A WEAK FORM OF SOME TYPES OF CONTINUOUS MULTIFUNCTIONS

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ABSTRACT. The aim of this paper is to introduce and study upper and lower almost γ-continuous multifunctions as a generalization of some types of continuous multifunctions including almost continuity, almost α-continuity, almost precontinuity, almost quasi-continuity and γ-continuity. Furthermore, basic characterizations, preservation theorems and several properties concerning upper and lower almost γ-continuous multifunctions are investigated. The relationships between almost γ-continuous multifunctions and the other types of continuity are also discussed.
Keywords: open, regular open, continuity, γ-continuity, almost γ-continuity, multifunction, almost continuity, almost α-continuity, almost quasi-continuity, almost precontinuity.

1. Introduction

Many authors have researched and studied several stronger and weaker forms of continuous functions and multifunctions. It is well known that continuity and multifunctions are basic topics in general topology and in set valued analysis and in several branches of mathematics. Multifunctions and of course continuous multifunctions stand among the most important and most researched points in the whole of the Mathematical Science. Many different forms of continuous multifunctions have been introduced over the years. Some of them are semi-continuity [22], α-continuity [16], precontinuity [26], quasi-continuity [25], γ-continuity [2] and δ-precontinuity [21].

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The purpose of this paper is to give a new weaker form of some types of continuity including almost continuity [24], almost α-continuity [30], almost precontinuity [31], almost quasi-continuity [19, 29] and γ-continuity. In this paper, almost γ-continuity is introduced and studied. Moreover, basic properties and preservation theorems of almost γ-continuous multifunctions are investigated and relationships between almost γ-continuous multifunctions and the other types of continuity are investigated.

In Section 3, the notion of almost γ-continuous multifunctions is introduced and characterizations and some relationships of almost γ-continuous multifunctions and basic properties of almost γ-continuous multifunctions are investigated and obtained. Furthermore, the relationships almost γ-continuity and the other types of continuity are investigated. In Section 4, various relationships are investigated. In Section 5, the relationships between almost γ-continuity and graphs, product spaces are obtained. In Section 6, the other several properties of almost γ-continuity are investigated.

2. Preliminaries

In this paper, spaces (X, τ) and (Y, υ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $cl(A)$ and $int(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ, respectively.

A subset A of a space X is said to be regular open (respectively regular closed) if $A = int(cl(A))$ (respectively $A = cl(int(A))$) [35].

The δ-interior [36] of a subset A of X is the union of all regular open sets of X contained in A is denoted by $\delta - int(A)$. A subset A is called δ-open [36] if $A = \delta - int(A)$, i. e., a set is δ-open if it is the union of regular open sets. The complement of δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [36] if $A = \delta - cl(A)$, where $\delta - cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$.

A subset A of a space X is said to be α-open [17] (resp. semi-open [18], preopen [14], b-open [4] or γ-open [11] or sp-open [10], δ-preopen [33], β-open [1] or semi-preopen [3]) if $A \subset int(cl(int(A)))$ (resp. $A \subset cl(int(A))$, $A \subset int(cl(A))$, $A \subset cl(int(A)) \cup int(cl(A))$, $A \subset int(\delta - cl(A))$, $A \subset cl(int(cl(A)))$). The family of all open (resp. α-open, semi-open, preopen, γ-open, δ-preopen, β-open) sets of X containing a point $x \in X$ is denoted by $O(X, x)$ (resp. $\alpha O(X, x)$, $SO(X, x)$, $PO(X, x)$, $\gamma O(X, x)$, $\delta PO(X, x)$, $\beta O(X, x)$).

The complement of a semi-open (resp. α-open, preopen, β-open) set is said to be semi-closed [9] (resp. α-closed [15], preclosed [12], β-closed [1]).

The complement of a γ-open set is said to be γ-closed [11]. The intersection of all γ-closed sets of X containing A is called the γ-closure [11] of A and is denoted by $\gamma - cl(A)$. The union of all γ-open sets of X contained A is called γ-interior of A and is denoted by $\gamma - int(A)$. A subset U of X

is called a γ-neighborhood of a point $x \in X$ if there exists a γ-open set V such that $x \in V \subset U$.

Similarly, $s - cl(A)$, $\alpha - cl(A)$ and $p - cl(A)$ are defined in [9], [17] and [12], respectively.

The family of all α-open (resp. γ-open, γ-closed, regular open, regular closed, β-open) sets of X is denoted by $\alpha O(X)$ (resp. $\gamma O(X)$, $\gamma C(X)$, $RO(X)$, $RC(X)$, $\beta O(X)$).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [5, 7] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surjection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover, $F : X \to Y$ is called upper semi continuous (resp. lower semi continuous) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [22].

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$ [34].

Definition 1. *Let (X, τ) and (Y, σ) be topological spaces. A multifunction $F : X \to Y$ is said to be:*

1. Upper almost continuous [24] (resp. upper almost α-continuous [30], upper almost quasi-continuous [19, 29], upper almost precontinuous [31], upper almost β-continuous [20, 32]) at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in O(X, x)$ (resp. $U \in \alpha O(X, x)$, $U \in SO(X, x)$, $U \in PO(X, x)$, $U \in \beta O(X, x)$) such that $F(U) \subset int(cl(V))$.

Upper γ-continuous [2] (resp. upper δ-precontinuous [21]) at $x \in X$ if for each open set V of Y containing $F(x)$, there exists $U \in \gamma O(X, x)$ (resp. $U \in \delta PO(X, x)$) such that $F(U) \subset V$.

2. Lower almost continuous [24] (resp. lower almost α-continuous [30], lower almost quasi-continuous [19, 29], lower almost precontinuous [31], lower almost β-continuous [20, 32]) at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in O(X, x)$ (resp. $U \in \alpha O(X, x)$, $U \in SO(X, x)$, $U \in PO(X, x)$, $U \in \beta O(X, x)$) such that $F(u) \cap int(cl(V)) \neq \emptyset$ for every $u \in U$.

Lower γ-continuous [2] (resp. lower δ-precontinuous [21]) at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \gamma O(X, x)$ (resp. $U \in \delta PO(X, x)$) such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

3. Upper (lower) almost continuous (resp. upper (lower) almost α-continuous, upper (lower) almost quasi-continuous, upper (lower) almost

precontinuous, upper (lower) almost β-continuous, upper (lower) γ-continuous, upper (lower) δ-precontinuous) if it has this property at each point of X.

