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# A Wadge hierarchy for second countable spaces

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**Abstract** We define a notion of reducibility for subsets of a second countable $T_0$ topological space based on relatively continuous relations and admissible representations. This notion of reducibility induces a hierarchy that refines the Baire classes and the Hausdorff–Kuratowski classes of differences. It coincides with Wadge reducibility on zero dimensional spaces. However in virtually every second countable $T_0$ space, it yields a hierarchy on Borel sets, namely it is well founded and antichains are of length at most 2. It thus differs from the Wadge reducibility in many important cases, for example on the real line $\mathbb{R}$ or the Scott Domain $\mathcal{P}\omega$.

**Keywords** Wadge reducibility · Wadge hierarchy · Relatively continuous relation · Admissible representation

**Mathematics Subject Classification** 03E15 · 03D55 · 03F60

## 1 Introduction

The versatile concept of a topological space has proved valuable in various areas of mathematics. In many cases of interest, the spaces are second countable, i.e. their topology admits a countable basis. While separable metrisable spaces are of primary

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importance to Analysis [11], topological spaces that do not satisfy the Hausdorff separation property are central to Algebraic Geometry [6] and to Computer Science [7]. This paper considers without distinction all second countable spaces which satisfy the weakest separation property $T_0$, namely every two points which have exactly the same neighbourhoods are equal.

The very act of defining a topology on a set of objects consists in specifying simple, easily observable properties: the open sets. We are then interested in understanding the complexity of the other subsets relatively to the open sets. Already at the turn of the twentieth century, the French analysts—Baire, Borel and Lebesgue—stratified the Borel sets of a metric space into a transfinite hierarchy: the Baire classes $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ and $\boldsymbol{\Delta}^0_\alpha$. These classes are well-known to exhibit the following pattern:

$$\begin{array}{ccccccccccccc}
\boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & & & \boldsymbol{\Sigma}^0_\alpha & & \\
& \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq \;\cdots\; \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq \\
& & \boldsymbol{\Delta}^0_2 & & & & \boldsymbol{\Delta}^0_3 & \cdots & \boldsymbol{\Delta}^0_\alpha & & & & \boldsymbol{\Delta}^0_{\alpha+1} & \cdots \\
& \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq \;\cdots\; \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq \\
\boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & & & \boldsymbol{\Pi}^0_\alpha & &
\end{array}$$

Borel sets are thus classified according to the complexity of their definition from open sets along this transfinite ladder. This classification was further refined by Hausdorff, and later by Kuratowski, by identifying what is now called the difference hierarchies, consisting of the Haudorff-Kuratowski classes $D_\xi(\boldsymbol{\Sigma}^0_\alpha)$. Since for every map $f : X \to X$, the preimage function $f^{-1} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a complete Boolean homomorphism, it directly follows from their definition that the Baire classes and the Hausdorff–Kuratowski classes are *closed under continuous preimages*.[1]

Wadge in his Ph.D. thesis [25] was the first to investigate the quasiorder (qo) of *continuous reducibility* on the subsets of the Baire space $\omega^\omega$: for $A, B \subseteq \omega^\omega$ we say that $A$ is *Wadge reducible* to $B$, $A \leq_W B$, if and only if there exists a continuous $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(B) = A$. This quasiorder is remarkable. By considering suitable infinite games, called Wadge games, and using the determinacy of these games, which follows from Borel determinacy, this quasiorder turns out to be well founded and to admit antichains of size at most 2 on the Borel sets. As Andretta and Louveau [1] describe in introduction to [12]: "The Wadge hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity". While the Baire classes and the Hausdorff–Kuratowski classes are closed under continuous preimages, and therefore represent initial segments for $\leq_W$, there are in fact many more initial segments, so that the Wadge qo refines greatly these classical hierarchies. All these results for the Baire space easily apply to every Polish zero dimensional space.

However, when the space is not zero dimensional there may be very few continuous functions. Hertling [8] in his Ph.D. thesis showed that the qo of continuous reducibility of the Borel subsets of the real line exhibits a more complicated pattern than in the case of the Baire space. For example, Ikegami [9] showed in his Ph.D. thesis (see also [10]) that the powerset $\mathcal{P}(\omega)$ partially ordered by inclusion modulo finite (and hence any partial order of size $\aleph_1$) embeds in the qo of continuous reducibility of Borel sets of the real line (cf. Sect. 10). In a more general setting, Schlicht [20] showed that in any non zero dimensional metric space there is an antichain for the qo of continuous reducibility

[1] i.e. for every $A \subseteq X$ in the class and every continuous $f : X \to X$ the set $f^{-1}(A)$ belongs to the class.

of size continuum consisting of Borel sets. Selivanov ([21] and references there) and also Becher and Grigorieff [2] studied continuous reducibility in non Hausdorff spaces, where the situation is in general much less satisfactory than in the case of the Baire space.

In search for a useful notion of hierarchy outside Polish zero dimensional spaces, Motto Ros et al. [15] consider reductions by discontinuous functions. For example they obtain that the Borel subsets of the real line are well founded with antichains of size at most 2 when quasiordered by reducibility via functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $A \in \mathbf{\Sigma}^0_3(\mathbb{R})$ we have $f^{-1}(A) \in \mathbf{\Sigma}^0_3(\mathbb{R})$. They leave open the question whether $\mathbf{\Sigma}^0_3$ can be replaced by $\mathbf{\Sigma}^0_2$ in the above statement. Arguably one defect of this qo is that it does not refine the low level Baire classes, nor does it respect the Hausdorff hierarchy of the $\mathbf{\Delta}^0_2$.

Instead of considering reduction by discontinuous functions, we propose to keep continuity but release the second concept at stake, namely that of function.

Our approach is based on the simple and fundamental notion of *admissible representations* which is the starting point of the development of computable analysis from the point of view of Type-2 theory of effectivity [26]. The basic idea is to represent the points of a topological space $X$ by means of infinite sequences of natural numbers. Given such a representation of $X$, i.e. a partial surjective function $\rho :\subseteq \omega^\omega \to X$, an $\alpha \in \omega^\omega$ is a *name* for a point $x \in X$ when $\rho(\alpha) = x$. A function $f : X \to X$ is then said to be *relatively continuous* (resp. computable) with respect to $\rho$ if the function $f$ is continuous (or computable) in the $\rho$-names, i.e. there exists a continuous (resp. computable) $F : \operatorname{dom}\rho \to \operatorname{dom}\rho$ such that $f \circ \rho = \rho \circ F$. Of course the notion of relatively continuous function depends on the considered representation. However, for every second countable $T_0$ space $X$ there exists—up to equivalence—a greatest representation (see Theorem 1) among the continuous ones, called an *admissible representation* of $X$. The importance of admissible representations resides in the following fact (see Theorem 2): for an admissible representation $\rho$ of $X$, a function $f : X \to X$ is relatively continuous with respect to $\rho$ if and only if $f$ is continuous. Notice however that as long as the representation is *not* injective, there are in general many continuous transformations of the names which do not induce a map on the space $X$. Indeed different names $\alpha, \beta$ of some point $x$ can be sent by a continuous function $F$ onto names $F(\alpha)$, $F(\beta)$ representing different points, i.e. $\rho(F(\alpha)) \neq \rho(F(\beta))$. Such transformations are called *relatively continuous relations* (see Definition 3) and they were first investigated in a systematic manner by Brattka and Hertling [4].

We propose to consider reducibility by total relatively continuous relations. In Sect. 2, we observe that total relations account perfectly for the idea of reducibility in the abstract and in fact generalise the framework of reductions as functions. However when we fix an admissible representation $\rho$ of a second countable $T_0$ space $X$, it is natural to think of reductions by relatively continuous relations as "reductions in the names": if $A, B \subseteq X$, then $A$ reduces to $B$, in symbols $A \preccurlyeq_W B$, if and only if there exists a continuous function $F$ from the names to the names such that for every name $\alpha$, $\rho(\alpha) \in A \leftrightarrow \rho(F(\alpha)) \in B$. In other words, for every point $x$ and every name $\alpha$ for $x$, $F(\alpha)$ is the name of a point that belongs to $B$ if and only if $x$ belongs to $A$.

Tang [23] works with an admissible representation of the Scott domain $\mathcal{P}\omega$ and studies exactly the notion of reduction that we propose here in a more general setting.

But first, this study is antecedent to the introduction by Kreitz and Weihrauch [13] of admissible representations and Tang does not notice that his representation of $\mathcal{P}\omega$ is admissible, and thus canonical in a sense. Second, even though his paper is often cited, we have not found any other reference to his particular approach to reducibility on $\mathcal{P}\omega$.

Importantly, reducibility by relatively continuous relations coincides with the continuous reducibility on zero dimensional spaces. It can therefore be viewed as a generalisation of Wadge reducibility outside zero dimensional spaces. Notice however that it differs notably from the continuous reducibility in every separable metrisable space that is not zero dimensional (see Corollary 2, Sects. 10 and 11).

Moreover, it follows from a result by Saint Raymond [19] extended by de Brecht [5] that in every second countable $T_0$ space $X$, the Baire classes and the Kuratowski–Hausdorff classes are initial segments for $\preccurlyeq_W$. And therefore reducibility by relatively continuous relations refines these classical hierarchies.

Finally, using a variant of the Wadge game, it follows from Borel determinacy, by the same methods as in the case of the Baire space, that the qo $\preccurlyeq_W$ is well founded and satisfies the Wadge duality principle (in particular antichains are of size at most 2) on the Borel sets of any *Borel representable space*. Here a Borel representable space is simply a second countable $T_0$ space for which there exists an admissible representation whose domain is Borel in $\omega^\omega$. As in the case of the Baire space, this structural result depends on the determinacy of the games under consideration. In particular, under the Axiom of Determinacy, it extends to all subsets of every second countable $T_0$ space.

## 2 Reductions as total relations

The concept of reduction is used in several different fields, such as complexity theory, automata theory and descriptive set theory. While particular definitions relies on different concepts, they all share a general idea. If $X, Y$ are sets, $A \subseteq X$ and $B \subseteq Y$, a function $f : X \to Y$ is called a *reduction of $A$ to $B$* if $f^{-1}(B) = A$ or equivalently if

$$\forall x \in X\, (x \in A \leftrightarrow f(x) \in B)\,.$$

Let $\mathcal{F}$ be a class of functions from $X$ to $X$ that contains the identity on $X$ and that is closed under composition. For $A, B \subseteq X$ we say that $A$ is *reducible* to $B$ with respect to $\mathcal{F}$ if there exists $f \in \mathcal{F}$ such that $f$ is a reduction of $A$ to $B$. This defines a quasiorder, i.e. a reflexive and transitive relation, on the powerset of $X$.

