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A viscosity prescription for a self-gravitating accretion disc*

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Summary. We propose a prescription for treating the transfer of angular momentum within a gaseous differentially rotating disc subject to gravitational instability in terms of an effective kinematic viscosity. We show that under certain conditions this prescription allows a similarity solution and that all solutions tend towards this at large times.

1 Introduction

Observations of star-forming regions at infrared, millimetre and radio wavelengths have shown the existence of rotating discs of molecular material (see for example the reviews by Goldsmith & Arquilla 1985; Pudritz 1986). Although the specific angular momentum of matter in a typical disc is in general less than that which would arise if the disc condensed directly from the interstellar medium, it is still much greater than the value of specific angular momentum that can be accommodated by protostars. It is evident therefore that some redistribution of angular momentum must take place to enable star formation to occur. Disc-like shear flows are observed or suspected in a variety of astrophysical contexts (e.g. Pringle 1981) and there is observational evidence that redistribution of angular momentum does occur even though theoretical understanding of the viscous or dissipative mechanism responsible is still lacking. Some kind of hydrodynamic or hydromagnetic turbulence is usually envisaged and the strength of the mechanism is measured by the dimensionless parameter α (Shakura & Sunyaev 1973). In the context of molecular discs, however, it is evident from observational estimates that in some cases they are close to being unstable gravitationally (e.g. Bieging 1984; Güsten, Chini & Neckel 1984). This raises the possibility that it is a gravitational instability in the disc which is responsible for the redistribution of angular momentum and hence for the evolution of the disc as a whole. Larson (1984) and Boss (1984) have discussed this possibility. In this paper we propose a prescription for estimating the effect of the gravitational instabilities on the evolution of a disc.

In Section 2 we review briefly the nature of the instabilities due to self-gravity in differentially rotating discs and show how this might lead under certain conditions to treating the effect of the

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instabilities in terms of an effective kinematic viscosity. In Section 3 we show that in the range of validity of this prescription the disc evolution equation permits a similarity solution and that general solutions tend to the similarity solution at large times. We discuss our results in Section 4.

2 Gravitational instability and viscosity prescription

The basic principles underlying the gravitational instability of a thin rotating disc have been elucidated by Toomre (1964). If the disc has local surface density Σ and local angular frequency Ω (in this order of magnitude discussion, we take local shearing rate~local angular velocity~local epicyclic frequency) then disturbances of size L, greater than L_{crit}, where

$$L_{\mathrm{crit}} \sim \frac{G\Sigma}{\Omega^2} \tag{2.1}$$

are stabilized by the shear. If the disc has local dispersion velocity (e.g. sound speed) C_s, then disturbances of size L, less than L_J, where

$$L_J \sim \frac{C_s^2}{G\Sigma} \tag{2.2}$$

are stabilized. Thus in general only disturbances in the range $L_J < L < L_{\mathrm{crit}}$ are able to grow due to self-gravity. Conversely if the dimensionless number Q, where $Q^2 = L_J/L_{\mathrm{crit}}$, is greater than unity, the disc is stable with respect to self-gravity.

To set these quantities in the context of accretion discs we estimate the gravitating disc mass M (within radius R) by $\Omega^2 \sim GM/R^3$, and the mass of the disc, M_d, by $M_d \sim \Sigma R^2$. For the standard accretion discs with a central gravitating point mass we have $M_d \ll M$, and in general we expect $M_d \lesssim M$. The disc semi-thickness, H, is estimated by use of the equation of hydrostatic equilibrium in the direction perpendicular to the disc plane which yields $H \sim H_R \min(1, Q)$ where $H_R \sim C_s/\Omega$. We shall assume for the moment that we are working with thin discs so that $H \ll R$. In terms of these quantities we find

$$\frac{L_J}{H_R} \sim \frac{H_R}{R} \times \frac{M}{M_d} \sim Q \tag{2.3}$$

and

$$\frac{L_{\mathrm{crit}}}{H_R} \sim \frac{R}{H_R} \times \frac{M_d}{M} \sim Q^{-1}. \tag{2.4}$$

Thus if $Q<1$, disturbances over regions of size $Q < L/H_R < Q^{-1}$, grow with time, and since the instability is a dynamical one the characteristic growth time-scale for the instabilities is $\sim\Omega^{-1}$. The above considerations are borne out by detailed analysis in the linear regime (Toomre 1964, 1977).