3. ALMOST γ-CONTINUOUS MULTIFUNCTIONS

In this section, the notion of almost γ-continuous functions is introduced and characterizations and some relationships of almost γ-continuous multifunctions and basic properties of almost γ-continuous multifunctions are investigated and obtained. Furthermore, the relationships almost γ-continuity and the other types of continuity are investigated.

Definition 2. *A multifunction $F : X \to Y$ is said to be:*

1. Lower almost γ-continuous at a point $x \in X$ if for each open set V of Y such that $x \in F^-(V)$, there exists $U \in \gamma O(X, x)$ such that $U \subset F^-(int(cl(V)))$,

2. Upper almost γ-continuous at a point $x \in X$ if for each open set V of Y such that $x \in F^+(V)$, there exists $U \in \gamma O(X, x)$ such that $U \subset F^+(int(cl(V)))$.

3. Lower (upper) almost γ-continuous if F has this property at each point of X.

The following theorem give some characterizations of upper almost γ-continuous multifunction.

Theorem 3. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(1). F is upper almost γ-continuous multifunction,

(2). for each $x \in X$ and for each open set V such that $F(x) \subset V$, there exists $U \in \gamma O(X, x)$ such that if $y \in U$, then $F(y) \subset int(cl(V))$,

(3). for each $x \in X$ and for each regular open set G of Y such that $F(x) \subset G$, there exists $U \in \gamma O(X, x)$ such that $F(U) \subset G$,

(4). for each $x \in X$ and for each closed set K such that $x \in F^+(Y \backslash K)$, there exists a γ-closed set H such that $x \in X \backslash H$ and $F^-(cl(int(K))) \subset H$,

(5). $F^+(int(cl(V))) \in \gamma O(X)$ for any open set $V \subset Y$,

(6). $F^-(cl(int(K))) \in \gamma C(X)$ for any closed set $K \subset Y$,

(7). $F^+(G) \in \gamma O(X)$ for any regular open set G of Y,

(8). $F^-(K) \in \gamma C(X)$ for any regular closed set K of Y,

(9). for each point x of X and each neighbourhood V of $F(x)$, $F^+(int(cl(V)))$ is a γ-neighbourhood of x,

(10). for each point x of X and each neighbourhood V of $F(x)$, there exists a γ-neighbourhood U of x such that $F(U) \subset int(cl(V))$,

(11). $cl(int(F^-(cl(int(H))))) \cap int(cl(F^-(cl(int(H))))) \subset F^-(cl(int(cl(H))))$ for every subset H of Y,

(12). $F^+(int(cl(int(N)))) \subset int(cl(F^+(int(cl(N))))) \cup cl(int(F^+(int(cl(N)))))$ for every subset N of Y,

(13). $\gamma - cl(F^-(cl(int(H)))) \subset F^-(cl(int(cl(H))))$ *for every subset H of* Y,

(14). $F^+(int(cl(int(N)))) \subset \gamma - int(F^+(int(cl(N))))$ *for every subset N of Y.*

Proof. (1)$\Leftrightarrow$(2). Clear.

(2)$\Rightarrow$(3). Let $x \in X$ and G be a regular open set of Y such that $F(x) \subset G$. By (2), there exists $U \in \gamma O(X, x)$ such that if $y \in U$, then $F(y) \subset int(cl(G)) = G$. We obtain $F(U) \subset G$.

(3)$\Rightarrow$(2). Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $int(cl(V)) \in RO(Y)$. By (3), there exists $U \in \gamma O(X, x)$ such that $F(U) \subset int(cl(V))$.

(2)$\Rightarrow$(4). Let $x \in X$ and K be a closed set of Y such that $x \in F^+(Y \backslash K)$. By (2), there exists $U \in \gamma O(X, x)$ such that $F(U) \subset int(cl(Y \backslash K))$. We have $int(cl(Y \backslash K)) = Y \backslash cl(int(K))$ and

$$U \subset F^+(Y \backslash cl(int(K))) = X \backslash F^-(cl(int(K))).$$

We obtain $F^-(cl(int(K))) \subset X \backslash U$. Take $H = X \backslash U$. Then, $x \in X \backslash H$ and H is a γ-closed set.

(4)$\Rightarrow$(2). It can be obtained similarly as (2)$\Rightarrow$(4).

(1)$\Rightarrow$(5). Let V be any open set of Y and $x \in F^+(int(cl(V)))$. By (1), there exists $U_x \in \gamma O(X, x)$ such that $U_x \subset F^+(int(cl(V)))$. Therefore, we obtain

$$F^+(int(cl(V))) = \bigcup_{x \in F^+(int(cl(V)))} U_x.$$

Hence, $F^+(int(cl(V))) \in \gamma O(X)$.

(5)$\Rightarrow$(1). Let V be any open set of Y and $x \in F^+(V)$. By (5),

$$F^+(int(cl(V))) \in \gamma O(X).$$

Take $U = F^+(int(cl(V)))$. Then, $F(U) \subset int(cl(V))$. Hence, F is upper almost γ-continuous.

(5)$\Rightarrow$(6). Let K be any closed set of Y. Then, $Y \backslash K$ is an open set of Y. By (5), $F^+(int(cl(Y \backslash K))) \in \gamma O(X)$. Since $int(cl(Y \backslash K)) = Y \backslash cl(int(K))$, it follows that $F^+(int(cl(Y \backslash K))) = F^+(Y \backslash cl(int(K))) = X \backslash F^-(cl(int(K)))$. We obtain that $F^-(cl(int(K)))$ is γ-closed in X.

(6)$\Rightarrow$(5). It can be obtained similarly as (5)$\Rightarrow$(6).

(5)$\Rightarrow$(7). Let G be any regular open set of Y. By (5), $F^+(int(cl(G))) = F^+(G) \in \gamma O(X)$.

(7)$\Rightarrow$(5). Let V be any open set of Y. Then, $int(cl(V)) \in RO(Y)$. By (7), $F^+(int(cl(V))) \in \gamma O(X)$.

(6)$\Rightarrow$(8). It can be obtained similarly as (5)$\Rightarrow$(7).