We now observe that as far as reducibility is concerned reductions do not need to be functions. In fact one may as well consider *total relations* in place of functions.

We say $R \subseteq X \times Y$ is a *(total) relation from $X$ to $Y$*, in symbols $R : X \rightrightarrows Y$, if for all $x \in X$ there exists $y \in Y$ with $(x, y) \in R$. We also write $R(x, y)$ in place of $(x, y) \in R$. If $A \subseteq X$ and $B \subseteq Y$ we say that a *reduction of $A$ to $B$* is a total relation $R : X \rightrightarrows Y$ such that

$$\forall x \in X\, \forall y \in Y \left[ R(x, y) \to (x \in A \leftrightarrow y \in B) \right]. \tag{1}$$

One can also view a relation $R \subseteq X \times Y$ as the function

$$\begin{aligned} R^{\to} : X &\longrightarrow \mathcal{P}(Y) \\ x &\longmapsto R^{\to}(x) = \{y \in Y \mid R(x, y)\}. \end{aligned}$$

From this point of view, $R$ is total from $X$ to $Y$ if and only if $R^{\to}(x) \neq \emptyset$ for all $x \in X$, and (1) can be stated as follows:

$$\forall x \in X \left[ \left(x \in A \wedge R^{\to}(x) \subseteq B\right) \vee \left(x \in A^{\complement} \wedge R^{\to}(x) \subseteq B^{\complement}\right) \right]. \tag{2}$$

Of course, for every function $f : X \to Y$, $f$ is a reduction of $A$ to $B$ if and only if its graph $\{(x, f(x)) \mid x \in X\}$, as a total relation from $X$ to $Y$, is a reduction of $A$ to $B$. So our notion of reduction as total relations subsumes the notion of reduction as functions.

Observe also that it follows directly from (2) that a total relation $R$ is a reduction of $A$ to $B$ if and only if it is a reduction from $A^{\complement}$ to $B^{\complement}$.

Two total relations $R : X \rightrightarrows Y$ and $S : Y \rightrightarrows Z$ compose to yield the total relation $S \circ R : X \rightrightarrows Z$ in the expected way

$$S \circ R = \left\{(x, z) \in X \times Z \,\middle|\, \exists y \in Y\ R(x, y) \wedge S(y, z)\right\}.$$

**Fact 1** *If $A \subseteq X$, $B \subseteq Y$, $C \subseteq Z$, $R : X \rightrightarrows Y$ is a reduction of $A$ to $B$ and $S : Y \rightrightarrows Z$ is a reduction of $B$ to $C$, then $S \circ R : X \rightrightarrows Z$ is a reduction of $A$ to $C$.*

Let $\mathcal{R}$ be a class of total relations from $X$ to $X$ that contains the diagonal $\{(x, x) \mid x \in X\}$ and that is closed under composition. For $A, B \subseteq X$ we say that $A$ is reducible to $B$ with respect to $\mathcal{R}$ if there $R \in \mathcal{R}$ such that $R$ is a reduction of $A$ to $B$. Again this defines a quasiorder on the powerset of $X$ that we call $\mathcal{R}$-reducibility.

The following fact follows immediately from (1).

**Fact 2** *Let $R, S : X \rightrightarrows Y$, $A \subseteq X$, $B \subseteq Y$. If $R \subseteq S$ and $S$ is a reduction of $A$ to $B$, then $R$ is also a reduction of $A$ to $B$.*

Consequently, for a class $\mathcal{R}$ as above, if we consider the the upward closure of $\mathcal{R}$ defined by

$$\overline{\mathcal{R}} = \{S : X \rightrightarrows X \mid \exists R \in \mathcal{R}\ R \subseteq S\},$$

then the $\mathcal{R}$-reducibility equals the $\overline{\mathcal{R}}$-reducibility. Therefore as far as reducibility is concerned, we gain generality by considering classes of total relations instead of classes of functions, and we can always consider classes of total relations that are upward closed.

## 3 Admissible representations

In this section we briefly review the notion of admissible representation of a topological space. This notion is fundamental to the approach to computable analysis known as Type-2 Theory of Effectivity (see [26]).

A topological space $X$ is called *second countable* if it admits a countable basis of open sets. It satisfies the separation axiom $T_0$ if every two distinct points are topologically distinguishable, i.e. for any two distinct points $x$ and $y$ there is an open set which contains one of these points and not the other. It is called 0-*dimensional* if it admits a basis of clopen sets, i.e. of simultaneously open and closed sets. The Baire space is denoted by $\omega^\omega$. Recall that a space is second countable and 0-dimensional if and only if it is homeomorphic to a subset of $\omega^\omega$. A Polish space is a second countable completely metrisable topological space, the Baire space is a crucial example of Polish space. Recall [11, (3.11), p.17] that a subspace of a Polish space is Polish if and only if it is $\boldsymbol{\Pi}^0_2$, i.e. a countable intersection of open sets.

Let $X, Y$ be second countable $T_0$ spaces. If $A \subseteq X$ and $f : A \to Y$ is a function, $f$ is called a *partial function* from $X$ to $Y$, in symbols $f :\subseteq X \to Y$, and we refer to $A$ as the domain of $f$, denoted by $\operatorname{dom} f$. A partial function $f :\subseteq X \to Y$ is continuous if it is continuous on its domain for the subspace topology on $\operatorname{dom} f$, i.e. if for every open $U$ of $Y$ there is an open $V$ of $X$ such that $f^{-1}(U) = V \cap \operatorname{dom} f$.

We quasiorder the partial functions from $\omega^\omega$ into $X$ by saying that for $f, g :\subseteq \omega^\omega \to X$

$$f \leq_C g \longleftrightarrow \begin{cases} \text{there exists a continuous } h : \operatorname{dom} f \to \operatorname{dom} g \\ \text{with } g \circ h(\alpha) = f(\alpha) \text{ for all } \alpha \in \operatorname{dom} f. \end{cases}$$

Clearly, if $g$ is continuous and $f \leq_C g$, then $f$ is continuous too. Hence the set of partial continuous functions from $\omega^\omega$ into $X$ is downward closed with respect to $\leq_C$.

**Definition 1** [26] Let $X$ be second countable $T_0$. A partial continuous function $\rho :\subseteq \omega^\omega \to X$ is called an *admissible representation* of $X$ if it is a $\leq_C$-greatest element among partial continuous functions to $X$, i.e. $f \leq_C \rho$ holds for every partial continuous $f :\subseteq \omega^\omega \to X$.

Observe that an admissible representation $\rho$ of $X$ is necessarily onto $X$, since for every point $x \in X$, we have $c_x \leq_C \rho$ where $c_x : \omega^\omega \to X, \alpha \mapsto x$ is the constant function.

*Remark 1* Since the subspaces of $\omega^\omega$ are up to homeomorphism the second countable 0-dimensional spaces, an admissible representation of $X$ is also a continuous map $\rho : D \to X$ from some second countable 0-dimensional space $D$ such that for every continuous map $g : E \to X$ from a second countable 0-dimensional space $E$ there exists a continuous map $h : E \to D$ such that $\rho \circ h = g$.

It is well known that every second countable $T_0$ space $X$ has an admissible representation. As this is crucial for the sequel, we now explain this simple fact. So let $X$ be a second countable $T_0$ space and $(V_n)_{n\in\omega}$ be a countable basis of open sets for $X$.

We define the *standard representation* of $X$ with respect to $(V_n)$ to be the partial map $\rho :\subseteq \omega^\omega \to X$ defined by

$$\rho(\alpha) = x \longleftrightarrow \operatorname{Im}\alpha = \{n \mid \exists k\ \alpha(k) = n\} = \{n \mid x \in V_n\},$$

for $\alpha : \omega \to \omega$ and $x \in X$. Note that $\rho$ is indeed a function on its domain because $X$ is $T_0$. An $\alpha \in \omega^\omega$ codes via $\rho$ a point $x \in X$ if and only if $\alpha$ enumerates the indices of all the $V_n$'s to which $x$ belongs.

**Theorem 1** *For every second countable $T_0$ space $X$ there exists an admissible representation $\rho :\subseteq \omega^\omega \to X$. Moreover it can be chosen such that*

1. *$\rho$ is open,*
2. *for every $x \in X$, the fibre $\rho^{-1}(x)$ is Polish.*

*Proof* Let $(V_n)$ be a countable basis for $X$ and let $\rho :\subseteq \omega^\omega \to X$ be the standard representation of $X$ with respect to $(V_n)$. It is enough to show that $\rho$ satisfies all the requirements.

Continuity: Note that $\rho^{-1}(V_n) = \{\alpha \in \operatorname{dom}\rho \mid \exists k\ \alpha(k) = n\}$ is open in $\omega^\omega$ for every $n$, so $\rho$ is continuous.

Openness: For every basic $N_s = \{x \in \omega^\omega \mid s \subseteq x\}$, $s \in \omega^{<\omega}$, we have $\rho(N_s) = \bigcap_{k<|s|} V_{s_k}$ which is open in $X$, so $\rho$ is an open map.

Polish fibres: For every point $x \in X$

$$\rho(\alpha) = x \longleftrightarrow \forall n\big[(\exists k\ \alpha(k) = n) \leftrightarrow x \in V_n\big]$$

is a $\boldsymbol{\Pi}^0_2$ definition of the fibre in $x$, so $\rho$ has Polish fibres.

