} \leq C \left(\|u - u_I\|_{1,\Omega} + |u - u_\pi|_{1,h} \right),$$

with u_I and u_π as in that lemma. Thus, the result follows from Propositions 4.1 and 4.2, and Lemma 4.4. □

As a consequence of this lemma, a spectral convergence result analogous to Theorem 4.1 holds for $\widetilde{T}_h$ and $\widetilde{T}$. Moreover, we are in a position to establish the following estimate.

Theorem 4.4. *Let w_h be an eigenfunction of T_h associated with the eigenvalue $\mu_h^{(i)}$, $1 \leq i \leq m$, with $\|w_h\|_{0,\Gamma_0} = 1$. Then, there exists an eigenfunction w of T associated with μ such that, for all $r \in [\frac{1}{2}, r_\Omega)$, there exists $C > 0$ such that*

$$\|w - w_h\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}}.$$

Proof. Thanks to Lemma 4.5, Theorem 7.1 from [4] yields spectral convergence of $\widetilde{T}_h$ to $\widetilde{T}$. In particular, because of the relation between the eigenfunctions of T and T_h with those of $\widetilde{T}$ and $\widetilde{T}_h$, respectively, we have that $w_h|_{\Gamma_0} \in \widetilde{\mathcal{E}}_h$ and there exists $w \in \mathcal{E}$ such that

$$\|w - w_h\|_{0,\Gamma_0} \leq C \sup_{\widetilde{f}\in\widetilde{\mathcal{E}}:\ \|\widetilde{f}\|_{0,\Gamma_0}=1} \left\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\right\|_{0,\Gamma_0}. \tag{4.16}$$

On the other hand, because of Lemma 4.3, for all $\widetilde{f} \in \widetilde{\mathcal{E}}$, if $f \in \mathcal{E}$ is such that $\widetilde{f} = f|_{\Gamma_0}$, then

$$\left\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\right\|_{0,\Gamma_0} = \|(T - T_h)f\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}} \|f\|_{1,\Omega}.$$

Now, for $f \in \mathcal{E}$, $Tf = \mu f$. Hence, $\|f\|_{1,\Omega} = \frac{1}{\mu}\|Tf\|_{1,\Omega} \leq C\|f\|_{0,\Gamma_0}$ (cf. Lemma 4.4). Thus, substituting this expressions into the previous inequality, we have that

$$\left\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\right\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}} \left\|\widetilde{f}\right\|_{0,\Gamma_0},$$

which together with (4.16) allow us to conclude the proof. □

5. Numerical results

We report in this section a couple of tests which have allowed us to assess the theoretical results proved above. With this aim, we have implemented in a MATLAB code a lowest-order VEM ($k = 1$) on arbitrary polygonal meshes, by following the ideas proposed in [9].

To complete the choice of the VEM, we had to fix the bilinear forms $S^K(\cdot,\cdot)$ satisfying (3.2) to be used. To do this, we have proceeded as in [7]: for each polygon K with vertices $P_1, \ldots, P_{N_K}$, we have used

$$S^K(u, v) := \sum_{r=1}^{N_K} u(P_r)v(P_r), \qquad u, v \in V_1^K.$$

As stated in [7, Section 4.6], under assumption **AO.3**, this choice of $S^K(\cdot,\cdot)$ satisfies (3.2).

5.1. Test 1: Sloshing in a square domain.

In this test, we have taken $\Omega := (0, 1)^2$, with Γ_0 and Γ_1 as shown in Figure 1.

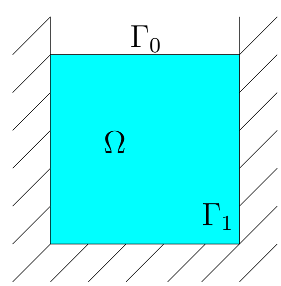

Figure 1: Sloshing in a square domain.

