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AW^*-ALGEBRAS WITH MONOTONE CONVERGENCE PROPERTY AND EXAMPLES BY TAKENOUCHI AND DYER

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In 1951, I. Kaplansky [6] introduced a class of C^*-algebras called AW^*-algebras to separate the discussion of the internal structure of a W^*-algebra(or von Neumann algebra) from the action of its elements on a Hilbert space and showed that much of the "non-spatial theory" of W^*-algebras can be extended to AW^*-algebras.

Every W^*-algebra is of course AW^*, however, the converse is not true as was shown by Dixmier [3] with an abelian example (the algebra of all bounded Baire functions on the real line modulo the set of first category is a non-W^*, AW^*-algebra). I. Kaplansky [7] proved that an AW^*-algebra of type I is a W^*-algebra if and only if its center is a W^*-algebra and conjectured that the theorem is true without the assumption of "type I". In 1970, O. Takenouchi [12] and Dyer [2], independently, showed this to be false by counter examples (non-W^*, AW^*-factors). In 1976, J. D. Maitland Wright [16, 18] defined a *regular σ-completion* (some kind of Dedekind cut completion) of a separable C^*-algebra and proved that the regular σ-completion of an infinite dimensional simple separable C^*-algebra is a type III, non-W^*, σ-finite AW^*-factor with the monotone convergence property (see the definition below).

In this paper, the author will give a modification of a J. D. M. Wright's theorem and using this, will show that the non-W^*, AW^*-factors given by Takenouchi and Dyer are σ-finite, type III AW^*-factors. The key point of the proof is, roughly speaking, to construct a faithful state on these factors. To do this, a J. D. Maitland Wright's theorem plays an essential role. He states that the *pure state space* of the regular σ-completion $\widehat{C[0, 1]}$ (which is essentially the same as $\mathfrak{A}$ in section 1) of the C^*-algebra $C[0, 1]$ of continuous complex functions on [0, 1] is *separable* ([18, p. 85]).

The AW^*-factor given by Takenouchi is a "weakly closed" (in the sense of [13]) AW^*-subalgebra of type I AW^*-algebra $\mathfrak{B}(\mathfrak{M})$ of all bounded module endomorphisms of some AW^*-module $\mathfrak{M}$ over an abelian AW^*-algebra. The author believes that it is natural to represent AW^*-

algebras as "weakly closed" AW^*-subalgebra of some $\mathfrak{B}(\mathfrak{M})$. The author then will show that the AW^*-factor constructed by Dyer can be represented faithfully as a "weakly closed" AW^*-subalgebra of some $\mathfrak{B}(\mathfrak{M})$. Moreover, we shall remark that these factors are *monotone closed* (in the sense of [5]), simple and do not have any non-trivial separable representations.

1. AW^*-algebras with a monotone convergence property (M. C. P.). An AW^*-algebra M means that it is both a C^*-algebra and a Baer*-ring ([1], [6]). M has a *monotone convergence property* (M. C. P.) if for every increasing sequence $\{x_n\}$ of self-adjoint elements in M bounded above has the supremum x in the self-adjoint part of M (we simply denote $x_n \uparrow x$ or $\mathrm{Sup}_n x_n = x$).

First of all, we shall show the following technical lemma.

LEMMA. *Let M be an AW^*-algebra with* M. C. P. *For every increasing sequence $\{e_n\}$ of projections in M, let $\bigvee_{n=1}^{\infty} e_n$ be the supremum projection of $\{e_n\}$ in the projection of M. Then* $\mathrm{Sup}_n e_n = \bigvee_{n=1}^{\infty} e_n$. *Moreover, for any $a \in M$,*

$$\mathrm{Sup}_n\, a^* e_n a = a^*\Big(\bigvee_{n=1}^{\infty} e_n\Big)a .$$

PROOF. Put $b = \mathrm{Sup}_n e_n$, then $0 \leqq e_n \leqq b \leqq \bigvee_{n=1}^{\infty} e_n$ in M for each n. Thus $e_n \leqq \mathrm{LP}(b) \leqq \bigvee_{n=1}^{\infty} e_n$ for all n and hence $\mathrm{LP}(b) = \bigvee_{n=1}^{\infty} e_n$ where $\mathrm{LP}(b)$ is the left projection of b in M ([1, 6]). On the other hand, $e_n = be_n$ for every n implies by [6, Lemma 2.2] that $\mathrm{LP}(b) = b\,\mathrm{LP}(b) = b$ and $\mathrm{Sup}_n e_n = \bigvee_{n=1}^{\infty} e_n$.

Now arguments used in [5] tells us that for any $a \in M$, $a^* e_n a \uparrow a^*(\bigvee_{n=1}^{\infty} e_n)a$ in M. This completes the proof.

Using this, we have the following theorem which is a modification of a J. D. M. Wright's result ([17, Theorem 6]).