Admissibility: Let $f :\subseteq \omega^\omega \to X$ be continuous and fix some enumeration $\pi : \omega \to \omega$ of $\omega$ where each natural number appears infinitely often. Let us consider the monotone map $h^* : \omega^{<\omega} \to \omega^{<\omega}$ defined by induction on the length $|s|$ of $s \in \omega^{<\omega}$ by:

$$h^*(\emptyset) = \emptyset$$

$$h^*(s^\frown m) = \begin{cases} h^*(s)^\frown \pi(|s|) & \text{if } f\left(N_{s^\frown m}\right) \subseteq V_{\pi(|s|)} \\ h^*(s) & \text{otherwise,} \end{cases}$$

where, for $s \in \omega^{<\omega}$, $f(N_s)$ denotes the set $\{f(\alpha) \mid \alpha \in \operatorname{dom} f \text{ and } s \subseteq \alpha\}$. We let $h : \operatorname{dom} f \to \operatorname{dom}\rho$ be the continuous function defined by $h(\alpha) = \bigcup_{l\in\omega} h^*(\alpha\restriction_l)$ (see [11, (2.6), p.8]). We claim that $h$ is a witness for the fact that $f \le_C \rho$, namely that for every $\alpha \in \operatorname{dom} f$ and every $n \in \omega$ we have $n \in \operatorname{Im} h(\alpha)$ if and only if $f(\alpha) \in V_n$.
Let $\alpha \in \operatorname{dom} f$ and $n \in \omega$ and assume that $f(\alpha) \in V_n$. Then by continuity of $f$ there exists a $k \in \omega$ such that $f(N_{\alpha\restriction_{k+1}}) \subseteq V_n$ and $\pi(k) = n$. It follows that $n$ belongs to the image of $h^*(\alpha\restriction_{k+1})$ and therefore to the image of $h(\alpha)$.

Conversely, let $\alpha \in \operatorname{dom} f$ and assume that $n$ belongs to the image of $h(\alpha)$. Then for $l$ minimal such that $n$ belongs to the image of $h^*(\alpha\restriction_{l+1})$, and by definition of $h^*$ this means that we have $f(N_{\alpha\restriction_{l+1}}) \subseteq V_{\pi(l)}$ and $\pi(l) = n$. Therefore $f(\alpha) \in V_n$ and this concludes the proof. □

Importantly, Brattka [3, Corollary 4.4.12] showed that every Polish space $X$ has a *total admissible representation*, i.e. an admissible representation $\rho :\subseteq \omega^\omega \to X$ with $\operatorname{dom} \rho = \omega^\omega$. As an easy consequence one gets that for every second countable $T_0$ space $X$: there exists an admissible representation of $X$ with a Polish domain if and only if there exists a total admissible representation of $X$. Motivated by the rich theory of Polish spaces, it is natural to consider the class of those second countable $T_0$ spaces which have a total admissible representation. As a matter of fact de Brecht [5] showed that this class coincides with the class of *quasi-Polish* spaces that he recently introduced. Moreover he showed that many classical results of descriptive set theory can be generalised to this large class of non necessarily Hausdorff spaces and that the metrisable quasi-Polish spaces are exactly the Polish spaces.

The real line $\mathbb{R}$ will serve as an example along the paper and we now introduce two different admissible representations for it.

*Example 1* Let $(q_n)_{n\in\omega}$ be an enumeration of the rationals and let $I_n = (q_{n_0}, q_{n_1})$ be an enumeration of the non empty intervals of the real line $\mathbb{R}$ with rational endpoints.

We define $\rho_{\mathbb{R}} :\subseteq \omega^\omega \to \mathbb{R}$ as the standard representation relatively to the enumerated basis $(I_n)_{n\in\omega}$, i.e.

$$\rho_{\mathbb{R}}(\alpha) = x \longleftrightarrow \operatorname{Im} \alpha = \{n \mid x \in I_n\},$$

so that $\alpha \in \omega^\omega$ codes $x \in \mathbb{R}$ if and only if $\alpha$ enumerates all the intervals with rational endpoints to which $x$ belongs.

The second admissible representation is based on Cauchy sequences and it works mutatis mutandis for every separable complete metric space.

*Example 2* Let $(q_n)_{n\in\omega}$ be an enumeration of the rationals, and let $d$ be the euclidean metric on $\mathbb{R}$. A sequence $(x_k)_{k\in\omega}$ is said to be *rapidly Cauchy* if for every $i, j \in \omega$, $i < j$ implies $d(x_i, x_j) \leq 2^{-i}$. The Cauchy representation $\sigma_{\mathbb{R}} :\subseteq \omega^\omega \to \mathbb{R}$ of the real line is defined by

$$\sigma_{\mathbb{R}}(\alpha) = x \longleftrightarrow \left(q_{\alpha(k)}\right)_{k\in\omega} \text{ is rapidly Cauchy and } \lim_{k\to\infty} q_{\alpha(k)} = x.$$

This is an admissible representation of $\mathbb{R}$.

As an example of a non metrisable space we consider the Scott Domain $\mathcal{P}\omega$, namely the powerset of $\omega$ partially ordered by inclusion and endowed with the Scott topology. A basis of $\mathcal{P}\omega$ is given by sets of the form $O_F = \{X \subseteq \omega \mid F \subseteq X\}$ for some finite $F \subseteq \omega$. This space is universal for the second countable $T_0$ spaces. Indeed for every such space $X$ with some basis $(V_n)_{n\in\omega}$ the map $e : X \to \mathcal{P}\omega, x \mapsto \{n \mid x \in V_n\}$ is an embedding.

*Example 3* The enumeration representation of $\mathcal{P}\omega$ is the total function $\rho_{\mathrm{En}} : \omega^\omega \to \mathcal{P}\omega$ defined by

$$\rho_{\mathrm{En}}(\alpha) = \{n \mid \exists k\ \alpha(k) = n + 1\}.$$

It is easy to see that $\rho_{\mathrm{En}}$ is an open admissible representation with Polish fibres.

As another example of an admissible representation of $\mathcal{P}\omega$ consider:

*Example 4* Let $(s_n)_{n\in\omega}$ be an enumeration of the finite subsets of $\omega$. We define $\rho_{<\infty} :\subseteq \omega^\omega \to \mathcal{P}\omega$ by

$$\rho_{<\infty}(\alpha) = x \longleftrightarrow \forall n \in \omega\ s_{\alpha(n)} \subseteq s_{\alpha(n+1)} \quad \text{and} \quad \bigcup_{n\in\omega} s_{\alpha(n)} = x.$$

The domain of $\rho_{<\infty}$ is closed and $\rho_{<\infty}$ is clearly continuous. The map $\rho_{<\infty}$ is also an admissible representation of the space $\mathcal{P}\omega$ since it is continuous and $\rho_{\mathrm{En}} \leq_C \rho_{<\infty}$, as witnessed by the continuous $f : \omega^\omega \to \mathrm{dom}\,\rho_{<\infty}$ defined by

$$f(\alpha)(n) = k, \quad \text{where } s_k = \{m \mid \exists j \leq n\ \alpha(j) = m + 1\}.$$

## 4 Relative continuity

The importance of admissible representations stems from the fact that continuity of a function between second countable $T_0$ spaces can be accounted for "in the codes".

**Definition 2** Let $X, Y$ be second countable $T_0$ spaces. We say that a total function $f : X \to Y$ is *relatively continuous* if for some (any) admissible representations $\rho_X$ and $\rho_Y$ of $X$ and $Y$ respectively, there exists a continuous $g : \mathrm{dom}\,\rho_X \to \mathrm{dom}\,\rho_Y$, called a *continuous realiser of f*, such that $f \circ \rho_X(\alpha) = \rho_Y \circ g(\alpha)$ for ever $\alpha \in \mathrm{dom}\,\rho_X$.

Using the maximality property of admissible representations, it is easy to see that a function $f : X \to Y$ admits a continuous realiser for some choice of admissible representations of $X$ and $Y$ if and only if it admits a continuous realiser for any choice of admissible representations.

**Theorem 2** *Let $X, Y$ be second countable $T_0$ spaces. A* total *function $f : X \to Y$ is relatively continuous if and only if $f$ is continuous.*

*Proof* Let $\rho_X$ and $\rho_Y$ be open admissible representations of $X$ and $Y$ respectively.

If $f : X \to Y$ is continuous, then $f \circ \rho_X : \mathrm{dom}\,\rho_X \to Y$ is continuous. Since $\rho_Y$ is admissible, there exists a continuous $g : \mathrm{dom}\,\rho_X \to \mathrm{dom}\,\rho_Y$ ($\mathrm{dom}\, f \circ \rho_X = \mathrm{dom}\,\rho_X$) with $f \circ \rho_X = \rho_Y \circ g$ on the domain of $\rho_X$, so $f$ is relatively continuous.

Conversely, if $f : X \to Y$ is relatively continuous there exists a partial continuous $g : \mathrm{dom}\,\rho_X \to \mathrm{dom}\,\rho_Y$ with $f \circ \rho_X = \rho_Y \circ g$ on $\mathrm{dom}\, f \circ \rho_X = \mathrm{dom}\,\rho_X$. Therefore $f \circ \rho_X : \mathrm{dom}\,\rho_X \to Y$ is continuous. So the proof will be finished once we have proved the following fact: if $g :\subseteq X \to Y$ is continuous, surjective and open map,

$f : Y \to Z$ is any function and $f \circ g :\subseteq X \to Z$ is continuous, then $f$ is continuous. To see this, let $U$ be open in $Z$. Then

$$\begin{aligned} f^{-1}(U) &= \{g(x) \mid x \in \operatorname{dom} g \wedge g(x) \in f^{-1}(U)\} && \text{since } g \text{ is onto,} \\ &= g\big((f \circ g)^{-1}(U)\big) \end{aligned}$$

is open in $Y$ since $g$ is an open map and $f \circ g$ is continuous. □

## 5 Injective admissible representations and dimension

For an admissible representation $\rho :\subseteq \omega^\omega \to X$ and a point $x \in X$, one can think of $\alpha \in \omega^\omega$ with $\rho(\alpha) = x$ as a "code" or "name" for $x$. It is natural to ask what are the spaces which possess an injective admissible representation. It is actually simple to see that these spaces are exactly those of dimension 0. We now show this fact.

Recall the following fact on the cardinality of a basis.