Toomre (1964) speculates that in the case when the disc consists of stars, and the corresponding C_s is simply the stellar dispersion velocity, the effect of the instabilities is to heat the disc locally (i.e. increase C_s and hence H_R) until Q is large enough that the disc becomes stable. An accretion disc, however, in which C_s corresponds to a thermal velocity, is able to cool locally and one must therefore take into account the balance between heating due to the instability and cooling due to radiation. Paczynski (1978) and Kozlowski, Wiita & Paczynski (1979) constructed models of steady accretion discs in which this balance was assumed to occur precisely when the disc was on the border of instability, i.e. when $Q=1$.

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In the context of star formation Larson (1984) has used the formula given by Lynden-Bell & Kalnajs (1972) for the torque produced by a particular spiral pattern and strength to estimate the torque which can be achieved. He concluded that such torques could be strong enough to dominate the angular momentum transfer in such discs. However without knowledge of the non-linear growth and the mutual interaction of the unstable spiral modes, the net dynamical effect of the instability is not readily amenable to analysis.

In this paper we address ourselves to the long-term dynamical evolution of a disc subject to these instabilities. In order that the instabilities give rise to local heating, they must also give rise to transfer of mass and angular momentum within the disc (Lynden-Bell & Pringle 1974). In this sense, therefore, provided that the maximum size of unstable regions L_{crit} is such that $L_{\text{crit}} \ll R$, the instabilities may be regarded as giving rise to a local effective kinematic viscosity, ν_{eff}. Since the size of region over which angular momentum is transferred is $\sim L_{\text{crit}}$ and the time-scale $\sim \Omega^{-1}$, we propose a prescription for the effective kinematic viscosity of the form

$$\nu_{\text{eff}} \sim \frac{L_{\text{crit}}^2}{\Omega^{-1}} \sim Q^{-2} H_{\text{R}}^2 \Omega. \tag{2.5}$$

The prescription would be valid for $Q<1$ (so that the disc is unstable) and for $Q^2>H/R$, so that $L_{\text{crit}}<R$, or equivalently so that the 'viscous' time-scale in the disc

$$t_\nu \sim \frac{R^2}{\nu_{\text{eff}}} \sim \left(\frac{M}{M_{\text{d}}}\right)^2 \Omega^{-1}$$

is more than the dynamical time-scale in the disc $t_{\text{d}} \sim \Omega^{-1}$. If $Q>1$, then the disc is stable against self-gravity, $\nu_{\text{eff}}=0$ and any transfer of angular momentum within the disc must be due to other effects. We speculate below on what happens if $Q^2<H/R$. In contrast to previous work we do not insist that $Q \simeq 1$.

It follows immediately that in the absence of any other form of momentum transport, once a disc has become self-gravitating it remains so since as $Q \to 1$, $\nu_{\text{eff}} \to 0$ and so the heating, but not the cooling, tends to zero. We also note that this prescription for the effective viscosity is equivalent to the standard α parameter formulation of Shakura & Sunyaev (1973) with $\alpha \sim Q^{-2} > 1$.

3 Similarity solution

In any particular circumstance the detailed local disc structure, and consequent value of Q, must be determined using the relevant equation of state, cooling functions, opacity etc. in the standard manner. In this section we consider the particular case of a disc for which $M_{\text{d}} \ll M$ so that we may take $\Omega^2 = GM/R^3$ with M constant, and for which throughout the disc at all times we may take $H_{\text{R}}/R<Q<1$, so that $\nu_{\text{eff}} = Q^{-2} H_{\text{R}}^2 \Omega$. We show that in this case a similarity solution may be constructed.