(8)$\Rightarrow$(6). It can be obtained similarly as (7)$\Rightarrow$(5).

(5)$\Rightarrow$(9). Let $x \in X$ and V be a neighbourhood of $F(x)$. Then there exists an open set G of Y such that $F(x) \subset G \subset V$. therefore, we obtain $x \in F^+(G) \subset F^+(V)$. Since $F^+(int(cl(G))) \in \gamma O(X)$, $F^+(int(cl(V)))$ is a γ-neighbourhood of x.

(9)$\Rightarrow$(10). Let $x \in X$ and V be a neighbourhood of $F(x)$. By (9), $F^+(int(cl(V)))$ is a γ-neighbourhood of x. Take $U = F^+(int(cl(V)))$. Then $F(U) \subset int(cl(V))$.

(10)$\Rightarrow$(1). Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighbourhood of $F(x)$. By (10), there exists a γ-neighbourhood U of x such that $F(U) \subset int(cl(V))$. Therefore, there exists $G \in \gamma O(X)$ such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset int(cl(V))$. We obtain that F is upper almost γ-continuous.

(6)$\Rightarrow$(11). For any subset H of Y, $cl(H)$ is closed in Y. By (6),

$$F^-(cl(int(cl(H)))) \in \gamma C(X).$$

This means

$$int(cl(F^-(cl(int(H))))) \cap cl(int(F^-(cl(int(H)))))$$
$$\subset int(cl(F^-(cl(int(cl(H)))))) \cap cl(int(F^-(cl(int(cl(H))))))$$
$$\subset F^-(cl(int(cl(H)))).$$

(11)$\Rightarrow$(12). By replacing $Y \backslash H$ instead of H in (11), we have

$$int(cl(F^-(cl(int(Y\backslash H))))) \cap cl(int(F^-(cl(int(Y\backslash H)))))$$
$$= int(cl(F^-(Y\backslash int(cl(H))))) \cap cl(int(F^-(Y\backslash int(cl(H)))))$$
$$\subset F^-(cl(int(cl(Y\backslash H)))).$$

and therefore,

$$F^+(int(cl(int(H)))) \subset int(cl(F^+(int(cl(H))))) \cup cl(int(F^+(int(cl(H))))).$$

(12)$\Rightarrow$(5). Let V be any open set of Y. Then by using (12) we have

$$F^+(int(cl(V))) \subset int(cl(F^+(int(cl(V))))) \cup cl(int(F^+(int(cl(V))))).$$

Hence, we obtain $F^+(int(cl(V))) \in \gamma O(X)$.

(6)$\Rightarrow$(13). For any subset H of Y, $cl(H)$ is closed in Y. By (6),

$$F^-(cl(int(cl(H))))$$

is γ-closed in X. Therefore, we obtain

$$\gamma - cl(F^-(cl(int(H)))) \subset F^-(cl(int(cl(H)))).$$

(13)$\Rightarrow$(6). Let K be any closed set of Y. Then we have

$$\gamma - cl(F^-(cl(int(K)))) \subset F^-(cl(int(cl(K)))) = F^-(cl(int(K))).$$

Thus, $F^-(cl(int(K)))$ is γ-closed in X.

(5)$\Rightarrow$(14). For any subset N of Y, $int(N)$ is open in Y. By (5),

$$F^+(int(cl(int(N))))$$

is γ-open in X. Therefore, we obtain

$$F^+(int(cl(int(N)))) \subset \gamma - int(F^+(int(cl(N)))).$$

(14)$\Rightarrow$(5). Let V be any open set of Y. Then we have $F^+(int(cl(V))) \subset \gamma - int(F^+(int(cl(V))))$. Hence, $F^+(int(cl(V)))$ is γ-open in X. $\qquad\square$

Remark 4. *For a multifunction $F : X \to Y$ from a topological space (X, τ) to a topological space (Y, υ), the following implications hold:*

$$upper\ almost\ continuity$$
$$\Downarrow$$
$$upper\ almost\ \alpha\text{-}continuity \quad \Rightarrow \quad upper\ almost\ precontinuity$$
$$\Downarrow \qquad\qquad\qquad \Downarrow$$
$$upper\ almost\ quasi\text{-}continuity \quad \Rightarrow \quad upper\ almost\ \gamma\text{-}continuity$$
$$\Downarrow$$
$$upper\ almost\ \beta\text{-}continuity$$

Note that none of these implications is reversible. We give examples for the last three implications as follows. The other examples can be obtained in [19, 24, 29, 30, 31].

Example 5. *Let $X = Y = \{a, b, c, d\}$. Let τ and σ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b, d\}, \{a, b, d\}\}$. Define the multifunction $F : X \to Y$ by $F(x) = \{x\}$ for each $x \in X$. Then F is upper almost β-continuous but not upper almost γ-continuous, since $\{b, d\} \in RO(Y)$ and $F^+(\{b, d\}) = \{b, d\}$ is not γ-open in (X, τ).*

Example 6. *Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$. Let τ and σ be respectively topologies on X and on Y given by $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Define the multifunction $F : X \to Y$ by $F(a) = \{1\}$, $F(b) = \{3, 4, 5\}$ and $F(c) = \{2\}$. Then F is upper almost γ-continuous but not upper almost precontinuous.*

Example 7. *Let $X = \{1, 2, 3, 4, 5\}$. Let τ be a topology on X given by $\tau = \{\emptyset, X, \{1, 2\}, \{3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Define the multifunction $F : X \to X$ by $F(1) = \{1\}$, $F(2) = \{3\}$, $F(3) = \{2\}$, $F(4) = \{4\}$ and $F(5) = \{5\}$. Then F is upper almost γ-continuous but not upper almost quasi-continuous.*

The following theorem give some characterizations of lower almost γ-continuous multifunction.