**Lemma 1** *Let $X$ be second countable. For every basis $\mathcal{C}$, there is a countable basis $\mathcal{C}' \subseteq \mathcal{C}$.*

*Proof* Let $(V_n)$ be countable basis for $X$. Whenever possible choose $C_{n,m} \in \mathcal{C}$ with $V_n \subseteq C_{n,m} \subseteq V_m$. Then the countable family of the $C_{n,m}$'s is a basis for $X$. Indeed for every $x \in V_m$ there is a $C \in \mathcal{C}$ with $x \in C \subseteq V_m$ (since $\mathcal{C}$ is a basis), and furthermore there exists $n$ with $x \in V_n \subseteq C \subseteq V_m$ (since $(V_n)_{n\in\omega}$ is a basis), hence $x \in C_{n,m} \subseteq V_m$. □

**Lemma 2** *Let $X$ be a second countable $T_0$ space and $\sigma :\subseteq \omega^\omega \to X$ be an admissible representation of $X$. Then there is $A \subseteq \operatorname{dom} \sigma$ such that $\sigma\restriction_A$ is an open admissible representation of $X$.*

*Proof* Let $\rho :\subseteq \omega^\omega \to X$ be an open admissible representation of $X$ which exists by Theorem 1. There exists a continuous $g : \operatorname{dom} \rho \to \operatorname{dom} \sigma$ that witnesses $\rho \leq_C \sigma$. We claim that $A = \{g(\alpha) \mid \alpha \in \operatorname{dom} \rho\}$ works. Indeed $\rho \leq_C \sigma\restriction_A$ as $g$ also witnesses, and for every open $O \subseteq \omega^\omega$ we have

$$\sigma\restriction_A(O) = \{\sigma \circ g(\alpha) \mid \alpha \in \operatorname{dom} \rho\} = \rho(O),$$

which concludes the proof. □

**Proposition 1** *Let $X$ be a second countable $T_0$ space. The following are equivalent:*

1. *$X$ is 0-dimensional,*
2. *there exists an injective admissible representation of $X$.*

*Proof*

1→2: By Lemma 1, $X$ admits a countable basis $(V_n)$ consisting in clopen subsets of $X$, and for simplicity we may assume further that the basis is closed under complements, i.e. for every $n$ there exists $m$ with $X \setminus V_n = V_m$.

Let $\sigma :\subseteq \omega^\omega \to X$ be the partial map defined by $\sigma(\alpha) = x$ if and only if $\alpha : \omega \to 2$ is the characteristic function of $\{n \in \omega \mid x \in V_n\}$. Clearly $\sigma$ is injective and continuous. To see that $\sigma$ is admissible, it is enough to show that $\rho \leq_C \sigma$ where $\rho$ is the standard representation of $X$ with respect to $(V_n)$. This is witnessed by the continuous function $g : \operatorname{dom}\rho \to \operatorname{dom}\sigma$ defined by

$$g(\alpha)(n) = \begin{cases} 1 & \text{if there exists } k \text{ with } \alpha(k) = n+1, \\ 0 & \text{if there exists } k \text{ with } \alpha(k) = m+1 \text{ and } V_m = X \setminus V_n. \end{cases}$$

2→1: If $\rho$ is an injective admissible representation of $X$, then by Lemma 2 there exists $A \subseteq \operatorname{dom}\rho$ such that $\rho\restriction_A$ is open and admissible. But since an admissible representation is surjective and $\rho$ is injective, we must have $A = \operatorname{dom}\rho$. Therefore $\rho$ is an homeomorphism, and so $X$ is homeomorphic to $\operatorname{dom}\rho$, hence $X$ is 0-dimensional. □

## 6 Relatively continuous relations

We have seen that a function $f : X \to Y$ between second countable $T_0$ spaces is continuous if and only if it is induced by some continuous function "in the codes". Moreover we have seen that when $X$ is not 0-dimensional, then no admissible representation of $X$ is injective, and so necessarily some points are to receive several codes. Since different codes of the same point can be sent onto codes of different points, a continuous function in the codes may very well induce a relation which is not functional on the spaces. Even though the resulting "transformations" of the space are not necessarily functional, they are still continuous in a sense. They are called *relatively continuous relations*, and were first studied in [4].

**Definition 3** Let $X, Y$ be second countable $T_0$ spaces. A total relation $R : X \rightrightarrows Y$ is said to be *relatively continuous* if, for some (any) admissible representations $\rho_X$ and $\rho_Y$ of $X$ and $Y$ respectively, there exists a *continuous realiser* $f : \operatorname{dom}\rho_X \to \operatorname{dom}\rho_Y$ such that for every $\alpha \in \operatorname{dom}\rho_X$ we have

$$\big(\rho_X(\alpha), \rho_Y \circ f(\alpha)\big) \in R.$$

*Remark 2* Suppose $R : X \rightrightarrows Y$ is relatively continuous with respect to $\rho_X$ and $\rho_Y$ as witnessed by some continuous $f : \operatorname{dom}\rho_X \to \rho_Y$ and let $\sigma_X, \sigma_Y$ be admissible representations of $X$ and $Y$ respectively. Since $\sigma_X \leq_C \rho_X$ and $\rho_Y \leq_C \sigma_Y$ there are continuous $g : \operatorname{dom}\sigma_X \to \operatorname{dom}\rho_X$ and $h : \operatorname{dom}\rho_Y \to \operatorname{dom}\sigma_Y$ with $\rho_X \circ g = \sigma_X$ and $\sigma_Y \circ h = \rho_Y$. Therefore if we set $f' : \operatorname{dom}\sigma_X \to \operatorname{dom}\sigma_Y$ to be $f' = h \circ f \circ g$ we obtain that for every $\alpha \in \operatorname{dom}\sigma_X$

$$\sigma_Y \circ f'(\alpha) = \sigma_Y \circ h \circ f \circ g(\alpha) = \rho_Y \circ f \circ g(\alpha).$$

Now since $\sigma_X(\alpha) = \rho_X \circ g(\alpha)$ if we let $\beta = g(\alpha)$ we have

$$\big(\sigma_X(\alpha), \sigma_Y \circ f'(\alpha)\big) = \big(\rho_X(\beta), \rho_Y \circ f(\beta)\big) \in R,$$

so $R$ is relatively continuous with respect to $\sigma_X$ and $\sigma_Y$.

Clearly a function $f : X \to Y$ is (relatively) continuous if and only if its graph is relatively continuous as a total relation from $X$ to $Y$. Moreover it is easily seen that the class of relatively continuous total relations is closed under composition.

Observe also that the definition directly implies that if $R : X \rightrightarrows Y$ is relatively continuous and $R \subseteq S : X \rightrightarrows Y$, then $S$ is relatively continuous too.

Let $X, Y$ be second countable $T_0$ spaces together with admissible representations $\rho_X, \rho_Y$. Every continuous function $f : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ induces a total relation $R_f^{\rho_X,\rho_Y} : X \rightrightarrows Y$ defined by

$$x \; R_f^{\rho_X,\rho_Y} \; y \longleftrightarrow \exists \alpha \in \operatorname{dom} \rho_X \; (\rho_X(\alpha) = x \wedge \rho_Y \circ f(\alpha) = y).$$

The function $f$ witnesses that $R_f^{\rho_X,\rho_Y}$ is relatively continuous. In fact, $f$ witnesses that some $R : X \rightrightarrows Y$ is relatively continuous if and only if $R_f^{\rho_X,\rho_Y} \subseteq R$. Therefore we have the following.

**Fact 3** *Let $X, Y$ be second countable $T_0$ and $\rho_X, \rho_Y$ be admissible representations of $X$ and $Y$ respectively. A total relation $R : X \rightrightarrows Y$ is relatively continuous if and only if there exists a continuous $f : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ such that $R_f^{\rho_X,\rho_Y} \subseteq R$.*

From Proposition 1 and the previous fact, it follows that the relatively continuous relations from a 0-dimensional spaces are simply the continuously uniformisable relations.

**Corollary 1** *Let $X, Y$ be second countable $T_0$ with $X$ zero dimensional. A total relation $R$ from $X$ to $Y$ is relatively continuous if and only if it admits a continuous uniformising function, i.e. there exists a continuous $f : X \to Y$ with $R(x, f(x))$ for all $x \in X$.*

It is an interesting problem to look for an intrinsic characterisation of the relatively continuous total relations, that is, one which does not rely on the notion of admissible representation. Partial answers were obtained in [4,18]. However, to our knowledge, the general problem is still open. We conclude this section with some known results in the direction.

Let us say that $R : X \rightrightarrows Y$ *preserves open sets* if $R^{-1}(O) = \{x \in X \mid \exists y \in O \; R(x, y)\}$ is open in $X$ for ever open set $O$ of $Y$.

**Proposition 2** [4, Proposition 4.5] *Let $X, Y$ be second countable $T_0$ spaces. There exists a class $\mathcal{R}$ of total relations which preserves open sets such that for every $S : X \rightrightarrows Y$*

$$S \text{ is relatively continuous} \longleftrightarrow \exists R \in \mathcal{R} \; R \subseteq S.$$

*Proof* Let $\rho_X, \rho_Y$ be admissible representations of $X, Y$ with $\rho_X$ an open map. Let $\mathcal{R}$ be the family of total relations $R_f^{\rho_X,\rho_Y}$ where $f : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ is continuous. By Fact 3, it only remains to prove that $R_f^{\rho_X,\rho_Y}$ preserves open sets for every continuous $f$.

Indeed for every continuous $f : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ and every open $O$ of $Y$

$$(R_f^{\rho_X,\rho_Y})^{-1}(O) = \{\rho_X(x) \mid x \in (\rho_Y \circ f)^{-1}(O)\} = \rho_X[(\rho_Y \circ f)^{-1}(O)],$$

which is open since $\rho_Y \circ f$ is continuous and $\rho_X$ is open. □

Moreover, in the case of a Polish codomain, Brattka and Hertling [4] showed the following.

**Theorem 3** *Let $X$ be second countable $T_0$, $Y$ be Polish, and $R : X \rightrightarrows Y$ be such that $R^{\to}(x)$ is closed for every $x \in X$. Then $R$ is relatively continuous if and only if there exists $S : X \rightrightarrows Y$ that preserves open sets and such that $S \subseteq R$.*

One should notice that preserving open sets is not a sufficient condition for the relative continuity of a total relation. Consider for example the partition of $\omega^\omega$ into

$$F = \{\alpha \in \omega^\omega \mid \exists n \, \forall k \geq n \; \alpha(k) = 0\} \quad \text{and} \quad F = \omega^\omega \setminus G.$$

Clearly $G$ and $F$ are both dense in $\omega^\omega$. Moreover it is well known that $F \in \mathbf{\Sigma}_2^0 \setminus \mathbf{\Pi}_2^0$. Consider the total relation $R = (G \times F) \cup (F \times G)$. Then $R^{-1}(O) = \omega^\omega$ for every non empty open set $O$, but $R$ not relatively continuous. Indeed any function $f : \omega^\omega \to \omega^\omega$ which uniformises $R$ needs to verify $f^{-1}(G) = F$, and since $F$ is not $\mathbf{\Pi}_2^0$, $f$ cannot be continuous.

