

A Visualization System for Correctness Proofs of Graph Algorithms

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Abstract

In this paper we describe a system for visualizing correctness proofs of graph algorithms. The system has been demonstrated for a greedy algorithm, Prim's algorithm for finding a minimum spanning tree of an undirected, weighted graph. We believe that our system is particularly appropriate for greedy algorithms, though much of what we discuss can guide visualization of proofs in other contexts. While an example is not a proof, our system provides concrete examples to illustrate the operation of the algorithm. These examples can be referred to by the user interactively and alternatively with the visualization of the proof where the general case is portrayed abstractly.

1. INTRODUCTION

We address the problem of visualizing proofs of graph algorithms. In particular we look at correctness proofs of certain greedy graph algorithms. To the best of our knowledge, no previous work on visualizing proofs appears in the literature, despite the considerable body of research in algorithm and scientific visualization, and the significance of understanding an algorithm's proof of correctness.

Visualizing the proof of an algorithm is different from visualizing the algorithm itself, as is done by various systems (Baecker, 1969; Brown and Sedgewick, 1985; Brown, 1988; Duisberg, 1986; Gloor, 1992; Helttula et al., 1989; Stasko, 1990). An algorithm visualization system is composed of an integrated set of multimedia tools (such as graphics, animation, text, video, code, etc.) which simulate how the algorithm works using abstractions of the data, operations, and semantics of the program behind the algorithm. Algorithm visualization may involve program views that have no direct correspondence to the program's data or execution. Visualization of a proof is an algorithm visualization system which goes one step further: it can be composed of an algorithm visualization system plus a proof-dialog which extracts in a temporal fashion and with visual means the correctness of the algorithm. The main problem of all these systems is how to simplify the production of the algorithm visualization. In this paper we provide an easy to produce framework that defines three primary components for such a system: the definition of the abstract operations necessary for a proof of correctness visualization, the design of operations for the proof process or dialog and the correspondence between the proof visualization and the algorithm visualization.

The correctness proofs we consider are at the level that such proofs are customarily done in a classroom and thus should not be confused with *program verification*. The latter usually refers to a formal proof that examines the actual code implementing some algorithm in a step-like fashion and shows that the program actually achieves the goal. Our endeavor is to illustrate *why* the

algorithm works; an interactive system that deals with the algorithm at a high level and with the fundamental ideas underlying the algorithm.

There is yet another class of proofs that we will not address in this paper, though our work may have application to this class. This class includes the complexity proofs, i.e. proofs of the worst-case running time claimed by some algorithm.

The lack of previous work in this area is partly due to the fact that proofs often involve hypothetical and abstract objects created for the purpose of the proof. Such objects are difficult to visualize. We make use of the fact that there is a question-answer process that goes on when one reads or is told a proof. We call this process the *proof-dialog*. During the proof-dialog, the reader/listener seeks answers from the text or the speaker on several questions in order to become persuaded of the proof's correctness. These questions mainly attempt to find counter-examples in the arguments presented. It is this proof process that we try to simulate with visualization.

The proof-dialog operates on two levels, those of concrete and abstract objects. Our database consists, correspondingly, of two kinds of objects. Concrete graph objects such as vertices, edges, adjacency matrices and linked lists that represent the primitives and the data structures implementing them, and abstract objects that, for example, are hypothetical or of indefinite size or structure. Concrete objects are shown explicitly. Abstract objects must of necessity have a symbolic representation.

2. SYSTEM OVERVIEW

We describe a system capable of visualizing correctness proofs for graph algorithms. In this section we present the components of such a system and in section 4 we go through a detailed example that will clarify our description.

We have done our prototype implementation in HyperCard on a Macintosh. The proof visualization system is composed of:

1. abstract graph objects, for example graphs, subgraphs and paths that are hypothetical, of indefinite size or structure.
2. concrete graph objects, such as vertices and edges of a given graph.
3. descriptions of theorems to be proven and invariants examined for validity during the proof.
4. dynamic help mode based on hypertext that provides updatable information about the objects appearing on the screen.
5. the proof-dialog description.

In the description below we will distinguish between the *user* and the *programmer*. The first term refers to the person that uses the system in order to get a better understanding of the proof,

while the second term refers to the programmer who will use the system to compose a new proof visualization.