Theorem 8. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(1). F is lower almost γ-continuous multifunction,

(2). for each $x \in X$ and for each open set V such that $F(x) \cap V \neq \emptyset$, there exists a $U \in \gamma O(X, x)$ such that if $y \in U$, then $F(y) \cap int(cl(V)) \neq \emptyset$,

(3). for each $x \in X$ and for each regular open set G of Y such that $F(x) \cap G \neq \emptyset$, there exists a $U \in \gamma O(X, x)$ such that if $y \in U$, then $F(y) \cap G \neq \emptyset$,

(4). for each $x \in X$ and for each closed set K such that $x \in F^-(Y \backslash K)$, there exists a γ-closed set H such that $x \in X \backslash H$ and $F^+(cl(int(K))) \subset H$,

(5). $F^-(int(cl(V))) \in \gamma O(X)$ for any open set $V \subset Y$,

(6). $F^+(cl(int(K))) \in \gamma C(X)$ for any closed set $K \subset Y$,

(7). $F^-(G) \in \gamma O(X)$ for any regular open set G of Y,

(8). $F^+(K) \in \gamma C(X)$ for any regular closed set K of Y,

(9). $cl(int(F^+(cl(int(H))))) \cap int(cl(F^+(cl(int(H))))) \subset$ $F^+(cl(int(cl(H))))$ for every subset H of Y,

(10). $F^-(int(cl(int(N)))) \subset int(cl(F^-(int(cl(N)))))\cup$ $cl(int(F^-(int(cl(N)))))$ for every subset N of Y,

(11). $\gamma - cl(F^+(cl(int(H)))) \subset F^+(cl(int(cl(H))))$ for every subset H of Y,

(12). $F^-(int(cl(int(N)))) \subset \gamma - int(F^-(int(cl(N))))$ for every subset N of Y.

Proof. It can be obtained similarly as the previous theorem.   □

Theorem 9. *The following properties are equivalent for a multifunction $F : X \to Y$:*

(1) F is upper almost γ-continuous,

(2) $\gamma - cl(F^-(V)) \subset F^-(cl(V))$ for every $V \in \beta O(Y)$,

(3) $\gamma - cl(F^-(V)) \subset F^-(cl(V))$ for every $V \in SO(Y)$,

(4) $F^+(V) \subset \gamma - int(F^+(int(cl(V))))$ for every $V \in PO(Y)$.

Proof. It is analogous with them of Theorem 1 of [19], Theorem 5 of [31] and Theorem 5 of [20].   □

Theorem 10. *The following properties are equivalent for a multifunction $F : X \to Y$:*

(1) F is lower almost γ-continuous,

(2) $\gamma - cl(F^+(V)) \subset F^+(cl(V))$ for every $V \in \beta O(Y)$,

(3) $\gamma - cl(F^+(V)) \subset F^+(cl(V))$ for every $V \in SO(Y)$,

(4) $F^-(V) \subset \gamma - int(F^-(int(cl(V))))$ for every $V \in PO(Y)$.

Proof. It is analogous with them of Theorem 2 of [19], Theorem 6 of [31] and Theorem 6 of [20].   □

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition 11. *Let (X, τ) be a topological space and let (x_α) be a net in X. It is said that the net (x_α) γ-converges to x if for each γ-open set G containing x in X, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.*

Theorem 12. *Let $F : X \to Y$ be a multifunction. If F is lower (upper) almost γ-continuous multifunction, then for each $x \in X$ and for each net (x_α) which γ-converges to x in X and for each open set $V \subseteq Y$ such that $x \in F^-(V)$ (resp. $x \in F^+(V)$), the net (x_α) is eventually in $F^-(int(cl(V)))$ (resp. $F^+(int(cl(V)))$).*

Proof. Let (x_α) be a net which γ-converges to x in X and let V be any open set in Y such that $x \in F^-(V)$. Since F is lower almost γ-continuous multifuction, it follows that there exists a γ-open set U in X containing

x such that $U \subset F^{-}(int(cl(V)))$. Since (x_α) γ-converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^{-}(int(cl(V)))$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^{-}(int(cl(V)))$.

The proof of the upper almost γ-continuity of F is similar to the above. $\square$

Lemma 13. *Let A and X_0 be subsets of a space (X, τ). If $A \in \gamma O(X)$ and $X_0 \in \alpha O(X)$, then $A \cap X_0 \in \gamma O(X_0)$ [4, 11].*

Lemma 14. *Let $A \subset X_0 \subset X$. If $X_0 \in \gamma O(X)$ and $A \in \gamma O(X_0)$, $A \in \gamma O(X)$ [11].*

Theorem 15. *Let $F : X \rightarrow Y$ be a multifunction and let U be a α-open set in X. If F is a lower (upper) almost γ-continuous, then the restriction multifunction $F \mid_U : U \rightarrow Y$ is a lower (resp. upper) almost γ-continuous.*

Proof. Suppose that V is an open set in Y. Let $x \in U$ and let $x \in (F \mid_U)^{-}(V)$. Since F is lower almost γ-continuous multifunction, it follows that there exists a γ-open set G such that $x \in G \subset F^{-}(int(cl(V)))$. By Lemma 13, we obtain that $x \in G \cap U \in \gamma O(U)$ and $G \cap U \subset (F \mid_U)^{-}(int(cl(V)))$. Thus, we show that the restriction multifunction $F \mid_U$ is a lower almost γ-continuous.

The proof of the upper almost γ-continuity of $F \mid_U$ is similar to the above. $\square$

Theorem 16. *Let $\{U_\lambda : \lambda \in \Lambda\}$ be a α-open cover of a space X. Then a multifunction $F : X \rightarrow Y$ is upper almost γ-continuous (resp. lower almost γ-continuous) if and only if the restriction $F \mid_{U_\lambda} : U_\lambda \rightarrow Y$ is upper almost γ-continuous (resp. lower almost γ-continuous) for each $\lambda \in \Lambda$.*

Proof. We prove only the case for F upper almost γ-continuous, the proof for F lower almost γ-continuous being analogous.

($\Rightarrow$) Let $\lambda \in \Lambda$ and V be any open set of Y. Since F is upper almost γ-continuous, $F^{+}(int(cl(V)))$ is γ-open in X. By Lemma 13, $(F \mid_{U_\lambda})^{+}(int(cl(V))) = F^{+}(int(cl(V))) \cap U_\lambda$ is γ-open in U_λ and hence $F \mid_{U_\lambda}$ is upper almost γ-continuous.