The equation governing the evolution of the surface density, Σ, of an accretion disc with a central, dominant gravitating point mass and with kinematic viscosity ν is (Pringle 1981)

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left[R^{1/2} \frac{\partial}{\partial R} (\nu \Sigma R^{1/2}) \right]. \tag{3.1}$$

For the case we are considering we have using (2.1) and (2.5), that

$$\nu = \Sigma^2 R^6 \Omega / M^2 \tag{3.2}$$

which is independent of C_{s}. Thus since M is a constant, ν is of the form $\nu \propto \Sigma^a R^b$ for some a, b. By

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analogy with solutions of the thermal conduction equation with a non-linear, but power-law, coefficient of conductivity given by Zel'dovich & Raizer (1966), Pringle (1974, 1981) pointed out that such an equation has solutions of similarity form. There are two basic solutions. One solution has constant disc mass, but steadily increasing disc angular momentum and corresponds to the disc being dispersed to infinite radius by a torque supplied at the origin. The other solution, which we consider here, has zero torque at the origin and is given by

$$\frac{\Sigma}{\Sigma_0}=\left(\frac{R}{R_0}\right)^{-3/2}\left(\frac{t}{t_0}\right)^{-2/5}\left[1-\left(\frac{R}{R_f}\right)^{1/2}\right]^{1/2} \tag{3.3}$$

where

$$\frac{R_f}{R_0}=\left(\frac{45}{8}\right)^2\left(\frac{\Sigma_0R_0^2}{M}\right)^4 t_0^2\,\frac{GM}{R_0^3}\left(\frac{t}{t_0}\right)^{2/5}, \tag{3.4}$$

and R_0, Σ_0, t_0 are arbitrary constants.

At any given time, t, the mass of the disc is given by

$$M_{\rm disc}=\int_0^{R_f}\Sigma 2\pi R\,dR$$

$$=\Sigma_0R_0^2\frac{15\pi}{2}\left(\frac{\Sigma_0R_0^2}{M}\right)^2 t_0\left(\frac{GM}{R_0^3}\right)^{1/2}\left(\frac{t}{t_0}\right)^{-1/5} \tag{3.5}$$

whereas the total disc angular momentum

$$J_{\rm disc}=\int_0^{R_f}\Sigma R^2\Omega\times 2\pi R\,dR$$

$$=\sqrt{GMR_0}\times\Sigma_0R_0^2\times 6\pi\left(\frac{\Sigma_0R_0^2}{M}\right)^4 t_0^2\,\frac{GM}{R_0^3}$$

$$=\text{constant}. \tag{3.6}$$

The evolution of the surface density distribution as a function of time is shown in Fig. 1. At time $t=0$, all the mass (an infinite amount in order to provide a finite amount of angular momentum) is at the origin. As time evolves the outer disc radius $R_f\propto t^{2/5}$. Each element of mass in the disc moves outwards initially, and then reverses direction and collapses back to the origin. Since $M_{\rm disc}\propto t^{-1/5}$, the mass flux on to the origin is of the form $\dot M\propto t^{-6/5}$. At large times, almost all the mass has returned to the origin with an ever-diminishing amount of mass going to ever-larger radii carrying all of the initial angular momentum. We note that this kind of similarity behaviour has been discussed by Lin & Papaloizou (1985). The arbitrariness of the constants t_0, Σ_0 and R_0 can be used to express the solution more succinctly. In particular we note that if we take $R_0=R_f$ when $t=t_0$, we have $M_{\rm disc}(t=t_0)=(4\pi/3)\Sigma_0R_0^2$, and thus

$$M_{\rm disc}(t)=\frac{4\pi}{3}\,\Sigma_0R_0^2\left(\frac{t}{t_0}\right)^{-1/5} \tag{3.7}$$

We also find that in this case

$$t_0\Omega(R_0)=\frac{1}{10\pi^2}\left[\frac{M}{M_{\rm disc}(t=t_0)}\right]^2, \tag{3.8}$$