## 7 Reduction by relatively continuous relations

We recall the classical (see e.g. [11, (21.13), p. 156])

**Definition 4** If $X, Y$ are topological spaces, $A \subseteq X$ and $B \subseteq Y$, we say that $A$ is *Wadge reducible* to $B$, in symbols $A \leq_W B$, if there exists a continuous function $f : X \to Y$ that is a reduction of $A$ to $B$.

We propose to make the following definition.

**Definition 5** If $X, Y$ are second countable $T_0$ spaces, $A \subseteq X$ and $B \subseteq Y$, we say that $A$ is *reducible* to $B$, in symbols $A \preccurlyeq_W B$, if there exists a total relatively continuous $R : X \rightrightarrows Y$ that is a reduction of $A$ to $B$.

Notice that strictly speaking we should write $(A, X) \preccurlyeq_W (B, Y)$ in place of $A \preccurlyeq_W B$, but the spaces are usually understood.

Since the class of relatively continuous relations between second countable $T_0$ spaces contains the identity functions (in fact all relatively continuous functions) and is closed under composition, $\preccurlyeq_W$ is a quasi-order.

For second countable $T_0$ spaces $X, Y$, $(A, X) \leq_W (B, Y)$ implies $(A, X) \preccurlyeq_W (B, Y)$, by Theorem 2. However, the qo $\preccurlyeq_W$ should not be confused with the Wadge qo $\leq_W$.

By, when we fix admissible representations, then reducibility by relatively continuous relations simply amounts to continuous reducibility in the codes.

**Lemma 3** *Let $X, Y$ be second countable $T_0$ spaces with fixed admissible representations $\rho_X, \rho_Y$. For every $A \subseteq X$ and $B \subseteq Y$ the following are equivalent*

1. $A \preccurlyeq_W B$,
2. $(\rho_X^{-1}(A), \operatorname{dom} \rho_X) \leq_W (\rho_Y^{-1}(B), \operatorname{dom} \rho_Y)$.

**Proposition 3** *Let $X$ be a separable 0-dimensional metrisable space. Then $\leq_W$ and $\preccurlyeq_W$ coincide on subsets of $X$.*

*Proof* It directly follows from Fact 2 and Corollary 1. □

## 8 Baire classes and Hausdorff–Kuratowski classes

The classical approach initiated by the French analysts Baire, Borel, and Lebesgue to the classification of the subsets of a metric space is more descriptive in nature. Sets are classified according to the complexity of their definition from open sets. This approach was continued later by Luzin, Suslin, Hausdorff, Sierpiński and Kuratowski.

As observed in [22,23], the classical definition of Baire classes in metric spaces is not satisfactory for non metrisable spaces. Following [5,21] we use the following slightly modified definition of Baire classes in an arbitrary topological space.

**Definition 6** Let $X$ be a topological space. For each positive ordinal $\alpha < \omega_1$ we define by induction

$$\begin{aligned}
\boldsymbol{\Sigma}^0_1(X) &= \{O \subseteq X \mid X \text{ is open}\},\\
\boldsymbol{\Sigma}^0_\alpha(X) &= \left\{ \bigcup_{i\in\omega} B_i \cap C_i^{\complement} \;\middle|\; B_i, C_i \in \bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_\beta \text{ for each } i \in \omega \right\},\\
\boldsymbol{\Pi}^0_\alpha(X) &= \{A^{\complement} \mid A \in \boldsymbol{\Sigma}^0_\alpha\},\\
\boldsymbol{\Delta}^0_\alpha(X) &= \boldsymbol{\Sigma}^0_\alpha(X) \cap \boldsymbol{\Pi}^0_\alpha(X).
\end{aligned}$$

**Proposition 4** *For any topological space $X$ and any $\alpha > 0$:*

1. *$\boldsymbol{\Sigma}^0_\alpha(X)$ is closed under countable union and finite intersection;*
2. *$\boldsymbol{\Pi}^0_\alpha(X)$ is closed under countable intersection and finite union;*
3. *$\boldsymbol{\Delta}^0_\alpha(X)$ is closed under finite union and intersection as well as under complementation.*

**Proposition 5** *If $\alpha < \beta$, then $\boldsymbol{\Sigma}^0_\alpha \cup \boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Delta}^0_\beta$. So the following diagram of inclusion holds between Baire classes:*

$$\begin{array}{ccccccccccccc}
 & & \boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & & & \boldsymbol{\Sigma}^0_\alpha & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & & \subsetneq & \subsetneq & & \subsetneq & \\
\boldsymbol{\Delta}^0_1 & & & & \boldsymbol{\Delta}^0_2 & & & & \boldsymbol{\Delta}^0_3 & \cdots & \boldsymbol{\Delta}^0_\alpha & & & & \boldsymbol{\Delta}^0_{\alpha+1} & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & & & \subsetneq & \subsetneq & & \subsetneq & \\
 & & \boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & & & \boldsymbol{\Pi}^0_\alpha & &
\end{array}$$

**Proposition 6** *If $\alpha > 2$, then*

$$\boldsymbol{\Sigma}^0_\alpha(X) = \left\{ \bigcup_{i\in\omega} B_i \;\middle|\; B_i \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta(X) \text{ for each } i \in \omega \right\}.$$

*And if $X$ is metrisable the previous statement holds also for $\alpha = 2$, i.e.*

$$\mathbf{\Sigma}^0_2(X) = \left\{ \bigcup_{i \in \omega} B_i \;\middle|\; B_i \in \mathbf{\Pi}^0_1(X) \textit{ for each } i \in \omega \right\}.$$

Hausdorff and later Kuratowski refined the Baire classes by introducing the so called Difference Hierarchy. Recall that any ordinal $\alpha$ can uniquely be expressed as $\alpha = \lambda + n$ where $\lambda$ is limit or equal to 0, and $n < \omega$. The ordinal $\alpha$ is said to be *even* if $n$ is even, otherwise $\alpha$ is said to be *odd*.

**Definition 7** Let $\xi \geq 1$ be a countable ordinal. For any sequence $(C_\eta)_{\eta<\xi}$ with $\alpha < \beta < \xi$ implies $C_\alpha \subseteq C_\beta$, the set $A = D_\xi((C_\eta)_{\eta<\xi})$ is defined by

$$A = \begin{cases} \bigcup\{C_\eta \setminus \bigcup_{\eta'<\eta} C_{\eta'} \mid \eta \text{ odd}, \eta < \xi\} & \text{for } \xi \text{ even}, \\ \bigcup\{C_\eta \setminus \bigcup_{\eta'<\eta} C_{\eta'} \mid \eta \text{ even}, \eta < \xi\} & \text{for } \xi \text{ odd}. \end{cases}$$

For a topological space $X$, and $0 < \alpha, \xi < \omega_1$ we let $D_\xi(\mathbf{\Sigma}^0_\alpha(X))$ be the class of all sets $D_\xi((C_\eta)_{\eta<\xi})$ where $(C_\eta)_{\eta<\xi}$ is an increasing sequence in $\mathbf{\Sigma}^0_\alpha(X)$.

Of course if $f : X \to Y$ is continuous map and $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(Y))$, then $f^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(X))$. This straightforward observation is crystallised in the definition of (boldface) *pointclass*, that is a collection of subsets of the Baire space closed under continuous preimages, or in other words, an initial segment of the Wadge quasiorder on the Baire space.

In fact in an arbitrary second countable $T_0$ space $X$ the classes $D_\xi(\mathbf{\Sigma}^0_\alpha)$ enjoy the stronger and less straightforward property of being initial segments of the quasiorder $\preccurlyeq_W$.

**Theorem 4** *Let $X, Y$ be second countable $T_0$ spaces and $A \subseteq X$, $B \subseteq Y$. For $1 \leq \alpha, \xi < \omega_1$, if $B \in D_\xi(\mathbf{\Sigma}^0_\alpha(Y))$ and $A \preccurlyeq_W B$, then $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(X))$.*

This proposition is a consequence of the following theorem due to de Brecht [5, Theorem 78].

**Theorem 5** (de Brecht) *Let $X$ be a second countable $T_0$ space, $\rho :\subseteq \omega^\omega \to X$ an admissible representation of $X$. For any countable $\alpha, \xi > 0$ and $A \subseteq X$*

$$A \in D_\xi(\mathbf{\Sigma}^0_\alpha(X)) \longleftrightarrow \rho^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\operatorname{dom} \rho)).$$

Here is the proof of Theorem 4.

*Proof (of Theorem 4)* Let $B \in D_\xi(\mathbf{\Sigma}^0_\alpha(Y))$ and suppose that $A \subseteq X$ satisfies $A \preccurlyeq_W B$. Let $\rho_X, \rho_Y$ be admissible representations of $X, Y$ respectively. Since $A \preccurlyeq_W B$, there exists a continuous $f : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ with $(\rho_Y \circ f)^{-1}(B) = \rho_X^{-1}(A)$. By continuity, $\rho_X^{-1}(A) = (\rho_Y \circ f)^{-1}(B) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\operatorname{dom} \rho_X))$, and so by Theorem 5 $A$ is $D_\xi(\mathbf{\Sigma}^0_\alpha)$ in $X$. □

For the convenience of the reader we devote the rest of this section to the proof of Theorem 5. The main ingredient is the following proposition which is a slightly modified version of a result by Saint Raymond [19, Lemma 17]. Its relevance in our context was first observed by de Brecht [5]. It is based on Baire category and we refer the reader to [11, Section 8] for definitions and results.

**Proposition 7** (Saint Raymond) *Let $X, Y$ be topological spaces, with $X$ metrisable. Let $\varphi : X \to Y$ be an open, continuous map with Polish fibres, i.e. $\varphi^{-1}(y)$ is Polish for all $y \in Y$. Define for $Z \subseteq X$*

$$N_0(Z) = \{y \in Y \mid Z \cap \varphi^{-1}(y) \text{ is non meagre in } \varphi^{-1}(y)\},$$
$$N_1(Z) = \{y \in Y \mid Z \cap \varphi^{-1}(y) \text{ is comeagre in } \varphi^{-1}(y)\}.$$

*Then for every positive ordinal $\xi < \omega_1$,*

1. *If $Z \in \mathbf{\Sigma}^0_\xi(X)$, then $N_0(Z) \in \mathbf{\Sigma}^0_\xi(Y)$,*
2. *If $Z \in \mathbf{\Pi}^0_\xi(X)$, then $N_1(Z) \in \mathbf{\Pi}^0_\xi(Y)$.*

*Therefore, if $\varphi$ is surjective then for every $A \subseteq Y$ and every $\alpha > 0$*

1. *$\varphi^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X) \longleftrightarrow A \in \mathbf{\Sigma}^0_\alpha(Y)$,*
2. *$\varphi^{-1}(A) \in \mathbf{\Pi}^0_\alpha(X) \longleftrightarrow A \in \mathbf{\Pi}^0_\alpha(Y)$.*

*Proof* Since $N_1(X\setminus Z) = Y\setminus N_0(Z)$ for every $\xi$ both statements are equivalent. Let $(V_k)_{k\in\omega}$ be a countable basis for the topology of $X$. We proceed by induction on $\xi$.