($\Leftarrow$) Let V be any open set of Y. Since $F \mid_{U_\lambda}$ is upper almost γ-continuous for each $\lambda \in \Lambda$, $(F \mid_{U_\lambda})^{+}(int(cl(V))) = F^{+}(int(cl(V))) \cap U_\lambda$ is γ-open in U_λ. By Lemma 14, $(F \mid_{U_\lambda})^{+}(int(cl(V)))$ is γ-open in X for each $\lambda \in \Lambda$. We obtain that $F^{+}(int(cl(V))) = \underset{\lambda \in \Lambda}{\cup} (F \mid_{U_\lambda})^{+}(int(cl(V)))$ is γ-open in X. Hence F is upper almost γ-continuous. $\square$

4. γ-CONTINUITY AND ALMOST γ-CONTINUITY

In this section, various relationships are investigated.

Definition 17. *Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semiregularization.*
In case when $\tau = \tau^*$, the space (X, τ) is called semi-regular [35].

Theorem 18. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a semi-regular topological space (Y, υ). F is lower almost γ-continuous multifunction if and only if F is lower γ-continuous.*

Proof. Let $x \in X$ and let V be an open set such that $x \in F^-(V)$. Since (Y, υ) is a semi-regular space, there exist regular open sets U_i for $i \in I$ such that $V = \bigcup_{i \in I} U_i$. We have $F^-(V) = F^-(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} F^-(U_i)$. By Theorem 8, $F^-(U_i) \in \gamma O(X)$ for $i \in I$. We obtain $F^-(V) \in \gamma O(X)$. Hence, by Theorem 3.2 in [2], F is lower γ-continuous.

Conversely, obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 19. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then $F : (X, \tau) \to (Y, \upsilon)$ is lower almost γ-continuous multifunction if and only if $F : (X, \tau) \to (Y, \upsilon^*)$ is lower γ-continuous.*

Definition 20. *A space X is said to be:*
  (1) submaximal [8] if each dense subset of X is open in X,
  (2) extremely disconnected [8] if the closure of each open set of X is open in X.

Theorem 21. *If (X, τ) a submaximal extremely disconnected space and (Y, σ) is a semi-regular space, then the following are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*
  (1) F is lower almost γ-continuous;
  (2) F is lower semi-continuous.

Proof. (1)$\Rightarrow$(2): Let $x \in X$ and let V be an open set such that $x \in F^-(V)$. Since (Y, σ) is a semi-regular space, there exist regular open sets U_i for $i \in I$ such that $V = \bigcup_{i \in I} U_i$. We have $F^-(V) = F^-(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} F^-(U_i)$. By Theorem 8, $F^-(U_i) \in \gamma O(X)$ for $i \in I$. We obtain $F^-(V) \in \gamma O(X)$. Since (X, τ) is a submaximal extremely disconnected space, then $\tau = \gamma O(X)$. We have $F^-(V) \in \tau$. Hence, F is lower semi-continuous.

(2)$\Rightarrow$(1): Obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 22. *Let Y be a semi-regular space and X be a submaximal extremely disconnected space. The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*
  (1) F is lower almost γ-continuous;
  (2) F is lower semi-continuous;
  (3) F is lower α-continuous;
  (4) F is lower precontinuous;
  (5) F is lower quasi-continuous;
  (6) F is lower γ-continuous.

Suppose that (X, τ), (Y, υ) and (Z, ω) are topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the composite

multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 23. *Let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. The following statements hold:*

(1) If F is upper (lower) γ-continuous and G is upper (lower) semi-continuous, then $G \circ F : X \to Z$ is a upper (lower) almost γ-continuous multifunction.

(2) If F is upper (lower) precontinuous and G is upper (lower) semicontinuous, then $G \circ F : X \to Z$ is a upper (lower) almost γ-continuous multifunction.

(3) If F is upper (lower) quasi-continuous and G is upper (lower) semicontinuous, then $G \circ F : X \to Z$ is a upper (lower) almost γ-continuous multifunction.

(4) If F is upper (lower) α-continuous and G is upper (lower) semicontinuous, then $G \circ F : X \to Z$ is a upper (lower) almost γ-continuous multifunction.

(5) If F is upper (lower) semi-continuous and G is upper (lower) semicontinuous, then $G \circ F : X \to Z$ is a upper (lower) almost γ-continuous multifunction.

Proof. (1) Let $V \subset Z$ be any regular open set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V))$). Since G is upper (lower) semi continuous multifunction, it follows that $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is upper (lower) γ-continuous multifunction, it follows that $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is a γ-open set. It shows that $G \circ F$ is a upper (resp. lower) almost γ-continuous multifunction.

The other proofs can be obtained similarly. $\qquad\square$

Definition 24. *A multifunction $F : X \to Y$ is said to be:*

1. Lower weakly γ-continuous at a point $x \in X$ if for each open set V of Y such that $x \in F^-(V)$, there exists $U \in \gamma O(X,x)$ such that $U \subset F^-(cl(V))$,

2. Upper weakly γ-continuous at a point $x \in X$ if for each open set V of Y such that $x \in F^+(V)$, there exists $U \in \gamma O(X,x)$ such that $U \subset F^+(cl(V))$.

3. Lower (upper) weakly γ-continuous if F has this property at each point of X.

Definition 25. *A subset A of a topological space X is said to be α-paracompact [37] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X.*

Furthermore, a multifunction $F : (X,\tau) \to (Y,\sigma)$ is called punctually α-paracompact [28] if $F(x)$ is α-paracompact for each point $x \in X$.

Definition 26. *A subset A of a topological space X is said to be α-regular [13] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset cl(G) \subset U$.*

Lemma 27. *If A is an α-regular α-paracompact set of a topological space X and U is an open neighbourhood of A, then there exists an open set G of X such that $A \subset G \subset cl(G) \subset U$ [13].*

Theorem 28. *For a multifunction $F : X \to Y$ such that $F(x)$ is an α-regular α-paracompact set for each $x \in X$, the following are equivalent:*
   (1) F is upper weakly γ-continuous,
   (2) F is upper almost γ-continuous,
   (3) F is upper γ-continuous.

Proof. It is analogous with them of Theorem 15 of [20]. □

Lemma 29. *If A is an α-regular set of X, then for every open set G which intersects A, there exists an open set D such that $A \cap D \neq \varnothing$ and $cl(D) \subset G$ [27].*

Theorem 30. *For a multifunction $F : X \to Y$ such that $F(x)$ is an α-regular set of Y for each $x \in X$, the following are equivalent:*
   (1) F is lower weakly γ-continuous,
   (2) F is lower almost γ-continuous,
   (3) F is lower γ-continuous.