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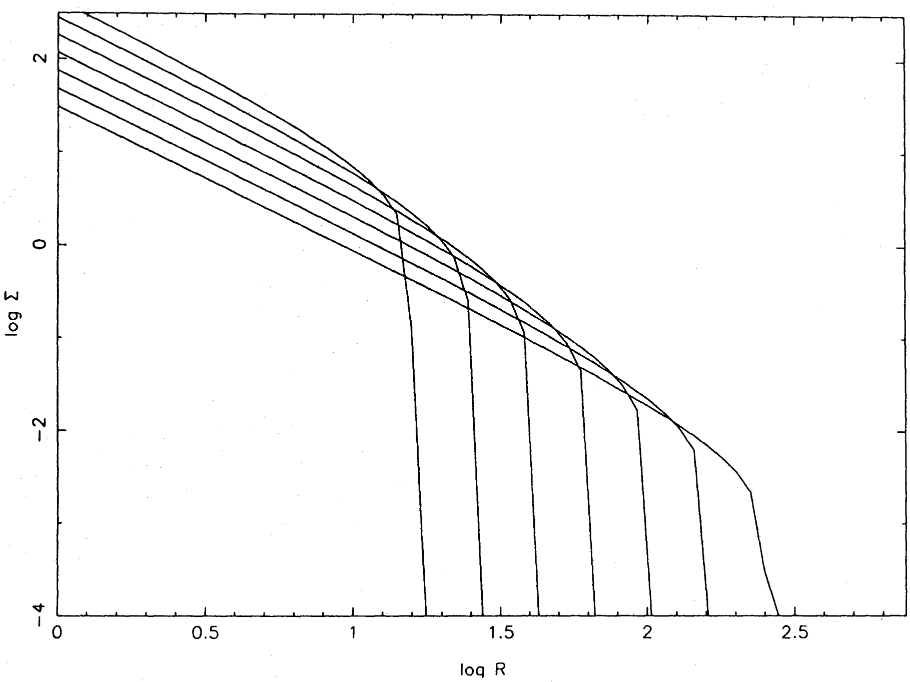

Figure 1. A log–log plot of surface density Σ versus radius R for the similarity solution given by equations (3.3) and (3.4). Σ and R are in arbitrary units. The outer disc radius increases with time, and time increases by a factor of $\sqrt{10}$ for each model.