Though our system could be stand-alone, we suggest that it be used in conjunction with the following parts: (a) a proof orientation (introduction) describing the problem and the algorithm through pseudocode that explains the important steps, and (b) an integrated animation of the algorithm. The term “integrated” refers to an animation that operates on the same pseudocode and uses the same concrete objects as the proof system. Users are urged to examine and understand the animation first, so they have some intuition of *how* the algorithm works. As we mentioned, the aim of the proof system is to show *why* the proof works.

A typical session is composed of the following steps: In the beginning users go through the introduction and the interactive visualization of the algorithm. When they are familiar with the algorithm and the *pseudocode* that implements the algorithm, they are told the *theorem* along with a set of *invariants* that should be satisfied during the run of the algorithm. Special attention must be paid in choosing these three components, in order to satisfy the following objective: When presented to the users it must be clear that, if the invariants are satisfied during the run of the algorithm, then the theorem is true.

Users are then guided through the proof step-by-step. The proof visualization is controlled by the proof-dialog in the following form: A sequence of statements is presented to the users and they can either accept them or ask for more explanation of any of these statements. In the latter case, the system gives further explanation. Moreover, users can at any time jump back into the algorithm animation and examine more examples to improve the understanding of the algorithm, before resuming the proof session. The session ends when a complete set of statements has been presented and has been understood by the users. Of course, users can go through the proof-dialog as many times as they wish. They also have the option of stepping backwards to take a closer look at some concepts presented earlier.

The programmer who uses the system has to provide the following things:

1. The introduction part, in which the definition of the problem, its significance and other similar information appears.
2. An animation of the algorithm that solves the problem. (As our prototype system has been implemented in HyperCard, the animation ideally should be implemented in HyperCard, too. But with enough technical knowledge it is possible to integrate animation in any format into our system using digitized video.)
3. The proof-dialog in terms of a text file having a special format.
4. The animation of the basis and inductive proof steps.
6. A help file that contains information about the objects.

3. EDUCATIONAL ASPECTS

To present complex concepts, we are using a three-step process called *master teaching* that has already been applied successfully in other educational multimedia projects (Dartmouth Interactive Media Lab). The three steps are defined as follows:

1. Master Teaching:

In the first step a teacher who is a master in teaching as well as a master in the methods he teaches, presents the subject to the students. Ideally, this is done interactively by the human teacher. Unfortunately, master teachers are a scarce resource, and it is desirable to replace or augment the master teacher. In the simplest case this is done by a text provided by the master teacher, but with today's multimedia capabilities this can be done much more effectively by including video segments showing the master teacher teaching.

2. Experiential Learning:

The second step includes practicing the concepts taught by the master teacher. "Learning by doing" is still one of the most effective ways to understand new ideas. The learning effect can be improved if the new ideas are presented, and can be practiced, in different ways.

3. Reflective Learning:

The third step provides for an in depth understanding of the taught and practiced concepts. By combining their own experiences with the methods taught by the master teacher, students build their own mental model of the concepts.

This three-step-learning process can easily be adapted for the multimedia enhanced teaching of computer science algorithms. It is particularly well suited for teaching proofs of theorems. Fig. 1 contains the overview map of an animated learning environment for Prim's algorithm for finding a Minimum Cost Spanning Tree (MST) (Prim, 1957).

Fig. 1 shows the three main parts of our learning environment. These correspond directly to the three steps of master teaching. The "introduction" part explains the basic concepts as they would be explained by the master teacher. The "algorithm" part contains an interactive animation environment corresponding to the "experiential learning" chapter in the master teaching method. Fig. 2 shows a particular instance of the animation of Prim's algorithm.

The interactive environment in Fig. 2 allows the student to experiment with Prim's algorithm, while the animation is running synchronously with different pseudo code representations. This visual-with-code representation of the algorithm allows an intuitive and, ideally, thorough understanding of the algorithm which, in turn, leads to a better understanding of how the proof works. The presentation of the proof corresponds to the third "reflective learning" step in the master teaching method. In this step the student is expected to understand the proof based on an understanding of the algorithm gained in the previous two steps.

4. CORRECTNESS PROOF FOR PRIM'S MINIMUM SPANNING TREE ALGORITHM.

In this section we go over an example that will better illustrate our approach. The algorithm we want to visualize is the well-known Prim's MST algorithm (Prim, 1957) (also known as the *snowball* algorithm). This algorithm computes, for any given weighted undirected connected graph G , a spanning tree whose sum of edge weights is minimum over all spanning trees. There are several ways to go about proving this algorithm; we choose one that we find appropriate for visualization. We adopt an approach similar to the one described by Tarjan (Tarjan, 1983).