Proof. It is analogous with them of Theorem 16 of [20]. □

Theorem 31. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following are equivalent:*
   (1) F is upper weakly γ-continuous,
   (2) F is upper almost γ-continuous,
   (3) F is upper γ-continuous.

Proof. It is analogous with them of Theorem 18 of [20]. □

5. Graphs and product spaces

In this section, the relationships between almost γ-continuity and graphs, product spaces are obtained.

Lemma 32. *The intersection of an open set and a γ-open set is a γ-open set [4].*

Lemma 33. *For a multifunction $F : X \to Y$, the following hold:*
   (1) $G_F^+(A \times B) = A \cap F^+(B)$,
   (2) $G_F^-(A \times B) = A \cap F^-(B)$
   for any subsets $A \subset X$ and $B \subset Y$ [18].

Theorem 34. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then the graph multifunction of F is upper almost γ-continuous if and only if F is upper almost γ-continuous.*

Proof. ($\Rightarrow$). Suppose that $G_F : X \to X \times Y$ is upper almost γ-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is

open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \gamma O(X, x)$ such that $G_F(U) \subset int(cl(X \times V)) = X \times int(cl(V))$. By the previous lemma, we have $U \subset G_F^+(X \times int(cl(V))) = F^+(int(cl(V)))$ and $F(U) \subset int(cl(V))$. This shows that F is upper almost γ-continuous.

($\Leftarrow$): Suppose that $F : X \to Y$ is upper almost γ-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family of $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, it follows that there exists a finite number of points, says y_1, y_2, y_3,..., y_n in $F(x)$ such that $F(x) \subset \bigcup\{V(y_i) : i = 1, 2, ..., n\}$. Take $U = \bigcap\{U(y_i) : i = 1, 2, ..., n\}$ and $V = \bigcup\{V(y_i) : i = 1, 2, ..., n\}$. Then U and V are open in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is upper almost γ-continuous, there exists $U_0 \in \gamma O(X, x)$ such that $F(U_0) \subset int(cl(V))$. By the previous lemma, we have $U \cap U_0 \subset U \cap F^+(int(cl(V))) = G_F^+(U \times int(cl(V))) \subset G_F^+(int(cl(U \times V))) \subset G_F^+(int(cl(W)))$. Therefore, we obtain $U \cap U_0 \in \gamma O(X, x)$ and $G_F(U \cap U_0) \subset int(cl(W))$. This shows that G_F is upper almost γ-continuous. $\square$

Theorem 35. *A multifunction $F : X \to Y$ is lower almost γ-continuous if and only if $G_F : X \to X \times Y$ is lower almost γ-continuous.*

Proof. ($\Rightarrow$) Suppose that F is lower almost γ-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets U and V of X and Y, respectively. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \gamma O(X, x)$ such that $G \subset F^-(int(cl(V)))$. By Lemma 33, $U \cap G \subset U \cap F^-(int(cl(V))) = G_F^-(U \times int(cl(V))) \subset G_F^-(int(cl(W)))$. Furthermore, $x \in U \cap G \in \gamma O(X)$ and hence G_F is lower almost γ-continuous.

($\Leftarrow$) Suppose that G_F is lower almost γ-continuous. Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower almost γ-continuous, there exists a γ-open set U containing x such that $U \subset G_F^-(int(cl(X \times V)))$. Since $G_F^-(int(cl(X \times V))) = G_F^-(X \times int(cl(V)))$, by Lemma 33, we have $U \subset F^-(int(cl(V)))$. This shows that F is lower almost γ-continuous. $\square$

Definition 36. *Let $F : X \to Y$ be a multifunction. The multigraph $G(F)$ is said to be γ-graph in $X \times Y$ if for each $(x, y) \notin G(F)$, there exist γ-open set U and open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.*

Theorem 37. *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper almost γ-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y, σ). Then the multigraph $G(F)$ of F is a γ-graph in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y, σ) is a Hausdorff space, then for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$. The

family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members U_{β_1}, U_{β_2},..., U_{β_n} of Φ. Choose y_1, y_2,..., y_n in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $i = 1, 2, ..., n$ and set $W = W_0 \cap (\overset{n}{\underset{i=1}{\cap}} W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\underset{\beta \in I}{\cup} U_\beta) = \emptyset$, which implies that $W \cap int(cl(\underset{\beta \in I}{\cup} U_\beta)) = \emptyset$. By the upper almost γ-continuity of F, there exists a $U \in \gamma O(X, x_0)$ such that $F(U) \subset int(cl(\underset{\beta \in I}{\cup} U_\beta))$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore, the graph $G(F)$ is a γ-graph in $X \times Y$. $\square$

Theorem 38. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection for each $\alpha \in J$. If F is upper (lower) almost γ-continuous multifunction, then $P_\alpha \circ F$ is upper (resp. lower) almost γ-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a open set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(int(cl(V_{\alpha_0})) = F^+(P_{\alpha_0}^+(int(cl(V_{\alpha_0}))) = F^+(int(cl(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively,

$$(P_{\alpha_0} \circ F)^-(int(cl(V_{\alpha_0})) = F^-(P_{\alpha_0}^-(int(cl(V_{\alpha_0}))) = F^-(int(cl(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha)).$$

Since F is upper (resp. lower) almost γ-continuous multifunction and since $int(cl(V_{\alpha_0}) \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a regular open set, it follows that $F^+(int(cl(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $F^-(int(cl(V_{\alpha_0})) \times \prod_{\alpha \neq \alpha_0} X_\alpha))$ is γ-open in (X, τ). It shows that $P_{\alpha_0} \circ F$ is upper (lower) almost γ-continuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is upper (lower) almost γ-continuous multifunction for each $\alpha \in J$. $\square$

Theorem 39. *Suppose that for each $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \upsilon_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is upper (lower) almost γ-continuous multifunction, then each F_α is upper (resp. lower) almost γ-continuous multifunction for each $\alpha \in J$.*

Proof. Let $V_\alpha \subseteq Y_\alpha$ be a open set. Then $int(cl(V_\alpha)) \times \prod_{\alpha \neq \beta} Y_\beta$ is a regular open set. Since F is upper (lower) almost γ-continuous multifunction, it follows that $F^+(int(cl(V_\alpha)) \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^+(int(cl(V_\alpha))) \times \prod_{\alpha \neq \beta} X_\beta$ (resp.