THEOREM 1. *Let M be an AW^*-factor with* M. C. P. *Suppose that M has a faithful state (not necessarily normal) ϕ and is semi-finite, then M is a σ-finite W^*-algebra. The assumption of semi-finiteness cannot be dropped.*

REMARK. Maitland Wright proved, without the assumption of M. C. P., however under the condition that M is *finite*, that the above proposition holds.

PROOF OF THEOREM 1. For any non-zero finite projection e (note that M is semi-finite), put $N=eMe$, then N is a finite AW^*-factor with the faithful positive functional ψ where $\psi(exe) = \phi(exe)$ for $x \in M$. J. D. M. Wright's theorem [17, Theorem 6] tells us that N is a W^*-algebra, that is, there exists a faithful W^*-representation π_e of N on some Hilbert space $\mathfrak{H}\pi_e$ ($\pi_e(N)$ is a weakly closed $*$-subalgebra with the identity of $\mathfrak{B}(\mathfrak{H}\pi_e)$ (the algebra of all bounded linear operators on $\mathfrak{H}\pi_e$)). Next we shall show that for any $\xi \in \mathfrak{H}\pi_e$, the positive functional $\phi(e, \xi)$ on M (where $\phi(e, \xi)(x) = (\pi_e(exe)\xi, \xi)$, $x \in M$) is completely additive on projections. To prove this we have only to show that for any decreasing *sequence* $\{e_n\}$ of projections in M with $e_n \downarrow 0$, $\phi(e, \xi)(e_n) \downarrow 0$ $(n \to \infty)$, because M is σ-finite (note that M has a faithful state). Let $\{e_n\}$ be any decreasing sequence of projections in M with $e_n \downarrow 0$, then by the above lemma, $\mathrm{Inf}_n\, ee_ne = 0$ in the self-adjoint part of N. Since $\{\pi_e(ee_ne)\}$ is a decreasing sequence in the non-negative portion of $\mathfrak{B}(\mathfrak{H}\pi_e)$, there is $A \in \mathfrak{B}(\mathfrak{H}\pi_e)$ such that $\pi_e(ee_ne) \downarrow A$ (*strongly*). The strong closedness of $\pi_e(N)$ implies $A \in \pi_e(N)$. Hence there is $a \in N(a \geqq 0)$ such that $A = \pi_e(a)$. The faithfulness of π_e implies that $a = 0$, that is, $\pi_e(ee_ne) \downarrow 0$ strongly. Thus $\phi(e, \xi)$ is completely additive on projections of M. The semi-finiteness of M tells us that $\{\phi(e, \xi)$; e is any non-zero finite projection, $\xi \in \mathfrak{H}\pi_e$ where π_e is a faithful W^*-representation of $eMe\}$ is a separating family of positive functionals on M which are completely additive on projections of M. Hence by ([10], Theorem 5.2, see also [9]) M is a semi-finite W^*-algebra. Non W^*, AW^*-factors constructed by Takenouchi and Dyer have the M. C. P. and faithful states (see the next section), thus the assumption of semi-finiteness cannot be dropped. This completes the proof of Theorem 1.

REMARK. In the above proof, we suppose that M has the M. C. P., however, Theorem 1 still holds under a nominally weaker assumption such that for any increasing sequence $\{e_n\}$ of projections in M and for any projection e in M, $\mathrm{Sup}_n\, ee_ne$ exists in the self-adjoint portion of M and $\mathrm{Sup}_n\, ee_ne = e(\bigvee_{n=1}^{\infty} e_n)e$.

The above theorem implies that if non-W^*, AW^*-factor with M. C. P. has a faithful state, then it is of type III ([6, p. 241 Definition]).

In the rest of this section, we treat with examples of abelian AW^*-algebras with groups of $*$-automorphisms of them which are needed in the later sections.

Let $B^\infty[0, 1)$ be the algebra of all bounded Baire functions on $[0, 1)$ and let $\mathfrak{A}$ be the algebra $B^\infty[0, 1)$ modulo the set of first category. Then one can easily check that $\mathfrak{A}$ is a non-W^*, abelian AW^*-algebra which is $*$-isomorphic with the regular σ-completion of a separable abelian C^*-algebra ([2], [18], p. 86). J. D. Maitland Wright proved also that $\mathfrak{A}$ *has* *a faithful state* because the pure state space of $\mathfrak{A}$ is separable [18, Proposition A, Corollary D].