This problem corresponds to the computation of the sloshing modes of a two-dimensional fluid contained in Ω with a horizontal free surface Γ_0. The analytical solutions of this problem are

$$\lambda_n = n\pi \tanh(n\pi), \qquad w_n(x, y) = \cos(n\pi x)\sinh(n\pi y), \qquad n \in \mathbb{N}.$$

We have used three different families of meshes (see Figure 2):

- $\mathcal{T}_h^1$: triangular meshes, considering the middle point of each edge as a new degree of freedom;
- $\mathcal{T}_h^2$: trapezoidal meshes which consist of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertexes $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{2}{3})$, and $(0,\frac{1}{3})$;
- $\mathcal{T}_h^3$: meshes built from $\mathcal{T}_h^1$ with the edge midpoint moved randomly; note that these meshes contain non-convex elements.

The refinement parameter N used to label each mesh is the number of elements on each edge.

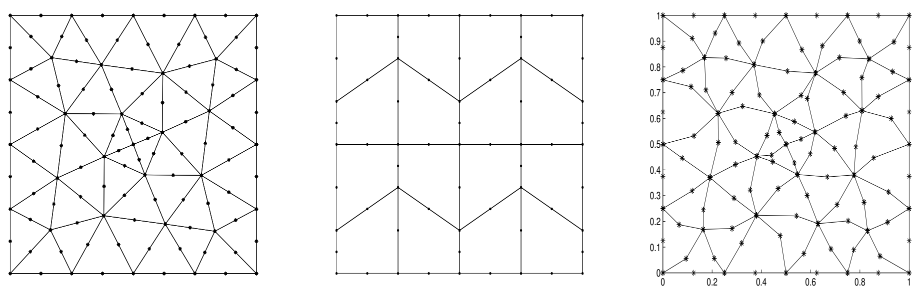

Figure 2: Test 1. Sample meshes: $\mathcal{T}_h^1$ (left), $\mathcal{T}_h^2$ (middle) and $\mathcal{T}_h^3$ (right) for $N = 4$.

We report in Table 1 the lowest eigenvalues λ_{hi} computed with this method. The table also includes the estimated orders of convergence, as well as more accurate values of the eigenvalues extrapolated from the computed ones by means of a least-squares fitting. The exact eigenvalues are also reported in the last column to allow for comparison.

Table 1: Test 1. Computed lowest eigenvalues λ_{hi}, $1 \leq i \leq 3$, on different meshes.

$\mathcal{T}_h$	λ_{hi}	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Extrap.	λ_i
	λ_{h1}	3.1330	3.1306	3.1301	3.1299	2.03	3.1299	3.1299
$\mathcal{T}_h^1$	λ_{h2}	6.3095	6.2894	6.2846	6.2835	2.07	6.2832	6.2831
	λ_{h3}	9.5183	9.4459	9.4298	9.4260	2.09	9.4250	9.4248
	λ_{h1}	3.1424	3.1331	3.1307	3.1301	1.98	3.1299	3.1299
$\mathcal{T}_h^2$	λ_{h2}	6.3765	6.3095	6.2900	6.2849	1.92	6.2825	6.2831
	λ_{h3}	9.6929	9.5092	9.4475	9.4306	1.85	9.4205	9.4248
	λ_{h1}	3.1331	3.1308	3.1301	3.1299	2.03	3.1299	3.1299
$\mathcal{T}_h^3$	λ_{h2}	6.3105	6.2896	6.2847	6.2835	2.05	6.2832	6.2831
	λ_{h3}	9.5193	9.4470	9.4300	9.4261	2.06	9.4248	9.4248

It can be seen from Table 1 that the computed eigenvalues converge to the exact ones with an optimal quadratic order as predicted by the theory.

We report in Table 2 the $L^2(\Gamma_0)$-errors of the eigenfunctions corresponding to the lowest eigenvalue for each family of meshes and different refinement levels. We also include in this table the estimated orders of convergence.

Table 2: Test 1. Errors $\|w - w_h\|_{0,\Gamma_0}$ of the vibration mode for the lowest eigenvalue λ_{h1} on different meshes.

$\mathcal{T}_h$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	Order
$\mathcal{T}_h^1$	0.003633	0.0008715	0.0002265	0.0000557	2.00
$\mathcal{T}_h^2$	0.025074	0.0059387	0.0014453	0.0003558	2.05
$\mathcal{T}_h^3$	0.004559	0.0009943	0.0002576	0.0000659	2.03

We observe from this table a clear quadratic order of convergence. Let us remark that this is the optimal order attainable with the virtual elements used, which is actually larger than the order $\mathcal{O}(h^{3/2})$ predicted by the theory.