Let us define a *blue tree* $T=(V_T, E_T)$ on node set V_T and edge set E_T to be an MST on V_T . A trivial blue tree is one that contains a single vertex and no edges. The objective of the algorithm is to divide up the set of edges E of the graph into two sets; a set of *blue edges* (accepted by the algorithm) that are contained in an MST, and a set of *red edges* that are not part of any MST. A *cut* is a set of edges whose removal disconnects the given graph G .

Prim's algorithm starts off with a trivial blue tree and computes an MST by applying the following rule repeatedly:

Blue rule: *Select a cut where every edge has exactly one endpoint in the blue tree. Among the edges of the cut, select one with minimum cost and color it blue.*

Now, Prim's algorithm can be stated as follows:

Given a connected graph $G=(V,E)$ on $|V|=n$ vertices and $|E|=m$ edges, find a MST $T' = (V, E_{T'})$ by starting from some arbitrary vertex s and repeating the following inductive step $n-1$ times:

Let $T=(V_T, E_T)$ be a blue tree containing s . Select the minimum-cost uncolored edge (v,w) incident to T , and color it blue. Augment T by edge (v,w) .

In light of the previous discussion, the appropriate invariant is the following:

Invariant: At the beginning of each step, (V_T, E_T) is a blue tree on the subgraph of the given graph G induced by V_T .

It is now straightforward to see, that if we show that the invariant holds in the beginning of every step of the algorithm, and T grows at each step, then we have proven the following:

Theorem: Prim's algorithm terminates and correctly computes a MST T' of G .

The proof-dialog proceeds by presenting a sequence of statements that will eventually lead to a complete proof of the theorem.

The user can challenge the validity of these statements at any point and ask for additional explanation. The proof-dialog proceeds as follows. After presenting the theorem, the step invariant, and the pseudocode, the system presents the first statement:

Statement 1. *Examine the theorem and the invariant. If we show that (a) the algorithm terminates and (b) the invariant is satisfied at every step, then the theorem is true.*

At this point, the program prompts for the user's opinion on this statement. The user can either agree on the validity of this statement by clicking on the "next proof step" button appearing on the screen or ask for more explanation by clicking on the "explain this step" button. In the second case the dialog provides the following explanation:

(explanation of a.) *To see that the algorithm terminates, observe that the induction step (step 2) of the algorithm adds one vertex per iteration. Therefore, starting with one vertex, it terminates after $n-1$ steps.*

(explanation of b.) *If the invariant is satisfied on every step, then it is satisfied on the last step as well, i.e., in the step that T' spans the vertices of G . Since T' is a blue tree, i.e. a MST of the whole graph, then the theorem holds true.*

When the user agrees on the validity of the statement, the proof-dialog presents the next statements:

Statement 2. *We will prove the theorem by induction on the number of steps required.*

Statement 3. *The invariant holds when $T = \{s\}$.*

The sequence of statements and explanations has been specified by the programmer in the proof-dialog document. In our example the proof-dialog document is as follows:

1. Examine the theorem and the invariant. If we show that (a) the algorithm terminates and (b) the invariant is satisfied at every step, then the theorem is true.

1.1 (explanation of a.) *To see that the algorithm terminates, observe that the induction step (step 2) of the algorithm adds one vertex per iteration. Therefore, starting with one vertex, it terminates after $n-1$ steps.*

(explanation of b.) *If the invariant is satisfied on every step, then it is satisfied on the last step as well, i.e., in the step that T' spans the vertices of G . Since T' is a blue tree, i.e. a MST of the whole graph, then the theorem holds true.*

2. We will prove the theorem by induction on the number of steps required.

3. The invariant holds true for $T=\{s\}$.

3.1 The MST of a graph containing one vertex s and no edges is just **that vertex**.

4. Assumption hypothesis:

assume the invariant holds at the k -th step and let's assume that the invariant is violated for the first time in the $(k+1)$ st step with the incorporation of the cut-edge (v,w) and vertex w into the blue tree T .

5. There are two possible cases:

5a. There exists a blue tree T' on $V_T \cup \{w\}$ with exactly one edge (x,w) incident on w . (Note that x is in V_T .)