Let G_θ (resp. G_0) be the group of translations on $[0, 1)$ by an irrational number $\theta(\text{mod}\, 1)$ (resp. by all dyadic rationals in $[0, 1)$ (mod 1)). Denote for each $\sigma \in G_\theta$ (resp. G_0), $\sigma(t) = t+\sigma$ (mod 1), $f^\sigma(t) = f(\sigma(t))$ for all $t \in [0, 1)$, $f \in B^\infty[0, 1)$ and $a^\sigma = \tilde{f}^\sigma$ where f belongs to a coset $a(a = \tilde{f})$, $f \in B^\infty[0, 1)$ for all $a \in \mathfrak{A}$. Then both G_θ and G_0 naturally induce groups of $*$-automorphisms ($a \to a^\sigma$ $a \in \mathfrak{A}$) of $\mathfrak{A}$ (we denote them by the same notations G_θ and G_0 since any confusion does not occur). It is easy to check that G_θ and G_0 act *freely* and *ergodically* on $\mathfrak{A}$.

2. **Types of the AW^*-factors constructed by Takenouchi.** First, we shall sketch briefly the construction of AW^*-factors of [12]. Let Z be an abelian AW^*-algebra, G be an abelian group of $*$-automorphisms of Z with an action $a \to a^g$ $(a \in Z,\ g \in G)$. One can construct a faithful AW^*-module ([8]) $\mathfrak{M}$ over Z as follows: Let $\mathfrak{M}$ be the set $l^2(G, Z)$ of all sequences $\{x_g\}$ of elements in Z with the indices $g \in G$ such that $\sum_{g \in G} x_g{}^* x_g$ is in Z (the supremum of the family of finite sums). Then $\mathfrak{M}$ is a faithful AW^*-module over Z and the set $\mathfrak{B}(\mathfrak{M})$ of all bounded module endomorphisms (we simply call them "operators") of $\mathfrak{M}$ is a type I AW^*-algebra with center Z.

Define, for any $a \in Z$ and $h \in G$, the following types of "operators" on $\mathfrak{M}$:

$$\left.\begin{array}{l} L_a\colon \{x_g\} \to \{a^g x_g\} \\ U_h\colon \{x_g\} \to \{y_g\} \quad \text{where} \quad y_g = x_{g-h} \end{array}\right\} \quad \text{for} \quad \{x_g\} \in \mathfrak{M} .$$

Then one can easily show that $a \to L_a$ is a $*$-isomorphism of Z into $\mathfrak{B}(\mathfrak{M})$ and $h \to U_h$ is a unitary representation of G into $\mathfrak{B}(\mathfrak{M})$ such that $U_h{}^* L_a U_h = L_{a^h}$ for all $a \in Z$ and $h \in G$.

Next, for any $h \in G$, we introduce the following *linear* operator (note that this is not a module endomorphism of $\mathfrak{M}$) on $\mathfrak{M}$:

$$V_h\colon \{x_g\} \to \{y_g\} \quad \text{where} \quad y_g = (x_{g+h})^{-h} \quad \text{for} \quad \{x_g\} \in \mathfrak{M} .$$

For every "operator" on $\mathfrak{M}$ has a matrix representation $A \sim \langle a_{g,h} \rangle$ where $a_{g,h} = (Au_h, u_g)$ $(g, h \in G)$ (where $u_h = \{\delta_{g,h}\}$ $(h \in G)$ and $\delta_{g,h}$ is the Kronecker's delta).

Let $M(Z, G) = \{A \in \mathfrak{B}(\mathfrak{M});\ A \sim \langle a_{g,h} \rangle$ where $a_{g,h} = (a_{g-h,e})^h$ for any pair g, h in G(e is a unit of G)$\}$, then $A \in M(Z, G)$ if and only if $AV_h = V_h A$ for all $h \in G$ and $M(Z, G)$ is an AW^*-subalgebra of $\mathfrak{B}(\mathfrak{M})$ which contains all U_h and L_a, where an AW^*-subalgebra means that the structure of an AW^*-algebra of $M(Z, G)$ is compatible with that of $\mathfrak{B}(\mathfrak{M})$ in the sense of [1, 7].

Takenouchi showed under the condition that the action of G on Z is *free* and *ergodic*, $M(Z, G)$ is an AW^*-factor such that $\{L_a; a \in Z\} = \widetilde{Z}$ is a maximal abelian $*$-subalgebra (whose proof is analogous to that of Murray-von Neumann's) and gave an example of (Z, G) as $(\mathfrak{A}, G_\theta)$ in section 1. If $M(\mathfrak{A}, G_\theta)$ is a W^*-algebra, then $\widetilde{Z}$ ($*$-isomorphic with $\mathfrak{A}$) is a W^*-algebra. This is a contradiction and hence $M(\mathfrak{A}, G_\theta)$ is a *non-*W^*, AW^*-factor.

The rest of this section is devoted to prove

THEOREM 2. *$M(\mathfrak{A}, G_\theta)$ is a σ-finite, type* III, *non-W^*, AW^*-factor with* M. C. P. (*more precisely, $M(\mathfrak{A}, G_\theta)$ is "weakly closed" $*$-subalgebra of $\mathfrak{B}(\mathfrak{M})$ ($\mathfrak{M} = l^2(G_\theta, \mathfrak{A})$) in the sense of H. Widom* [13], *and that, it is monotone closed in the sense that in its self-adjoint part, every norm-bounded increasing net has a least upper bound*).