For $\xi = 1$ let $Z \in \mathbf{\Sigma}^0_1$, since $\varphi$ is assumed to be open we have $\varphi(Z)$ is open in $Y$. Since $\varphi^{-1}(y)$ is a Baire space for all $y \in Y$, the open subset $Z \cap \varphi^{-1}(y)$ of $\varphi^{-1}(y)$ is non meagre if and only if it is non empty. So $N_0(Z) = \varphi(Z) \in \mathbf{\Sigma}^0_1(Y)$.

So assume now that both statements are true for every $\xi' < \xi$ and let $Z \in \mathbf{\Sigma}^0_\xi$. Since $X$ is metrizable, $Z$ is the union of a countable family $(Z_n)_{n\in\omega}$ with $Z_n \in \mathbf{\Sigma}^0_{\xi_n}$ for some $\xi_n < \xi$. For any point $y \in Y$, using the fact that any Borel subset of a Polish space has the Baire Property, we have the following equivalences:

$$\begin{aligned}
&Z \cap \varphi^{-1}(y) \text{ is non meagre in } \varphi^{-1}(y)\\
&\quad\leftrightarrow \text{ some } Z_n \cap \varphi^{-1}(y) \text{ is non meagre in } \varphi^{-1}(y)\\
&\quad\leftrightarrow \text{ some } Z_n \cap \varphi^{-1}(y) \text{ is comeagre in some non empty open subset of } \varphi^{-1}(y)\\
&\quad\leftrightarrow \exists n\, \exists k\ (Z_n \cup V_k^{\complement}) \cap \varphi^{-1}(y) \text{ is comeagre in } \varphi^{-1}(y) \text{ and } V_k \cap \varphi^{-1}(y) \neq \emptyset.
\end{aligned}$$

Therefore,

$$N_0(Z) = \bigcup_{n,k} N_1(Z_n \cup V_k^{\complement}) \cap \varphi(V_k).$$

Now $Z_n \cup V_k^{\complement} \in \mathbf{\Pi}^0_{\xi_n}$, and so $N_1(Z_n \cup V_k^{\complement}) \in \mathbf{\Pi}^0_{\xi_n}$ by the induction hypothesis. Moreover $\varphi(V_k) \in \mathbf{\Sigma}^0_1$ since $\varphi$ is an open map. It follows that $N_0(Z)$ is $\mathbf{\Sigma}^0_\xi$ according to Definition 6.

For the second claim, it is enough to notice that if $\varphi$ is surjective and $A \subseteq Y$ then for $Z = \varphi^{-1}(A)$ we have $A = N_0(Z) = N_1(Z)$. □

Building on Proposition 7 and using the same technique de Brecht [5, see Theorem 78] showed:

**Proposition 8** (de Brecht) *Let $\varphi : X \to Y$ be an open and continuous map with Polish fibres, and $X$ metrisable. If $\varphi^{-1}(A) = D_\xi((C_\eta)_{\eta<\xi})$ with $(C_\eta)_{\eta<\xi}$ an increasing sequence in $\mathbf{\Sigma}^0_\alpha$, then $A = D_\xi(N_0(C_\eta)_{\eta<\xi})$. So $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(Y))$ if and only if $\varphi^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(X))$.*

*Proof* Let $B_\eta = N_0(C_\eta)$. First let $y \in A$. Since $\varphi^{-1}(y) = \bigcup_{\eta<\xi} C_\eta \cap \varphi^{-1}(y)$ and $\varphi^{-1}(y)$ is Polish and non empty, there exists a least $\eta_y < \xi$ such that $C_{\eta_y} \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$, i.e. $y \in B_{\eta_y}$. In particular, $C_{\eta'} \cap \varphi^{-1}(y)$ is meagre in $\varphi^{-1}(y)$ for all $\eta' < \eta_y$, hence $\bigcup_{\eta'<\eta_y} C_{\eta'} \cap \varphi^{-1}(y)$ is meagre in $\varphi^{-1}(y)$. It follows that $(C_{\eta_y} \setminus \bigcup_{\eta'<\eta_y} C_{\eta'}) \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$, so in particular it contains some $x \in X$. Since $x \in \varphi^{-1}(A) = D_\xi((C_\eta)_{\eta<\xi})$ the parity of $\xi$ must differ from that of $\eta_y$. Therefore $y \in D_\xi((B_\eta)_{\eta<\xi})$.

Conversely let $y \in D_\xi((B_\eta)_{\eta<\xi})$. There exists $\eta_y < \xi$ whose parity is different from that of $\xi$ such that $y \in B_{\eta_y} \setminus \bigcup_{\eta'<\eta_y} B_{\eta'}$. Since $B_\eta = N_0(A_\eta)$, $C_{\eta_y} \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$, and $\bigcup_{\eta'<\eta_y} C_{\eta'} \cap \varphi^{-1}(y)$ is meagre in $\varphi^{-1}(y)$. As before $(C_{\eta_y} \setminus \bigcup_{\eta'<\eta_y} C_{\eta'}) \cap \varphi^{-1}(y)$ is non meagre in $\varphi^{-1}(y)$ and so in particular it must contain some point $x \in X$. We have $x \in D_\xi((C_\eta)_{\eta<\xi}) = \varphi^{-1}(A)$ and so $y = \varphi(x) \in A$. □

Using the fact that every second countable $T_0$ space has an admissible representation which is open and has Polish fibres, we can now conclude the proof of Theorem 5.

*Proof (of Theorem 5)* The left to right implication follows from the continuity of the admissible representation and the fact that preimage maps are complete Boolean homomorphism.

For the right to left implication, it is enough by Propositions 7 and 8 to show that we can assume $\rho$ to be open with Polish fibres—since such an admissible representation always exists by Theorem 1. So let $\delta :\subseteq \omega^\omega \to X$ be any admissible representation of $X$, then there exists a continuous $f : \operatorname{dom}\rho \to \operatorname{dom}\sigma$ with $\delta \circ f = \rho$ on the domain of $\rho$. If $\delta^{-1}(A) \in D_\xi(\mathbf{\Sigma}^0_\alpha(\operatorname{dom}\delta))$ then as in the first implication we have $\rho^{-1}(S) = f^{-1}(\delta^{-1}(S)) \in D_\alpha(\mathbf{\Sigma}^0_\theta(\operatorname{dom}\rho))$. This concludes the claim. □

## 9 A reduction game

We now show that on virtually every second countable $T_0$ space—for instance on every quasi-Polish space—the reducibility $\preccurlyeq_W$ is well founded and satisfies the Wadge duality principle, in particular antichains have size at most 2.

Suppose that $\rho :\subseteq \omega^\omega \to X$ is an admissible representation of a second countable $T_0$ space $X$ then by Lemma 3 the preimage map

$$\rho^{-1} : (\mathcal{P}(X), \preccurlyeq_W) \longrightarrow (\mathcal{P}(\operatorname{dom} \rho), \leq_W)$$
$$A \longmapsto \rho^{-1}(A)$$

is an embedding of quasiorders. Therefore if we know that the quasi-order $\leq_W$ on the 0-dimensional space dom $\rho$ is well founded and satisfies the Wadge duality principle then we can conclude that the quasi-order $\preccurlyeq_W$ also enjoys these properties on $X$. In particular if $X$ is quasi-Polish then we can choose $\rho$ such that dom $\rho = \omega^\omega$ and therefore we directly get that the quasi-order $\preccurlyeq_W$ is well founded and satisfies the Wadge duality principle on Borel subsets of $X$ from the corresponding facts for the Baire space.

We consider a simple generalisation of the game first introduced by Wadge to study continuous reducibility on the Baire space in order to account for the structural properties of the reducibility by relatively continuous relations on an arbitrary second countable $T_0$ space.

Let $X, Y$ be second countable $T_0$ spaces, $\rho_X$, $\rho_Y$ admissible representations of $X$ and $Y$ respectively, and $A \subseteq X$, $B \subseteq Y$. We define a perfect information two players game $G^{\rho_X,\rho_Y}(A, B)$ as follows

$$\begin{array}{lccccccc} \mathrm{I}: & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & & \alpha \\ \mathrm{II}: & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots & & \beta \end{array}$$

Player I starts by choosing some $\alpha_0 \in \omega$ and then Player II chooses some $\beta_0 \in \omega$, then Player I choose some $\alpha_1 \in \omega$, so on and so forth. Player II wins a game $(\alpha, \beta)$ if and only if either $\alpha \notin$ dom $\rho_X$, or $\alpha \in$ dom $\rho_X$, $\beta \in$ dom $\rho_Y$ and

$$\rho_X(\alpha) \in A \longleftrightarrow \rho_Y(\beta) \in B.$$

This is the Lipschitz Wadge game with the additional condition that if Player I plays in dom $\rho_X$, then Player II must play in dom $\rho_Y$. When $\rho_X = \rho_Y$ we write $G^{\rho_X}(A, B)$ instead of $G^{\rho_X,\rho_X}(A, B)$.

The game $G^{\rho_X,\rho_Y}(A, B)$ is tightly related to our notion of reducibility.

**Lemma 4** *Let $X, Y$ be second countable $T_0$ spaces, $\rho_X$, $\rho_Y$ admissible representations of $X, Y$ respectively. Then for all $A \subseteq X$ and $B \subseteq Y$:*

1. *If Player* II *has a winning strategy in $G^{\rho_X,\rho_Y}(A, B)$, then $A \preccurlyeq_W B$,*
2. *If Player* I *has a winning strategy in $G^{\rho_X,\rho_Y}(A, B)$, then $B \preccurlyeq_W A^{\complement}$.*

*Proof* A winning strategy for Player II induces a total continuous function $f : \omega^\omega \to \omega^\omega$ such that for every $\alpha \in$ dom $\rho_X$, $f(\alpha) \in$ dom $\rho_Y$ and

$$\alpha \in \rho_X^{-1}(A) \longleftrightarrow f(\alpha) \in \rho_Y^{-1}(B).$$

Therefore if Player II has a winning strategy in $G^{\rho_X,\rho_Y}(A, B)$, then

$$(\rho_X^{-1}(A), \mathrm{dom}\,\rho_X) \leq_W (\rho_Y^{-1}(B), \mathrm{dom}\,\rho_Y)$$

and so $A \preccurlyeq_W B$.