5b. Every blue tree on $V_T \cup \{w\}$ has two (or more) edges incident on w .

6. (explanation of 5a) By the invariant at step k , $\text{cost}(T' - (x, w)) = \text{cost}(T)$. By the algorithm, $\text{cost}(v, w) < \text{cost}(x, w)$, since x is in V_T . This implies $\text{cost}(T' - (v, w)) < \text{cost}(T)$, a contradiction.
7. (explanation of 5b) Let the two edges incident on w be (x, w) and (y, w) , where x and y are in V_T . Now consider the subgraph $H = T - (x, w) - (y, w)$. H will have exactly one cycle C , and C will contain edges (x, w) and (y, w) .
8. (explanation of 5b) If no edge in C is more costly than one of (x, w) and (y, w) , then one of (x, w) and (y, w) can be deleted from H to give an MST with only one edge incident on w . But this contradicts the assumption of case 5b.
9. (explanation of 5b) So, without loss of generality, assume that x was admitted to the blue tree before y , and let (u, y) be in C . Since both (x, w) and (y, w) are less costly than (u, y) , edges (x, w) and (y, w) would have to have been brought into the blue tree before edge (u, y) was a least costly cut-edge. But this did not happen, so the assumption of 5b is false.
10. Since neither case 5a nor 5b can happen, $T - \{v, w\}$ is a MST.

Basically, our proof visualization system demands two input sets:

- an animation of the graph:
This animation can be used as a repository for the concrete and abstract proof objects. In addition the animation also contains an educationally suitable pseudocode description of the algorithm that is employed in the theorem.
- a text file containing the basis for the proof-dialog.
This text file contains:
 - theorem
 - set of invariants
 - set of expandable statements

Obviously the set of expandable statements represents the core of our animation system. That means that our system still is based on a textual description of the text. But the usefulness of the text alone gets vastly expanded by adding hypertext links on a micro level (Gloor, 1991). The hypertext links allow direct access to any points where the teacher thinks that a student might have difficulties in understanding, but they do not interrupt the flow of the proof because the student is not forced to follow the links.

For a sequence of screen dumps showing how the proof visualization works, see the Appendix.

5. CONCLUSIONS

We have presented a system for visualization of correctness proofs for greedy graph algorithms. There is considerable research activity in the field of algorithm animation, but so far nobody has

ever tried to combine hypertext concepts with algorithm animation to visualize a correctness proof of an algorithm. The following points summarize our experience:

- The method for animating proofs is basically a guided walkthrough of slides to be shown one after the other.
- Emphasis is on the hypertext aspect of the system because access to additional information can be given at the location where it is actually needed.
- The animation for the proof is based on an animation of the algorithm to be presented before the proof. Although it is rather simple, it vastly improves the student's understanding of the complex subject matter.
- Color is very useful for emphasizing the actual/important point of a slide.

Obviously the above listed points can be used to visualize any proof, not only induction based correctness proofs of greedy algorithms. Because graph algorithms are particularly well suited for animation, the integration of the algorithm animation into the visualization of the correctness proof is especially straightforward. This is the first time such a system is introduced, but we hope that it will stimulate research in the area and influence the generation of more systems for proof visualization.

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APPENDIX

The following screen shots describe on Prim's algorithm how our proof visualization system works.