PROOF. First of all, we shall show that $M(\mathfrak{A}, G_\theta)$ is "weakly closed" subalgebra of $\mathfrak{B}(\mathfrak{M})$ where $\mathfrak{M} = l^2(G_\theta, \mathfrak{A})$ in the sense that for any net $\{A_\alpha\}$ in $M(\mathfrak{A}, G_\theta)$ such that $(A_\alpha\xi, \eta) \to (A\xi, \eta)$ (order convergence in $\mathfrak{A}$) [13], for some $A \in \mathfrak{B}(\mathfrak{M})$, $A \in M(\mathfrak{A}, G_\theta)$. In fact, putting $A_\alpha \sim \langle a^\alpha_{g,h} \rangle$ $A \sim \langle a_{g,h} \rangle$, then $a^\alpha_{g,h} \to a_{g,h}$ (order convergence in $\mathfrak{A}$) for each pair g and h in G_θ. Thus $a_{g,h} = (a_{g-h,e})^h$ for $g, h \in G_\theta$ and $A \in M(\mathfrak{A}, G_\theta)$. In particular, $M(\mathfrak{A}, G_\theta)$ has M. C. P. In fact, let $\{A_n\}$ be an increasing sequence of self-adjoint elements of $M(\mathfrak{A}, G_\theta)$ bounded above by $B \in M(\mathfrak{A}, G_\theta)$, then $A_n \uparrow A$ "weakly" for some A (where A is the supremum of $\{A_n\}$ in the self-adjoint part of $\mathfrak{B}(\mathfrak{M})$), in $\mathfrak{B}(\mathfrak{M})$ ([13, Lemma 1.4]). It follows by the above argument that $A \in M(\mathfrak{A}, G_\theta)$ and $A \leqq B$ and hence $M(\mathfrak{A}, G_\theta)$ has M. C. P. By the same way, we can easily show that $M(\mathfrak{A}, G_\theta)$ is monotone closed.

Next, we shall show that $M(\mathfrak{A}, G_\theta)$ has a faithful positive projection map onto $\widetilde{\mathfrak{A}}(=\{L_a, a \in \mathfrak{A}\})$. In fact, for any $A \in M(\mathfrak{A}, G_\theta)$, let $\Phi(A) = L_{a_{e,e}}$ where $A \sim \langle a_{g,h} \rangle$, then one can easily check that Φ is a positive projection map of $M(\mathfrak{A}, G_\theta)$ onto $\widetilde{\mathfrak{A}}$. To prove the faithfulness of Φ, we argue as follows. For any $A \in M(\mathfrak{A}, G_\theta)$ with $A \sim \langle a_{g,h} \rangle$, noting that, $(A^*A)_{e,e} = \sum_{g \in G} a_{g,e}{}^* a_{g,e}$, we have $\Phi(A^*A) = 0$ implies $a_{g,e} = 0$ for all $g \in G$ and hence $A = 0$ because $a_{g,h} = (a_{g-h,e})^h = 0$ for all $g, h \in G$.

Let ψ be a faithful state on $\mathfrak{A}$ in section 1, and let $\phi = \psi \circ \Phi$, then ϕ is a faithful state on $M(\mathfrak{A}, G_\theta)$. Assume that $M(\mathfrak{A}, G_\theta)$ is semi-finite, then by Theorem 1, $M(\mathfrak{A}, G_\theta)$ is a W^*-algebra, however this is a contradiction because $M(\mathfrak{A}, G_\theta)$ is non-W^*. Hence $M(\mathfrak{A}, G_\theta)$ is of type III. Since $M(\mathfrak{A}, G_\theta)$ has a faithful state ϕ, we can easily show that $M(\mathfrak{A}, G_\theta)$ is σ-finite. This completes the proof.

3. Dyer's example. In this section, we shall sketch briefly the construction by Dyer [3] and then show that the Dyer's example is a σ-finite, non-W^*, type III AW^*-factor with M. C. P. Moreover we shall prove that it can be represented faithfully as $M(\mathfrak{A}, G_0)$ in section 2. Thus Dyer's factor is also monotone closed.

Let $\mathfrak{H}$ be a Hilbert space with an orthonormal basis $\{e_x; 0 \leqq x < 1, x$: a real number$\}$. Every bounded linear operator A on $\mathfrak{H}$ has a matrix representation $A_{x,y} = (Ae_y, e_x)$ for x and $y \in [0, 1)$. Let $\mathfrak{A}_1$ (respectively $\mathfrak{J}_1$) denote the algebra of operators A such that $A_{x,y} = \delta_{x,y} f(x)$ for any x, y where $f \in B^\infty[0, 1)$ and $\delta_{x,y}$ is a Kronecker's delta (resp. $\{x; 0 \leqq x < 1, f(x) \neq 0\}$ is contained in a set of 1st category in $[0, 1)$).