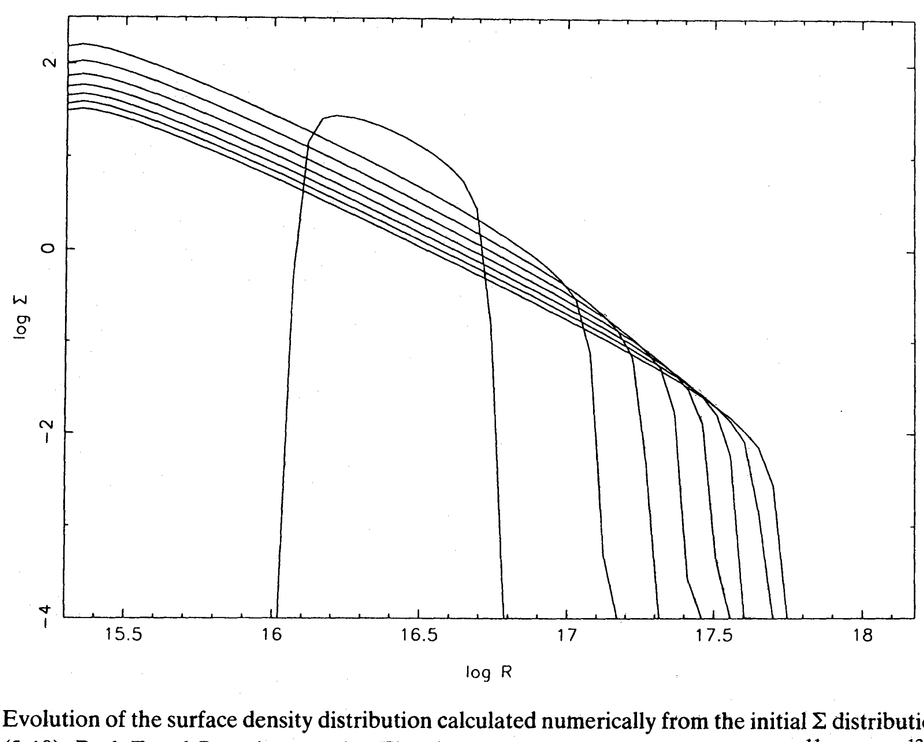

Figure 2. Evolution of the surface density distribution calculated numerically from the initial Σ distribution given by equation (3.10). Both Σ and R are in cgs units. The eight curves represent times of 1.1×10^{11}, 6.9×10^{12}, 2.5×10^{13}, 6.1×10^{13}, 1.2×10^{14}, 2.1×10^{14}, 3.6×10^{14} and 5.6×10^{14} s with the outer disc radius a monotonically increasing function of time.

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and hence

$$J_{\rm disc}=M_{\rm disc}(t_0)\times\sqrt{GMR_0}\times\frac{3^6}{2^{11}5^2\pi^8}$$
$$=1.5\times10^{-6}M_{\rm disc}(t_0)\sqrt{GMR_0}. \tag{3.9}$$

Although the similarity solution (3.3), (3.4) is not a general solution, there are grounds for expecting solutions of the equations to tend towards the similarity solutions at large times (e.g. Lin & Papaloizou 1985). We have been able to demonstrate this explicitly by calculating the solution numerically with various initial conditions.

To illustrate this we take a particular case with $M=20\,M_\odot$ and consider radii $R_{\rm in}=2\times10^{15}$ cm$\leqslant R\leqslant R_{\rm out}=5\times10^{17}$ cm. We set $\Sigma=0$ interior to the inner boundary $R_{\rm in}$ and integrate the equation (3.1) using a standard explicit to first-order finite-difference scheme. We take 60 radial zones in a geometric progression between $R_{\rm in}$ and $R_{\rm out}$. We limit ourselves to times for which the surface density Σ vanishes at some radius less than $R_{\rm out}$, so that the boundary condition applied at $R_{\rm out}$ is not relevant. For the case we illustrate (Fig. 2) we chose

$$\Sigma(R,t=0)=\Sigma_0\exp\left[-(R-R_a)^2/(\Delta R)^2\right] \tag{3.10}$$

with $R_a=2.8\times10^{16}$, $\Delta R=0.1R_a$ and $\Sigma_0=100\,{\rm g\,cm^{-1}}$.

It is evident from the figure that the density distribution rapidly becomes of the form $\Sigma\propto R^{-3/2}$ over most of the radius in line with the expectation of the similarity solution.

4 Discussion

We have put forward a prescription for the calculation of the time-evolution of discs subject to gravitational instability. Implicit within that prescription is the assumption that even when matter in the disc is subject to self-gravitation, the instability does not necessarily lead directly to condensation of parts of the disc into distinct self-gravitating bodies. In line with previous authors, we have assumed rather that the instability gives rise to density waves of some kind which are able to transfer angular momentum, and therefore, dissipate energy and cause internal heating of the disc. As the disc evolves however, the strength of the instability measured by Q also evolves and at some stage in the evolution it is likely that Q^2 will become less than H/R.

We note that in the centre of the similarity solution of Section 3, $\Sigma\rightarrow\infty$ as $R\rightarrow0$. This means that unless $C_{\rm s}\rightarrow\infty$ there, $Q\rightarrow0$ and the validity of the equation (2.5) for $\nu_{\rm eff}$ is violated. If Q becomes too small in the centre of the disc it is reasonable to assume that a self-gravitating body is formed there. Since Q depends through $C_{\rm s}$ on the thermal state of the disc at each radius, and since the thermal state depends on the opacity which is in general not a monotonic function of radius, it may be that with certain initial conditions, Q will become less than $H_{\rm R}/R$, and the instability will become fully dynamical at a particular finite radius of the disc. One might speculate that in this case a self-gravitating ring is formed at around that radius, or perhaps a ring of N self-gravitating entities where $N\sim R/H_{\rm R}\sim Q^{-1}$. Further computations are required to determine the circumstances under which this might happen.

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