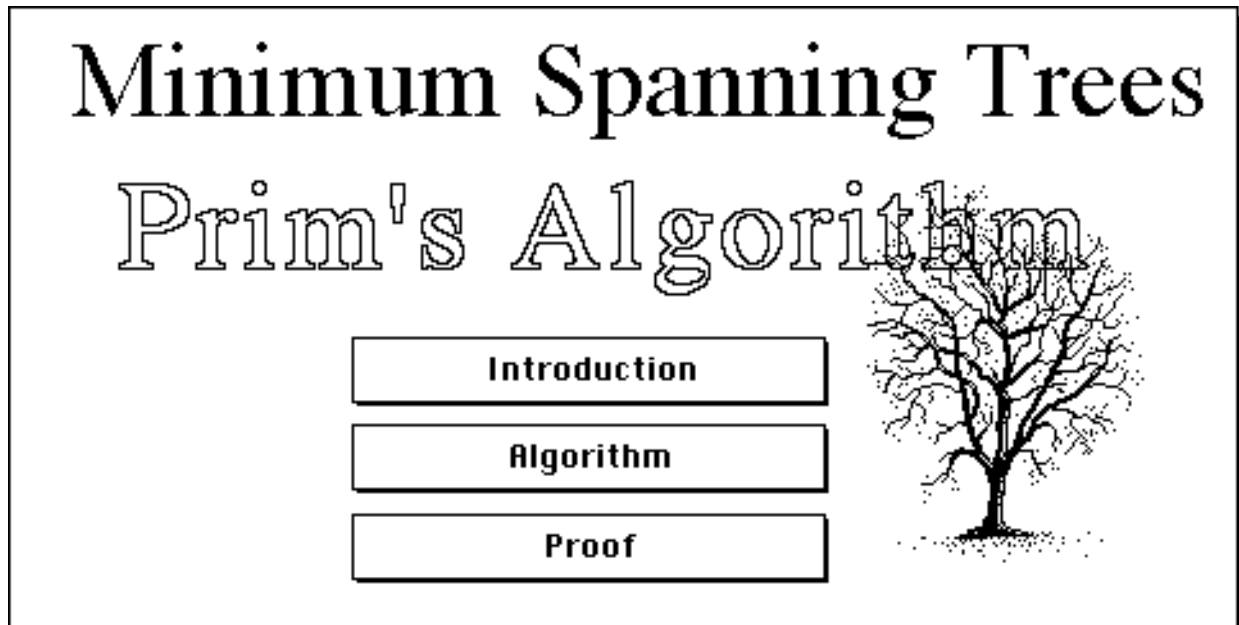


Fig. 1 overview map of Prim's algorithm/theorem

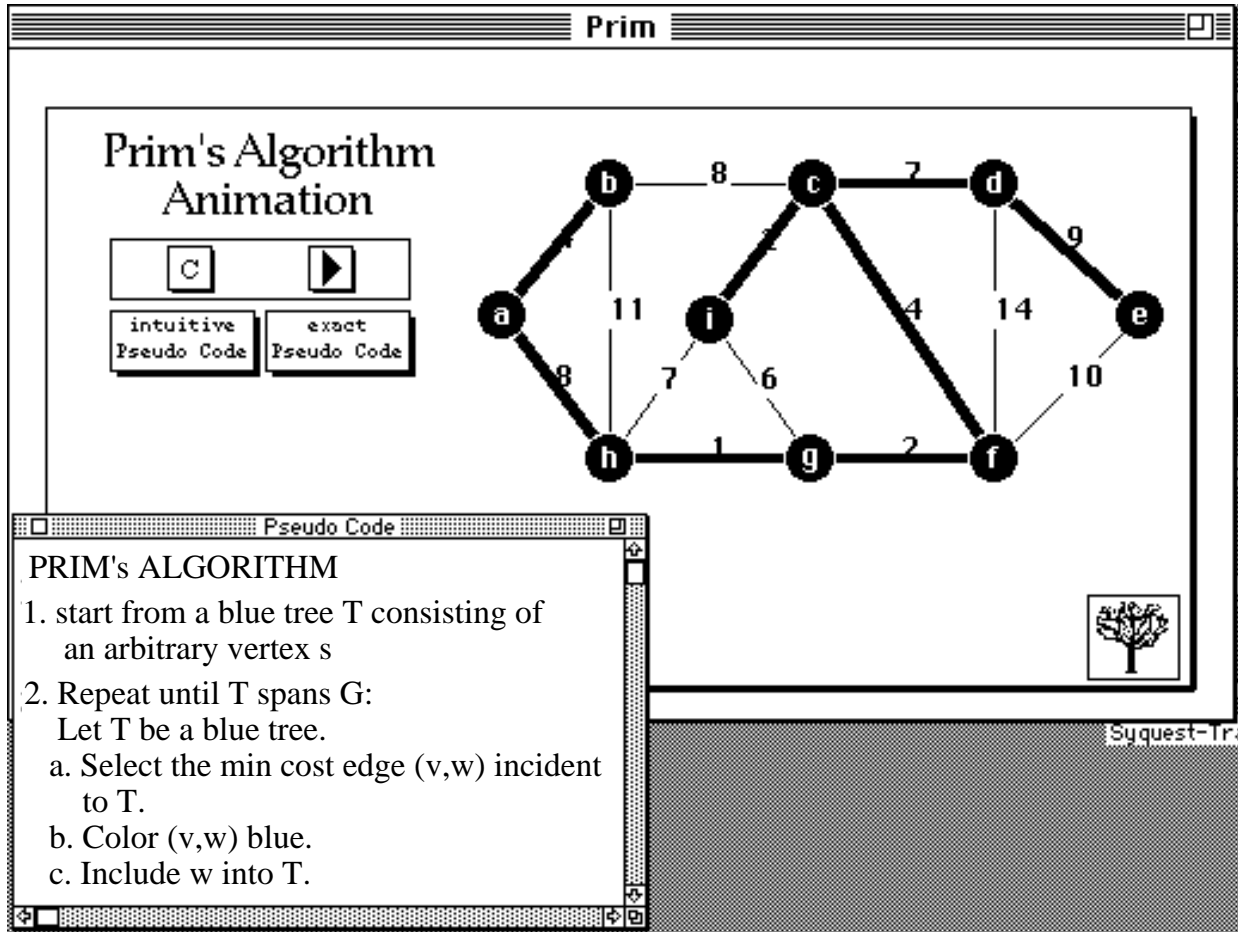


Fig. 2 Animation of Prim's Algorithm

Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: At the beginning each step, (V_T, E_T) is a **blue tree** on the subgraph of G .

Definitions: A blue tree $T=(V_T, E_T)$ on V_T nodes and E_T edges is a m.s.t. on V_T .

1. Examine the theorem and the invariant.
 If we show that (a) the algorithm terminates and (b) the invariant is satisfied at every step, then the theorem is true.


explain this step **next proof step** 

Fig. 3. first step in proof of Prim's algorithm


Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: At the beginning each step, (V_T, E_T) is a **blue tree** on the subgraph of G .

Definitions: A blue tree $T=(V_T, E_T)$ on V_T nodes and E_T edges is a m.s.t. on V_T .

3. The invariant holds true for $T=\{s\}$



number of vertices in graph = 1


explain this step **next proof step** 

Fig. 4 step 3 in proof of Prim's algorithm

Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: At the beginning each step, (V_t, E_t) is a **blue tree** on the subgraph of G .

Definitions: A blue tree $T=(V_t, E_t)$ on V_t nodes and E_t edges is a m.s.t. on V_t .

4. Assumption hypothesis: assume the invariant holds at the k -th step and let's assume that the invariant is violated for the first time in the $(k+1)$ st step with the incorporation of the **cut-edge** (v, w) and vertex w into the blue tree V_t .

T is a blue tree

next proof step

Fig. 5 step 4 in proof of Prim's algorithm

Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: At the beginning each step, (V_t, E_t) is a **blue tree** on the subgraph of G .

Definitions: A blue tree $T=(V_t, E_t)$ on V_t nodes and E_t edges is a m.s.t. on V_t .

5. Depending on how w is connected to V_t there are two possible cases:

a. There exists a blue tree T' on $V_t \cup \{w\}$ with exactly one edge (x, w) incident on w . (Note that x is in V_t).

b. Every blue tree on $V_t \cup \{w\}$ has two (or more) edges incident on w .