Let $\mathfrak{A}_0$ (resp. $\mathfrak{J}_0$) be the set of operators A on $\mathfrak{H}$ with matrices $A_{x,y}$ with

(1) $A_{x,y} = 0$ except when $y - x = j2^{-k}$ for some $k \geqq 1$ and $-2^k < j < 2^k$ (integer).

(2) For $k \geqq 1$ and $0 \leqq i, j < 2^k$, the function defined for $x \in [0, 1)$ by $f(x) = A_{2^{-k}(i+x), 2^{-k}(j+x)}$ is a bounded Baire function (resp. $\{x; 0 \leqq x < 1, f(x) \neq 0\}$ is contained in a set of 1st category in $[0, 1)$).

Dyer [3] proved that $\mathfrak{A}_0$ (resp. $\mathfrak{A}_1$) is a C^*-algebra with a closed two-sided ideal $\mathfrak{J}_0$ (resp. $\mathfrak{J}_1$) and the quotient algebra $\mathfrak{A}_0/\mathfrak{J}_0$ is a non-W^*, AW^*-factor of which $\mathfrak{A}_1/\mathfrak{J}_1$ is a maximal abelian $*$-subalgebra (note that $\mathfrak{A}_1/\mathfrak{J}_1$ is $*$-isomorphic with $\mathfrak{A}$ in section 2).

By the above construction, a straightforward verification tells us that $\mathfrak{A}_0$ has M. C. P. and $\mathfrak{J}_0$ is a σ-ideal in the sense that for every increasing sequence $\{A_n\}$ of self-adjoint elements in $\mathfrak{J}_0$ which converges strongly to some operator A, $A \in \mathfrak{J}_0$. Now by the arguments of J. D. M. Wright [15] it follows that $\mathfrak{A}_0/\mathfrak{J}_0$ has M. C. P. Moreover, $\mathfrak{A}_0/\mathfrak{J}_0$ has a faithful positive projection Ψ onto $\mathfrak{A}_1/\mathfrak{J}_1$. In fact, for any $A \in \mathfrak{A}_0$ with $A \sim \langle A_{x,y} \rangle$, put $B \sim \langle \delta_{x,y} A_{x,y} \rangle$ ($B \in \mathfrak{A}_1$) and consider the following mapping $\Psi: A + \mathfrak{J}_0 \to B + \mathfrak{J}_1$ of $\mathfrak{A}_0/\mathfrak{J}_0$ onto $\mathfrak{A}_1/\mathfrak{J}_1$. Then it is easy to check that Ψ is a projection map of $\mathfrak{A}_0/\mathfrak{J}_0$ onto $\mathfrak{A}_1/\mathfrak{J}_1$. For any $A \in \mathfrak{A}_0$ with $A \sim \langle A_{x,y} \rangle$, we have that $(A^*A)_{x,x} = \sum_{0 \leqq z < 1} |A_{z,x}|^2$ for all x. This implies that Ψ is positive and faithful. Since $\mathfrak{A}_1/\mathfrak{J}_1$ is $*$-isomorphic with $\mathfrak{A}$ in section 1, $\mathfrak{A}_1/\mathfrak{J}_1$ has a faithful state and then by the same reasoning as in Theorem 2, $\mathfrak{A}_0/\mathfrak{J}_0$ has a faithful state and thus by Theorem 1 we have

THEOREM 3. *$\mathfrak{A}_0/\mathfrak{J}_0$ is a σ-finite, non-W^*, type* III *AW^*-factor.*

The rest of this section is devoted to prove the following:

THEOREM 4. *$\mathfrak{A}_0/\mathfrak{J}_0$ is $*$-isomorphic with $M(\mathfrak{A}, G_0)$ in section* 2.