Now a winning strategy for Player I induces a continuous function $g : \omega^\omega \to \omega^\omega$ such that whenever $\alpha \in \mathrm{dom}\,\rho_Y$ then $g(\alpha) \in \mathrm{dom}\,\rho_X$ and

$$g(\alpha) \notin \rho_X^{-1}(A) \longleftrightarrow \alpha \in \rho_Y^{-1}(B)$$

or equivalently,

$$\rho_Y(\alpha) \in B \longleftrightarrow \rho_X \circ g(\alpha) \in A^{\complement}.$$

Therefore if Player I has a winning strategy in $G^{\rho_X,\rho_Y}(A, B)$, then we have both $B \preccurlyeq_W A^{\complement}$ and $B^{\complement} \preccurlyeq_W A$. □

As long as dom $\rho_X$, dom $\rho_Y$, and $A \subseteq X$, $B \subseteq Y$ are all Borel, it is easy to see that $G^{\rho_X,\rho_Y}(A, B)$ is a Gale-Stewart game with Borel payoff set, and thus is determined by Martin's Borel determinacy. We are naturally led to the following definition.

**Definition 8** A second countable $T_0$ space $X$ is called *Borel representable* if there exists an admissible representation $\rho :\subseteq \omega^\omega \to X$ of $X$ such that dom $\rho$ Borel in $\omega^\omega$.

From Lemma 4 and the Borel determinacy we obtain the following.

**Theorem 6** *Let $X$ be a Borel representable space. The quasi-order $\preccurlyeq_W$ satisfies the Wadge Duality principle on Borel sets of $X$, i.e. for all Borel $A, B \subseteq X$ either $A \preccurlyeq_W B$, or $B \preccurlyeq_W A^{\complement}$.*

Of course assuming the Axiom of Determinacy (AD), the general structural result holds, i.e. assuming AD, if $X$ is a second countable $T_0$ space, $A, B \subseteq X$, then either $A \preccurlyeq_W B$, or $B^{\complement} \preccurlyeq_W A$.

Following the exact same proof by Martin and Monk as in the case of Wadge reducibility in 0-dimensional Polish spaces, see for example [11, (21.15) p.158], we obtain:

**Theorem 7** *Let $X$ be a Borel representable space. The quasi-order $\preccurlyeq_W$ is well founded on the Borel subsets of $X$.*

Again, assuming AD, this result extends to all subsets of every second countable $T_0$ space.

These positive results on the structure of the quasi-order $\preccurlyeq_W$ also imply that $\preccurlyeq_W$ often differs with the quasi-order of continuous reducibility $\leq_W$. Indeed Schlicht [20] showed that in every non 0-dimensional metric space there exists an antichain of the size of the continuum for the continuous reducibility. Using this result and Proposition 3, we see that in the separable metrisable case the two reducibilities differ as soon as we leave the zero-dimensional framework.

**Corollary 2** *Let $X$ be a metrisable and Borel representable space. Then $\preccurlyeq_W$ and $\leq_W$ coincide on subsets of $X$ if and only if $X$ is 0-dimensional.*

As it is observed in [14,24] the fact that the Wadge hierarchy of Borel sets in Polish 0-dimensional is well founded and has finite antichains can be generalised and its proof simplified by considering the notion of a *better quasiorder* introduced by Nash-Williams [16]. Let $(Q, \leq_Q)$ be a quasiorder, and let $X$ be a second countable $T_0$ space. Define a quasi-order on

$$Q_X^* = \{l : X \to Q \mid \operatorname{Im} l \text{ is countable and } l^{-1}(\{q\}) \text{ is Borel } \forall q \in Q\}$$

by letting $l_1 \preccurlyeq_W^* l_2$ if and only if there exists a total relatively continuous relation $R : X \rightrightarrows X$ such that

$$\forall x, y \in X \Big[ x \; R \; y \to l_1(x) \leq_Q l_2(y) \Big].$$

Then for an admissible representation $\rho :\subseteq \omega^\omega \to X$ consider the Lipschitz game $G^\rho(l_1, l_2)$ where Player I and II play alternatively in $\omega$ eventually determining $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$. We say that Player II wins the run $(\alpha, \beta)$ if and only if $\alpha \notin \operatorname{dom} \rho$ or $\alpha, \beta \in \operatorname{dom} \rho$ and $l_1(\rho(\alpha)) \leq_Q l_2(\rho(\beta))$.

It is easy to see that the existence of a winning strategy for Player II in $G^\rho(l_1, l_2)$ implies that $l_1 \preccurlyeq_W^* l_2$. Moreover as along as $l_1, l_2 \in Q_X^*$ and $\operatorname{dom} \rho$ is Borel, the game $G^\rho(l_1, l_2)$ the game is Borel, and thus determined. Therefore if $l_1 \not\preccurlyeq_W^* l_2$ then Player I has a winning strategy in $G^\rho(l_1, l_2)$.

The exact same proof as in [24, Theorem 3.2] or as in [14, Theorem 3] yields the following.

**Theorem 8** *Let $X$ be Borel representable. If $Q$ is a better quasi-order, then $(Q_X^*, \preccurlyeq_W^*)$ is a better quasi-order.*

Notice that in the case of a quasi-Polish space $X$ Theorem 8 can also be viewed as a consequence of the van Engelen–Miller–Steel Theorem [24, Theorem 3.2].

## 10 An example: the Real Line

In [9] (see also [10]) shows the existence of an embedding from $(\mathcal{P}(\omega), \subseteq_{\text{fin}})$—the subsets of $\omega$ quasiordered by inclusion modulo finite, i.e. $x \subseteq_{\text{fin}} y \leftrightarrow x \setminus y$ is finite—into the differences of two open sets of the real line equipped with the Wadge quasiorder. We now recall this construction.

Take increasing sequences of real numbers $\langle a_\alpha, b_\alpha \mid \alpha < \omega^\omega \rangle$ indexed by the ordinal $\omega^\omega$ and $\langle c_n \mid n \geq 1 \rangle$ with

$$\begin{aligned} & a_\alpha < b_\alpha < a_{\alpha+1} && \text{for each } \alpha < \omega^\omega \\ & a_\lambda^- := \sup\{a_\alpha \mid \alpha < \lambda\} < a_\lambda && \text{for each limit } \lambda < \omega^\omega \\ & a_{\omega^n}^- < c_n < a_{\omega^n} && \text{for each } n \in \omega. \end{aligned}$$

Now for $X \subseteq \omega \backslash \{0\}$ we let

$$D_X = \bigcup_{\alpha < \omega^\omega} [a_\alpha, b_\alpha) \cup \{c_n \mid n \notin X\}.$$

Clearly $D_X$ is a difference of two open sets for all $X \subseteq \omega \backslash \{0\}$.

**Theorem 9** [9] *For every $X, Y \subseteq \omega \backslash \{0\}$,*

$$X \subseteq_{\text{fin}} Y \longleftrightarrow D_X \leq_W D_Y.$$

By Parovičenko's Theorem [17], any poset of size $\aleph_1$ embeds into the partially ordered set $(\mathcal{P}(\omega), \subseteq_{\text{fin}})$, hence there are long infinite descending chains and long antichains for the Wadge reducibility, already among the difference of two open sets of the real line.

As an example, we now give winning strategies witnessing $D_X \preccurlyeq_W D_Y$ for every $X, Y \subseteq \omega \backslash \{0\}$.

**Proposition 9** *For every $X, Y \subseteq \omega \backslash \{0\}$, we have $D_X \preccurlyeq_W D_Y$.*

*Proof* Let $\rho_{\mathbb{R}}$ be the admissible representation of the real line from Example 1.

We choose for every $x \in \mathbb{R}$ a particular code via $\rho_{\mathbb{R}}$ by setting $\alpha^x : \omega \to \omega$ to be the increasing enumeration of $\{n \in \omega \mid x \in I_n\}$.

Now fix $X, Y \subseteq \omega \backslash \{0\}$. We describe a winning strategy $\sigma = \sigma_{X,Y}$ for player II in the game $G^{\rho_{\mathbb{R}}}(D_X, D_Y)$. Let $J_k$ be the open interval $(a^-_{\omega^k}, a_{\omega^k})$. And note that we only need to consider positions where Player I has played $(n_0, n_1, \ldots, n_j)$ with $\bigcap_{i=0}^j I_{n_i}$ is non empty. Let $X \triangle Y$ denote the symmetric difference of $X$ and $Y$, i.e.

$$X \triangle Y = \{x \in \omega \backslash \{0\} \mid \neg(x \in X \leftrightarrow x \in Y)\}.$$

Our winning strategy $\sigma : \omega^\omega \to \omega$ for Player II in $G^{\rho_{\mathbb{R}}}(D_X, D_Y)$ goes as follows:

As long as Player I is in a position where he has played $(n_0, n_1, \ldots, n_j)$ such that $I^j = \bigcap_{i=0}^j I_{n_i} \nsubseteq J_k$ for all $k \in X \triangle Y$, $\sigma$ consists simply in copying Player I's last move: $n_j$. Therefore $\sigma$ will induce the identity function outside the $J_k$'s for which $k \notin X \triangle Y$.

Now consider Player I has played $(n_0, n_1, \ldots, n_j)$ such that there exists $k \in X \triangle Y$ with $I^j = \bigcap_{i=0}^j \subseteq J_k$ and let $l$ be the least integer with $I^l = \bigcap_{i=0}^l I_{n_i} \subseteq J_k$. We distinguish several cases:

1. if $c_k \in D_Y \backslash D_X$: then for $\sigma$ to be winning for Player II, it must eventually make him play the code of a point outside of $D_Y$ and it cannot be $c_k$.
   Now since $I^l \subseteq J_k$, say $I^l = (r_0, r_1)$, we can for example choose

$$y = \frac{r_0 + \min\{r_1, c_k\}}{2}, \text{ if } r_0 < c_k, \text{ or } \quad y = \frac{\max\{r_0, c_k\} + r_1}{2}, \text{ if } c_k \leq r_0,$$

   and play $\alpha^y(j - l)$.