$F^-(int(cl(V_\alpha)) \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^-(int(cl(V_\alpha))) \times \prod_{\alpha \neq \beta} X_\beta)$ is a γ-open set. Conse-
quently, we obtain that $F_\alpha^+(int(cl(V_\alpha)))$ (resp. $F_\alpha^-(int(cl(V_\alpha))))$ is a γ-open
set. Thus, we show that F_α is upper (resp. lower) almost γ-continuous
multifunction. $\square$

Theorem 40. *Suppose that (X, τ), (Y, υ), (Z, ω) are topological spaces and*
$F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be
a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each
$x \in X$. If $F_1 \times F_2$ is upper (lower) almost γ-continuous multifunction, then
F_1 and F_2 are upper (resp. lower) almost γ-continuous multifunctions.

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$
and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so
$F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$.
Since $F_1 \times F_2$ is upper almost γ-continuous multifunction, it follows that
there exists a γ-open set U containing x such that $U \subset (F_1 \times F_2)^+(int(cl(K \times H)))$. We obtain that $U \subset F_1^+(int(cl(K)))$ and $U \subset F_2^+(int(cl(H)))$. Thus,
we obtain that F_1 and F_2 are upper almost γ-continuous multifunctions.

The proof of the lower almost γ-continuity of F_1 and F_2 is similar to the
above. $\square$

6. Several properties

In this section, the other several properties of almost γ-continuity
are investigated.

Recall that a multifunction $F : X \to Y$ is said to be punctually connected
if, for each $x \in X$, $F(x)$ is connected.

Definition 41. *A space X is called γ-connected [11] provided that X is not*
the union of two disjoint nonempty γ-open sets.

Theorem 42. *Let F be a multifunction from a γ-connected topological space*
X onto a topological space Y such that F is punctually connected. If F is
upper almost γ-continuous multifunction, then Y is a connected space.

Proof. Let $F : X \to Y$ be a upper almost γ-continuous multifunction from
a γ-connected topological space X onto a topological space Y. Suppose
that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then
both H and K are open and closed subsets of Y. Since F is upper almost
γ-continuous multifunction, $F^+(H)$ and $F^+(K)$ are γ-open subsets of X.
In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is punctually
connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the
γ-connectedness of X. Hence, it is obtained that Y is a connected space. $\square$

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if,
for each $x \in X$, $F(x)$ is closed.

Theorem 43. *Let F be an upper almost γ-continuous punctually closed multifunction and G be an upper almost continuous punctually closed multifunction from a space X to a normal space Y. Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is γ-closed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a normal space, it follows that there exists disjoint open sets U and V containing $F(x)$ and $G(x)$ respectively. Since F and G are upper almost γ-continuous and upper almost continuous, respectively the sets $F^+(int(cl(U)))$ and $G^+(int(cl(V)))$ are γ-open and open, respectively such that contain x. Let $H = F^+(int(cl(U))) \cap G^+(int(cl(V)))$. Then H is a γ-open set containing x and $H \cap K = \emptyset$. Hence, K is γ-closed in X. $\square$

Definition 44. *A space X is said to be γ-T_2 (γ-Hausdorff) if for each pair of distinct points x and y in X, there exist disjoint γ-open sets U and V in X such that $x \in U$ and $y \in V$.*

Theorem 45. *Let $F : X \to Y$ be an upper almost γ-continuous multifunction and punctually closed from a topological space X to a normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a γ-Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a normal space, it follows that there exists disjoints open sets U and V containing $F(x)$ and $F(y)$ respectively. Thus $F^+(int(cl(U)))$ and $F^+(int(cl(V)))$ are disjoint γ-open sets containing x and y respectively. Thus, it is obtained that X is γ-Hausdorff. $\square$

For a multifunction $F : X \to Y$, by $clF : X \to Y$ [5] we denote a multifunction defined as follows: $(clF)(x) = cl(F(x))$ for each point $x \in X$. Similarly, we can define $\gamma - clF : X \to Y$, $s - clF : X \to Y$, $p - clF : X \to Y$ and $\alpha - clF : X \to Y$.

Lemma 46. *If $F : X \to Y$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each regular open set V of Y, $G^+(V) = F^+(V)$, where G denotes $\gamma - clF$, $s - clF$, $p - clF$, $\alpha - clF$ or clF.*

Proof. Let V be any regular open set of Y and $x \in G^+(V)$. Thus $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$.

Conversely, let $x \in F^+(V)$, then $F(x) \subset V$. By Lemma 27, there exists an open set W of Y such that $F(x) \subset W \subset cl(W) \subset V$; hence $G(x) \subset cl(W) \subset V$. Therefore, we have $x \in G^+(V)$ and $F^+(V) \subset G^+(V)$. $\square$

Theorem 47. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is a α-paracompact α-regular for each $x \in X$. Then the following are equivalent:*
  (1) *F is upper almost γ-continuous;*
  (2) *$\gamma - clF$ is upper almost γ-continuous;*
  (3) *$s - clF$ is upper almost γ-continuous;*

(4) $p - clF$ is upper almost γ-continuous;
(5) $\alpha - clF$ is upper almost γ-continuous;
(6) clF is upper almost γ-continuous.

Proof. Take $G = \gamma - clF$, $s - clF$, $p - clF$, $\alpha - clF$ or clF. Suppose that F is upper almost γ-continuous. Let $x \in X$ and V be any regular open set of Y containing $G(x)$. By Lemma 46, we have $x \in G^+(V) = F^+(V)$ and hence there exists $U \in \gamma O(X, x)$ such that $F(U) \subset V$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, by Lemma 27, there exists an open set W such that $F(u) \subset W \subset cl(W) \subset V$; hence $G(u) \subset cl(W) \subset V$ for each $u \in U$. Therefore, we obtain $G(U) \subset V$. This shows that G is upper almost γ-continuous.