PROOF. For any $A+\mathfrak{J}_0 \in \mathfrak{A}_0/\mathfrak{J}_0$ $(A \in \mathfrak{A}_0)$, let $A \sim \langle A_{x,y}\rangle$, then $A_{x,y}=0$ except when $y-x=j\cdot 2^{-k}$ for some $k \geqq 1$, $-2^k<j<2^k$ and for $k \geqq 1$, $0 \leqq i$, $j<2^k$ the function $x \rightarrow f(x)=A_{2^{-k}(i+x),2^{-k}(j+x)}$ $(0 \leqq x<1)$ is in $B^\infty[0,1)$. Keeping the notations in the last paragraph of section 1, for any $g \in G_0$, $x \rightarrow A_{\sigma_g(x),x}$ is a bounded Baire function on $[0,1)$. Let ϕ be the canonical map of $B^\infty[0,1)$ onto $\mathfrak{A}$ and $a_{g,e}=\phi(x \rightarrow A_{\sigma_g(x),x})$ for $g \in G_0$. Note that $a_{g,e}$ does not depend on the choice of particular $A \in A+\mathfrak{J}_0$. In fact, if $A, B \in A+\mathfrak{J}_0$, then $A-B \in \mathfrak{J}_0$ and hence $\phi(x \rightarrow A_{\sigma_g(x),x})=\phi(x \rightarrow B_{\sigma_g(x),x})$ for all $g \in G_0$. Let $a_{g,h}=(a_{g-h,e})^h$ for any g and $h \in G_0$, then $(a_{g,h})$ defines an "operator" $\psi(A+\mathfrak{J}_0)$ on $\mathfrak{M}=l^2(G_0,\mathfrak{A})$ such that $\psi(A+\mathfrak{J}_0)_{g,h}=a_{g,h}$ for all $g, h \in G_0$.

Observe that $a_{g,h} \in \mathfrak{A}$ is the canonical image of $x \rightarrow A_{\sigma_g(x),\sigma_h(x)}$ for any $g, h \in G_0$.

Since $\sum_{g \in G_0}|A_{\sigma_g(x),x}|^2=\sum_{g \in G_0}|(Ae_x, e_{\sigma_g(x)})|^2 \leqq \sum_{0 \leqq y<1}|(Ae_x, e_y)|^2=||Ae_x||^2 \leqq ||A||^2$ for all $x \in [0,1)$, we have that $\sum_{g \in G_0}|a_{g,e}|^2 \leqq ||A||^2\cdot 1$ on $\mathfrak{A}$. Thus, for any $\xi=(x_g) \in \mathfrak{M}$, $\sum_{g \in G_0}|x_g a_{g,h}| \leqq ||\xi||\cdot||A||$. This implies that $\sum_{g \in G_0} x_g a_{g,h}$ is order convergent in $\mathfrak{A}$ [13]. Put $\eta_h=\sum_{g \in G_0} x_g a_{g,h} \in \mathfrak{A}$, we can show that $\sum_{h \in G_0}|\eta_h|^2 \in \mathfrak{A}$ (order convergent in $\mathfrak{A}$). In fact, let $\hat{x}_g$ be the inverse image of x_g by ϕ in $B^\infty[0,1)$, then $\sum_{g \in G_0}|x_g(x)|^2 \leqq ||\xi||^2$ except on a set of first category. Hence it follows that

$$\sum_{h \in G_0}\Big|\sum_{g \in G_0}\hat{x}_g(x)A_{\sigma_g(x),\sigma_h(x)}\Big|^2=\sum_{h \in G_0}\Big|\sum_{g \in G_0}\hat{x}_g(x)(Ae_{\sigma_h(x)}, e_{\sigma_g(x)})\Big|^2$$
$$=\sum_{h \in G_0}\Big|\sum_{0 \leqq y<1}\hat{x}_y(x)(Ae_{\sigma_h(x)}, e_y)\Big|^2$$

(where $\hat{x}_y(x)=0$ if $y \neq \sigma_g(x)$ for any $g \in G_0$ and $\hat{x}_y(x)=\hat{x}_g(x)$ if $y=\sigma_g(x)$ $g \in G_0$)

$$=\sum_{h \in G_0}|(Ae_{\sigma_h(x)}, (\hat{x}_y(x)))|^2 \quad (\text{where } (\hat{x}_y(x)) \in \mathfrak{H})$$
$$=\sum_{h \in G_0}|(e_{\sigma_h(x)}, A^*(\hat{x}_y(x)))|^2 \leqq ||A||^2\cdot||(\hat{x}_y(x))||^2=||A||^2\cdot||\xi||^2$$

except on a set of first category. Thus $\sum_{h \in G_0}|\sum_{g \in G_0} x_g a_{g,h}|^2 \leqq ||A||^2\cdot||\xi||^2$ and $\sum_{h \in G_0}|\eta_h|^2 \in \mathfrak{A}$. Hence let $\psi(A+\mathfrak{J}_0)\xi=(\sum_{g \in G_0} x_g a_{g,h}) \in \mathfrak{M}$, then $\psi(A+\mathfrak{J}_0) \in \mathfrak{B}(\mathfrak{M})$ and $||\psi(A+\mathfrak{J}_0)|| \leqq ||A+\mathfrak{J}_0||$. $\psi(A+\mathfrak{J}_0)_{g,h}=a_{g,h}$ for all g, h implies that $\psi(A+\mathfrak{J}_0) \in M(\mathfrak{A}, G_0)$. Thus ψ is a bounded $*$-linear map of $\mathfrak{A}_0/\mathfrak{J}_0$ into $M(\mathfrak{A}, G_0)$. Next we shall show that ψ is a $*$-isomorphism. For any $A+\mathfrak{J}_0$, $B+\mathfrak{J}_0 \in \mathfrak{A}_0/\mathfrak{J}_0$ $(A, B \in \mathfrak{A}_0)$,