Figure 3 shows the eigenfunctions on Γ_0 corresponding to the three lowest eigenvalues. Let us remark that, in the sloshing problem, this corresponds to the shape of the fluid free surface ($\partial_n w = \lambda w$) for each sloshing mode.

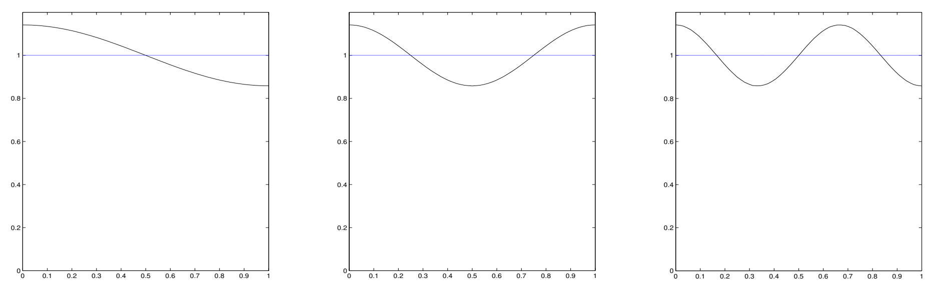

Figure 3: Test1. Sloshing modes: u_{h1} (left), u_{h2} (middle) and u_{h3} (right) computed with $N = 256$.

5.2. Test 2: Circular Domain.

In this test, we have taken as domain the unit circle $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with $\Gamma_0 = \partial\Omega$ and $\Gamma_1 = \emptyset$.

It is easy to check that any homogeneous harmonic polynomial of degree n satisfies $\partial_n w = nw$ on $\partial\Omega$. Therefore, for all $n \in \mathbb{N}$, $\lambda = n$ is an eigenvalue of this problem and the corresponding eigenspace is the set of homogeneous harmonic polynomials of degree n, whose dimension is 2.

We have used polygonal meshes constructed as those in $\mathcal{T}_h^3$ on the previous test (see Figure 4). The refinement parameter N used to label each mesh is now the number of elements on the whole boundary.

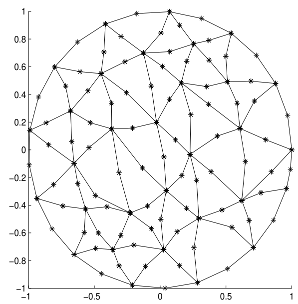

Figure 4: Test 2. Sample mesh for $N = 13$

We report in Table 3 the four lowest eigenvalues λ_{hi} computed with this method. The table also includes the estimated orders of convergence, as well as more accurate values of the eigenvalues extrapolated from the computed ones by means of a least-squares fitting. The last column shows the exact eigenvalues.

Table 3: Test 2. Computed lowest eigenvalues λ_{hi}, $1 \leq i \leq 4$.

λ_{hi}	N=26	N=51	N=76	N=101	Order	Extrap.	λ_i
λ_{h1}	1.000630	1.000131	1.000070	1.000040	1.98	1.000025	1
λ_{h2}	1.000663	1.000142	1.000074	1.000041	1.99	1.000022	1
λ_{h3}	2.007673	2.001942	2.000889	2.000498	1.97	2.000040	2
λ_{h4}	2.008321	2.002018	2.000907	2.000503	2.02	2.000053	2

Once more, a quadratic order of convergence can be clearly appreciated from Table 3.

Finally, Figure 5 shows a plot of the third eigenfunction on the whole domain.

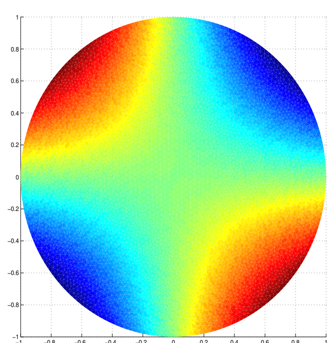

Figure 5: Test 2. Eigenfunction u_{h3} for $N = 214$.

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