In other words, if Player I enters some $J_k$ with $c_k \in D_Y \setminus D_X$, then $\sigma$ consists in playing the code of some $y \in J_k$ different from $c_k$, where $y$ depends on the first position where Player I enters $J_k$.

2. if $c_k \in D_X \setminus D_Y$ and $c_k \in I^j$: then as long as $c_k \in I^j$, $\sigma$ must consist in playing as if Player I was going to play $c_k$, i.e. describe step by step a point belonging to $D_Y$ and it cannot be $c_k$.
   Now since $I^{l-1} \not\subseteq J_k$ (if $l = 0$ set $I^{l-1} = \mathbb{R}$), we choose some $y \in D_Y \cap I_{l-1}$ as follows:
   (a) if $a_{\omega^k} \in I^{l-1}$, then set $y = a_{\omega^k}$,
   (b) otherwise there is a minimal $\beta < \omega^k$ with $a_\beta \in I^{l-1}$, set $y = a_\beta$,
   and we play $\alpha^y(j - l)$.
3. if $c_k \in D_X \setminus D_Y$ and $c_k \notin I^j$: then for $\sigma$ to be winning for Player II, it must eventually make him play the code of a point which is outside of $D_Y$, but we must be careful to be consistent with what Player II has already played until that point. Let $p$ be the least integer such that $c_k \notin I^k$. First if $p \leq l$, i.e. at the first position where Player I entered $J_k$ we already knew he was not going to play $c_k$, so we can just copy its last move $n_j$. Otherwise $l < p$ so $c_k \in I^l$ and we must distinguish two cases:
   (a) if $a_{\omega^k} \in I^{l-1}$, then according to our previous case, at round $p$, Player II has so far played according to $\sigma$:
   $$t = (n_0, n_1, \ldots, n_{l-1}, \alpha^{a_{\omega^k}}(0), \alpha^{a_{\omega^k}}(1), \ldots, \alpha^{a_{\omega^k}}(p - l - 1)).$$
   so $\bigcap_{i=0}^{p-1} I_{t(i)}$ is an open interval $(r_0, r_1)$ with rational endpoints satisfying $r_0 < a_{\omega^k} < r_1$, so we can take
   $$z = \frac{\max\{a^-_{\omega^k}, r_0\} + a_{\omega^k}}{2}$$
   and play $\alpha^z(j - p)$.
   (b) Otherwise according to our previous case, up to round $p$, player II's moves according to $\sigma$ are
   $$t = (n_0, n_1, \ldots, n_{l-1}, \alpha^{a_\beta}(0), \alpha^{a_\beta}(1), \ldots, \alpha^{a_\beta}(p - l - 1)).$$
   where $\beta$ is the minimal ordinal with $a_\beta \in I^{l-1}$. Again $\bigcap_{i=0}^{p-1} I_{t(i)}$ is an open interval $(r_0, r_1)$ with rational endpoints satisfying $r_0 < a_\beta < r_1$, so we can take
   $$z = \frac{\max\{a^-_\beta, r_0\} + a_\beta}{2}$$
   where $a^-_\beta$ stands for $b_{\beta-1}$ if $\beta$ is successor, and we play $\alpha^z(j - p)$.

It should be clear that $\sigma$ is a winning strategy for Player II in $G^{\rho_\mathbb{R}}(D_X, D_Y)$. So $D_X \preccurlyeq_W D_Y$. □

If $X \not\subseteq_{\text{fin}} Y$, then $X \not\leq_W Y$ by Theorem 9 and so the winning strategy for II in $G^{\rho_{\mathbb{R}}}(D_X, D_Y)$ described in the previous proof induces a continuous $f_{X,Y} : \omega^\omega \to \omega^\omega$. The relation

$$R^{\rho_{\mathbb{R}}}_{f_{X,Y}}(x, y) \longleftrightarrow \exists \alpha \in \operatorname{dom} \rho_{\mathbb{R}} \left(\rho_{\mathbb{R}}(\alpha) = x \wedge y = \rho_{\mathbb{R}}(f_{X,Y}(\alpha))\right)$$

is therefore a relatively continuous relation from $\mathbb{R}$ to $\mathbb{R}$ with no continuous uniformising function. Indeed any function uniformising $R^{\rho_{\mathbb{R}}}_{f_{X,Y}}$ is a reduction of $X$ to $Y$ and since $D_X \not\leq_W D_Y$ there is no such continuous function.

## 11 An example: the Scott domain

We now give a simple example in the space $\mathcal{P}\omega$ of a case where $\leq_W$ differs from $\preccurlyeq_W$. Consider $\big\{\{0\}\big\}, \{\omega\} \subseteq \mathcal{P}\omega$, we first show that $\big\{\{0\}\big\} \not\leq_W \{\omega\}$. To see this, recall that continuous functions on $\mathcal{P}\omega$ are the Scott continuous functions with respect to inclusion, so in particular they are monotone for inclusion. Now since $\omega$ is the top element, any monotone map $f : \mathcal{P}\omega \to \mathcal{P}\omega$ with $f(\{0\}) = \omega$ has to send every $x \subseteq \omega$ with $0 \in x$ onto $\omega$ too, so that $f^{-1}(\omega) \supseteq O_{\{0\}}$. Therefore no Scott continuous function is a reduction from $\big\{\{0\}\big\}$ to $\{\omega\}$.

While we have $\big\{\{0\}\big\} \not\leq_W \{\omega\}$, we actually have $\big\{\{0\}\big\} \preccurlyeq_W \{\omega\}$, i.e. there exists a relatively continuous $R : \mathcal{P}\omega \rightrightarrows \mathcal{P}\omega$ such that for all $x, y \in \mathcal{P}\omega$

$$x \ R \ y \longrightarrow (x = \{0\} \leftrightarrow y = \omega).$$

Clearly any such relation $R$ cannot be uniformised by a Scott continuous function. Indeed such a Scott continuous function would be a reduction between the considered sets, and we know there is none.

The desired relation $R$ can be given as a strategy in the Lipschitz Wadge game $G^{\rho_{\text{En}}}(\big\{\{0\}\big\}, \{\omega\})$. Since $\rho_{\text{En}}$ is total, we know by Lemma 3 that $\big\{\{0\}\big\} \preccurlyeq_W \{\omega\}$ if and only if $A \leq_W B$ for

$$A = \rho^{-1}(\big\{\{0\}\big\}) = \{\alpha \in \omega^\omega \mid \alpha \in 2^\omega \wedge \exists k \ \alpha(k) = 1\}$$
$$\text{and} \qquad B = \rho^{-1}(\{\omega\}) = \{\alpha \in \omega^\omega \mid \alpha : \omega \to \omega \text{ is surjective}\}.$$

A winning strategy for Player II is for example given by the function $\sigma : \omega^{<\omega} \to \omega$ defined by

$$\sigma(s) = \begin{cases} 0 & \text{if } s \in \{0\}^{<\omega} \text{ or } \exists k < |s| \ s_k \neq 0, 1, \\ n & \text{if } s \in 2^\omega \text{ and } n = |s| - \min\{k \mid s_k = 1\}. \end{cases}$$

It is easily seen that this strategy induces a continuous function $f : \omega^\omega \to \omega^\omega$ witnessing the relative continuity of the relation $R : \mathcal{P}\omega \rightrightarrows \mathcal{P}\omega$ given by

$$\begin{aligned} R(\{0\}) &= \{\omega\} \\ R(x) &= \{\emptyset\} && \text{if } 0 \notin x \\ R(x) &= \{n \mid n \in \omega\} && \text{if } \{0\} \subset x. \end{aligned}$$

where $n = \{0, \dots, n-1\}$ and $\subset$ denotes strict inclusion.

*A complete* $\mathbf{\Sigma}^0_2$ *in* $\mathcal{P}\omega$. Recall (e.g. [11]) that $F = \{\alpha \in \omega^\omega \mid \exists n\, \forall k \geq n\, \alpha(k) = 0\}$ is complete for $\mathbf{\Sigma}^0_2(\omega^\omega)$, i.e. $F \in \mathbf{\Sigma}^0_2(\omega^\omega)$ and for every $A \in \mathbf{\Sigma}^0_2(\omega^\omega)$ we have $A \leq_W F$.

The set $\mathcal{P}_{<\infty}(\omega)$ of finite subsets of $\omega$ is $\mathbf{\Sigma}^0_2$ in $\mathcal{P}\omega$. It is shown in [2, Theorem 5.10] that it is not complete for the Scott continuous reducibility in the class $\mathbf{\Sigma}^0_2(\mathcal{P}\omega)$, i.e. there exists $G \in \mathbf{\Sigma}^0_2(\mathcal{P}\omega)$ such that $G \not\leq_W \mathcal{P}_{<\infty}(\omega)$. In contrast

**Proposition 10** *We have* $\mathbf{\Sigma}^0_2(\mathcal{P}\omega) = \{A \subseteq \mathcal{P}\omega \mid A \preccurlyeq_W \mathcal{P}_{<\infty}(\omega)\}$.

*Proof* Let us use the admissible representation $\rho_{\mathrm{En}} : \omega^\omega \to \mathcal{P}\omega$ from Example 3. Let

$$\widetilde{F} = \rho^{-1}_{\mathrm{En}}(\mathcal{P}_{<\infty}(\omega)) = \{\alpha \in \omega^\omega \mid \exists n\, \forall k\, \alpha(k) \leq n\}.$$

Clearly $\widetilde{F}$ is $\mathbf{\Sigma}^0_2$ in $\omega^\omega$. So the right to left inclusion follows from Theorem 4.

Now we have $F \leq_W \widetilde{F}$ as the continuous function $f : \omega^\omega \to \omega^\omega$, $f(\alpha)(n) = \mathrm{Card}\{k < n \mid \alpha(k) \neq 0\}$ clearly witnesses. Therefore for any $\mathbf{\Sigma}^0_2$ set $A \subseteq \mathcal{P}\omega$, there is a continuous function $f : \omega^\omega \to \omega^\omega$ which reduces $\rho^{-1}_{\mathrm{En}}(A)$ to $\rho^{-1}_{\mathrm{En}}(\mathcal{P}_{<\infty}(\omega))$, and so $A \preccurlyeq_W \mathcal{P}_{<\infty}(\omega)$. ◻

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