$$(AB)_{\sigma_g(x),\sigma_h(x)}=\sum_{k \in G_0} A_{\sigma_g(x),\sigma_k(x)}B_{\sigma_k(x),\sigma_h(x)}$$

for all $0 \leqq x<1$. Thus $\sum_{k \in G_0} a_{g,k}b_{k,h}$ is order convergent to $\phi(x \rightarrow (AB)_{\sigma_g(x),\sigma_h(x)})$ in $\mathfrak{A}$. Hence $\psi(AB+\mathfrak{J}_0)=\psi(A+\mathfrak{J}_0)\psi(B+\mathfrak{J}_0)$. If

$\psi(A+\mathfrak{J}_0)=0$ $(A\in\mathfrak{A}_0)$, then $\{x; 0\leqq x<1, A_{\sigma_g(x),\sigma_h(x)}\neq 0\}$ is contained in a set of 1st category in $[0,1)$. Thus for all $k\geqq 1$, $0\leqq i$, $j<2^k$, $x\rightarrow A_{2^{-k}(i+x),2^{-k}(j+x)}$ has a first category support, and hence $A\in\mathfrak{J}_0$, that is, $A+\mathfrak{J}_0=0$. This implies that ψ is a $*$-isomorphism of $\mathfrak{A}_0/\mathfrak{J}_0$ into $M(\mathfrak{A}, G_0)$.

Next, we shall show that the map ψ is onto. To do this we argue as follows: Let $A\in M(\mathfrak{A}, G_0)$ with $A\sim\langle a_{g,h}\rangle$. Then one can choose for any g and $h\in G_0$, a function $a_{g,h}(x)\in B^\infty[0,1)$ such that there is a Baire set contained in a set of 1st category I in $[0,1)$ such that

$$|\sum_{g,h\in G_0}a_{g,h}(x)\xi_h\bar{\eta}_g|\leqq ||A||\cdot(\sum_{h\in G_0}|\xi_h|^2)^{1/2}(\sum_{g\in G_0}|\eta_g|^2)^{1/2}$$

for all $\{\xi_h\}$, $\{\eta_h\}\in l^2(G_0)$ and for all $x\in[0,1)\backslash I$ where $\bar{c}$ is the complex conjugate of a complex number c. Replacing $a_{g,h}(x)$ by $a_{g,h}'(x)$ with the function $a_{g,h}'(x)$ defined to be zero if $x\in I$ and equal to $a_{g,h}(x)$ otherwise, we have that for any $\{\xi_h\}$, $\{\eta_h\}\in l^2(G_0)$,

$$|\textstyle\sum_{g,h\in G_0}a_{g,h}'(x)\xi_h\bar{\eta}_g|\leqq ||A||(\sum_{h\in G_0}|\xi_h|^2)^{1/2}(\sum_{g\in G_0}|\eta_g|^2)^{1/2}$$

for all x. Now we shall define $\langle A_{x,y}\rangle$ as follows: $A_{x,y}=0$ except when $x-y=j\cdot 2^{-k}$ for some $k\geqq 1$, $-2^k<j<2^k$, $A_{\sigma_g(x),x}=a_{g,e}(x)$ $0\leqq x<1$, $g\in G_0$, then $x\rightarrow A_{2^{-k}(i+x),2^{-k}(j+x)}$ is a bounded Baire function on $[0,1)$. To see that $\langle A_{x,y}\rangle$ determines a bounded linear operator B on $\mathfrak{H}$, we have only to show that $|\sum_{0\leqq x,y<1}A_{x,y}\bar{\xi}_x\eta_y|\leqq ||A||\cdot||\xi||\cdot||\eta||$ for any $\xi=\{\xi_x\}$, $\eta=\{\eta_x\}$ in $\mathfrak{H}$. In fact,