T

show definition of cut

next proof step

Fig. 6 step 5 in proof of Prim's algorithm

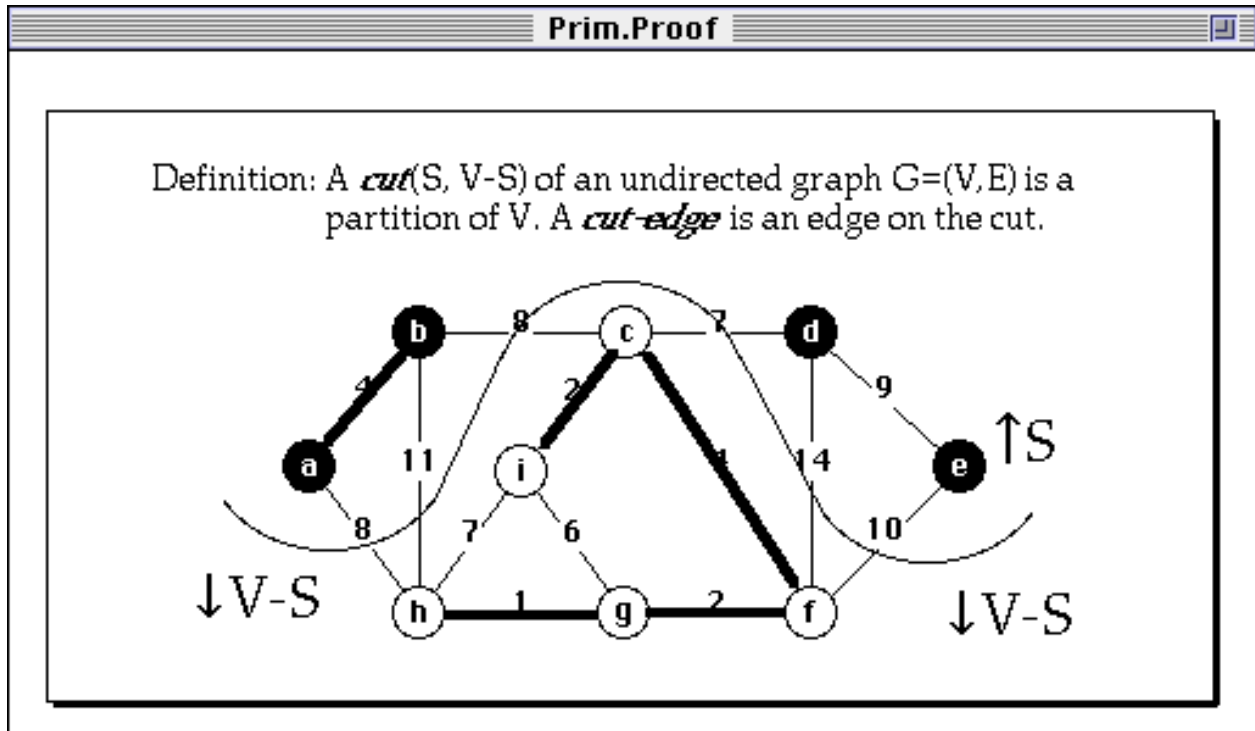


Fig. 7 definition of a "cut"

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: There exists a blue tree T' on $V_T \cup \{w\}$ with exactly one edge (x,w) incident on w .

Definition: T' is a m.s.t. on V_T .

6. (explanation of 5a)
 By the invariant at step k , $\text{cost}(T' - (x,w)) = \text{cost}(T)$, and by the algorithm, $\text{cost}(v,w) \leq \text{cost}(x,w)$, since x is in V_T .

show algorithm
 $\text{cost}(v,w) \leq \text{cost}(x,w)$, since x is in V_T .

show $\exists v$
 $\Rightarrow \text{cost}(T' \cup (v,w)) \leq \text{cost}(T')$, a contradiction.

next proof step

We choose a node as v that fulfills the requirements of the proof.

Fig. 8 step 6 in proof of Prim's algorithm

Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: blue tree on the subgraph of G .

Definition: Every blue tree on $V_T \cup \{w\}$ has two (or more) edges incident on w .

7. (explanation)

Let the two edges incident on w be (x,w) and (y,w) , where x and y are in V_T . Now consider the subgraph $H = T \cup (x,w) \cup (y,w)$. H will have exactly one cycle C, and C will contain edges (x,w) and (y,w) .

next proof step

Fig. 9 step 7 in proof of Prim's algorithm

Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: At the beginning each step, (V_r, E_r) is a **blue tree** on the subgraph of G .

Definitions: A blue tree $T=(V_r, E_r)$ on V_r nodes and E_r edges is a m.s.t. on V_r .

9. **5b** Without loss of generality, assume that x was admitted to the blue tree before y , and let (u,y) be in C .

show a 'u' Since both (x,w) and (y,w) are less costly than (u,y) , edges (x,w) and (y,w) would have to have been brought into the blue tree before edge (u,y) was a least costly cut-edge. => contradiction.

next proof step

Fig. 10 step 9 in proof of Prim's algorithm

Prim.Proof

THEOREM: Prim's algorithm terminates and correctly finds a MST T' of G .

Invariant: At the beginning each step, (V_r, E_r) is a **blue tree** on the subgraph of G .

Definitions: A blue tree $T=(V_r, E_r)$ on V_r nodes and E_r edges is a m.s.t. on V_r .

10. Since neither case **5a** nor **5b** can happen, $T \cup \{v,w\}$ is a MST.

repeat proof

Fig. 11 step 10 in proof of Prim's algorithm