Conversely, suppose that G is upper almost γ-continuous. Let $x \in X$ and V be any regular open set of Y containing $F(x)$. By Lemma 46, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in \gamma O(X, x)$ such that $G(U) \subset V$. Therefore, we obtain $U \subset G^+(V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper almost γ-continuous. $\square$

Lemma 48. *If $F : X \to Y$ is a multifunction, then for each regular open set V of Y, $G^-(V) = F^-(V)$, where G denotes $\gamma - clF$, $s - clF$, $p - clF$, $\alpha - clF$ or clF.*

Proof. Let V be any regular open set of Y and $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$ since V is open. Thus, we obtain $x \in F^-(V)$ and hence $G^-(V) \subset F^-(V)$.

Conversely, let $x \in F^-(V)$. Then we have $F(x) \cap V \neq \emptyset$ and $F(x) \cap V \subset G(x) \cap V$ and hence $x \in G^-(V)$. Thus, we have $F^-(V) \subset G^-(V)$.

Consequently, we obtain $G^-(V) = F^-(V)$. $\square$

Theorem 49. *Let $F : X \to Y$ be a multifunction. Then the following are equivalent:*
(1) F is lower almost γ-continuous;
(2) $\gamma - clF$ is lower almost γ-continuous;
(3) $s - clF$ is lower almost γ-continuous;
(4) $p - clF$ is lower almost γ-continuous;
(5) $\alpha - clF$ is lower almost γ-continuous;
(6) clF is lower almost γ-continuous.

Proof. By using Lemma 48, this is shown similarly to that of Theorem 47.
$\square$

Definition 50. *The γ-frontier of a subset A of a space X, denoted by $\gamma - Fr(A)$, is defined by $\gamma - Fr(A) = \gamma - cl(A) \cap \gamma - cl(X \backslash A) = \gamma - cl(A) \backslash \gamma - int(A)$ [2].*

Theorem 51. *The set all points of X at which a multifunction $F : X \to Y$ is not upper almost γ-continuous (lower almost γ-continuous) is identical with the union of the γ-frontier of the upper (lower) inverse images of regular open sets containing (meeting) $F(x)$.*

Proof. Let $x \in X$ at which F is not upper almost γ-continuous. Then there exists a regular open set V of Y containing $F(x)$ such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in \gamma O(X, x)$. Therefore, we have $x \in \gamma - cl(X \backslash F^+(V)) = X \backslash \gamma - int(F^+(V))$ and x$\in F^+(V)$. Thus, we obtain $x \in \gamma - Fr(F^+(V))$.

Conversely, suppose that V is a regular open set of Y containing $F(x)$ such that $x \in \gamma - Fr(F^+(V))$. If F is upper almost γ-continuous at x, then there exists $U \in \gamma O(X, x)$ such that $U \subset F^+(V)$; hence $x \in \gamma - int(F^+(V))$. This is a contradiction and hence F is not upper almost γ-continuous at x.

The case for lower almost γ-continuous is similarly shown. $\qquad \square$

In the following $(D, >)$ is a directed set, (F_λ) is a net of multifunction $F_\lambda : X \to Y$ for every $\lambda \in D$ and F is a multifunction from X into Y.

Definition 52. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from X to Y. A multifunction $F^* : X \to Y$ is defined as follows: for each $x \in X$, $F^*(x) = \{y \in Y : \text{for each open neighborhood } V \text{ of } y \text{ and each } \mu \in D, \text{ there exists } \lambda \in D \text{ such that } \lambda > \mu \text{ and } V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit of the net $(F_\lambda)_{\lambda \in D}$ [6].*

Definition 53. *A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper almost γ-continuous at $x_0 \in X$ if for every open set V_λ containing $F_\lambda(x_0)$, there exists a γ-open set U containing x_0 such that $F_\lambda(U) \subset int(cl(V_\lambda))$ for all $\lambda \in D$.*

Theorem 54. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from a space X into a compact space Y. If the following are satisfied:*
(1) $\bigcup\{F_\mu(x) : \mu > \lambda\}$ is closed in Y for each $\lambda \in D$ and each $x \in X$,
(2) $(F_\lambda)_{\lambda \in D}$ is equally upper almost γ-continuous on X,
then F^* is upper almost γ-continuous on X.

Proof. We have $F^*(x) = \bigcap\{(\bigcup\{F_\mu(x) : \mu > \lambda\}) : \lambda \in D\}$. Since the net $(\bigcup\{F_\mu(x) : \mu > \lambda\})_{\lambda \in D}$ is a family of closed sets having the finite intersection property and Y is compact, $F^*(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let V be a proper open subset of Y such that $F^*(x_0) \subset V$. Since $F^*(x_0) \cap (Y \backslash V) = \emptyset$, $F^*(x_0) \neq \emptyset$ and $Y \backslash V \neq \emptyset$, $\bigcap\{(\bigcup\{F_\mu(x_0) : \mu > \lambda\}) : \lambda \in D\} \cap (Y \backslash V) = \emptyset$ and hence $\bigcap\{(\bigcup\{F_\mu(x_0) \cap (Y \backslash V) : \mu > \lambda\}) : \lambda \in D\} = \emptyset$. Since Y is compact and the family $\{(\bigcup\{F_\mu(x_0) \cap (Y \backslash V) : \mu > \lambda\}) : \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that $F_\mu(x_0) \cap (Y \backslash V) = \emptyset$ for each $\mu \in D$ with $\mu > \lambda$. Since the net $(F_\lambda)_{\lambda \in D}$ is equally upper almost γ-continuous on X, there exists a γ-open set U containing x_0 such that $F_\mu(U) \subset int(cl(V))$ for each $\mu > \lambda$, i. e., $F_\mu(x) \cap (Y \backslash int(cl(V))) = \emptyset$ for each $x \in U$. Then we have $\bigcup\{F_\mu(x) \cap (Y \backslash int(cl(V))) : \mu > \lambda\} = \emptyset$ and hence $\bigcap\{(\bigcup\{F_\mu(x) : \mu > \lambda\}) : \lambda \in D\} \cap (Y \backslash int(cl(V))) = \emptyset$. This implies that $F^*(U) \subset int(cl(V))$. If $V = Y$, then it is clear that for each γ-open set U containing x_0 we have $F^*(U) \subset int(cl(V))$. Hence F^* is upper almost γ-continuous at x_0. Since x_0 is arbitrary, the proof completes. $\qquad \square$

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