$$\begin{aligned}
&\sum_{0\leqq x<1}\sum_{g\in G_{k_0}}A_{\sigma_g(x),x}\bar{\xi}_{\sigma_g(x)}\eta_x \quad (\text{where } G_{k_0}=\{\sigma_{2^{-k_0}i}; i=0,1,2,\cdots,2^{k_0}-1\})\\
&=\sum_{0\leqq x<2^{-k_0}}\sum_{h\in G_{k_0}}\sum_{g\in G_{k_0}}A_{\sigma_{g+h}(x),\sigma_h(x)}\overline{\xi_{\sigma_{g+h}(x)}}\eta_{\sigma_h(x)}\\
&=\sum_{0\leqq x<2^{-k_0}}\sum_{h\in G_{k_0}}\{\sum_{g\in G_{k_0}}A_{\sigma_g(x),\sigma_h(x)}\overline{\xi_{\sigma_g(x)}}\eta_{\sigma_h(x)}\}\\
&\leqq\sum_{0\leqq x<2^{-k_0}}||A||\cdot(\sum_{g\in G_{k_0}}|\xi_{\sigma_g(x)}|^2)^{1/2}(\sum_{h\in G_{k_0}}|\eta_{\sigma_h(x)}|^2)^{1/2}\\
&\leqq||A||\cdot(\sum_{0\leqq x<2^{-k_0}}\sum_{g\in G_{k_0}}|\xi_{\sigma_g(x)}|^2)^{1/2}(\sum_{0\leqq x<2^{-k_0}}\sum_{h\in G_{k_0}}|\eta_{\sigma_h(x)}|^2)^{1/2}\\
&=||A||\cdot||\xi||\cdot||\eta||
\end{aligned}$$

for all $k_0\geqq 1$. Since $\bigcup_{k=1}^\infty G_k=G_0$ and $G_k\subset G_{k+1}$ for all k, we have $|\sum_{0\leqq x,y<1}A_{x,y}\bar{\xi}_x\eta_y|\leqq ||A||\cdot||\xi||\cdot||\eta||$ and hence there is a $B\in\mathfrak{A}_0$ such that $(Be_y, e_x)=A_{x,y}$ for all x, y. By the construction, it is easy to check that $\psi(B+\mathfrak{J}_0)=A$. Thus ψ is onto. Hence $\mathfrak{A}_0/\mathfrak{J}_0\cong M(\mathfrak{A}, G_0)$. This completes the proof of Theorem 4.

4. **Remarks.** (1) We shall remark first that every σ-finite type III

AW^*-factor is simple. Certain standard arguments tells us that for any pair e and f of non-zero projections in each σ-finite type III AW^*-factor M, $e \sim f$ in M. In fact, since "comparability theorem" of projections and "additivity of equivalence" of projections hold in any AW^*-algebra ([6]), we can easily show that for any non-zero projection e in M, there exists a mutually orthogonal sequence of projections $\{e_i\}_{i=1}^{\infty}$ in M such that $e = \sum_{i=1}^{\infty} e_i$, $e \sim e_i$ for all i. Let $\{f_j\}_{j\in J}$ be a maximal family of orthogonal projections such that $f_j \prec e$ for all j. Then the σ-finiteness of M implies that the cardinal of J is at most countable. The maximality of $\{f_j\}_{j\in J}$ tells us that $1 - \sum_{j\in J} f_j = 0$. Thus $1 = \sum_{j\in J} f_j \prec \sum_{i=1}^{\infty} e_i = e$ and $e \sim 1$ in M.

Now let I be any non-zero uniformly closed two-sided ideal of M, then by F. B. Wright's theorem [14], I contains a non-zero projection e. Thus, by the above argument, $e \sim 1$ and $1 \in I$, that is, $I = M$ and M is simple.

(2) We note also that every type I_∞ or type II_∞ AW^*-factor is not simple because the uniformly closed two-sided ideal generated by all finite projections in it is non-trivial.

Using this, the regular σ-completion $\hat{A}$ of a simple, infinite dimensional, separable unital C^*-algebra A is neither of type I_∞ nor of type II_∞ (because $\hat{A}$ is simple), that is, $\hat{A}$ is of type II_1 or of type III. Since $\hat{A}$ has a faithful state ([18, Theorem M]), [17, Theorem 6] tells us that $\hat{A}$ is of type III.

(3) Next we shall show that for any σ-finite, type III *non-*W^*, AW^*-factor M, M does not have any non-trivial *separable* representations. Suppose, on the contrary, that M has a non-trivial separable representation $(\pi, \mathfrak{H}_\pi)$ ($\mathfrak{H}_\pi$ is separable). Then we may assume without loss of generality that $\pi(1) = 1_{\mathfrak{H}_\pi}$ (the identity operator on $\mathfrak{H}_\pi$). Feldman and Fell [4] state that π is completely additive on projections and by the argument in (1) (M is simple), π is faithful. This implies that M has sufficiently many c.a. states. Thus M is a W^*-algebra by [9]. This is a contradiction and M has no non-trivial separable representations.

Thus the examples $M(\mathfrak{A}, G_\theta)$, $M(\mathfrak{A}, G_0)$ and $\hat{A}$ are simple and do not have any non-trivial separable representations.

We note that the above statements also hold for any σ-finite, properly infinite AW^*-algebra without any W^*-direct summands, but we will omit the details.

(4) We shall also remark that there is a monotone closed C^*-factor which is not a W^*-algebra ($M(\mathfrak{A}, G_\theta)$, $M(\mathfrak{A}, G_0)$, $\hat{A}$) see [5, Corollary 3.10].

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